ON A POLYNOMIAL SEQUENCE RELATED TO THE DITKIN-PRUDNIKOV PROBLEM

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ABSTRACT. This article spotlights the problem of characterizing a non-orthogonal polynomial sequence that is simultaneously connected to the moments of the modified Bessel function and the Ditkin-Prudnikov weight function. Integral relations with the Bernoulli numbers and the Euler polynomials, along with an analog of the Rodrigues type formula and generating functions, will as well be established.

1. INTRODUCTION AND PRELIMINARY RESULTS

Throughout the text, $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, whereas $\mathbb{R}$ and $\mathbb{C}$ the field of the real and complex numbers, respectively. The notation $\mathbb{R}_+$ corresponds to the set of all positive real numbers. The present investigation is primarily targeted at analysis of sequences of polynomials whose degrees equal its order, which will be shortly called as PS. Whenever the leading coefficient of each of its polynomials equals 1, the PS is said to be a MPS (*monic polynomial sequence*). A PS or a MPS forms a basis of the vector space of polynomials with coefficients in $\mathbb{C}$, here denoted as $\mathscr{P}$. Further notations are introduced as needed.

The operational calculus associated to the differential operator $\frac{d}{dt}$ gives rise to the Laplace transform having the exponential function as a kernel, which we are going to represent in terms of the Mellin-Barnes integral [7, Vol.I]

$$\mathrm{e}^{-x} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma(s) x^{-s} ds\,, \quad x, a > 0.$$

Meanwhile the operator $\frac{d}{dt} t \frac{d}{dt}$ leads to the Meijer transform [26] involving the modified Bessel function (also known as MacDonald's function) $2K_0(2\sqrt{x})$ as a positive kernel given by the formula [7, Vol.II]

$$2K_0(2\sqrt{x}) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma^2(s) x^{-s} ds\,, \quad x, a > 0.$$

In the seminal work of 1966 [5], bearing in mind that e^{-x} is the weight function of the very classical Laguerre polynomials (of parameter 0) [18, 12], Ditkin and Prudnikov posed the problem to find a new sequence of orthogonal polynomials related to the weight $2K_0(2\sqrt{x})$. The problem at issue encompasses the characterization of an Orthogonal Polynomial Sequence (OPS) (or a Monic Orthogonal Polynomial Sequence - MOPS, when $\deg(V_n(x) - x^n) = n-1$ for $n \in \mathbb{N}$ and $V_0(x) = 1$), say $\{V_n\}_{n\geqslant 0}$, such that

$$\int_0^\infty 2K_0(2\sqrt{x}) V_m(x) V_n(x) dx = N_n \delta_{n,m}\,, \quad n, m \in \mathbb{N}_0,$$

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where $\delta_{n,m}$ represents the *Kronecker symbol*, and with $N_n > 0$. Hereafter, we will shortly refer to $\{V_n\}_{n\geqslant 0}$ as the Ditkin-Prudnikov MOPS.

This problem was later on broaden to other ultra-exponential weights (see [14]) and in the sequel, other approaches were considered, largely focused on the investigation of polynomial sequences being either multiple orthogonal [19] or d-orthogonal [3] with respect to these ultra-exponential weights.

However we do not unravel the characterization of this Ditkin-Prudnikov MOPS, in the present work we enlighten the investigation of a MPS $\{P_n\}_{n\geqslant 0}$ that is simultaneously associated to the moments of the MacDonald function of pure imaginary subscript, $K_{i\tau}(2\sqrt{x})$ where $x,\tau > 0$, and to the Ditkin-Prudnikov problem. The function $K_{i\tau}(2\sqrt{x})$ is real valued and can be defined by integrals of Fourier type

$$K_{i\tau}(2\sqrt{x}) = \int_0^\infty \mathrm{e}^{-2\sqrt{x}\cosh(u)} \cos(\tau u) du \ , \ x \in \mathbb{R}_+, \ \tau \in \mathbb{R}. \tag{1.1}$$

In addition, it is an eigenfunction of the operator

$$\mathscr{A} \ = \ x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} - x \ = \ x\frac{d}{dx}x\frac{d}{dx} - x \tag{1.2}$$

insofar as

$$\mathscr{A} K_{i\tau}(2\sqrt{x}) = -\left(\frac{\tau}{2}\right)^2 K_{i\tau}(2\sqrt{x}) \ . \tag{1.3}$$

Entailed in the analysis on the moments of this function, comes out the central factorial numbers of even order (also called as modified Stirling numbers or Jacobi-Stirling numbers), which will be the issue on §2. This triggers the introduction of the MPS $\{P_n\}_{n\geqslant 0}$ whose elements have as coefficients these central factorial numbers, up to a sign. In §3 the arisen MPS will be thoroughly expounded in several aspects, beginning with (in §3.1) the characterization of its elements by means of the integral composite powers of the Bessel operator (1.2), a relation resembling a Rodrigues-type formula, which also compels a differential-difference equation. Later on, in §3.2, we will bring integral representations for this MPS along with the representation via generating function (in §3.3), which will enable us to relate it to the Bernoulli numbers and the Euler polynomials, in §3.4. Behind these procedures, the modified Kontorovich-Lebedev transform is an asset which has $K_{i\tau}(2\sqrt{x})$ as a kernel [23, 26].

The last section §4 is devoted to explain the connection between the MPS $\{P_n(x)\}_{n\geqslant 0}$ and the Ditkin-Prudnikov MOPS $\{V_n(x)\}_{n\geqslant 0}$, which arises while analyzing the canonical element of their dual sequences. Regarding this and for a more clear understanding, we recall a few concepts, of the utmost importance for this goal.

The dual sequence $\{u_n\}_{n\geqslant 0}$ of a given MPS $\{P_n(x)\}_{n\geqslant 0}$, whose elements are called forms (or *linear functionals*) belong to the dual space $\mathscr{P}'$ of $\mathscr{P}$ and are defined according to

$$\langle u_n, P_k\rangle := \delta_{n,k}, \ n,k \geqslant 0,$$

where $\delta_{n,k}$ represents the *Kronecker delta* function. Its first element, u_0, earns the special name of *canonical form* of the MPS. Here, by $\langle u, f\rangle$ we mean the action of $u \in \mathscr{P}'$ over $f \in \mathscr{P}$, but a special notation is given to the action over the elements of the canonical sequence $\{x^n\}_{n\geqslant 0}$ – the *moments of* $u \in \mathscr{P}'$: $(u)_n := \langle u, x^n\rangle, n \geqslant 0$. Any element u of $\mathscr{P}'$ can be written in a series of any dual sequence $\{\mathbf{v}_n\}_{n\geqslant 0}$ of a MPS $\{P_n\}_{n\geqslant 0}$ [11]:

$$u = \sum_{n\geqslant 0} \langle u, P_n\rangle \ u_n \ . \tag{1.4}$$

Differential equations or other kind of linear relations realized by the elements of the dual sequence can be deduced by transposition of those relations fulfilled by the elements of the corresponding MPS, insofar as a linear operator $T : \mathscr{P} \to \mathscr{P}$ has a transpose ${}^tT : \mathscr{P}' \to \mathscr{P}'$ defined by

$$\langle {}^tT(u), f\rangle = \langle u, T(f)\rangle \ , \quad u \in \mathscr{P}', \ f \in \mathscr{P}. \tag{1.5}$$

For example, for any form u and any polynomial g, let $Du = u'$ and gu be the forms defined as usual by $\langle u', f\rangle := -\langle u, f'\rangle$, $\langle gu, f\rangle := \langle u, gf\rangle$, where D is the differential operator [11]. Thus, D on forms is minus the transpose of the differential operator D on polynomials.

The investigation about the orthogonality of a MPS can be performed in a purely algebraic point of view. Precisely, a form $v \in \mathscr{P}'$ is said to be *regular* if we can associate a PS $\{Q_n\}_{n\geqslant 0}$ such that $\langle v, Q_n Q_m\rangle = k_n \delta_{n,m}$ with $k_n \neq 0$ for all $n,m \in \mathbb{N}_0$ [11, 12]. The PS $\{Q_n\}_{n\geqslant 0}$ is then said to be orthogonal with respect to v and we can assume the system (of orthogonal polynomials) to be monic. Therefore, there exists a dual sequence $\{v_n\}_{n\geqslant 0}$ and the original form is proportional to v_0. The remaining elements of the corresponding dual form can be represented by

$$v_{n+1} = \left(\langle v_0, Q_{n+1}^2(\cdot)\rangle\right)^{-1} Q_{n+1}(x) v_0 \ , \ n \in \mathbb{N}_0. \tag{1.6}$$

When $v \in \mathscr{P}'$ is regular, let Φ be a polynomial such that $\Phi v = 0$, then $\Phi = 0$ [11, 12].

This unique MOPS $\{Q_n(x)\}_{n\geqslant 0}$ with respect to the regular form v can be characterized by the popular second order recurrence relation

$$\begin{cases} Q_0(x) = 1 \quad ; \quad Q_1(x) = x - \beta_0 \\ Q_{n+2}(x) = (x - \beta_{n+1}) Q_{n+1}(x) - \gamma_{n+1}\, Q_n(x) \ , \quad n \in \mathbb{N}_0. \end{cases} \tag{1.7}$$

As it will be concluded in the last section 4 the MPS $\{P_n(x)\}_{n\geqslant 0}$, despite non-orthogonal, share with the Ditkin-Prudnikov MOPS $\{V_n(x)\}_{n\geqslant 0}$ the same canonical form. An analogous relation occurs between the Bernoulli polynomials, which also happen to be non-orthogonal, and the (orthogonal) Legendre polynomials [13].

2. On the moment sequence of $K_{i\tau}(2\sqrt{x})$

When evaluating the Mellin transform of the function $K_{i\tau}(2\sqrt{x})$ at positive integer values (*i.e.*, the moments of this function), the output is a product of an elementary function by a polynomial whose degree is exactly the order of the moment increased by one unity. To be more specific, for positive real values of τ, we recall relation (2.16.2.2) in [15]

$$\int_0^\infty K_{i\tau}(2\sqrt{x}) x^n dx = \frac{\pi}{\tau \sinh(\pi\tau/2)} \prod_{\sigma=0}^{n} \left(\sigma^2 + \frac{\tau^2}{4}\right) \ , \ n \in \mathbb{N}_0, \tag{2.1}$$

while as $\tau \to 0$, the moments become much more simpler

$$\int_0^\infty K_0(2\sqrt{x}) x^n dx = \frac{1}{2}\,(n!)^2 \ . \tag{2.2}$$

By means of the central factorial numbers of even orders, expounded in the book [17, Ch. 6], we will bring to analysis a MPS, denoted by $\{P_n\}_{n\in\mathbb{N}_0}$, that is in deep connection with the moments of $K_{i\tau}(2\sqrt{x})$.

Thus we consider the polynomials in the variable $z = \tau^2/4$ of degree k, to what we call as the *k-th order modified rising factorial* of the variable z, denoted by $[z]_k$ and defined by

$$[z]_k := \begin{cases} 1 & if \quad k = 0, \\ \displaystyle\prod_{\nu=0}^{k-1} (z + \nu^2) & if \quad k \in \mathbb{N}\ . \end{cases} \tag{2.3}$$

The integral relation (2.1) may now be re-expressed as follows

$$\int_0^\infty K_{i\tau}(2\sqrt{x}) x^n dx = \frac{\pi}{\tau \sinh(\pi\tau/2)} [\tau^2/4]_{n+1} \quad , \quad n \in \mathbb{N}_0. \tag{2.4}$$

Just like the set of the standard rising factorials (also commonly called as *Pochammer*) $\{(z)_k\}_{n\geqslant 0}$, the set of polynomials $\{[z]_k\}_{n\geqslant 0}$ forms a basis of $\mathscr{P}$. Notwithstanding their resemblance, they are considerably different specially in what concerns the connection coefficients between these two MPS and the canonical sequence $\{z^k\}_{n\geqslant 0}$. The pair of Stirling numbers (of first and second kind), widely known, is the bridge between $\{(z)_k\}_{n\geqslant 0}$ and $\{z^k\}_{n\geqslant 0}$. To this purpose we refer to [4, Ch.V][16, 17], not disregarding the extensive list of work within this matter. Indeed, the connection coefficients, *i.e.* the set of numbers permitting to pass from one basis to the other, between $\{[z]_k\}_{n\geqslant 0}$ and $\{z^k\}_{n\geqslant 0}$ mimics the pair of Stirling numbers, as they also fulfill a triangular relation of the same type but with slight differences. These coefficients are a specialization of a wider class of coefficients, depending on a complex parameter α, that were considered in [10] but also in [8]. They arose in the expansion of the integral composite powers of a second order differential operator having the classical polynomials of Jacobi as eigenfunctions and for this reason the authors in [8] decided to denominate them as the pair of *Jacobi-Stirling numbers*. However, as pointed out in [10], such numbers also perform an analogous situation for the classical Bessel polynomials, so the author in [10] dubbed this pair "α-modified stirling numbers".

Regarding the problem we are handling, the choice of $\alpha = 0$ is required (which, in the light of [8], we should be dealing with the *Tchebyshev of first kind Stirling numbers*). Following the notation carried in [10], we will denote this pair of connection coefficients by $(\widehat{s}_0(n,k), \widehat{S}_0(n,k))$. To set in concrete what we have just described, they realize the (inverse) relations

$$[z]_k = \sum_{\nu=0}^{k} (-1)^{k+\nu} \widehat{s}_0(k,\nu)\, z^\nu \;,\; k \in \mathbb{N}, \tag{2.5}$$

$$z^k = \sum_{\nu=0}^{k} (-1)^{k+\nu} \widehat{S}_0(k,\nu)\, [z]_\nu \;,\; k \in \mathbb{N}, \tag{2.6}$$

which naturally compel the pair $(\widehat{s}_0(n,k), \widehat{S}_0(n,k))$ to fulfill the triangular relations

$$\begin{cases} \widehat{s}_0(k+1,\nu+1) = \widehat{s}_0(k,\nu) - k^2\, \widehat{s}_0(k,\nu+1)\;, \\ \widehat{s}_0(k,0) = \widehat{s}_0(0,k) = \delta_{k,0} \quad , \quad \widehat{s}_0(k,\nu) = 0\;,\; \nu \geqslant k+1\;, \end{cases} \tag{2.7}$$

and

$$\begin{cases} \widehat{S}_0(k+1,\nu+1) = \widehat{S}_0(k,\nu) + (\nu+1)^2 \widehat{S}_0(k,\nu+1)\;, \\ \widehat{S}_0(k,0) = \widehat{S}_0(0,k) = \delta_{k,0} \quad , \quad \widehat{S}_0(k,\nu) = 0\;,\; \nu \geqslant k+1\;, \end{cases} \tag{2.8}$$

for $k,\nu \in \mathbb{N}_0$. A closed form expression for the latter set of numbers was also revealed [10], namely,

$$\widehat{S}_0(n,k) := T(2n,2k) = \sum_{\sigma=1}^{k} \frac{2(-1)^{k+\sigma}\sigma^{2n}}{(k+\sigma)!(k-\sigma)!}\;, \quad 0 \leqslant k \leqslant n. \tag{2.9}$$

Actually, the set of these 0-modified Stirling numbers is a subset of the *central factorial numbers*: a thorough account on this matter may be followed in [17, pp. 212-216], not disregarding the comments pointed out in [9] (specially Theorem 1 therein). They are indeed the central factorial numbers of even indices, namely $\widehat{s}_0(n,k) = t(2n,2k)$ while $\widehat{S}_0(n,k)) = T(2n,2k)$, where $t(n,k)$ and $T(n,k)$ are, respectively, the central factorial number of first and second kind. Another relation between the central factorial numbers and the standard Stirling numbers of second kind, $S(n,k)$, can be reached in [1, p.824][17, (30), p. 216], namely

$$\widehat{S}_0(n,k) := T(2n,2k) = \sum_{\sigma=1}^{n-k} \binom{2n}{\sigma} (-2)^\sigma S(2n-\sigma,2k) k^\sigma \;,\; 0 \leqslant k \leqslant n \;,\; n,k \in \mathbb{N}_0.$$

The Jacobi-Stirling numbers have received a combinatorial interpretation, first for the specialization of the value $\alpha = 1$ – the so called Legendre-Stirling numbers – taken in [2], and later on, to the wider class given in [9], with the pair $(\widehat{s}_0(n,k), \widehat{S}_0(n,k))$ included.

3. A MPS connected to the moments of the modified Macdonald function

Bearing these arguments in mind, we are able to thoroughly expound the polynomial sequence $\{P_n\}_{n\in\mathbb{N}_0}$ given by

$$P_n(x) = \sum_{\nu=0}^{n} (-1)^{n-k}\, \widehat{S}_0(n+1,k+1)\, x^k \ , \quad n \in \mathbb{N}_0, \tag{3.1}$$

not (regularly) orthogonal, as we will conclude later on, but whose corresponding canonical form is the one of the Ditkin-Prudnikov MOPS, ergo regular.

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x-1\\
P_2(x) &= x^2-5x+1\\
P_3(x) &= x^3-14x^2+21x-1\\
P_4(x) &= x^4-30x^3+147x^2-85x+1\\
P_5(x) &= x^5-55x^4+627x^3-1408x^2+341x-1\\
P_6(x) &= x^6-91x^5+2002x^4-11440x^3+13013x^2-1365x+1\\
P_7(x) &= x^7-140x^6+5278x^5-61490x^4+196053x^3-118482x^2+5461x-1\\
P_8(x) &= x^8-204x^7+12138x^6-251498x^5+1733303x^4-3255330x^3+1071799x^2-21845x+1\\
P_9(x) &= x^9-285x^8+25194x^7-846260x^6+10787231x^5-46587905x^4+53157079x^3-9668036x^2+87381x-1\\
P_{10}(x) &= x^{10}-385x^9+48279x^8-2458676x^7+52253971x^6-434928221x^5+1217854704x^4-860181300x^3+87099705x^2-349525x+1
\end{aligned}$$

TABLE 1. List of the first elements of the MPS $\{P_n\}_{n\in\mathbb{N}_0}$.

Meanwhile, the inverse relation of (3.1) can be achieved based on the properties of the modified Stirling pair of numbers, namely, on account of (2.5)-(2.6), we obtain

$$x^n = \sum_{k=0}^{n} (-1)^{n+k} \widehat{s}_0(n+1,k+1) P_k(x) \ , \quad n \in \mathbb{N}_0. \tag{3.2}$$

Remark 3.1. The analysis of the integrals $\int_0^\infty K_{i\tau}(2\sqrt{x})x^{n+1/2}dx$, $n \in \mathbb{N}_0$, instead of (2.1), would then trigger the study of a MPS $\{\widehat{P}_n\}_{n\geqslant 0}$ similar to $\{P_n\}_{n\geqslant 0}$ but with the central factorial numbers of odd order playing the role of the ones of even order. Naturally this gives rise to other developments. We will defer this to a further work, where the analysis of the functions resulting from the Mellin transform of $K_{i\tau}(2\sqrt{x})$, will be the foremost investigation issue.

3.1. **Representation by means of integral composite powers of the modified Bessel operator.** The elements of the MPS $\{P_n\}_{n\in\mathbb{N}_0}$ that we are dealing with are actually represented by the action of the integral composite powers of the operator $\mathscr{A}$ in (1.2) acting over x, after a division by x.

Lemma 3.2. *The MPS $\{P_n\}_{n\geqslant 0}$ whose elements are defined in* (3.1) *can be equivalently represented by*

$$P_n(x) = (-1)^n \frac{1}{x} \mathscr{A}^n x^n \quad , \quad n \in \mathbb{N}_0. \tag{3.3}$$

which amounts to the same as

$$P_{n+1}(x) = -x^2 P_n''(x) - 3xP_n'(x) - (1-x)P_n(x) \ , \quad n \geqslant 0. \tag{3.4}$$

with $P_0(x) = 1$.

Proof. We begin by showing that the elements of the MPS $\{P_n\}_{n\geqslant 0}$ defined by (3.1) can be represented by (3.3) to afterwards prove that they necessarily realize (3.4), which, in turn implies (3.1).

Let $\{P_n\}_{n\geqslant 0}$ be given by (3.1). Clearly, (3.3) is fulfilled for $n = 0,1,2$. Owing to the relation on the modified Stirling numbers (2.8), by induction, we prove that (3.3) holds for any $n \geqslant 0$. Indeed, we successively have

$$\begin{aligned} P_{n+1}(x) &= \sum_{k=0}^{n+1} (-1)^{n+k+1} \widehat{S}_0(n+2,k+1)\, x^k \\ &= \sum_{k=0}^{n+1} (-1)^{n+k+1} \left(\widehat{S}_0(n+1,k) + (k+1)^2 \widehat{S}_0(n+1,k+1)\right) x^k \\ &= \sum_{k=0}^{n} (-1)^{n+k} \widehat{S}_0(n+1,k+1) x^{k+1} - \sum_{k=0}^{n} (-1)^{n+k} \widehat{S}_0(n+1,k+1) \left(\frac{d}{dx} x \frac{d}{dx} x\right) x^k, \quad n \in \mathbb{N}_0, \end{aligned}$$

because $\widehat{S}_0(n+1,0) = \widehat{S}_0(n+1,n+2) = 0$. Now, under the assumption we deduce from the latter equalities

$$P_{n+1}(x) = \left(x - \frac{d}{dx} x \frac{d}{dx} x\right)\left((-1)^n \frac{1}{x} \mathscr{A}^n x\right) = (-1)^{n+1} \frac{1}{x}\left(x - x\frac{d}{dx} x \frac{d}{dx}\right) \mathscr{A}^n x = (-1)^{n+1} \frac{1}{x} \mathscr{A}^{n+1} x, \quad n \in \mathbb{N}_0,$$

whence the conclusion.

The relation (3.4) is a consequence of (3.3) regarding the successive identities

$$P_{n+1}(x) \;=\; (-1)^{n+1} \frac{1}{x} \mathscr{A}\; x\; \frac{1}{x}\; \mathscr{A}^n x = -\frac{1}{x} \mathscr{A}\left(xP_n(x)\right) = -x^2 P_n''(x) - 3xP_n'(x) - (1-x)P_n(x) \ , \ n \geqslant 0,$$

By equating the first and last members, we obtain (3.4).

At last, if a polynomial sequence $\{P_n\}_{n\geqslant 0}$, with $P_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$, realizes the condition (3.4) with $P_0(x) = 1$, then it is necessarily a MPS and its coefficients fulfill the relation

$$c_{n+1,k} = c_{n,k-1} - (k+1)^2 c_{n,k} \quad , \quad 0 \leqslant k \leqslant n, \quad n,k \in \mathbb{N}_0,$$

where $c_{n,-1} = c_{n,k} = 0$ whenever $k > n$ and with the initial conditions $c_{n,0} = P_n(0) = (-1)^n$ for $n \in \mathbb{N}$ and $c_{n,n} = 1$ for $n \in \mathbb{N}_0$. So necessarily we have $c_{n,k} = (-1)^{n-k} \widehat{S}_0(n+1,k+1)$ for any $n,k \in \mathbb{N}_0$. □

3.2. The Kontorovich-Lebedev transform of the MPS.

A sharp instrument for the forthcoming results is the modified Kontorovich-Lebedev transform given by the formula

$$F(\tau) = \sinh \frac{\pi\tau}{2} \int_0^\infty K_{i\tau}(2\sqrt{x}) f(x) dx \ , \tag{3.5}$$

which has an inverse defined through

$$x \, \frac{f(x-0)+f(x+0)}{2} = \frac{2}{\pi^2} \lim_{\lambda \to \frac{\pi}{2}-} \int_0^\infty \tau \cosh(\lambda\tau) K_{i\tau}(2\sqrt{x}) F(\tau) d\tau \tag{3.6}$$

being valid for $f \in L_1\left(\mathbb{R}_+, K_0(2\mu\sqrt{x})dx\right)$, $0 < \mu < 1$, in a neighborhood of each $x \in \mathbb{R}_+$ where $f(x)$ has bounded variation. For a thorough account on this matter, we refer to [26, Th. 6.3].

Besides, $K_\nu(2\sqrt{x})$ reveals the asymptotic behaviour with respect to x [7, Vol. II][24]

$$K_\nu(2\sqrt{x}) = \frac{\sqrt{\pi}}{2\,x^{1/4}}\mathrm{e}^{-2\sqrt{x}}[1+O(1/\sqrt{x})], \quad x\to+\infty, \tag{3.7}$$

$$K_\nu(2\sqrt{x}) = O(x^{-\Re(\nu)/2})\ ,\ K_0(2\sqrt{x}) = O(\log x)\ , \quad x\to 0. \tag{3.8}$$

Lemma 3.3. *The MPS* $\{P_n\}_{n\geqslant 0}$ (3.1) *fulfills*

$$\int_0^\infty K_{i\tau}(2\sqrt{x})P_n(x)dx = \frac{\pi}{2\sinh(\pi\tau/2)}\left(\frac{\tau}{2}\right)^{2n+1}\ ,\ n\in\mathbb{N}_0\ , \tag{3.9}$$

while

$$xP_n(x) = \lim_{\lambda\to\frac{\pi}{2}-}\frac{2}{\pi}\int_0^\infty K_{i\tau}(2\sqrt{x})\cosh(\lambda\tau)\left(\frac{\tau}{2}\right)^{2n+2}d\tau\ , \quad n\in\mathbb{N}_0. \tag{3.10}$$

Proof. The modified Kontorovich-Lebedev transform of each polynomial P_n may be easily computed via the moments of $K_{i\tau}(2\sqrt{x})$ given in (2.4) and, afterwards by taking into account (2.6). This procedure provides (3.9).

On the other hand, via the inversion relation (3.6) of the Kontorovich-Lebedev transform applied to (3.9), we derive (3.10), taking into account that any polynomial plainly belongs to $L_1(\mathbb{R}_+, K_0(2\mu\sqrt{x})dx)$ with $0<\mu<1$. □

Remark 3.4. Following a similar procedure of the one taken in [23], we come out with

$$\int_0^\infty u(x)\Big(\mathscr{A}\varphi(x)\Big)\frac{dx}{x} = \int_0^\infty\Big(\mathscr{A}u(x)\Big)\varphi(x)\frac{dx}{x}.$$

Therefore, from (3.3), we can successively write

$$\int_0^\infty K_{i\tau}(2\sqrt{x})P_n(x)dx = (-1)^n\int_0^\infty K_{i\tau}(2\sqrt{x})\Big(\mathscr{A}^{n+1}1\Big)\frac{dx}{x} = \left(\frac{\tau}{2}\right)^{2n}\int_0^\infty K_{i\tau}(2\sqrt{x})dx \quad , \quad n\in\mathbb{N}_0,$$

because of (1.3). Appealing now to (2.1) with $n=0$, we recover relation (3.9).

Lemma 3.5. *The polynomial sequence* $\{P_n\}_{n\geqslant 0}$ *can be represented by*

$$2^{2n+2}xP_n(x) = \lim_{\lambda\to\pi/2-}\frac{\partial^{2n+2}}{\partial\lambda^{2n+2}}\mathrm{e}^{-2\sqrt{x}\cos(\lambda/2)}\ , \quad n\in\mathbb{N}_0. \tag{3.11}$$

Proof. Considering that

$$\cosh(\lambda\tau)\left(\frac{\tau}{2}\right)^{2n+2} = 2^{-(2n+2)}\frac{\partial^{2n+2}}{\partial\lambda^{2n+2}}\cosh(\lambda\tau),$$

it is reasonable to rewrite (3.10) as follows

$$xP_n(x) = 2^{-(2n+2)}\lim_{\lambda\to\frac{\pi}{2}-}\frac{2}{\pi}\frac{\partial^{2n+2}}{\partial\lambda^{2n+2}}\int_0^\infty K_{i\tau}(2\sqrt{x})\cosh(\lambda\tau)d\tau \tag{3.12}$$

motivated by the absolute and uniform convergence by $\lambda\in[0,\pi/2-\varepsilon]$, for a small positive ε. Meanwhile, from (1.1) and the inversion formula of the cosine Fourier transform, we deduce

$$\frac{2}{\pi}\int_0^\infty K_{i\tau}(2\sqrt{x})\cosh(\lambda\tau)d\tau = \mathrm{e}^{-2\sqrt{x}\cos(\lambda)}\ , \quad x>0,$$

permitting to attain (3.11) from (3.12). □

3.3. **The generating function.** As aforementioned, the coefficients of the elements of the MPS $\{P_n(x)\}_{n\geqslant 0}$ are essentially (up to a sign) the even order central factorial numbers (of second kind), $T(2n+2,2k+2)$, which were treated in [17, Ch. 6]. Therein we may as well read that the polynomials $T_n(x)=\sum_{k=0}^{n} T(n,k)x^k$ are generated by the function $f(x,u)=\mathrm{e}^{2x\sinh(u/2)}$ [17, Ch. 6, p.214], so that we may write

$$\mathrm{e}^{2x\sinh(u/2)}=\sum_{n\geqslant 0} T_n(x)\frac{u^n}{n!}. \tag{3.13}$$

Inasmuch as, $T(2n+2,2k+1)=T(2n+2,0)=T(2n+1,2k)=0$, $n,k\in\mathbb{N}_0$, after a few steps of computations we deduce that

$$T_{2n+2}(i\sqrt{x})=(-1)^{n+1}xP_n(x)\ ,\ n\in\mathbb{N}_0, \tag{3.14}$$

whereas

$$T_{2n+1}(i\sqrt{x})=i\sqrt{x}\sum_{k=0}^{n} T(2n+1,2k+1)(-1)^k x^k\ ,\ n\in\mathbb{N}_0. \tag{3.15}$$

The three relations (3.13)-(3.15) readily provide

$$-\frac{\partial^2}{\partial u^2}\mathrm{e}^{2i\sqrt{x}\sinh(u/2)}=\sum_{n\geqslant 0}(-1)^n\, xP_n(x)\frac{u^{2n}}{(2n)!}-i\sqrt{x}\sum_{n\geqslant 0}\left(\sum_{k=0}^{n+1}T(2n+3,2k+1)(-1)^k x^k\right)\frac{u^{2n+1}}{(2n+1)!}\ .$$

Thus, a generating function for the PS $\{(-1)^nP_n(x)\}_{n\geqslant 0}$ comes out:

$$\frac{\partial^2}{\partial u^2}\frac{-1}{x}\cos(2\sqrt{x}\sinh u/2)=\sum_{n\geqslant 0}(-1)^nP_n(x)\frac{u^{2n}}{(2n)!}. \tag{3.16}$$

Remark 3.6. Actually,by taking into account

$$\mathscr{A}_x\cos(2\sqrt{x}\sinh u/2)=\frac{\partial^2}{\partial u^2}\cos(2\sqrt{x}\sinh u/2)\quad ;\qquad \lim_{u\to 0}\frac{\partial^2}{\partial u^2}\cos(2\sqrt{x}\sinh u/2)=x,$$

where $\mathscr{A}_x=x^2\frac{\partial^2}{\partial x^2}+x\frac{\partial}{\partial x}-x$, and then using (3.3) as a key ingredient, we can recover (3.16).

3.4. **Relationship with the Genocchi numbers and the Euler polynomials.** The even order Genocchi numbers, G_{2n} [4, p.49] are the coefficients in the Taylor series expansion of the function $\frac{2t}{\mathrm{e}^t+1}$. Apart from their definition via the even order Bernoulli numbers B_{2n}, namely

$$G_{2n}=\left(2^{2n}-1\right)B_{2n}$$

they admit as well the integral representation [7, Vol.I]

$$G_{2n+2}=(-1)^n(n+1)\int_0^\infty\frac{1}{\sinh(\pi\tau/2)}\left(\frac{\tau}{2}\right)^{2n+1}d\tau\ ,\ n\in\mathbb{N}_0. \tag{3.17}$$

This representation endows an integral meaning for the elements of the MPS $\{P_n\}_{n\geqslant 0}$.

Corollary 3.7. *The MPS $\{P_n\}_{n\geqslant 0}$ is connected with the Genochi numbers of even order via*

$$\frac{\pi(-1)^n}{2n+2}G_{2n+2}=\int_0^\infty \mathrm{e}^{-2\sqrt{x}}P_n(x)dx\ ,\quad n\in\mathbb{N}_0. \tag{3.18}$$

Proof. We integrate both sides of (3.9) over $\mathbb{R}_+$ by τ and we change the order of integration in the left hand-side of the equality, according to Fubini's theorem and on the grounds of the inequality [20]

$$\left|\frac{\partial^m K_{i\tau}(x)}{\partial x^m}\right| \leqslant \mathrm{e}^{-\delta\tau} K_m(x\cos\delta), \quad x>0,\ \tau>0,\ m\in\mathbb{N}_0 \tag{3.19}$$

with $\delta\in(0,\pi/2)$ and $m=0$. By virtue of the relation

$$\int_0^\infty K_{i\tau}(2\sqrt{x})d\tau = \frac{\pi}{2}\mathrm{e}^{-2\sqrt{x}}$$

and on account of (3.17) the identity (3.18) holds. □

Furthermore, we can also deduce a riveting integral relationship between $\{P_n\}_{n\geqslant 0}$ and the well known Euler polynomials, $E_n(x)$, commonly defined by [1, (23.1.1)][4, 6]

$$\frac{2\mathrm{e}^{tx}}{\mathrm{e}^t+1} = \sum_{n\geqslant 0} E_n(x)\frac{t^n}{n!}\ .$$

Lemma 3.8. *The MPS $\{P_n\}_{n\geqslant 0}$ and the subsequence of even order of the Euler polynomials $\{E_{2n}(x)\}_{n\geqslant 0}$ satisfy the following formulas*

$$E_{2n+2}\left(\frac{1-i\tau}{2}\right) = \frac{2}{\pi}(-1)^n\cosh(\pi\tau/2)\int_0^\infty \sqrt{x}P_n(x)\ K_{i\tau}(2\sqrt{x})dx\ , \quad n\in\mathbb{N}_0, \tag{3.20}$$

and

$$x^{3/2}P_n(x) = \frac{(-1)^{n+1}}{\pi}\lim_{\lambda\to\frac{\pi}{2}-}\int_0^\infty \cosh(\lambda\tau)\tanh\left(\frac{\pi\tau}{2}\right)\ E_{2n+2}\left(\frac{1-i\tau}{2}\right)K_{i\tau}(2\sqrt{x})d\tau\ ,\ n\in\mathbb{N}_0. \tag{3.21}$$

Proof. The even order Euler polynomials admit the integral representation [7, p. 43, Vol.I][6, (24.7.9)]

$$\begin{aligned} E_{2n}\left(-\tfrac{y}{2}\right) &= 2(-1)^{n+1}\sin(\pi y/2)\int_0^\infty \frac{(\tau/2)^{2n}\cosh(\pi\tau/2)}{\cosh(\pi\tau)-\cos(\pi y)}d\tau \\ &= 2(-1)^{n+1}\sin(\pi y/2)\lim_{\lambda\to\pi/2^-}\int_0^\infty \frac{(\tau/2)^{2n}\cosh(\lambda\tau)}{\cosh(\pi\tau)-\cos(\pi y)}d\tau\ ,\ n\in\mathbb{N}_0\ ,\ -2<\Re(y)<0, \end{aligned}$$

which can be transformed into

$$E_{2n}\left(-\frac{y}{2}\right) = \frac{4}{\pi^2}(-1)^{n+1}\sin(\pi y/2)\lim_{\lambda\to\pi/2^-}\int_0^\infty\int_0^\infty (\tau/2)^{2n}\cosh(\lambda\tau)K_{i\tau}(2\sqrt{x})K_{y+1}(2\sqrt{x})\frac{dx}{\sqrt{x}}d\tau \tag{3.22}$$

owing to the relation (2.16.33.2) in [15, Vol.II] followed by a change of variable. Precisely, we obtain

$$\frac{1}{\cosh(\pi\tau)-\cos(\pi y)} = \frac{2}{\pi^2}\int_0^\infty K_{i\tau}(2\sqrt{x})K_{y+1}(2\sqrt{x})\frac{dx}{\sqrt{x}}\ , \quad -2<\Re(y)<0.$$

The absolute and uniform convergence by $\lambda\in[0,\pi/2-\varepsilon]$, for small positive ε, of the inner integral with respect to x in (3.22), justify the equality

$$\begin{aligned} E_{2n}\left(-\frac{y}{2}\right) &= \frac{4}{\pi^2}(-1)^{n+1}\sin\left(\frac{\pi y}{2}\right)2^{-2n-2} \\ &\times\lim_{\lambda\to\pi/2^-}\frac{\partial^{2n+2}}{\partial\lambda^{2n+2}}\int_0^\infty\int_0^\infty \cosh(\lambda\tau)K_{i\tau}(2\sqrt{x})K_{y+1}(2\sqrt{x})\frac{dx}{\sqrt{x}}d\tau\ ,\ n\in\mathbb{N}_0. \end{aligned} \tag{3.23}$$

In the light of Fubini's theorem, we change the order of integration in (3.23). The equality [24]

$$\frac{2}{\pi}\int_0^\infty \cosh(\lambda\tau)K_{i\tau}(2\sqrt{x})d\tau = \mathrm{e}^{-2\sqrt{x}\cos(\lambda/2)}$$

(*i.e.*, the inverse relation of (1.1)) provides (3.23) to become

$$E_{2n}\left(-\frac{y}{2}\right)=\frac{2}{\pi}(-1)^{n+1}\sin\left(\frac{\pi y}{2}\right)2^{-2n-2}\lim_{\lambda\to\pi/2^-}\frac{\partial^{2n+2}}{\partial\lambda^{2n+2}}\int_0^\infty e^{-2\sqrt{x}\cos(\lambda/2)}K_{y+1}(2\sqrt{x})\frac{dx}{\sqrt{x}}. \tag{3.24}$$

Moreover, it is straightforward to verify that $\left|\frac{\partial^{2n+2}}{\partial\lambda^{2n+2}}e^{-2\sqrt{x}\cos(\lambda/2)}\right|\leqslant C_n x^n$ for $x>0$ and $\lambda\in[0,\pi/2)$, where C_n depends exclusively on $n\in\mathbb{N}_0$. This argument combined with the fact that $\left|K_{y+1}(2\sqrt{x})\right|\leqslant K_{\Re(y)+1}(2\sqrt{x})$ and

$$\int_0^\infty K_{\Re(y)+1}(2\sqrt{x})x^n dx<+\infty,$$

gives grounds for the passage of the limit under the outer integral in (3.24). Appealing to (3.11) and as long as $-2<\Re y<0$, we derive

$$E_{2n}\left(-\frac{y}{2}\right)=\frac{2}{\pi}(-1)^{n+1}\sin(\pi y/2)\int_0^\infty\sqrt{x}P_{n-1}(x)\;K_{y+1}(2\sqrt{x})dx\;,\;n\in\mathbb{N},$$

which yields (3.20) after taking $y=-1+i\tau$.

Within the scope of the Kontorovich-Lebedev transform (according to (3.5)-(3.6)), it is possible to invert (3.20), which results in (3.21). □

Remark 3.9. It is worth to compare (3.21) with the representation given in (3.10).

4. The dual sequence and the Ditkin-Prudnikov problem

The properties of the MPS $\{P_n\}_{n\geqslant0}$ trigger those of the corresponding dual sequence. Basically, after analyzing them we will bring to light the connection between this MPS (non-orthogonal) $\{P_n\}_{n\geqslant0}$ and the Ditkin-Pridnikov orthogonal polynomial sequence: they share the same canonical form.

Lemma 4.1. *The elements of* $\{u_n\}_{n\geqslant0}$ *are solution of the following differential equations*

$$(x^2u_0)''-3(xu_0)'+(1-x)u_0=0, \tag{4.1}$$

$$(x^2u_{n+1})''-3(xu_{n+1})'+(1-x)u_{n+1}=-u_n\;,\;n\geqslant0\;. \tag{4.2}$$

Moreover, the moments of u_0 *are* $(u_0)_n=(n!)^2$ *for* , $n\in\mathbb{N}_0$.

Upon rearrangement, the relations (4.1)-(4.2) can be written

$$xDxDu_0-x\,u_0=0,$$
$$xDxD\,u_{n+1}-x\,u_{n+1}=-u_n\;,\;n\geqslant0\;.$$

Proof. The action of u_0 over (3.4) is given by

$$\langle u_0,x^2P_n''(x)+3xP_n'(x)+(1-x)P_n(x)\rangle=0\;,\;n\in\mathbb{N}_0,$$

which, by transposition, on account of (1.5), is equivalent to

$$\langle\big(x^2u_0\big)''-3\big(xu_0\big)'+(1-x)u_0,P_n\rangle=0\;,\;n\in\mathbb{N}_0,$$

providing (4.1). Likewise, the action of u_{k+1} over (3.4) yields

$$\langle u_{k+1},x^2P_n''(x)+3xP_n'(x)+(1-x)P_n(x)\rangle=-\delta_{n,k}\;,\;n,k\in\mathbb{N}_0,$$

and, due to (1.5), we may write this latter as

$$\langle\big(x^2u_{k+1}\big)''-3\big(xu_{k+1}\big)'+(1-x)u_{k+1},P_n\rangle=-\delta_{n,k}\;,\;n,k\in\mathbb{N}_0.$$

By virtue of (1.4), the relation (4.2) is then a consequence of this latter equality.

The action of both sides of (4.1) over the sequence $\{x^n\}_{n\geqslant 0}$ permits to obtain the relation for the moments of u_0, since we have

$$(u_0)_{n+1} = (n+1)^2 (u_0)_n \ , \ n \in \mathbb{N}_0$$

and thereby $(u_0)_n = (n!)^2 (u_0)_0 = (n!)^2$ for , $n \in \mathbb{N}_0$. □

Remark 4.2. Any MPS $\{B_n\}_{n\geqslant 0}$ such that $B_0 = 1$ and $B_{n+1}(0) = 0$ has the Dirac delta δ as canonical form. For this reason, the sequence $\{B_n(x) := \sum_{k=0}^{n} \widehat{S}_0(n,k) x^k\}_{n\geqslant 0}$ has δ as its canonical form.

Corollary 4.3. *The sequence of the moments of the elements of the dual sequence $\{(u_n)_k\}_{0\leqslant n\leqslant k}$, coincides with the sequence of the modified Stirling numbers of first kind, namely*

$$(u_n)_k = (-1)^{n+k} \widehat{s}_0(k+1, n+1)$$

with $(u_0)_k = (k!)^2$ for any $k \in \mathbb{N}_0$.

Proof. Regarding the definition of a dual sequence it is clear that $(u_n)_k = 0$, whenever $0 \leqslant k < n$. By virtue of (3.2), as long as $k \geqslant n \geqslant 0$, it follows

$$(u_n)_k = \sum_{\nu=0}^{k} (-1)^{k+\nu} \widehat{s}_0(k+1, \nu+1) \langle u_n, P_k \rangle = \sum_{\nu=0}^{k} (-1)^{k+\nu} \widehat{s}_0(k+1, \nu+1) \delta_{n,\nu}$$

whence the result. □

It is now to clarify whether or not the MPS $\{P_n(x)\}_{n\geqslant 0}$ is orthogonal. Somehow expected, we have the following result.

Lemma 4.4. *The MPS $\{P_n(x)\}_{n\geqslant 0}$ cannot be regularly orthogonal.*

Proof. Under the assumption of the orthogonality of the MPS $\{P_n(x)\}_{n\geqslant 0}$, the elements of the corresponding dual sequence can be written as a product of P_n by the canonical form u_0, according to (1.6), and (4.2) is then given as well by

$$\Big(x^2 P_{n+1} u_0\Big)'' - 3\Big(x P_{n+1} u_0\Big)' + (1-x) P_{n+1} u_0 = -\gamma_{n+1} P_n u_0 \ , \ n \in \mathbb{N}_0,$$

with $\gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2 \rangle}{\langle u_0, P_n^2 \rangle} \neq 0$. We collect it in terms of the derivatives of P_{n+1}, that is,

$$P_{n+1}\Big((x^2 u_0)'' - 3(x u_0)' + (1-x) u_0\Big) + P'_{n+1}\Big(2(x^2 u_0)' - 3x u_0\Big) + P''_{n+1} x^2 u_0 = -\gamma_{n+1} P_n u_0 \ , \ n \in \mathbb{N}_0,$$

and we invoke (4.1) in order to simplify it to

$$P'_{n+1}\Big(2(x^2 u_0)' - 3x u_0\Big) + P''_{n+1} x^2 u_0 = -\gamma_{n+1} P_n u_0 \ , \ n \in \mathbb{N}_0.$$

The particular choice of $n = 0$ brings

$$2(x^2 u_0)' - 3x u_0 = -\gamma_1 u_0$$

permitting to deduce the equality

$$\Big(-\gamma_1 P'_{n+1} + x^2 P''_{n+1} + \gamma_{n+1} P_n\Big) u_0 = 0 \ , \ n \in \mathbb{N}_0.$$

Now the regularity of u_0 compels the identity

$$-\gamma_1 P'_{n+1} + x^2 P''_{n+1} + \gamma_{n+1} P_n = 0 \ , \ n \in \mathbb{N}_0,$$

contradicting the fact that $\deg P_n = n$ for all $n \in \mathbb{N}_0$, whence the conclusion. □

Remark 4.5. Actually, the non-orthogonality of this MPS could be observed directly from Table 3, after a few steps of computation, regarding, for instance, the fact that P_3 do not fulfill a second order recurrence relation, namely,

$$P_3(x) = (x-9)P_2(x) - 25P_1(x) - 17P_0(x) \ .$$

Arguing in a similar way, we may as well conclude the non-ocurrence of other kinds of orthogonality, like the d-orthogonality for at least $d = 5$, insofar as there are polynomials not fulfilling a $(d+1)$th order recurrence relation. For a compendium of results about the d-orthogonality, related to the present work whatsoever, we refer the reader to [3].

Despite the non-(regular)orthogonality of $\{P_n\}_{n\geqslant 0}$ with respect to the form u_0, we cannot exclude the existence of an orthogonal polynomial sequence, say $\{V_n\}_{n\geqslant 0}$, with respect to u_0, which amounts to the same as ensuring the regularity of u_0. This question is handled in the next section.

Proposition 4.6. *The canonical form u_0 is definite positive and admits the integral representation*

$$\langle u_0, f\rangle = 2\int_0^\infty K_0(2\sqrt{x})f(x)dx \ , \ \forall f\in\mathscr{P}. \tag{4.3}$$

Proof. We seek a function $U(x)$ such that

$$\langle u_0, f\rangle = \int_C U(x)f(x)dx$$

holds in a certain domain C. Since $\langle u_0, 1\rangle = 1 \neq 0$, we must have

$$\int_C U(x)dx = 1 \neq 0 \tag{4.4}$$

Consider the three polynomials presented in (4.1)

$$\phi(x) = x^2 \ , \ \psi(x) = -3x \ , \ \chi(x) = 1-x \ .$$

By virtue of (4.1), we have, for any $f\in\mathscr{P}$

$$\begin{aligned} 0 &= \langle\Big((\phi(x)u_0)' + \psi(x)u_0\Big)' + \chi(x)u_0, f(x)\rangle = \langle u_0, \phi(x)f''(x) + \psi(x)f'(x) + \chi(x)f(x)\rangle \\ &= \int_C \big((\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x)\big) f(x)dx \\ &\quad - \Big(\phi(x)U(x)f'(x) - (\phi(x)U(x))'f(x) - \psi(x)U(x)f(x)\Big)\Big|_C \end{aligned}$$

therefore, $U(x)$ is a function simultaneously fulfilling

$$\int_C \big((\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x)\big) f(x)dx = 0 \quad , \quad \forall f\in\mathscr{P} \tag{4.5}$$

$$\Big(\phi(x)U(x)f'(x) - (\phi(x)U(x))'f(x) - \psi(x)U(x)f(x)\Big)\Big|_C = 0 \quad , \quad \forall f\in\mathscr{P} \ . \tag{4.6}$$

The first equation implies

$$(\phi(x)U(x))'' + (\psi(x)U(x))' + \chi(x)U(x) = \lambda g(x)$$

where λ is a complex number and $g(x)\neq 0$ is a function representing the null form, that is, a function such that

$$\int_C g(x)f(x)dx = 0 \quad , \quad \forall f\in\mathscr{P}.$$

We begin by choosing $\lambda = 0$ and we search a regular solution of the differential equation

$$(x^2U(x))'' - (3xU(x))' + (1-x)U(x) = 0$$

whose general solution is: $y(x) = c_1 I_0(2\sqrt{x}) + c_2 K_0(2\sqrt{x})$, $x \geqslant 0$, for some arbitrary constants c_1, c_2 and $y(x) = 0$ when $x < 0$ [7]. As a consequence $U(x) = \left\{ c_1 I_0(2\sqrt{x}) + c_2 K_0(2\sqrt{x}) \right\}$, $x \geqslant 0$. Insofar as $U(x)$ must be a rapidly decreasing sequence (that is, such that $\lim_{x\to+\infty} f(x)U(x) = 0$ for any polynomial f) simultaneously realizing the condition (4.4), we readily conclude that $U(x) = 2K_0(2\sqrt{x})$, considering its asymptotic behavior (3.7)-(3.8) together with the expression for its moments (2.2). Moreover, for every polynomial g that is not identically zero and is non-negative for all real x we have $\langle u_0, g\rangle = 2\int_0^\infty g(x)K_0(2\sqrt{x})\,dx > 0$ and therefore u_0 is a positive-definite form, which implies the existence of a corresponding MOPS (*i.e.*, u_0 is a regular form). □

The aforementioned MOPS is necessarily the Ditkin-Prudnikov MOPS $\{V_n\}_{n\geqslant 0}$, whose characterization is still an open problem. Needless to say that it is possible to compute the first recurrence coefficients, for instance

$$\beta_0 = 1\ ,\ \gamma_1 = 3\ ,\ \beta_1 = 29/3\ ,\ \gamma_2 = \frac{656}{9}\ ,\ \beta_2 = \frac{3467}{123}\ ,\ \gamma_3 = \frac{690363}{1681}\ ,\ \beta_3 = \frac{2196517}{38827}\ \ldots$$

and also

$$\beta_7 = \frac{36358573629873129006572781 1165063}{13526727898249762955774288 27577}$$

where β_n and γ_{n+1} are such that

$$V_0(x) = 1 \quad , \quad V_1(x) = x - \beta_0, \tag{4.7}$$

$$V_{n+2}(x) = (x - \beta_{n+1})V_{n+1}(x) - \gamma_{n+1}V_n(x)\ ,\ n \in \mathbb{N}_0. \tag{4.8}$$

Apparently this does not give any clue for the behavior of the remaining elements of the sequence. We did not reach this far further connections between these two polynomial sequences $\{P_n\}_{n\geqslant 0}$ and $\{V_n\}_{n\geqslant 0}$, capable to unravel the aforementioned problem. However we still work toward this goal, bring to light the properties of this MPS connected to the moments of $K_{i\tau}(2\sqrt{x})$ is worth by itself.

REFERENCES

1. M. Abramowitz, I. A. Stegun, Handbook Of Mathematical Functions, With Formulas, Graphs, And Mathematical Tables, New York: Dover Publications, 1972.
2. G. E. Andrews, L. L. Littlejohn, A combinatorial interpretation of the Legendre- Stirling numbers, Proc. Amer. Math. Soc. 137 (2009), 2581-2590.
3. Y. Ben Cheikh and K. Douak: On two-orthogonal polynomials realted to the Batemans $J_n^{u,v}$-fuction, Meth. Appl. Anal. 7 (2000), 641-662.
4. L. Comtet, Advanced Combinatorics - The art of finite and infinite expansions, Kluwer, Dordrecht, 1974.
5. V. A. Ditkin, A.P. Prudnikov, Integral transforms. Mathematical analysis, 1966, pp. 7-82. Akad. Nauk SSSR Inst. Nauchn. Informacii, Moscow, 1967 (in Russian).
6. Digital Library of Mathematical Functions. 2011-07-01. National Institute of Standards and Technology from `http://dlmf.nist.gov/24`.
7. A. Erdélyi,W. Magnus, F. Oberhettinger, and F.G. Tricomi, Higher Transcendental Functions,Vols. I and II, McGraw-Hill, NewYork, London, Toronto, 1953.
8. W. N. Everitt, K. H. Kwon, L. L. Littlejohn, R. Wellman, and G. J. Yoon, Jacobi-Stirling numbers, Jacobi polynomials, and the left-definite analysis of the classical Jacobi differential expression, J. Comput. Appl. Math., 208 (2007), 29-56.
9. Y. Gelineau, J. Zeng, Combinatorial interpretations of the Jacobi-Stirling numbers, Electron. J. Combin 17 (2) (2010) #R70.
10. A.F. Loureiro, New results on the Bochner condition about classical orthogonal polynomials, J. Math. Anal. Appl. 364 (2010) 307-323.
11. P. Maroni, Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques, C. Brezinski et al. (Eds.), Orthogonal Polynomials and their Applications, in: *IMACS Ann. Comput. Appl. Math.* 9 (1991), 95-130.
12. P. Maroni, Variations around classical orthogonal polynomials. Connected problems. *Journal of Comput. Appl. Math.*, 48 (1993), 133-155.

13. P. Maroni, M. Mejri, Generalized Bernoulli polynomials revisited and some other Appell sequences. Georgian Math. J. 12(4) (2005) pp. 697-716.
14. A.P. Prudnikov, Orthogonal polynomials with ultra-exponential weight functions, in: W. Van Assche (Ed.), Open Problems, J. Comput. Appl. Math. 48 (1993) 239-241.
15. A. P. Prudnikov, Yu. A. Brychkov and O.I. Marichev, Vol. II: Special Functions, Gordon and Breach, New York, London, 1986.
16. J. Riordan, An introduction to combinatorial analysis, Dover ed., New Yok, 2002.
17. J. Riordan, Combinatorial identities, Wiley, New Yok, 1968.
18. N. Temme, Special Functions, An Introduction to the Classical Functions of Mathematical Physics, Wiley Interscience, New York, 1996.
19. W. Van Assche and S.B. Yakubovich: Multiple orthogonal polynomials associated with Macdonald functions, Integral Transform. Spec. Funct. 9 (2000), 229244.
20. S. Yakubovich, A class of polynomials and discrete transformations associated with the Kontorovich-Lebedev operators, Integral Transforms Spec. Funct. 20 (2009), 551-567.
21. S.B. Yakubovich, On the least values of Lp-norms for the Kontorovich-Lebedev transform and its convolution. Journal of Approximation Theory, 131, 231-242 (2004)
22. S. B. Yakubovich, On the Kontorovich-Lebedev transformation, J. Integral Equations Appl. 15(1) (2003) pp. 95-112.
23. S. Yakubovich, On the index integral transformation with Nicholson's function as the kernel, J. Math. Anal. Appl. 269 (2002), no. 2, 689-701.
24. S. B. Yakubovich, Index Transforms, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1996.
25. S.B. Yakubovich, B. Fisher, On the theory of the Kontorovich-Lebedev transformation on distributions, Proc. of the Amer. Math. Soc., 122 (1994), 773-777.
26. Yakubovich, S. B. and Luchko, Yu. F., The Hypergeometric Approach to Integral Transforms and Convolutions, Kluwer Academic Publishers, Mathematics and Applications. Vol.287, 1994.

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