

HIGGS BUNDLES AND SURFACE GROUP REPRESENTATIONS IN THE REAL SYMPLECTIC GROUP

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ABSTRACT. In this paper we study the moduli space of representations of a surface group (i.e., the fundamental group of a closed oriented surface) in the real symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. The moduli space is partitioned by an integer invariant, called the Toledo invariant. This invariant is bounded by a Milnor–Wood type inequality. Our main result is a count of the number of connected components of the moduli space of maximal representations, i.e. representations with maximal Toledo invariant. Our approach uses the non-abelian Hodge theory correspondence proved in a companion paper [19] to identify the space of representations with the moduli space of polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. A key step is provided by the discovery of new discrete invariants of maximal representations. These new invariants arise from an identification, in the maximal case, of the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles with a moduli space of twisted Higgs bundles for the group $\mathrm{GL}(n, \mathbb{R})$.

1. INTRODUCTION

*Valeu a pena? Tudo vale a pena
Se a alma não é pequena.*

F. Pessoa

In this paper we study representations of the fundamental group of a compact oriented surface X in $\mathrm{Sp}(2n, \mathbb{R})$ — the group of linear transformations of \mathbb{R}^{2n} which preserve the standard symplectic form. By a representation we mean a homomorphism from $\pi_1(X)$ to $\mathrm{Sp}(2n, \mathbb{R})$. Given a representation of $\pi_1(X)$ in $\mathrm{Sp}(2n, \mathbb{R})$ there is an integer, often referred to as the *Toledo invariant*, associated to it. This integer can be obtained geometrically by considering the flat $\mathrm{Sp}(2n, \mathbb{R})$ -bundle corresponding to the representation and taking a reduction of the structure group of the underlying smooth vector bundle to $\mathrm{U}(n)$ — a maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$. The degree of the resulting $\mathrm{U}(n)$ -bundle is the Toledo invariant. As shown by Turaev [46] the Toledo invariant d of a representation

Date: 14 August 2012.

2010 Mathematics Subject Classification. Primary 14H60; Secondary 53C07, 58D29.

Key words and phrases. Representations of surface groups, real symplectic group, character varieties, moduli spaces, Higgs bundles.

Members of the Research Group VBAC (Vector Bundles on Algebraic Curves). Research partially supported by Ministerio de Educación y Ciencia, CSIC, Conselho de Reitores das Universidades Portuguesas and FCT (Portugal) through Spain–Portugal bilateral research projects. First and Third authors partially supported by Ministerio de Educación y Ciencia (Spain) through Projects MTM2004-07090-C03-01 and MTM2007-67623. Second author partially supported by the FCT (Portugal) with EU (FEDER/COMPETE) and national funds through the projects PTDC/MAT/099275/2008 and PTDC/MAT/098770/2008, and through Centro de Matemática da Universidade do Porto (PEst-C/MAT/UI0144/2011).

satisfies the inequality

$$(1.1) \quad |d| \leq n(g-1),$$

where g is the genus of the surface. When $n = 1$, one has $\mathrm{Sp}(2, \mathbb{R}) \simeq \mathrm{SL}(2, \mathbb{R})$, the Toledo invariant coincides with the Euler class of the $\mathrm{SL}(2, \mathbb{R})$ -bundle, and (1.1) is the classical inequality of Milnor [33] which was later generalized by Wood [48]. We shall follow custom and refer to (1.1) as the Milnor–Wood inequality.

Given two representations, a basic question to ask is whether one can be continuously deformed into the other. Put in a more precise way, we are asking for the connected components of the space of representations

$$\mathrm{Hom}(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R})).$$

As shown in [21], this space has the same number of connected components as the moduli space, or character variety,

$$\mathcal{R}(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R})) = \mathrm{Hom}^{\mathrm{red}}(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R})) / \mathrm{Sp}(2n, \mathbb{R})$$

of reductive representations $\rho: \pi_1(X) \rightarrow \mathrm{Sp}(2n, \mathbb{R})$, modulo the natural equivalence given by the action of $\mathrm{Sp}(2n, \mathbb{R})$ by overall conjugation.

The Toledo invariant descends to the quotient so, for any d satisfying (1.1), we can define

$$\mathcal{R}_d(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R})) \subset \mathcal{R}(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R}))$$

to be the subspace of representations with Toledo invariant d . For ease of notation, for the remaining part of the Introduction, we shall write \mathcal{R}_d for $\mathcal{R}_d(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R}))$ and \mathcal{R} for $\mathcal{R}(\pi_1(X), \mathrm{Sp}(2n, \mathbb{R}))$. Since the Toledo invariant varies continuously with the representation, the subspace \mathcal{R}_d is a union of connected components, and our basic problem is that of counting the number of connected components of \mathcal{R}_d for d satisfying (1.1). This has been done for $n = 1$ by Goldman [22, 25] and Hitchin [27], and for $n = 2$ in [26] (in the cases $d = 0$ and $|d| = 2g - 2$) and [21] (in the cases $|d| < 2g - 2$). In this paper we count the number of connected components of \mathcal{R}_d for $n > 2$ when $d = 0$ and $|d| = n(g - 1)$ — the maximal value allowed by the Milnor–Wood inequality. Our main result is the following (Theorem 8.7 below).

Theorem 1.1. *Let X be a compact oriented surface of genus g . Let \mathcal{R}_d be the moduli space of reductive representations of $\pi_1(X)$ in $\mathrm{Sp}(2n, \mathbb{R})$ with Toledo invariant d . Let $n \geq 3$. Then*

- (1) \mathcal{R}_0 is non-empty and connected;
- (2) $\mathcal{R}_{\pm n(g-1)}$ has $3 \cdot 2^{2g}$ non-empty connected components.

The main tool we employ to count connected components is the theory of Higgs bundles, as pioneered by Hitchin [27] for $\mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R})$. Fix a complex structure on X endowing it with a structure of a compact Riemann surface, which we will denote, abusing notation, also by X . An $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle over X is a triple (V, β, γ) consisting of a rank n holomorphic vector bundle V and holomorphic sections $\beta \in H^0(X, S^2V \otimes K)$ and $\gamma \in H^0(X, S^2V^* \otimes K)$, where K is the canonical line bundle of X . The sections β and γ are often referred to as Higgs fields. Looking at X as an algebraic curve, algebraic moduli spaces for $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle exist as a consequence of the work of Schmitt [40, 41]. Fixing $d \in \mathbb{Z}$, we denote by \mathcal{M}_d the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles on X with $\deg(V) = d$. As usual, one must introduce an appropriate stability condition (with related conditions of poly- and semistability) in order to have good moduli spaces. Thus \mathcal{M}_d

parametrizes isomorphism classes of polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. A basic result of non-abelian Hodge theory, growing out of the work of Corlette [13], Donaldson [15], Hitchin [27] and Simpson [42, 43, 44, 45], is the following (Theorem 2.5 below).

Theorem 1.2. *The moduli spaces \mathcal{R}_d and \mathcal{M}_d are homeomorphic.*

An essential part of the proof of this Theorem follows from a Hitchin–Kobayashi correspondence between polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles and solutions to certain gauge theoretic equations, known as Hitchin’s equations (see Theorem 2.6). In the generality required for stable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles, the Hitchin–Kobayashi correspondence is provided by the general theory of [6], together with the extension to the case of polystable (non-stable) pairs in general proved in [19]

Using the homeomorphism $\mathcal{R}_d \simeq \mathcal{M}_d$, our problem is reduced to studying the connectedness properties of \mathcal{M}_d . This is done by using the Hitchin functional. This is a non-negative function f which is defined on \mathcal{M}_d using the solution to Hitchin’s equations. This function arises as the moment map for the Hamiltonian circle action on the moduli space obtained by multiplying the Higgs field by an element of $U(1)$ and is (essentially) the L^2 -norm of the Higgs field. It was proved by Hitchin [27, 28] that f is proper, and this implies that f has a minimum on each connected component of \mathcal{M}_d . Using this fact, our problem essentially reduces to characterizing the subvariety of minima of the Hitchin functional and studying its connectedness properties.

While we characterize the minima for every value of d satisfying the Milnor–Wood inequality (see Theorem 5.10), we only carry out the full programme for $d = 0$ and $|d| = n(g - 1)$, the extreme values of d . For $d = 0$, the subvariety of minima of the Hitchin functional on \mathcal{M}_0 coincides with the set of Higgs bundles (V, β, γ) with $\beta = \gamma = 0$. This, in turn, can be identified with the moduli space of polystable vector bundles of rank n and degree 0. Since this moduli space is connected by the results of Narasimhan–Seshadri [35], \mathcal{M}_0 is connected and hence \mathcal{R}_0 is connected.

The analysis for the *maximal case*, $|d| = n(g - 1)$, is far more involved and interesting. It turns out that in this case one of the Higgs fields β or γ for a semistable Higgs bundle (V, β, γ) becomes an isomorphism. Whether it is β or γ , actually depends on the sign of the Toledo invariant. Since the map $(V, \beta, \gamma) \mapsto (V^*, \gamma^t, \beta^t)$ defines an isomorphism $\mathcal{M}_{-d} \simeq \mathcal{M}_d$, there is no loss of generality in assuming that $0 \leq d \leq n(g - 1)$. Suppose that $d = n(g - 1)$. Then $\gamma : V \rightarrow V^* \otimes K$ is an isomorphism (see Proposition 3.34). Since γ is furthermore symmetric, it equips V with a K -valued non-degenerate quadratic form. In order to have a proper quadratic bundle, we fix a square root $L_0 = K^{-1/2}$ of K^{-1} , and define $W := V \otimes L_0$. Then $q := \gamma \otimes I_{L_0} : W \rightarrow W^*$ is a symmetric isomorphism defining a non-degenerate symmetric bilinear form Q on W , in other words, (W, Q) is an $O(n, \mathbb{C})$ -holomorphic bundle. The K^2 -twisted endomorphism $\psi : W \rightarrow W \otimes K^2$ defined by $\psi := \beta \otimes I_{L_0^{-1}} \circ (\gamma \otimes I_{L_0})$ is Q -symmetric and hence (W, Q, ψ) defines what we call a K^2 -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair, from which we can recover the original $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle. The main result is the following (Theorem 4.3 below).

Theorem 1.3. *Let \mathcal{M}_{\max} be the moduli space of polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles with $d = n(g - 1)$, and let \mathcal{M}' be the moduli space of polystable K^2 -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pairs. The map $(V, \beta, \gamma) \mapsto (W, Q, \psi)$ defines an isomorphism of complex algebraic varieties*

$$\mathcal{M}_{\max} \simeq \mathcal{M}'.$$

We refer to this isomorphism as the *Cayley correspondence*. This name is motivated by the geometry of the bounded symmetric domain associated to the Hermitian symmetric space $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$. The Cayley transform defines a biholomorphism between this domain and a tube type domain defined over the symmetric cone $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$ — the Siegel upper half-space. In fact, there is a similar correspondence to that given in Theorem 1.3 for every semisimple Lie group G which, like $\mathrm{Sp}(2n, \mathbb{R})$, is the group of isometries of a Hermitian symmetric space of tube type (see [4] for a survey on this subject).

A key point is that the Cayley correspondence brings to the surface new topological invariants. These invariants, hidden a priori, are naturally attached to an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle with maximal Toledo invariant and generalize those obtained in the case $n = 2$ in [26]. The invariants are the first and second Stiefel-Whitney classes (w_1, w_2) of a reduction to $\mathrm{O}(n)$ of the $\mathrm{O}(n, \mathbb{C})$ -bundle defined by (W, Q) . It turns out that there is a connected component for each possible value of (w_1, w_2) , containing K^2 -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pairs (W, Q, ψ) with $\psi = 0$. This accounts for $2 \cdot 2^{2g}$ of the $3 \cdot 2^{2g}$ connected components of \mathcal{M}_{\max} . Thus it remains to account for the 2^{2g} “extra” components. As already mentioned, the group $\mathrm{Sp}(2n, \mathbb{R})$ is the group of isometries of a Hermitian symmetric space, but it also has the property of being a split real form. In fact, up to finite coverings, it is the only Lie group with this property. In [28] Hitchin shows that for every semisimple split real Lie group G , the moduli space of reductive representations of $\pi_1(X)$ in G has a topological component which is isomorphic to $\mathbb{R}^{\dim G(2g-2)}$, and which naturally contains Teichmüller space. Indeed, when $G = \mathrm{SL}(2, \mathbb{R})$, this component can be identified with Teichmüller space, via the Riemann uniformization theorem. Since $\mathrm{Sp}(2n, \mathbb{R})$ is split, the moduli space for $\mathrm{Sp}(2n, \mathbb{R})$ must have a Hitchin component. It turns out that there are 2^{2g} isomorphic Hitchin components (this is actually true for arbitrary n). As follows from Hitchin’s construction, the K^2 -twisted Higgs pairs (W, Q, ψ) in the Hitchin component all have $\psi \neq 0$.

As already mentioned, in the cases $n = 1$ (see Goldman [25] and Hitchin [27]) and $n = 2$ (see [21]) the subspaces \mathcal{R}_d are connected for $0 < d < n(g-1)$. It appears natural to expect that this should be true for general n . Indeed, given the analysis of the minima of the Hitchin functional carried out in this paper (cf. Theorem 5.10), this would follow if one could prove connectedness of the $\beta = 0$ locus of \mathcal{M}_d . Of course, this locus can be viewed as a moduli space of quadric bundles and as such is a natural object to study. However, not much is known about this question for general rank and we will leave a detailed study for another occasion.

A second reason for our focus on maximal representations in the present paper is that from many points of view they are the most interesting ones. They have been the object of intense study in recent years, using methods from diverse branches of geometry, and it has become clear that they enjoy very special properties. In particular, at least in many cases, maximal representations have a close relationship to geometric structures on the surface. The prototype of this philosophy is Goldman’s theorem [22, 24] that the maximal representations in $\mathrm{SL}(2, \mathbb{R})$ are exactly the Fuchsian ones. In the following, we briefly mention some results of this kind.

Using bounded cohomology methods, maximal representations in general Hermitian type groups have been studied by Burger–Iozzi [7, 8] and Burger–Iozzi–Wienhard [10, 11, 12]. Among many other results, they have given a very general Milnor–Wood inequality and

they have shown that maximal representations are discrete, faithful and completely reducible. One consequence of this is that the restriction to reductive representations is unnecessary in the case of the moduli space \mathcal{R}_{\max} of maximal representations. Building on this work and the work of Labourie [32], Burger–Iozzi–Labourie–Wienhard [9] have shown that maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$ are Anosov (in the sense of [32]). Furthermore, it has been shown that the action of the mapping class group on \mathcal{R}_{\max} is proper, by Wienhard [47] (for classical simple Lie groups of Hermitian type), and by Labourie [31] (for $\mathrm{Sp}(2n, \mathbb{R})$), who also proves further geometric properties of maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$.

From yet a different perspective, representations in the Hitchin component have been studied in the work on higher Teichmüller theory of Fock–Goncharov [17], using methods of tropical geometry. In particular, the fact that representations in the Hitchin component for $\mathrm{Sp}(2n, \mathbb{R})$ are faithful and discrete also follows from their work

Thus, while Higgs bundle techniques are very efficient in the study of topological properties of the moduli space (like counting components), these other approaches have been more powerful in the study of special properties of individual representations. It would be very interesting indeed to gain a better understanding of the relation between these distinct methods.

We describe now briefly the content of the different sections of the paper.

In Section 2 we review the general theory of L -twisted G -Higgs pairs, of which G -Higgs bundles are a particular case. We explain the general Hitchin–Kobayashi correspondence from [19] and the corresponding non-abelian Hodge theorem. We also review some general properties about moduli spaces and deformation theory of G -Higgs bundles.

In Section 3, we specialize the non-abelian Hodge theory correspondence of Section 2.2 to $G = \mathrm{Sp}(2n, \mathbb{R})$ — our case of interest in this paper. We recall some basic facts about the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles, including the Milnor–Wood inequality and we carry out a careful study of stable, non-simple $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. To do this, we study and exploit the relation between the polystability of a $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles and the $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundle naturally associated to it.

In Section 4 we study the Cayley correspondence between $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles with maximal Toledo invariant and K^2 -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pairs.

The rest of the paper is mostly devoted to the study of the connectedness properties of the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles and, in particular, to prove Theorem 8.3. In Section 5 we introduce the Hitchin functional on the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles and characterize its minima. We then use this and the Cayley correspondence of Section 4 to count the number of connected components of the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles for $d = 0$ and $|d| = n(g - 1)$. The proof of the characterization of the minima is split in two cases: the case of minima in the smooth locus of the moduli space, given in Section 6 and the case of the remaining minima, treated in Section 7.

The results of this paper have been announced in several conferences over the last few years, while several preliminary versions of this paper have been circulating. The main results, together with analogous results for other groups of Hermitian type have appeared in the review paper [4]. The authors apologize for having taken so long in producing this final version.

Acknowledgments. The authors thank Olivier Biquard, Steven Bradlow, Marc Burger, Bill Goldman, Nigel Hitchin, Alessandra Iozzi, François Labourie, S. Ramanan, Roberto Rubio, Domingo Toledo, and Anna Wienhard for numerous useful conversations and shared insights. We also thank Olivier Guichard for carefully reading an earlier version of this paper and pointing out a number of typos and mistakes.

2. L -TWISTED G -HIGGS PAIRS, G -HIGGS BUNDLES AND REPRESENTATIONS OF SURFACE GROUPS

2.1. L -twisted G -Higgs pairs, G -Higgs bundles, stability and moduli spaces. Let G be a real reductive Lie group, let $H \subset G$ be a maximal compact subgroup and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a Cartan decomposition, so that the Lie algebra structure on \mathfrak{g} satisfies

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

The group H acts linearly on \mathfrak{m} through the adjoint representation, and this action extends to a linear holomorphic action of $H^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m} \otimes \mathbb{C}$. This is the **isotropy representation**:

$$(2.2) \quad \iota: H^{\mathbb{C}} \rightarrow \mathrm{GL}(\mathfrak{m}^{\mathbb{C}}).$$

Furthermore, the Killing form on \mathfrak{g} induces on $\mathfrak{m}^{\mathbb{C}}$ a Hermitian structure which is preserved by the action of H .

Let X be a compact Riemann surface and let E be a holomorphic principal $H^{\mathbb{C}}$ -bundle on X . Let $E(\mathfrak{m}^{\mathbb{C}}) = E \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}}$ be the $\mathfrak{m}^{\mathbb{C}}$ -bundle associated to E via the isotropy representation. Let K be the canonical bundle of X and L be a holomorphic line bundle on X .

Definition 2.1. An L -twisted G -Higgs pair on X is a pair (E, φ) , where E is a holomorphic $H^{\mathbb{C}}$ -principal bundle over X and φ is a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes L$. A G -Higgs bundle on X is a K -twisted G -Higgs pair. Two L -twisted G -Higgs pairs (E, φ) and (E', φ') are **isomorphic** if there is an isomorphism $f: E \xrightarrow{\sim} E'$ such that $\varphi = f^* \varphi'$.

A general notion of (poly,semi)stability for G -Higgs bundles is given in [19]. This notion depends on an element $\alpha \in \sqrt{-1}\mathfrak{z}$, where \mathfrak{z} is the Lie algebra of the centre of H , and is formulated (similarly to [37]) in terms of parabolic subgroups of $H^{\mathbb{C}}$, antidominant characters of their Lie algebras and reductions of structure group of the $H^{\mathbb{C}}$ -bundle E to the parabolic subgroups. A workable notion can be obtained in terms of filtrations of a certain vector bundle associated to E . In particular, when $H^{\mathbb{C}}$ is a classical group—which is the situation for the particular groups considered in this paper—, this vector bundle is the one associated to E via the standard representation $\rho: H^{\mathbb{C}} \rightarrow \mathrm{GL}(n, \mathbb{C})$. For details, see Sections 2.8 and 4 of [19].

When studying G -Higgs bundles we shall mainly be interested in the case when $\alpha = 0$, since this is the relevant value for the applications to non-abelian Hodge theory. Thus we will talk about **stability** of a G -Higgs bundle, meaning 0-stability, and analogously for **semistability** and **polystability**.

Henceforth, we shall assume that G is connected. Then the topological classification of $H^{\mathbb{C}}$ -bundles E on X is given by a characteristic class $c(E) \in H^2(X, \pi_1(H^{\mathbb{C}})) = \pi_1(H^{\mathbb{C}}) = \pi_1(H) = \pi_1(G)$. For a fixed $d \in \pi_1(G)$, the **moduli space of polystable G -Higgs bundles** $\mathcal{M}_d(G)$ is the set of isomorphism classes of polystable G -Higgs bundles (E, φ)

such that $c(E) = d$. When G is compact, the moduli space $\mathcal{M}_d(G)$ coincides with $M_d(G^{\mathbb{C}})$, the moduli space of polystable $G^{\mathbb{C}}$ -bundles with topological invariant d .

The moduli space $\mathcal{M}_d(G)$ has the structure of a complex analytic variety. This can be seen by the standard slice method (see, e.g., Kobayashi [30]). Geometric Invariant Theory constructions are available in the literature for G real compact algebraic (Ramanathan [38]) and for G complex reductive algebraic (Simpson [44, 45]). The case of a real form of a complex reductive algebraic Lie group follows from the general constructions of Schmitt [40, 41]. We thus have the following.

Theorem 2.2. *The moduli space $\mathcal{M}_d(G)$ is a complex analytic variety, which is algebraic when G is algebraic.*

Remark 2.3. More generally, moduli spaces of L -twisted G -Higgs pairs can be constructed (see Schmitt [41]). We shall need this in Section 4 below.

2.2. Surface group representations and non-abelian Hodge theory. Let X be a closed oriented surface of genus g and let

$$\pi_1(X) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle$$

be its fundamental group. Let G be a connected reductive real Lie group and let $H \subseteq G$ be a maximal compact subgroup. By a **representation** of $\pi_1(X)$ in G we understand a homomorphism $\rho: \pi_1(X) \rightarrow G$. The set of all such homomorphisms, $\text{Hom}(\pi_1(X), G)$, can be naturally identified with the subset of G^{2g} consisting of $2g$ -tuples $(A_1, B_1, \dots, A_g, B_g)$ satisfying the algebraic equation $\prod_{i=1}^g [A_i, B_i] = 1$. This shows that $\text{Hom}(\pi_1(X), G)$ is a real analytic variety, which is algebraic if G is algebraic.

The group G acts on $\text{Hom}(\pi_1(X), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in G$, $\rho \in \text{Hom}(\pi_1(X), G)$ and $\gamma \in \pi_1(X)$. If we restrict the action to the subspace $\text{Hom}^{\text{red}}(\pi_1(X), G)$ consisting of reductive representations, the orbit space is Hausdorff (see Theorem 11.4 in [39]). By a **reductive representation** we mean one that composed with the adjoint representation in the Lie algebra of G decomposes as a sum of irreducible representations. If G is algebraic this is equivalent to the Zariski closure of the image of $\pi_1(X)$ in G being a reductive group. (When G is compact every representation is reductive.) Define the *moduli space of representations* of $\pi_1(X)$ in G to be the orbit space

$$\mathcal{R}(G) = \text{Hom}^{\text{red}}(\pi_1(X), G)/G.$$

One has the following (see e.g. Goldman [23]).

Theorem 2.4. *The moduli space $\mathcal{R}(G)$ has the structure of a real analytic variety, which is algebraic if G is algebraic and is a complex variety if G is complex.*

Given a representation $\rho: \pi_1(X) \rightarrow G$, there is an associated flat G -bundle on X , defined as $E_\rho = \tilde{X} \times_\rho G$, where $\tilde{X} \rightarrow X$ is the universal cover and $\pi_1(X)$ acts on G via ρ . We can then assign a topological invariant to a representation ρ given by the characteristic class $c(\rho) := c(E_\rho) \in \pi_1(G) \simeq \pi_1(H)$ corresponding to E_ρ . For a fixed $d \in \pi_1(G)$, the **moduli**

space of reductive representations $\mathcal{R}_d(G)$ with topological invariant d is defined as the subvariety

$$(2.3) \quad \mathcal{R}_d(G) := \{[\rho] \in \mathcal{R}(G) \mid c(\rho) = d\},$$

where as usual $[\rho]$ denotes the G -orbit $G \cdot \rho$ of $\rho \in \text{Hom}^{\text{red}}(\pi_1(X), G)$.

The non-abelian Hodge theorem states the following.

Theorem 2.5. *Let G be a connected semisimple real Lie group with maximal compact subgroup $H \subseteq G$. Let $d \in \pi_1(G) \simeq \pi_1(H)$. Then there is a homeomorphism $\mathcal{R}_d(G) \simeq \mathcal{M}_d(G)$.*

One half of this theorem is proved by solving Hitchin's equations. To explain this, let (E, φ) be a G -Higgs bundle over a compact Riemann surface X . By a slight abuse of notation, we shall denote the C^∞ -objects underlying E and φ by the same symbols. In particular, the Higgs field can be viewed as a $(1,0)$ -form: $\varphi \in \Omega^{1,0}(E(\mathfrak{m}^\mathbb{C}))$. Let $\tau: \Omega^1(E(\mathfrak{g}^\mathbb{C})) \rightarrow \Omega^1(E(\mathfrak{g}^\mathbb{C}))$ be the compact conjugation of $\mathfrak{g}^\mathbb{C}$ combined with complex conjugation on complex 1-forms. Given a reduction h of structure group to H in the smooth $H^\mathbb{C}$ -bundle E , we denote by F_h the curvature of the unique connection compatible with h and the holomorphic structure on E .

Theorem 2.6. *There exists a reduction h of the structure group of E from $H^\mathbb{C}$ to H satisfying the Hitchin equation*

$$F_h - [\varphi, \tau(\varphi)] = 0$$

if and only if (E, φ) is polystable.

Theorem 2.6 was proved by Hitchin [27] for $G = \text{SL}(2, \mathbb{C})$ and Simpson [42, 43] for an arbitrary semisimple complex Lie group G . The proof for an arbitrary reductive real Lie group G when (E, φ) is stable is given in [6, Theorem 2.13], and the general polystable case follows as a particular case of the more general Hitchin–Kobayashi correspondence given in [19] (see *loc. cit.* Theorem 3.21).

To explain the other half of Theorem 2.5, recall that $\mathcal{R}_d(G)$ can be identified with the moduli space of flat reductive connections on a fixed G -bundle of topological class $d \in \pi_1(G)$ (see, e.g., Simpson [44, 45]). Now any solution h to Hitchin's equations defines a flat reductive G -connection

$$(2.4) \quad D = D_h + \varphi - \tau(\varphi),$$

where D_h is the unique H -connection on E compatible with its holomorphic structure.

It is a fundamental theorem of Corlette [13] and Donaldson [15] (for $G = \text{SL}(2, \mathbb{C})$) that this process can be inverted: given a flat reductive connection D in a G -bundle E_G , there exists a *harmonic metric*, i.e. a reduction of structure group to $H \subset G$ corresponding to a harmonic section of $E_G/H \rightarrow X$. This reduction produces a solution h to Hitchin's equations such that (2.4) holds. (See [19, Section 3.6] for a fuller explanation).

Remark 2.7. These moduli spaces may be empty.

Remark 2.8. On the open subvarieties defined by the smooth points of \mathcal{R}_d and \mathcal{M}_d , this correspondence is in fact an isomorphism of real analytic varieties.

Remark 2.9. There is a similar correspondence when G is reductive but not semisimple. In this case, it makes sense to consider nonzero values of the stability parameter α . The

resulting Higgs bundles can be geometrically interpreted in terms of representations of the universal central extension of the fundamental group of X , and the value of α prescribes the image of a generator of the centre in the representation.

2.3. Deformation theory of G -Higgs bundles. In this section we recall some standard facts about the deformation theory of G -Higgs bundles, following Biswas–Ramanan [1]. The results summarized here are explained in more detail in [19, Section 3.3].

Definition 2.10. Let (E, φ) be a G -Higgs bundle. The *deformation complex* of (E, φ) is the following complex of sheaves:

$$(2.5) \quad C^\bullet(E, \varphi): E(\mathfrak{h}^{\mathbb{C}}) \xrightarrow{\text{ad}(\varphi)} E(\mathfrak{m}^{\mathbb{C}}) \otimes K.$$

This definition makes sense because φ is a section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$ and $[\mathfrak{m}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}] \subseteq \mathfrak{m}^{\mathbb{C}}$.

The following result generalizes the fact that the infinitesimal deformation space of a holomorphic vector bundle V is isomorphic to $H^1(\text{End } V)$.

Proposition 2.11. *The space of infinitesimal deformations of a G -Higgs bundle (E, φ) is naturally isomorphic to the hypercohomology group $\mathbb{H}^1(C^\bullet(E, \varphi))$.*

For any G -Higgs bundle there is a natural long exact sequence

$$(2.6) \quad \begin{aligned} 0 &\rightarrow \mathbb{H}^0(C^\bullet(E, \varphi)) \rightarrow H^0(E(\mathfrak{h}^{\mathbb{C}})) \xrightarrow{\text{ad}(\varphi)} H^0(E(\mathfrak{m}^{\mathbb{C}}) \otimes K) \\ &\rightarrow \mathbb{H}^1(C^\bullet(E, \varphi)) \rightarrow H^1(E(\mathfrak{h}^{\mathbb{C}})) \xrightarrow{\text{ad}(\varphi)} H^1(E(\mathfrak{m}^{\mathbb{C}}) \otimes K) \rightarrow \mathbb{H}^2(C^\bullet(E, \varphi)) \rightarrow 0. \end{aligned}$$

This justifies the following definition.

Definition 2.12. The **infinitesimal automorphism space** of (E, φ) is

$$\text{aut}(E, \varphi) = \mathbb{H}^0(C^\bullet(E, \varphi)).$$

Note that this agrees with the general notion of the infinitesimal automorphism space of a pair introduced in [19, Section 2.9].

Let $d\iota: \mathfrak{h}^{\mathbb{C}} \rightarrow \text{End}(\mathfrak{m}^{\mathbb{C}})$ be the derivative at the identity of the complexified isotropy representation $\iota = \text{Ad}_{|H^{\mathbb{C}}}: H^{\mathbb{C}} \rightarrow \text{Aut}(\mathfrak{m}^{\mathbb{C}})$ (cf. (2.2)). Let $\ker d\iota \subseteq \mathfrak{h}^{\mathbb{C}}$ be its kernel and let $E(\ker d\iota) \subseteq E(\mathfrak{h}^{\mathbb{C}})$ be the corresponding subbundle. Then there is an inclusion $H^0(E(\ker d\iota)) \hookrightarrow \mathbb{H}^0(C^\bullet(E, \varphi))$.

Definition 2.13. A G -Higgs bundle (E, φ) is said to be **infinitesimally simple** if the infinitesimal automorphism space $\text{aut}(E, \varphi)$ is isomorphic to $H^0(E(\ker d\iota \cap \mathfrak{z}))$.

Remark 2.14. If $\ker d\iota = 0$, then (E, φ) is infinitesimally simple if and only if the vanishing $\mathbb{H}^0(C^\bullet(E, \varphi)) = 0$ holds. A particular case of this situation is when the group G is a complex semisimple group: indeed, in this case the isotropy representation is just the adjoint representation.

Similarly, we have an inclusion $\ker \iota \cap Z(H^{\mathbb{C}}) \hookrightarrow \text{Aut}(E, \varphi)$.

Definition 2.15. A G -Higgs bundle (E, φ) is said to be **simple** if $\text{Aut}(E, \varphi) = \ker \iota \cap Z(H^{\mathbb{C}})$, where $Z(H^{\mathbb{C}})$ is the centre of $H^{\mathbb{C}}$.

Unlike the case of ordinary vector bundles, a stable G -Higgs bundle is not necessarily simple. However, we have the following infinitesimal result.

Proposition 2.16. *Any stable G -Higgs bundle (E, φ) with $\varphi \neq 0$ is infinitesimally simple.*

With respect to the question of the vanishing of \mathbb{H}^2 of the deformation complex, we have the following useful result.

Proposition 2.17. *Let G be a real semisimple group and let $G^{\mathbb{C}}$ be its complexification. Let (E, φ) be a G -Higgs bundle which is stable viewed as a $G^{\mathbb{C}}$ -Higgs bundle. Then the vanishing*

$$\mathbb{H}^0(C^\bullet(E, \varphi)) = 0 = \mathbb{H}^2(C^\bullet(E, \varphi))$$

holds.

The following result on smoothness of the moduli space can be proved, for example, from the standard slice method construction referred to above.

Proposition 2.18. *Let (E, φ) be a stable G -Higgs bundle. If (E, φ) is simple and*

$$\mathbb{H}^2(C^\bullet(E, \varphi)) = 0,$$

then (E, φ) is a smooth point in the moduli space. In particular, if (E, φ) is a simple G -Higgs bundle which is stable as a $G^{\mathbb{C}}$ -Higgs bundle, then it is a smooth point in the moduli space.

Suppose that G is semisimple and (E, φ) is stable and simple. If a (local) universal family exists then the dimension of the component of the moduli space containing (E, φ) equals the dimension of the infinitesimal deformation space $\mathbb{H}^1(C^\bullet(E, \varphi))$. We shall refer to this dimension as the **expected dimension** of the moduli space.

Moreover, since in this situation $\mathbb{H}^0(C^\bullet(E, \varphi)) = \mathbb{H}^2(C^\bullet(E, \varphi)) = 0$, the expected dimension can be calculated from Riemann–Roch as follows.

Proposition 2.19. *Let G be semisimple. Then the expected dimension of the moduli space of G -Higgs bundles is $(g - 1) \dim G^{\mathbb{C}}$.*

For a proper understanding of many aspects of the geometry of the moduli space of Higgs bundles, one needs to consider the moduli space as the gauge theory moduli space $\mathcal{M}_d^{\text{gauge}}(G)$. To define this, fix a C^∞ principal H -bundle \mathbf{E}_H with fixed topological class $d \in \pi_1(H)$ and consider the moduli space of solutions to **Hitchin's equations** for a pair (A, φ) consisting of an H -connection A and $\varphi \in \Omega^{1,0}(X, \mathbf{E}_H(\mathfrak{m}^{\mathbb{C}}))$:

$$(2.7) \quad \begin{aligned} F_A - [\varphi, \tau(\varphi)] &= 0 \\ \bar{\partial}_A \varphi &= 0. \end{aligned}$$

Here d_A is the covariant derivative associated to A and $\bar{\partial}_A$ is the $(0, 1)$ part of d_A , which defines a holomorphic structure on \mathbf{E}_H . The gauge group \mathcal{H} of \mathbf{E}_H acts on the space of solutions and the moduli space of solutions is

$$\mathcal{M}_d^{\text{gauge}}(G) := \{(A, \varphi) \text{ satisfying (2.7)}\} / \mathcal{H}.$$

Now, as a consequence of Theorem 2.6, we have that there is a homeomorphism $\mathcal{M}_d(G) \simeq \mathcal{M}_d^{\text{gauge}}(G)$.

This point of view is very relevant in particular for the Morse theoretic approach to the count of connected components, as explained in Section 5.1. Thus one should consider the infinitesimal deformation space of a solution (A, φ) to Hitchin's equations, which can be described as the first cohomology group of a certain elliptic deformation complex (cf.

Hitchin [27]). On the other hand, the formulation of the deformation theory in terms of hypercohomology is very convenient. Fortunately, at a smooth point of the moduli space, there is a natural isomorphism between the gauge theory deformation space and the infinitesimal deformation space $\mathbb{H}^1(C^\bullet(E, \varphi))$ (where the holomorphic structure on the Higgs bundle (E, φ) is given by $\bar{\partial}_A$). As in Donaldson–Kronheimer [16, § 6.4] this can be seen by using a Dolbeault resolution to calculate $\mathbb{H}^1(C^\bullet(E, \varphi))$ and using harmonic representatives of cohomology classes, via Hodge theory. For this reason we can freely apply the complex deformation theory described in this Section to the gauge theory situation.

3. $\mathrm{Sp}(2n, \mathbb{R})$ -HIGGS BUNDLES

3.1. Stability and simplicity of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. Let $G = \mathrm{Sp}(2n, \mathbb{R})$. The maximal compact subgroup of G is $H = \mathrm{U}(n)$ and hence $H^\mathbb{C} = \mathrm{GL}(n, \mathbb{C})$. Now, if $\mathbb{V} = \mathbb{C}^n$ is the fundamental representation of $\mathrm{GL}(n, \mathbb{C})$, then the isotropy representation space is:

$$\mathfrak{m}^\mathbb{C} = S^2\mathbb{V} \oplus S^2\mathbb{V}^*.$$

Let X be a compact Riemann surface. According to Definition 2.1, an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle over X is a triple (V, β, γ) consisting of a rank n holomorphic vector bundle V and holomorphic sections $\beta \in H^0(X, S^2V \otimes K)$ and $\gamma \in H^0(X, S^2V^* \otimes K)$, where K is the canonical line bundle of X . Some times we denote $\varphi = \beta + \gamma$.

Remark 3.1. When $H^\mathbb{C}$ is a classical group, like for $G = \mathrm{Sp}(2n, \mathbb{R})$, we prefer to work with the vector bundle V associated to the standard representation rather than the $H^\mathbb{C}$ -principal bundle. It is indeed in terms of V that we will describe the stability condition as we will see below.

Notation 3.2. Before giving a precise statement we introduce some notation. If W is a vector bundle and $W', W'' \subset W$ are subbundles, then $W' \otimes_S W''$ denotes the subbundle of the second symmetric power S^2W which is the image of $W' \otimes W'' \subset W \otimes W$ under the symmetrization map $W \otimes W \rightarrow S^2W$ (of course this should be defined in sheaf theoretical terms to be sure that $W' \otimes_S W''$ is indeed a subbundle, since the intersection of $W' \otimes W''$ and the kernel of the symmetrization map might change dimension from one fibre to the other). Also, we denote by $W'^\perp \subset W^*$ the kernel of the restriction map $W^* \rightarrow W'^*$.

According to [19, Theorem 4.9], the (semi)stability condition for an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle is equivalent to the following.

Definition 3.3. An $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, β, γ) over X is **semistable** if for any filtration of subbundles

$$0 \subset V_1 \subset V_2 \subset V$$

such that

$$(3.8) \quad \beta \in H^0(K \otimes (S^2V_2 + V_1 \otimes_S V)), \quad \gamma \in H^0(K \otimes (S^2V_1^\perp + V_2^\perp \otimes_S V^*)),$$

we have

$$(3.9) \quad \deg(V) - \deg(V_1) - \deg(V_2) \geq 0.$$

Furthermore, (V, β, γ) is **stable** if for any filtration as above, except the filtration $0 = V_1 \subset V_2 = V$ we have

$$(3.10) \quad \deg(V) - \deg(V_1) - \deg(V_2) > 0.$$

Remark 3.4. Note that when $\beta = \gamma = 0$, the semistability of (V, β, γ) is equivalent to the semistability of V with $\deg(V) = 0$.

Definition 3.5. A filtration $0 \subset V_1 \subset V_2 \subset V$ satisfying the first condition in (3.8) will be called β -**invariant**; if it satisfies the second condition will be called γ -**invariant**; and if it satisfies both will be said to be (β, γ) -**invariant**, or φ -**invariant**, where $\varphi = \beta + \gamma$.

The following will be useful many times below and explains the terminology just introduced.

Remark 3.6. Let V_i^\perp be the kernel of the projection $V^* \rightarrow V_i^*$ and view β and γ as symmetric maps $\beta : V^* \rightarrow K \otimes V$ and $\gamma : V \rightarrow K \otimes V^*$. If $0 \subset V_1 \subset V_2 \subset V$ is a filtration of vector bundles then for any $\beta \in H^0(K \otimes S^2 V)$ the condition $\beta \in H^0(K \otimes (S^2 V_2 + V_1 \otimes_S V))$ is equivalent to $\beta(V_2^\perp) \subset K \otimes V_1$ which, by symmetry of β , is equivalent to $\beta(V_1^\perp) \subset K \otimes V_2$. Similarly, for any $\gamma \in H^0(K \otimes S^2 V^*)$, the condition $\gamma \in H^0(K \otimes (S^2 V_1^\perp + V_2^\perp \otimes_S V^*))$ is equivalent to $\gamma(V_1) \subset K \otimes V_2^\perp$ which, by symmetry of γ , is equivalent to $\gamma(V_2) \subset K \otimes V_1^\perp$. Thus, if we use a local basis of V adapted to the filtration $0 \subseteq V_1 \subseteq V_2 \subseteq V$ and the dual basis of V^* , then the matrix of γ is of the form

$$\begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix},$$

while the matrix of β has the form

$$\begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix}.$$

Remark 3.7. There is yet another useful interpretation of (3.8) that will be used later. To explain this, let $Q_\gamma : V \times V \rightarrow K$ be the K -twisted symmetric bilinear pairing defined by γ as

$$Q_\gamma(u, v) := \gamma(v)(u), \quad \text{for } u, v \in V,$$

and denote, for a subbundle $V' \subset V$,

$$V'^{\perp\gamma} := \{v \in V \mid Q_\gamma(u, v) = 0 \text{ for every } u \in V'\} = \ker(V \rightarrow V'^* \otimes K; v \mapsto \gamma(v)|_{V'}).$$

Then it is immediate that

$$(3.11) \quad \gamma(V'^{\perp\gamma}) = V'^\perp \otimes K \subset V^* \otimes K.$$

From this it follows that, if we have a filtration $0 \subset V_1 \subset V_2 \subset V$ then the condition $\gamma(V_1) \subset K \otimes V_2^\perp$ is equivalent to $V_1 \subset V_2^{\perp\gamma}$. This is clearly equivalent to $V_2 \subset V_1^{\perp\gamma}$ which, in turn, is equivalent to $\gamma(V_2) \subset K \otimes V_1^\perp$. Similar reasoning applies to β .

The deformation complex (2.5) for an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle $(V, \varphi = \beta + \gamma)$ is

$$(3.12) \quad \begin{aligned} C^\bullet(V, \varphi) : \mathrm{End}(V) &\xrightarrow{\mathrm{ad}(\varphi)} S^2 V \otimes K \oplus S^2 V^* \otimes K \\ \psi &\mapsto (-\beta\psi^t - \psi\beta, \gamma\psi + \psi^t\gamma) \end{aligned}$$

Proposition 3.8. *An $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) is infinitesimally simple if and only if $\mathbb{H}^0(C^\bullet(V, \varphi)) = 0$. Equivalently, (V, φ) is infinitesimally simple if and only if there is a non-zero $\psi \in H^0(\mathrm{End}(V))$ such that*

$$\mathrm{ad}(\varphi)(\psi) = (-\beta\psi^t - \psi\beta, \gamma\psi + \psi^t\gamma) = (0, 0).$$

Proof. For $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles one has that $\ker(d\iota) = 0$. Thus the first statement is immediate from Definition 2.13. The equivalent statement now follows from the long exact sequence (2.6), recalling that in this case the deformation complex (2.5) is given by (3.12). \square

Proposition 3.9. *An $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) is simple if and only if $\mathrm{Aut}(V, \varphi) = \{\pm \mathrm{Id}\}$.*

Proof. Since $\lambda \in \mathbb{C}^* = Z(H^{\mathbb{C}})$ acts on the isotropy representation $\mathfrak{m}^{\mathbb{C}} = S^2\mathbb{V} \oplus S^2\mathbb{V}^*$ by $(\beta, \gamma) \mapsto (\lambda^2\beta, \lambda^{-2}\gamma)$ we have $\ker \iota \cap Z(H^{\mathbb{C}}) = \{\pm 1\}$, so the statement follows directly from Definition 2.15. \square

3.2. Stable and non-simple $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. The goal of this section is to obtain a complete understanding of how a stable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle can fail to be simple. The main result is Theorem 3.15. In order to state this theorem, we need to describe some special $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles arising from G -Higgs bundles associated to certain real subgroups $G \subseteq \mathrm{Sp}(2n, \mathbb{R})$.

The subgroup $G = \mathrm{U}(n)$. Observe that a $\mathrm{U}(n)$ -Higgs bundle is nothing but a holomorphic vector bundle V of rank n . The standard inclusion $v^{\mathrm{U}(n)}: \mathrm{U}(n) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$ gives the correspondence

$$(3.13) \quad V \mapsto v_*^{\mathrm{U}(n)}V = (V, 0)$$

associating the $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle $v_*^{\mathrm{U}(n)}V = (V, 0)$ to the holomorphic vector bundle V .

The subgroup $G = \mathrm{U}(p, q)$. In the following we assume that $p, q \geq 1$. As is easily seen, a $\mathrm{U}(p, q)$ -Higgs bundle (cf. [2, Definition 3.3]) is given by the data $(\tilde{V}, \tilde{W}, \tilde{\varphi} = \tilde{\beta} + \tilde{\gamma})$, where \tilde{V} and \tilde{W} are holomorphic vector bundles of rank p and q , respectively, $\tilde{\beta} \in H^0(K \otimes \mathrm{Hom}(\tilde{W}, \tilde{V}))$ and $\tilde{\gamma} \in H^0(K \otimes \mathrm{Hom}(\tilde{V}, \tilde{W}))$. Let $n = p + q$. The imaginary part of the standard indefinite Hermitian metric of signature (p, q) on \mathbb{C}^n is a symplectic form, and thus there is an inclusion $v^{\mathrm{U}(p, q)}: \mathrm{U}(p, q) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$. At the level of G -Higgs bundles, this gives rise to the correspondence

$$(3.14) \quad (\tilde{V}, \tilde{W}, \tilde{\varphi} = \tilde{\beta} + \tilde{\gamma}) \mapsto v_*^{\mathrm{U}(p, q)}(\tilde{V}, \tilde{W}, \tilde{\varphi}) = (V, \varphi = \beta + \gamma),$$

where

$$V = \tilde{V} \oplus \tilde{W}^*, \quad \beta = \begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{\beta} & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 0 \end{pmatrix}.$$

In the following we shall occasionally slightly abuse language, saying simply that $v_*^{\mathrm{U}(n)}V$ is a $\mathrm{U}(n)$ -Higgs bundle and that $v_*^{\mathrm{U}(p, q)}(\tilde{V}, \tilde{W}, \tilde{\varphi})$ is a $\mathrm{U}(p, q)$ -Higgs bundle.

Another piece of convenient notation is the following. Let (V_i, φ_i) be $\mathrm{Sp}(2n_i, \mathbb{R})$ -Higgs bundles and let $n = \sum n_i$. We can define an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) by setting

$$V = \bigoplus V_i \quad \text{and} \quad \varphi = \sum \varphi_i$$

by using the canonical inclusions $H^0(K \otimes (S^2V_i \oplus S^2V_i^*)) \subset H^0(K \otimes (S^2V \oplus S^2V^*))$. We shall slightly abuse language and write $(V, \varphi) = \bigoplus (V_i, \varphi_i)$, referring to this as **the direct sum** of the (V_i, φ_i) .

Remark 3.10. Note that $v_*^{\mathrm{U}(n)}V = (V, 0)$ is never simple as an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle, since its automorphism group contains the non-zero scalars \mathbb{C}^* . Similarly, the $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle $v_*^{\mathrm{U}(p,q)}(\tilde{V}, \tilde{W}, \tilde{\varphi})$ is not simple, since it has the automorphism $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We shall need a few lemmas for the proof of Theorem 3.15.

Lemma 3.11. *Let (V, φ) be an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle and assume that there is a non-trivial splitting $(V, \varphi) = (V_a \oplus V_b, \varphi_a + \varphi_b)$ such that $\varphi_\nu \in H^0(K \otimes (S^2V_\nu \oplus S^2V_\nu^*))$ for $\nu = a, b$. Assume that the $\mathrm{Sp}(2n_a, \mathbb{R})$ -Higgs bundle (V_a, φ_a) is not stable. Then (V, φ) is not stable.*

Proof. Since (V_a, φ_a) is not stable there is a filtration $0 \subset V_{a1} \subset V_{a2} \subset V_a$ such that

$$\beta \in H^0(K \otimes (S^2V_{a2} + V_{a1} \otimes_S V)), \quad \gamma \in H^0(K \otimes (S^2V_{a1}^\perp + V_{a2}^\perp \otimes_S V^*))$$

and

$$(3.15) \quad \deg(V_a) - \deg(V_{a1}) - \deg(V_{a2}) \leq 0.$$

Consider the filtration $0 \subset V_1 \subset V_2 \subset V$ obtained by setting

$$V_1 = V_{a1}, \quad V_2 = V_{a2} \oplus V_b.$$

Using Remark 3.6 one readily sees that this filtration satisfies the conditions (3.8). Since

$$\deg(V) - \deg(V_1) - \deg(V_2) = \deg(V_a) - \deg(V_{a1}) - \deg(V_{a2}),$$

it follows from (3.15) that (V, φ) is not stable. \square

Lemma 3.12. *Let (V, φ) be an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle and assume that there is a non-trivial splitting $V = V_a \oplus V_b$ such that $\varphi \in H^0(K \otimes (S^2V_a \oplus S^2V_a^*))$. In other words, $(V, \varphi) = (V_a \oplus V_b, \varphi_a + 0)$ with $(V_b, 0) = v_*^{\mathrm{U}(n_b)}V_b$. Then (V, φ) is not stable.*

Proof. It is immediate from Lemma 3.11 and Remark 3.4 that V_b is a stable vector bundle with $\deg(V_b) = 0$. Hence

$$\deg(V) = \deg(V_a).$$

Consider the filtration $0 \subset V_1 \subset V_2 \subset V$ obtained by setting $V_1 = 0$ and $V_2 = V_a$. As before this filtration satisfies (3.8). Therefore the calculation

$$\deg(V) - \deg(V_1) - \deg(V_2) = \deg(V) - \deg(V_a) = 0$$

shows that (V, φ) is not stable. \square

Lemma 3.13. *Let $(V, \varphi) = v_*^{\mathrm{U}(p,q)}(V_a, V_b^*, \tilde{\varphi})$ be an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle arising from a $\mathrm{U}(p, q)$ -Higgs bundle $(V_a, V_b^*, \tilde{\varphi})$ with $p, q \geq 1$. Then (V, φ) is not stable.*

Proof. The $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) is given by

$$V = V_a \oplus V_b, \quad \beta = \begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{\beta} & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 0 \end{pmatrix}.$$

Let $V_1 = V_2 = V_a$ and consider the filtration $0 \subset V_1 \subset V_2 \subset V$. Again this filtration satisfies the conditions (3.8). Thus, if (V, φ) is stable, we have from (3.10)

$$\deg(V) - 2\deg(V_a) < 0.$$

Similarly, considering $V_1 = V_2 = V_b$, we obtain

$$\deg(V) - 2\deg(V_b) < 0,$$

so we conclude that

$$\deg(V) = \deg(V_a) + \deg(V_b) < \deg(V),$$

which is absurd. □

Lemma 3.14. *Let $(\tilde{V}, \tilde{\varphi})$ be an $\mathrm{Sp}(2\tilde{n}, \mathbb{R})$ -Higgs bundle. Then the $\mathrm{Sp}(4\tilde{n}, \mathbb{R})$ -Higgs bundle $(\tilde{V} \oplus \tilde{V}, \tilde{\varphi} + \tilde{\varphi})$ is not stable.*

Proof. Consider the automorphism $f = \frac{1}{\sqrt{2i}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ of $V = \tilde{V} \oplus \tilde{V}$. Write $\beta = \begin{pmatrix} \tilde{\beta} & 0 \\ 0 & \tilde{\beta} \end{pmatrix}$ and $\gamma = \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & \tilde{\gamma} \end{pmatrix}$. Then we have that

$$(V, \varphi) \simeq (\tilde{V} \oplus \tilde{V}, f \cdot \beta + f \cdot \gamma),$$

where

$$f \cdot \beta = f\beta f^t = \begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{\beta} & 0 \end{pmatrix} \quad \text{and} \quad f \cdot \gamma = (f^t)^{-1}\gamma f^{-1} = \begin{pmatrix} 0 & \tilde{\gamma} \\ \tilde{\gamma} & 0 \end{pmatrix}.$$

We shall see that $(\tilde{V} \oplus \tilde{V}, f \cdot \beta + f \cdot \gamma)$ is not stable. To this end, consider the filtration $0 \subset V_1 \subset V_2 \subset \tilde{V} \oplus \tilde{V}$ obtained by setting $V_1 = V_2 = \tilde{V}$. This satisfies (3.8). But, on the other hand,

$$\deg(\tilde{V} \oplus \tilde{V}) - \deg(V_1) - \deg(V_2) = 0$$

so $(\tilde{V} \oplus \tilde{V}, f \cdot \beta + f \cdot \gamma)$ is not stable. □

Theorem 3.15. *Let (V, φ) be a stable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle. If (V, φ) is not simple, then one of the following alternatives occurs:*

- (1) *The vanishing $\varphi = 0$ holds and V is a stable vector bundle of degree zero. In this case, $\mathrm{Aut}(V, \varphi) \simeq \mathbb{C}^*$.*
- (2) *There is a nontrivial decomposition, unique up to reordering,*

$$(V, \varphi) = \left(\bigoplus_{i=1}^k V_i, \sum_{i=1}^k \varphi_i \right)$$

with $\varphi_i = \beta_i + \gamma_i \in H^0(K \otimes (S^2V_i \oplus S^2V_i^))$, such that each (V_i, φ_i) is a stable and simple $\mathrm{Sp}(2n_i, \mathbb{R})$ -Higgs bundle. Furthermore, each $\varphi_i \neq 0$ and $(V_i, \varphi_i) \not\cong (V_j, \varphi_j)$ for $i \neq j$. The automorphism group of (V, φ) is*

$$\mathrm{Aut}(V, \varphi) \simeq \mathrm{Aut}(V_1, \varphi_1) \times \cdots \times \mathrm{Aut}(V_k, \varphi_k) \simeq (\mathbb{Z}/2)^k.$$

Proof. First of all, we note that if $\varphi = 0$ then it is immediate from Remark 3.4 that alternative (1) occurs.

Next, consider the case $\varphi \neq 0$. Since (V, φ) is not simple, there is an automorphism $\sigma \in \mathrm{Aut}(V, \varphi) \setminus \{\pm 1\}$. We know from Lemma 2.25 in [19] that $\mathrm{Aut}(V, \varphi)$ is reductive. This implies that σ may be chosen to be semisimple, so that there is a splitting $V = \bigoplus V_i$ in eigenbundles of σ such that the action of σ on V_i is given by multiplication by some $\sigma_i \in \mathbb{C}^*$. If σ were a multiple of the identity, say $\sigma = \lambda \mathrm{Id}$ with $\lambda \in \mathbb{C}^*$, then it would act on $\varphi = \beta + \gamma$ by $\beta \mapsto \lambda^2 \beta$ and $\gamma \mapsto \lambda^{-2} \gamma$. Since $\varphi \neq 0$ this would force σ to be equal to 1 or -1 , in contradiction with our choice. Hence σ is not a multiple of the identity, so the

decomposition $V = \bigoplus V_i$ has more than one summand. The action of σ on $S^2V \oplus S^2V^*$ is given by

$$(3.16) \quad \sigma = \sigma_i \sigma_j: V_i \otimes V_j \rightarrow V_i \otimes V_j \quad \text{and} \quad \sigma = \sigma_i^{-1} \sigma_j^{-j}: V_i^* \otimes V_j^* \rightarrow V_i^* \otimes V_j^*.$$

If we denote by $\varphi_{ij} = \beta_{ij} + \gamma_{ij}$ the component of φ in $H^0(K \otimes (V_i \otimes V_j \oplus V_i^* \otimes V_j^*))$ (symmetrizing the tensor product if $i = j$), then

$$(3.17) \quad \sigma_i \sigma_j \neq 1 \implies \varphi_{ij} = 0.$$

Suppose that $\varphi_{i_0 j_0} \neq 0$ for some $i_0 \neq j_0$. From (3.17) we conclude that $\sigma_{i_0} \sigma_{j_0} = 1$. But then $\sigma_i \sigma_{j_0} \neq 1$ for $i \neq i_0$ and $\sigma_{i_0} \sigma_j \neq 1$ for $j \neq j_0$. Hence, again by (3.17), $\varphi_{ij_0} = 0 = \varphi_{i_0 j}$ if $i \neq i_0$ or $j \neq j_0$. Thus $(V_{i_0}, V_{j_0}^*, \varphi_{i_0 j_0})$ is a $U(p, q)$ -Higgs bundle and we have a non-trivial decomposition $(V, \varphi) = (V_a \oplus V_b, \varphi_a + \varphi_b)$ with $(V_a, \varphi_a) = v_*^{U(p, q)}(V_{i_0}, V_{j_0}^*, \varphi_{i_0 j_0})$. By Lemma 3.13 the $\text{Sp}(2n_a, \mathbb{R})$ -Higgs bundle (V_a, φ_a) is not stable so, by Lemma 3.11, (V, φ) is not stable. This contradiction shows that $\varphi_{ij} = 0$ for $i \neq j$.

It follows that $\varphi = \sum \varphi_i$ with $\varphi_i \in H^0(K \otimes (S^2V_i \oplus S^2V_i^*))$. By Lemma 3.11 each of the summands (V_i, φ_i) is a stable $\text{Sp}(2n, \mathbb{R})$ -Higgs bundle and by Lemma 3.12 each φ_i must be non-zero. Also, from (3.16), $\sigma \cdot \beta_i = \sigma_i^2 \beta_i$ and $\sigma \cdot \gamma_i = \sigma_i^{-2} \gamma_i$ so we conclude that the only possible eigenvalues of σ are 1 and -1 . Thus the decomposition $(V, \varphi) = \bigoplus (V_i, \varphi_i)$ has in fact only two summands and, more importantly, $\sigma^2 = 1$. This means that all non-trivial elements of $\text{Aut}(V, \varphi)$ have order two and therefore $\text{Aut}(V, \varphi)$ is abelian (indeed: if $\sigma, \tau \in \text{Aut}(V, \varphi)$ then we have $\sigma^2 = \tau^2 = (\tau\sigma)^2 = 1$, but $(\tau\sigma)^2 = \tau\sigma\tau\sigma = 1$ implies, multiplying both sides on the left by τ and then by σ , that $\tau\sigma = \sigma\tau$).

Now, the summands (V_i, φ_i) may not be simple but, applying the preceding argument inductively to each of the (V_i, φ_i) , we eventually obtain a decomposition $(V, \varphi) = (\bigoplus V_i, \sum \varphi_i)$ where each (V_i, φ_i) is stable and simple, and $\varphi_i \neq 0$. Since $\text{Aut}(V, \varphi)$ is abelian, the successive decompositions of V in eigenspaces can in fact be carried out simultaneously for all $\sigma \in \text{Aut}(V, \varphi) \setminus \{\pm 1\}$. From this the uniqueness of the decomposition and the statement about the automorphism group of (V, φ) are immediate.

Finally, Lemma 3.12 and Lemma 3.14 together imply that the (V_i, φ_i) are mutually non-isomorphic. \square

3.3. Polystable $\text{Sp}(2n, \mathbb{R})$ -Higgs bundles. According to [19, Theorem 4.9] the polystability condition for an $\text{Sp}(2n, \mathbb{R})$ -Higgs bundle is equivalent to the following.

Definition 3.16. A $\text{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) is **polystable** if is semistable and for any filtration of subbundles

$$0 \subset V_1 \subset V_2 \subset V$$

distinct from the filtration $0 = V_1 \subset V_2 = V$ such that

$$(3.18) \quad \beta \in H^0(K \otimes (S^2V_2 + V_1 \otimes_S V)), \quad \gamma \in H^0(K \otimes (S^2V_1^\perp + V_2^\perp \otimes_S V^*)),$$

and

$$\deg(V) - \deg(V_1) - \deg(V_2) = 0,$$

there exists an isomorphism of holomorphic vector bundles

$$\sigma: V \rightarrow V_1 \oplus V_2/V_1 \oplus V/V_2$$

satisfying the following properties:

$$(a) \quad V_1 = \sigma^{-1}(V_1), \quad V_2 = \sigma^{-1}(V_1 \oplus V_2/V_1),$$

- (b) $\beta \in H^0(K \otimes (S^2(\sigma^{-1}(V_2/V_1)) \oplus \sigma^{-1}(V_1) \otimes_S \sigma^{-1}(V/V_2)))$,
 (c) $\gamma \in H^0(K \otimes (S^2(\sigma^*(V_2/V_1)^*) \oplus \sigma^*(V_1^*) \otimes_S \sigma^*(V/V_2)^*))$.

Remark 3.17. The polystability condition can be rephrased by saying that there are subbundles F_1, F_2 and F_3 of V , such that $V = F_1 \oplus F_2 \oplus F_3$, with $V_1 = F_1, V_2 = F_1 \oplus F_2$, and $\beta(F_i^*) \subset F_{4-i} \otimes K$ and $\gamma(F_i) \subset F_{4-i}^* \otimes K$ for $i = 1, 2, 3$.

We have the following result [19, Proposition 4.14] on the structure of polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles.

Proposition 3.18. *A $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) is polystable if and only if there is a decomposition*

$$(V, \varphi) = (V_1, \varphi_1) \oplus \cdots \oplus (V_k, \varphi_k),$$

unique up to reordering, such that each (V_i, φ_i) is a stable and simple G_i -Higgs bundle of one of the following mutually exclusive types:

- (1) $G_i = \mathrm{Sp}(2n_i, \mathbb{R})$ and non-vanishing Higgs field;
- (2) $G_i = \mathrm{U}(p_i, q_i)$ with $\deg \tilde{V}_i + \deg \tilde{W}_i = 0$, where \tilde{V}_i and \tilde{W}_i are defined by (3.14);
- (3) $G_i = \mathrm{U}(n_i)$ and $\deg(V_i) = 0$.

3.4. L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pairs. We study now L -twisted G -Higgs pairs for $G = \mathrm{GL}(n, \mathbb{R})$. They will appear for $L = K$ in Section 3.5. When $L = K^2$, these will be related to maximal degree $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles as we will see in Section 4.

A maximal compact subgroup of $\mathrm{GL}(n, \mathbb{R})$ is $H = \mathrm{O}(n)$ and hence $H^{\mathbb{C}} = \mathrm{O}(n, \mathbb{C})$. Now, if \mathbb{W} is the standard n -dimensional complex vector space representation of $\mathrm{O}(n, \mathbb{C})$, and Q is the bilinear form defining $\mathrm{O}(n, \mathbb{C})$, then the isotropy representation space is:

$$\mathfrak{m}^{\mathbb{C}} = S_Q^2 \mathbb{W} := \{\xi \in \mathrm{End}(\mathbb{W}) \mid Q(\xi \cdot, \cdot) = Q(\cdot, \xi \cdot)\} \subset \mathrm{End}(\mathbb{W}).$$

Given a $\mathrm{O}(n, \mathbb{C})$ -bundle (W, Q) , denote by $S_Q^2 W$ the bundle of endomorphisms ξ of W which are symmetric with respect to Q i.e. such that $Q(\xi \cdot, \cdot) = Q(\cdot, \xi \cdot)$. An L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair over X is thus a triple (W, Q, ψ) consisting of a holomorphic $\mathrm{O}(n, \mathbb{C})$ -bundle, i.e. a rank n holomorphic vector bundle W over X equipped with a non-degenerate quadratic form Q , and a section

$$\psi \in H^0(L \otimes S_Q^2 W).$$

Note that when $\psi = 0$ a twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair is simply an orthogonal bundle.

Remark 3.19. Given a $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair (W, Q, ψ) , we can consider the symmetric isomorphism $q : W \xrightarrow{\cong} W^*$ given by $q(v) = Q(v, \cdot)$ and define the homomorphism

$$\tilde{\psi} : W^* \longrightarrow W \otimes L$$

by

$$\tilde{\psi} := \psi q^{-1}.$$

It follows from the symmetry of ψ with respect to Q that $\tilde{\psi}$ is symmetric i.e. $\tilde{\psi}^t \otimes I_L = \tilde{\psi}$. In other words,

$$\tilde{\psi} \in H^0(S^2 W \otimes L).$$

We can thus regard a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle as a triple $(W, Q, \tilde{\psi})$, where (W, Q) is a $\mathrm{O}(n, \mathbb{C})$ -bundle and $\tilde{\psi} \in H^0(S^2 W \otimes L)$.

Remark 3.20. Since the centre of $\mathfrak{o}(n)$ is trivial, $\alpha = 0$ is the only possible value for which stability of an L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair is defined.

According to [19, Theorem 4.17] the stability conditions (for $\alpha = 0$) for an L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair can be defined as follows.

Definition 3.21. Let (W, Q, ψ) be a L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair. Then (W, Q, ψ) is **semistable** if and only if $\deg(W') \leq 0$ for any isotropic and ψ -invariant subbundle $W' \subset W$. Furthermore, (W, Q, ψ) is **stable** if and only if it is semistable and $\deg(W') < 0$ for any isotropic and ψ -invariant strict subbundle $0 \neq W' \subset W$. Finally, (W, Q, ψ) is **polystable** if and only if it is semistable and, for any isotropic (respectively. coisotropic) and ψ -invariant strict subbundle $0 \neq W' \subset W$ such that $\deg(W') = 0$, there is another coisotropic (resp. isotropic) and ψ -invariant subbundle $0 \neq W'' \subset W$ such that $W \simeq W' \oplus W''$.

Remark 3.22. Let $W' \subset W$ be a subbundle and let W'^{\perp_Q} be its orthogonal complement with respect to Q . Then W' is isotropic if and only if W'^{\perp_Q} is coisotropic. Moreover, since ψ is symmetric with respect to Q , the subbundle W' is ψ -invariant if and only if W'^{\perp_Q} is ψ -invariant.

Proposition 3.23. *Let (W, Q, ψ) be a polystable L -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair and let $W' \subset W$ be an isotropic ψ -invariant subbundle such that $\deg(W') = 0$. Then there is a decomposition*

$$W \simeq W' \oplus W_2 \oplus \tilde{W}'$$

in which

- (1) each of the subbundles W' , W_2 and \tilde{W}' is ψ -invariant,
- (2) the quadratic form Q restricted to W_2 is non-degenerate, and
- (3) the subbundles W' and \tilde{W}' are isotropic and in duality with each other under Q .

Moreover, if we let $W_1 = W' \oplus \tilde{W}'$ and, for $i = 1, 2$, $\psi_i = \psi|_{W_i}$ and $Q_i = Q|_{W_i}$, then the (W_i, Q_i, ψ_i) are polystable L -twisted $\mathrm{GL}(n_i, \mathbb{R})$ -Higgs pairs and

$$(W, Q, \psi) = (W_1 \oplus W_2, Q_1 + Q_2, \psi_1 + \psi_2).$$

Proof. We have from Remark 3.22 that W'^{\perp_Q} is coisotropic and ψ -invariant. Thus, since (W, Q, ψ) is polystable, W'^{\perp_Q} has an isotropic ψ -invariant complement \tilde{W}' . Clearly, the subbundles W' and \tilde{W}' are isotropic and in duality with each other under Q , so that the restriction $Q_1 = Q|_{W_1}$ is non-degenerate (where $W_1 = W' \oplus \tilde{W}'$). Now define $W_2 = (W' \oplus W'^{\perp_Q})^{\perp_Q}$. Then, clearly, the restriction $Q_2 = Q|_{W_2}$ is non-degenerate and ψ -invariant.

For the final statement, it only remains to prove polystability of (W_i, Q_i, ψ_i) . But this follows from polystability of (W, Q, ψ) , noting the following facts: firstly, that any ψ -invariant isotropic subbundle of (W_i, Q_i, ψ_i) is also a ψ -invariant isotropic subbundle of (W, Q, ψ) and, secondly, that any ψ -invariant coisotropic subbundle of (W_i, Q_i, ψ_i) extends naturally to a ψ -invariant coisotropic subbundle of (W, Q, ψ) . \square

3.5. Comparison of stability conditions for $\mathrm{Sp}(2n, \mathbb{R})$ -, $\mathrm{Sp}(2n, \mathbb{C})$ - and $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundles. An $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle can be viewed as a Higgs bundle for the larger complex groups $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SL}(2n, \mathbb{C})$. The goal of this section is to understand the relation between the various corresponding stability notions. The main results are Theorems 3.26 and 3.27 below.

If $G = \mathrm{SL}(n, \mathbb{C})$ then the maximal compact subgroup of G is $H = \mathrm{SU}(n)$ and hence $H^{\mathbb{C}}$ coincides with $\mathrm{SL}(n, \mathbb{C})$. Now, if $\mathbb{W} = \mathbb{C}^n$ is the fundamental representation of $\mathrm{SL}(n, \mathbb{C})$, the isotropy representation space is given by the traceless endomorphisms of \mathbb{W}

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{sl}(\mathbb{W}) = \{\xi \in \mathrm{End}(\mathbb{W}) \mid \mathrm{Tr} \xi = 0\} \subset \mathrm{End} \mathbb{W},$$

so it coincides again with the adjoint representation of $\mathrm{SL}(n, \mathbb{C})$ on its Lie algebra. An $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundle is thus a pair consisting of a rank n holomorphic vector bundle W over X endowed with a trivialization $\det W \simeq \mathcal{O}$ and a holomorphic section

$$\Phi \in H^0(K \otimes \mathrm{End}_0 W),$$

where $\mathrm{End}_0 W$ denotes the bundle of traceless endomorphisms of W .

Again we refer the reader to [19] for the general statement of the stability conditions for $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles. (Semi)stability is equivalent in this case to the original notions given by Hitchin in [27] (see [19, Theorem 4.4]).

Definition 3.24. An $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundle (W, Φ) is **semistable** if and only if for any subbundle $W' \subset W$ such that $\Phi(W') \subset K \otimes W'$ we have $\deg W' \leq 0$. Furthermore, (W, Φ) is **stable** if for any nonzero and strict subbundle $W' \subset W$ such that $\Phi(W') \subset K \otimes W'$ we have $\deg W' < 0$. Finally, (W, Φ) is **polystable** if it is semistable and for each subbundle $W' \subset W$ such that $\Phi(W') \subset K \otimes W'$ and $\deg W' = 0$ there is another subbundle $W'' \subset W$ satisfying $\Phi(W'') \subset K \otimes W''$ and $W = W' \oplus W''$.

Consider now the case $G = \mathrm{Sp}(2n, \mathbb{C})$. A maximal compact subgroup of G is $H = \mathrm{Sp}(2n)$ and hence $H^{\mathbb{C}}$ coincides with $\mathrm{Sp}(2n, \mathbb{C})$. Now, if $\mathbb{W} = \mathbb{C}^{2n}$ is the fundamental representation of $\mathrm{Sp}(2n, \mathbb{C})$ and ω denotes the standard symplectic form on \mathbb{W} , the isotropy representation space is

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{sp}(\mathbb{W}) = \mathfrak{sp}(\mathbb{W}, \omega) := \{\xi \in \mathrm{End}(\mathbb{W}) \mid \omega(\xi \cdot, \cdot) + \omega(\cdot, \xi \cdot) = 0\} \subset \mathrm{End} \mathbb{W},$$

so it coincides with the adjoint representation of $\mathrm{Sp}(2n, \mathbb{C})$ on its Lie algebra. An $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle is thus a pair consisting of a rank $2n$ holomorphic symplectic vector bundle (W, Ω) over X (so Ω is a holomorphic section of $\Lambda^2 W^*$ whose restriction to each fibre of W is non degenerate) and a section

$$\Phi \in H^0(K \otimes \mathfrak{sp}(W)),$$

where $\mathfrak{sp}(W)$ is the vector bundle whose fibre over x is given by $\mathfrak{sp}(W_x, \Omega_x)$.

As for $\mathrm{SL}(n, \mathbb{C})$, we refer the reader to [19] for the general statement of the stability conditions for $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundles. We now have the following analogue of Proposition 3.3, which implies that the definition of (semi)stability from [19, Theorem 4.2] coincides with the usual one in the literature in the case $\Phi = 0$ (cf. Ramanathan [37]). Recall that if (W, Ω) is a symplectic vector bundle, a subbundle $W' \subset W$ is said to be isotropic if the restriction of Ω to W' is identically zero.

Definition 3.25. An $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle $((W, \Omega), \Phi)$ is **semistable** if and only if for any isotropic subbundle $W' \subset W$ such that $\Phi(W') \subset K \otimes W'$ we have $\deg W' \leq 0$. Furthermore, $((W, \Omega), \Phi)$ is **stable** if for any nonzero and strict isotropic subbundle $0 \neq W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ we have $\deg W' < 0$. Finally, $((W, \Omega), \Phi)$ is **polystable** if it is semistable and for any nonzero and strict isotropic (resp., coisotropic) subbundle $W' \subset W$ such that $\Phi(W') \subset L \otimes W'$ and $\deg W' = 0$ there is a coisotropic (resp., isotropic) subbundle $W'' \subset W$ such that $\Phi(W'') \subset L \otimes W''$ and $W = W' \oplus W''$.

Given an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) with $\varphi = (\beta, \gamma) \in H^0(K \otimes (S^2V \oplus S^2V^*))$ one can associate to it an $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle $((W, \Omega), \Phi)$ given by

$$(3.19) \quad W = V \oplus V^*, \quad \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad \text{and} \quad \Omega((v, \xi), (w, \eta)) = \xi(w) - \eta(v),$$

for local holomorphic sections v, w of V and ξ, η of V^* (i.e. Ω is the canonical symplectic structure on $V \oplus V^*$).

Since $\mathrm{Sp}(2n, \mathbb{C}) \subset \mathrm{SL}(2n, \mathbb{C})$, every $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle $((W, \Omega), \Phi)$ gives rise to an $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundle (W, Φ) . If $((W, \Omega), \Phi)$ is obtained from an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) we denote the associated $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundle by

$$H(V, \varphi) = (W, \Phi) = (V \oplus V^*, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}).$$

Theorem 3.26. *Let $(V, \varphi = (\beta, \gamma))$ be an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle and let $(W, \Phi) = H(V, \varphi)$ be the corresponding $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundle. Then*

- (1) *if (W, Φ) is stable then (V, φ) is stable;*
- (2) *if (V, φ) is stable and simple then (W, Φ) is stable unless there is an isomorphism $f: V \xrightarrow{\sim} V^*$ such that $\beta f = f^{-1}\gamma$, in which case (W, Φ) is polystable;*
- (3) *(W, Φ) is semistable if and only if (V, φ) is semistable.*
- (4) *(W, Φ) is polystable if and only if (V, φ) is polystable;*

In particular, if $\deg(V) \neq 0$ then (W, Φ) is stable if and only if (V, φ) is stable.

For the statement of the following theorem, recall from Section 3.4 that a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle is given by $((W, Q), \psi)$, where (W, Q) is rank n orthogonal bundle and $\psi \in H^0(K \otimes S^2W)$. The stability condition for $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles is given in Proposition 3.21.

Theorem 3.27. *Let (V, φ) be a stable and simple $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle. Then (V, φ) is stable as an $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle, unless there is a symmetric isomorphism $f: V \xrightarrow{\sim} V^*$ such that $\beta f = f^{-1}\gamma$. Moreover, if such an f exists, let $\psi = \beta = f^{-1}\gamma f^{-1} \in H^0(K \otimes S^2V)$. Then the $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle $((V, f), \psi)$ is stable, even as a $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle.*

The proof of Theorem 3.26 is given below in Section 3.6 and the proof of Theorem 3.27 is given below in Section 3.7.

The following observation is not essential for our main line of argument. We include it since it might be of independent interest.

Remark 3.28. Suppose we are in Case (2) of Theorem 3.26. Decompose $f = f_s + f_a: V \xrightarrow{\sim} V$ in its symmetric and anti-symmetric parts, given by $f_s = \frac{1}{2}(f + f^t)$ and $f_a = \frac{1}{2}(f - f^t)$. Let $V_a = \ker(f_s)$ and $V_s = \ker(f_a)$. Both V_a and V_s are vector bundles, since the ranks of f_s and f_a (which coincide with the multiplicities of -1 and 1 respectively as eigenvalues of f) are constant. There is then a decomposition $V = V_a \oplus V_s$ and f decomposes as

$$f = \begin{pmatrix} f_s & 0 \\ 0 & f_a \end{pmatrix} : V_s \oplus V_a \rightarrow V_s^* \oplus V_a^*.$$

Write $\gamma_{sa}: V_a \rightarrow V_s^* \otimes K$ for the component of γ in $H^0(K \otimes V_a^* \otimes V_s^*)$ and similarly for the other mixed components of β and γ . Since f intertwines β and γ , one has that $\gamma_{as} = f_a \beta_{as} f_s$. Hence

$$\gamma_{sa} = \gamma_{as}^t = f_s^t \beta_{as}^t f_a^t = -f_s \beta_{sa} f_a = -\gamma_{sa}.$$

It follows that $\gamma_{sa} = 0$ and similarly for the other mixed terms. Thus there is a decomposition $(V, \varphi) = (V_s \oplus V_a, \varphi_s + \varphi_a)$. If (V, φ) is simple then one of the summands must be trivial. The case when $(V, \varphi) = (V_s, \varphi_s)$ is the one covered in Theorem 3.27. In the other case, when $(V, \varphi) = (V_a, \varphi_a)$, the antisymmetric map f defines a symplectic form on V . If we let $\psi = \beta f = f^{-1}\gamma$, one easily checks that ψ is symplectic. Thus, in this case, (V, φ) comes in fact from an $\mathrm{Sp}(n, \mathbb{C})$ -Higgs bundle $((V, f), \psi)$. This is a stable $\mathrm{Sp}(n, \mathbb{C})$ -Higgs bundle, since (V, ψ) is a stable $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle (cf. the proof of Theorem 3.27 below).

3.6. Proof of Theorem 3.26. The proof of the theorem is split into several lemmas.

The following lemma proves (1) of Theorem 3.26.

Lemma 3.29. *Let $(V, \varphi = (\beta, \gamma))$ be an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle, and let*

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : V \oplus V^* \rightarrow K \otimes (V \oplus V^*).$$

The pair (V, φ) is semistable if and only if for any pair of subbundles $A \subset V$ and $B \subset V^$ satisfying $B^\perp \subset A$ (equivalently, $A^\perp \subset B$) and $\Phi(A \oplus B) \subset K \otimes (A \oplus B)$ we have $\deg(A \oplus B) \leq 0$.*

The pair (V, φ) is stable if and only if it is semistable and for any pair of subbundles $A \subset V$ and $B \subset V^$, such that $A \neq V$ and $B \neq V^*$, satisfying $B^\perp \subset A$ (equivalently, $A^\perp \subset B$) and $\Phi(A \oplus B) \subset K \otimes (A \oplus B)$, the inequality $\deg(A \oplus B) < 0$ holds.*

Proof. Suppose that $A \subset V$ and $B \subset V^*$ satisfy the conditions of the lemma. Then setting $V_2 := A$ and $V_1 := B^\perp$ we obtain a filtration $0 \subset V_1 \subset V_2 \subset V$ which, thanks to Remark 3.6, satisfies (3.8).

Conversely, given a filtration $0 \subset V_1 \subset V_2 \subset V$ for which (3.8) holds, we get subbundles $A := V_2 \subset V$ and $B := V_1^\perp \subset V^*$ satisfying the conditions of the lemma. Finally, we have

$$\deg(A \oplus B) = \deg(V_1^\perp \oplus V_2) = \deg(V_1) + \deg(V_2) - \deg(V),$$

so the lemma follows from Proposition 3.3. (For the case of stability, note that the condition $A \neq V$ and $B \neq V^*$ corresponds to the filtration $0 \subset V_1 \subset V_2 \subset V$ not being the trivial filtration $0 = V_1 \subset V_2 = V$.) \square

Remark 3.30. In the proof we have used the following formula: if $F \subset E$ is an inclusion of vector bundles, then $\deg F^\perp = \deg F - \deg E$. To check this, observe that there is an exact sequence $0 \rightarrow F^\perp \rightarrow E^* \rightarrow F^* \rightarrow 0$, and apply the additivity of the degree w.r.t. exact sequences together with $\deg E^* = -\deg E$ and $\deg F^* = -\deg F$.

The following lemma proves (2) of Theorem 3.26.

Lemma 3.31. *Suppose that (V, φ) is semistable, and define $\Phi: V \oplus V^* \rightarrow K \otimes (V \oplus V^*)$ as previously. Then any subbundle $0 \neq W' \subsetneq V \oplus V^*$ such that $\Phi(W') \subset K \otimes W'$ satisfies $\deg W' \leq 0$. Furthermore, if (V, φ) is stable and simple, one can get equality only if there is an isomorphism $f: V \rightarrow V^*$ such that $\beta f = f^{-1}\gamma$, and in this case $(W, \Phi) = H(V, \varphi)$ is polystable.*

Proof. Fix a subbundle $W' \subset V \oplus V^*$ satisfying $\Phi(W') \subset K \otimes W'$. We prove the lemma in various steps.

1. Denote by $p : V \oplus V^* \rightarrow V$ and $q : V \oplus V^* \rightarrow V^*$ the projections, and define subsheaves $A = p(W')$ and $B = q(W')$. It follows from $\Phi(W') \subset K \otimes W'$ that $\beta(B) \subset K \otimes A$ and $\gamma(A) \subset K \otimes B$ (for example, using that $\Phi p = q\Phi$ and $\Phi q = p\Phi$). Since both β and γ are symmetric we deduce that $\beta(A^\perp) \subset K \otimes B^\perp$ and $\gamma(B^\perp) \subset K \otimes A^\perp$ as well. It follows from this that if we define subsheaves

$$A_0 = A + B^\perp \subset V \quad \text{and} \quad B_0 = B + A^\perp \subset V^*$$

then we have $B_0^\perp \subset A_0$, $A_0^\perp \subset B_0$ and $\Phi(A_0 \oplus B_0) \subset K \otimes (A_0 \oplus B_0)$.

We can apply Lemma 3.29 also to subsheaves by replacing any subsheaf of V or V^* by its saturation, which is now a subbundle of degree not less than that of the subsheaf. Hence we deduce that

$$(3.20) \quad \deg A_0 + \deg B_0 = \deg(A + B^\perp) + \deg(B + A^\perp) \leq 0,$$

and equality holds if and only if $A + B^\perp = V$ and $B + A^\perp = V^*$.

Now we compute (using repeatedly the formula in Remark 3.30)

$$\begin{aligned} \deg(A + B^\perp) &= \deg A + \deg B^\perp - \deg(A \cap B^\perp) \\ &= \deg A + \deg B - \deg V^* - \deg((A^\perp + B)^\perp) \\ &= \deg A + \deg B - \deg V^* - \deg(A^\perp + B) + \deg V^* \\ &= \deg A + \deg B - \deg(A^\perp + B). \end{aligned}$$

Consequently $\deg A + \deg B = \deg(A + B^\perp) + \deg(A^\perp + B)$, so (3.20) implies that

$$(3.21) \quad \deg A + \deg B \leq 0,$$

with equality if and only if $A + B^\perp = V$ and $B + A^\perp = V^*$.

2. Let now $A' = W' \cap V$ and $B' = W' \cap V^*$. Using again that $\Phi(W') \subset K \otimes W'$ we prove that $\beta(B') \subset K \otimes A'$ and $\gamma(A') \subset K \otimes B'$. Now, the same reasoning as above (considering $(A' + B'^\perp) \oplus (B' + A'^\perp)$ and so on) proves that

$$(3.22) \quad \deg A' + \deg B' \leq 0,$$

with equality if and only if $A' + B'^\perp = V$ and $A'^\perp + B' = V^*$.

3. Observe that there are exact sequences of sheaves

$$0 \rightarrow B' \rightarrow W' \rightarrow A \rightarrow 0 \quad \text{and} \quad 0 \rightarrow A' \rightarrow W' \rightarrow B \rightarrow 0,$$

from which we obtain the formulas

$$\deg W' = \deg A + \deg B' \quad \text{and} \quad \deg W' = \deg B + \deg A'.$$

Adding up and using (3.21) together with (3.22) we obtain the desired inequality

$$\deg W' \leq 0.$$

4. Finally we consider the case when (V, φ) is stable and simple. Suppose that $\deg W' = 0$. Then we have equality both in (3.21) and in (3.22). Hence, $A + B^\perp = V$, $A^\perp + B = V^*$, $A' + B'^\perp = V$ and $A'^\perp + B' = V^*$. But $A^\perp + B = (A \cap B^\perp)^\perp$ and $A'^\perp + B' = (A' \cap B'^\perp)^\perp$, so we deduce that

$$A \oplus B^\perp = V \quad \text{and} \quad A' \oplus B'^\perp = V.$$

If one of these decompositions were nontrivial then V would not be simple, in contradiction with our assumptions. Consequently we must have $A = V$, $B^\perp = 0$ (because $W' \neq 0$) and similarly $A' = 0$, $B'^\perp = V^*$ (because $W' \neq V \oplus V^*$). This implies that the projections

$p : W' \rightarrow A$ and $q : W' \rightarrow B$ induce isomorphisms $u : W' \simeq V$ and $v : W' \simeq V^*$. Finally, defining $f := v \circ u^{-1} : V \rightarrow V^*$ we find an isomorphism which satisfies $\beta f = f^{-1} \gamma$ because $\Phi(W') \subset K \otimes W'$.

To prove that in this case $(W, \Phi) = H(V, \varphi)$ is strictly polystable just observe that $W' = \{(u, fu) \mid u \in V\}$ and define $W'' = \{(u, -fu) \mid u \in V\}$. It is then straightforward to check that $V \oplus V^* = W' \oplus W''$, that $\Phi(W') \subset K \otimes W'$ and that $\Phi(W'') \subset K \otimes W''$. Finally note that the Higgs bundle (W', Φ) is stable: any Φ -invariant subbundle $W_0 \subset W'$ is also a Φ -invariant subbundle of $(V \oplus V^*, \Phi)$. Hence, if $\deg W_0 = 0$ the argument of the previous paragraph shows that W_0 has to have the same rank as V , so $W_0 = W'$. Analogously, one sees that (W'', Φ) is a stable Higgs bundle. \square

Combining Lemmas 3.29 and 3.31 we get the proof of (3) of Theorem 3.26.

We conclude with the following lemma, which proves (4) of Theorem 3.26.

Lemma 3.32. *An $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle $(V, \varphi = (\beta, \gamma))$ is polystable if and only if $H(V, \varphi)$ is polystable.*

Proof. If (V, φ) is polystable then, by Proposition 3.18 there is a splitting $(V, \varphi) = (V_1, \varphi_1) \oplus \cdots \oplus (V_k, \varphi_k)$ such that each (V_i, φ_i) is a stable and simple G_i -Higgs bundle belonging to one out of three possible types of Higgs bundles, which are labeled in Proposition 3.18 as types (1), (2) or (3) (recall that type (1) corresponds $G_i = \mathrm{Sp}(2n_i, \mathbb{R})$, type (2) to $G_i = \mathrm{U}(p_i, q_i)$ and type (3) to $G_i = \mathrm{U}(n_i)$). By Lemma 3.31, for each summand (V_i, φ_i) belonging to type (1) the Higgs bundle $H(V_i, \varphi_i)$ is polystable. The corresponding statement for type (2) follows easily using the line of reasoning employed in [26, Section 2.3], cf. [2, Remark 3.8]. Finally, the statement for type (3) summands is immediate. This proves one direction of the equivalence.

We now prove the converse statement. Assume that $(W, \Phi) = H(V, \varphi)$ is polystable, so that $W = \bigoplus_{i=1}^N W_i$ with $\Phi W_i \subset K \otimes W_i$ and every $(W_i, \Phi|_{W_i})$ is stable with $\deg W_i = 0$.

1. We claim that for any subbundle $U \subset W$ satisfying $\deg U = 0$ and $\Phi(U) \subset K \otimes U$ there is an isomorphism $\psi : W \rightarrow W$ which commutes with Φ and a set $I \subset \{1, \dots, N\}$ such that $U = \psi(\bigoplus_{i \in I} W_i)$. To prove the claim we use induction on N (the case $N = 1$ being obvious). Let $W_{\geq 2} = \bigoplus_{i \geq 2} W_i$ and denote by $p_{\geq 2} : W \rightarrow W_{\geq 2}$ the projection. Then we have an exact sequence

$$0 \rightarrow W_1 \cap U \rightarrow U \rightarrow p_{\geq 2}(U) \rightarrow 0.$$

Since both $W_1 \cap U$ and $p_{\geq 2}(U)$ are invariant under Φ , by polystability their degrees must be ≤ 0 . And since according to the exact sequence above the sum of their degrees must be 0, the only possibility is that

$$\deg W_1 \cap U = 0 \quad \text{and} \quad \deg p_{\geq 2}(U) = 0.$$

Now we apply the induction hypothesis to the inclusion $p_{\geq 2}(U) \subset W_{\geq 2}$ and deduce that there is an isomorphism $\psi_2 : W_{\geq 2} \rightarrow W_{\geq 2}$ commuting with Φ and a subset $I_2 \subset \{2, \dots, N\}$ such that

$$p_{\geq 2}(U) = \psi_2 \left(\bigoplus_{i \in I_2} W_i \right).$$

Since $\deg W_1 \cap U = 0$ and W_1 is stable, only two things can happen. Either $W_1 \cap U = W_1$ or $W_1 \cap U = 0$. In the first case we have

$$U = W_1 \oplus \bigoplus_{i \in I_2} \psi(W_i),$$

so putting $I = \{1\} \cap I_2$ and $\psi = \text{diag}(1, \psi_2)$ the claim is proved. If instead $W_1 \cap U = 0$ then there is a map $\xi : p_{\geq 2}(U) \rightarrow W_1$ such that

$$U = \{(\xi(v), v) \in W_1 \oplus p_{\geq 2}\}.$$

Since U is Φ -invariant we deduce that ξ must commute with Φ . If we now extend ξ to $W_{\geq 2}$ by defining $\xi(\psi_2(W_j)) = 0$ for any $j \in \{2, \dots, N\} \setminus I_2$ then the claim is proved by setting $I = I_2$ and

$$\psi = \begin{pmatrix} 1 & \xi \circ \psi_2 \\ 0 & \psi_2 \end{pmatrix}.$$

2. Define for any $W' \subset W$ the subsheaves $R(W') = p(W') \oplus q(W')$ (recall that $p : W \rightarrow V$ and $q : W \rightarrow V^*$ are the projections) and $r(W') = (W' \cap V) \oplus (W' \cap V^*)$. Reasoning as in the first step of the proof of Lemma 3.31 we deduce that if W' is Φ -invariant then both $R(W')$ and $r(W')$ are Φ invariant, so in particular we must have $\deg R(W') \leq 0$ and $\deg r(W') \leq 0$. In case $\deg W' = 0$ these inequalities imply $\deg R(W') = \deg r(W') = 0$ (using the exact sequences $0 \rightarrow W' \cap V^* \rightarrow W' \rightarrow p(W') \rightarrow 0$ and $0 \rightarrow W' \cap V \rightarrow W' \rightarrow q(W') \rightarrow 0$).

Assume that there is some summand in $\{W_1, \dots, W_N\}$, say W_1 , such that $0 \neq r(W_1)$ or $R(W_1) \neq W$. Suppose, for example, that $W' := R(W_1) \neq W$ (the other case is similar). Let $A = p(W_1)$ and $B = q(W_1)$, so that $W' = A \oplus B$. By the observation above and the claim proved in **1** we know that there is an isomorphism $\psi : W \rightarrow W$ which commutes with Φ and such that, if we substitute $\{W_i\}_{1 \leq i \leq N}$ by $\{\psi(W_i)\}_{1 \leq i \leq N}$ and we reorder the summands if necessary, then we may write $W' = W_1 \oplus \dots \oplus W_k$ for some $k < N$. Now let $W'' = W_{k+1} \oplus \dots \oplus W_N$. We clearly have $W = W' \oplus W''$, so the inclusion of $W'' \subset W = V \oplus V^*$ composed with the projection $V \oplus V^* \rightarrow V/A \oplus V^*/B = W/W'$ induces an isomorphism. Consequently we have $V = A \oplus W'' \cap V$. Let us rename for convenience $V_1 := A$ and $V_2 := W'' \cap V$. Then, using the fact that each W_i is Φ -invariant we deduce that we can split both β and γ as

$$\begin{aligned} \beta &= (\beta_1, \beta_2) \in H^0(K \otimes S^2 V_1) \oplus H^0(K \otimes S^2 V_2), \\ \gamma &= (\gamma_1, \gamma_2) \in H^0(K \otimes S^2 V_1^*) \oplus H^0(K \otimes S^2 V_2^*). \end{aligned}$$

Hence, if we put $\varphi_i = (\beta_i, \gamma_i)$ for $i = 1, 2$ then we may write

$$(V, \varphi) = (V_1, \varphi_1) \oplus (V_2, \varphi_2).$$

3. Our strategy is now to apply recursively the process described in **2**. Observe that if $N \geq 3$ then for at least one i we have $R(W_i) \neq W$, because there must be a summand whose rank is strictly less than the rank of V . Hence the projection of this summand to V is not exhaustive.

Consequently, we can apply the process and split V in smaller and smaller pieces, until we arrive at a decomposition

$$(V, \varphi) = (V_1, \varphi_1) \oplus \dots \oplus (V_j, \varphi_j)$$

such that we can not apply **2** to any $H(V_i, \varphi_i)$. For each (V_i, φ_i) there are two possibilities. Either $H(V_i, \varphi_i)$ is stable, in which case (V_i, φ_i) is stable (by Lemma 3.29), or $H(V_i, \varphi_i)$

splits in two stable Higgs bundles $W'_i \oplus W''_i$ which satisfy:

$$R(W'_i) = R(W''_i) = W \quad \text{and} \quad r(W'_i) = r(W''_i) = 0.$$

But in this case it is easy to check that (V_i, φ_i) is also stable.

By the preceding lemma, (V, φ) is semistable. Suppose it is not stable. Then there is a filtration $0 \subset V_1 \subset V_2 \subset V$ such that $\Phi(V_2 \oplus V_1^\perp) \subset K \otimes (V_2 \oplus V_1^\perp)$ and $W' := V_2 \oplus V_1^\perp = 0$ has degree $\deg W' = 0$.

Define $W_{\geq 2} = \bigoplus_{i \geq 2} W_i$, and let $p_2 : W \rightarrow W_{\geq 2}$ denote the projection. We have an exact sequence

$$0 \rightarrow W' \cap W_1 \rightarrow W' \rightarrow p_2(W') \rightarrow 0.$$

It is easy to check that $\Phi(W' \cap W_1) \subset K \otimes (W' \cap W_1)$ and that $\Phi(p_2(W')) \subset K \otimes p_2(W')$. Since both W_1 and $W_{\geq 2}$ are polystable, we must have $\deg W' \cap W_1 \leq 0$ and $\deg p_2(W') \leq 0$. Finally, since $\deg W' = 0$, the exact sequence above implies that $\deg W' \cap W_1 = 0$ and $\deg p_2(W') = 0$. Now W_1 is stable, so $W' \cap W_1$ can only be either 0 or W_1 . Reasoning inductively with $p_2(W') \subset W_{\geq 2}$ in place of $W' \subset W$ we deduce that there must be some $I \subset \{1, \dots, k\}$ such that

$$W' = \bigoplus_{i \in I} W_i.$$

Since each $(W_i, \Phi|_{W_i})$ is stable, it is easy to check (for example using induction on N) that one must have $V_2 \oplus V_1^\perp = W_j$ for some j . This easily implies that $V_2 = V \cap W_j$ and if we define

$$V' = \bigoplus_{i \neq j} p(W_i)$$

then $V = V' \oplus V_2$. Applying the same process to V' and V_2 we arrive at the conclusion that (V, φ) is polystable. \square

3.7. Proof of Theorem 3.27. An $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle $((W, \Omega), \Phi)$ is stable if the $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundle (W, Φ) is stable. Thus the result is immediate from Theorem 3.26, unless we are in Case (2) of that Theorem. In that case, we have seen in the last paragraph of the proof of Lemma 3.31 that

- (1) There is an isomorphism f as stated, except for the symmetry condition.
- (2) There is an isomorphism $V \oplus V^* = W' \oplus W''$, where $W' = \{(u, f(u)) \mid u \in V\}$ and $W'' = \{(u, -f(u)) \mid u \in V\}$, and W' and W'' are Φ -invariant subbundles of W .
- (3) The $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundle (W, Φ) is strictly polystable, decomposing as the direct sum of stable $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundles:

$$(3.23) \quad (W, \Phi) = (W', \Phi') \oplus (W'', \Phi'').$$

Note also that $(W', \Phi') \simeq (W'', \Phi'')$.

Now, from Theorem 3.25 we have that for the $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle $((W, \Omega), \Phi)$ to be strictly semistable, it must have an isotropic Φ -invariant subbundle of degree zero. But the decomposition (3.23) shows that the only degree zero Φ -invariant subbundles are W' and W'' . The subbundle W' is isotropic if and only if, for all local sections u, v of V , we have

$$\Omega((u, f(u)), (v, f(v))) = 0 \iff \langle u, f(v) \rangle = \langle v, f(u) \rangle,$$

that is, if and only if f is symmetric. Analogously, W'' is isotropic if and only if f is symmetric. The first part of the conclusion follows.

For the second part, consider the $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle $((V, f), \beta f)$. This is stable as a $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle because $(V, \beta f) \simeq (W', \Phi')$, which is stable. Thus, in particular, $((V, f), \beta f)$ is stable as a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle (see Definition 3.21). \square

3.8. Milnor–Wood inequality and moduli space of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. The topological invariant attached to an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, β, γ) is an element in the fundamental group of $\mathrm{U}(n)$. Since $\pi_1(\mathrm{U}(n)) \simeq \mathbb{Z}$, this is an integer, which coincides with the degree of V .

We have the following Higgs bundle incarnation of the Milnor–Wood inequality (1.1) (see [26, Proposition 3.21] or [2, Corollary 3.27]).

Proposition 3.33. *Let (V, β, γ) be a semistable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle and let $d = \deg(V)$. Then*

$$(3.24) \quad d \leq \mathrm{rank}(\gamma)(g-1)$$

$$(3.25) \quad -d \leq \mathrm{rank}(\beta)(g-1),$$

This is proved by first using the equivalence between the semistability of (V, β, γ) and the $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundle (W, Φ) associated to it, and then applying the semistability numerical criterion to special Higgs subbundles defined by the kernel and image of Φ .

As a consequence of Proposition 3.33 we have the following.

Proposition 3.34. *Let (V, β, γ) be a semistable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle and let $d = \deg(V)$. Then*

$$|d| \leq n(g-1).$$

Furthermore,

- (1) $d = n(g-1)$ holds if and only if $\gamma: V \rightarrow V^* \otimes K$ is an isomorphism;
- (2) $d = -n(g-1)$ holds if and only if $\beta: V^* \rightarrow V \otimes K$ is an isomorphism.

Recall from our general discussion in Section 2.1 that $\mathcal{M}_d(\mathrm{Sp}(2n, \mathbb{R}))$ denotes the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles (V, β, γ) with $\deg(V) = d$. For brevity we shall henceforth write simply \mathcal{M}_d for this moduli space.

Combining Theorem 2.2 with Proposition 2.19 we have the following.

Proposition 3.35. *The moduli space \mathcal{M}_d is a complex algebraic variety. Its expected dimension is $(g-1)(2n^2+n)$.*

One has the following immediate duality result.

Proposition 3.36. *The map $(V, \beta, \gamma) \mapsto (V^*, \gamma^t, \beta^t)$ gives an isomorphism $\mathcal{M}_d \simeq \mathcal{M}_{-d}$.*

As a corollary of Proposition 3.34, we obtain the following.

Proposition 3.37. *The moduli space \mathcal{M}_d is empty unless*

$$|d| \leq n(g-1).$$

3.9. Smoothness and polystability of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. We study now the smoothness properties of the moduli space. As a corollary of Proposition 2.18 and Theorem 3.27 we have the following.

Proposition 3.38. *Let (V, φ) be an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle which is stable and simple and assume that there is no symmetric isomorphism $f: V \xrightarrow{\sim} V^*$ intertwining β and γ . Then (V, φ) represents a smooth point of the moduli space of polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles.*

So, a stable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) in \mathcal{M}_d with $d \neq 0$ can only fail to be a smooth point of the moduli space if it is not simple — this gives rise to an orbifold-type singularity — or if, in spite of being simple, there is an isomorphism $V \simeq V^*$ intertwining β and γ . Of course, this can only happen if $d = \deg(V) = 0$. Generally, if (V, φ) is polystable, but not stable it is also a singular point of \mathcal{M}_d .

We shall need the following analogue of Proposition 3.38 for $\mathrm{U}(n)$ -, $\mathrm{GL}(n, \mathbb{C})$ - $\mathrm{U}(p, q)$ - and $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles

Proposition 3.39. (1) *A stable $\mathrm{U}(n)$ -Higgs bundle represents a smooth point in the moduli space of $\mathrm{U}(n)$ -Higgs bundles.*
 (2) *A stable $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle represents a smooth point in the moduli space of $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundles.*
 (3) *A stable $\mathrm{U}(p, q)$ -Higgs bundle (V, W, β, γ) represents a smooth point of the moduli space of $\mathrm{U}(p, q)$ -Higgs bundles unless there exists an isomorphism $f: V \rightarrow W$ such that $\beta f = f^{-1}\gamma$. In this case $p = q$ and $(V, \beta f)$ defines a stable $\mathrm{GL}(p, \mathbb{C})$ -Higgs bundle.*
 (4) *A $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle which is stable as a $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle represents a smooth point in the moduli space of $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles.*

Proof. (1) A stable $\mathrm{U}(n)$ -Higgs bundle is nothing but a stable vector bundle, so this is classical.

(2) This is also classical ([27]).

(3) We first observe that a stable $\mathrm{U}(p, q)$ -Higgs bundle is simple. After this we prove that a stable $\mathrm{U}(p, q)$ -Higgs bundle is stable as a $\mathrm{GL}(p+q, \mathbb{C})$ -bundle unless there exists an isomorphism $f: V \rightarrow W$ such that $\beta f = f^{-1}\gamma$. The proof follows along the same lines as that of Theorem 3.27, although it is simpler since f is not symmetric. Now the stability as a $\mathrm{GL}(p+q, \mathbb{C})$ -Higgs bundle ensures that the appropriate \mathbb{H}^2 obstruction for smoothness vanishes.

(4) A stable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle is simple if and only if as a $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle it is stable. The rest follows by a similar argument to the last one in (3). \square

It will be convenient to make the following definition for $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles, analogous to the way we associate $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles to vector bundles and $\mathrm{U}(p, q)$ -Higgs bundles in (3.13) and (3.14), respectively (cf. Theorem 3.27). Given a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle $((W, Q), \psi)$, let $f: W \rightarrow W^*$ be the symmetric isomorphism associated to Q . Define an associated $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle

$$(3.26) \quad (V, \varphi) = v_*^{\mathrm{GL}(n, \mathbb{R})}((W, Q), \psi)$$

by setting

$$V = W, \quad \beta = \psi \quad \text{and} \quad \gamma = f\psi f.$$

Again we shall slightly abuse language, saying simply that $v_*^{\mathrm{GL}(n, \mathbb{R})}((W, Q), \psi)$ is a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle, whenever no confusion is likely to occur.

Putting everything together we obtain our main result of this section: a structure theorem for polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles.

Theorem 3.40. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle. Then there is a decomposition $(V, \varphi) = (V_1, \varphi_1) \oplus \cdots \oplus (V_k, \varphi_k)$, unique up to reordering, such that each of the $\mathrm{Sp}(2n_i, \mathbb{R})$ -Higgs bundles (V_i, φ_i) is one of the following:*

- (1) *A stable and simple $\mathrm{Sp}(2n_i, \mathbb{R})$ -Higgs bundle, which is stable as a $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle.*
- (2) *A stable $\mathrm{U}(p_i, q_i)$ -Higgs bundle with $n_i = p_i + q_i$, which is stable as a $\mathrm{GL}(p_i + q_i, \mathbb{C})$ -Higgs bundle*
- (3) *A stable $\mathrm{U}(n_i)$ -Higgs bundle.*
- (4) *A $\mathrm{GL}(n_i, \mathbb{R})$ -Higgs bundle which is stable as a $\mathrm{GL}(n_i, \mathbb{C})$ -Higgs bundle.*
- (5) *A stable $\mathrm{GL}(n_i, \mathbb{C})$ -Higgs bundle.*

Each (V_i, φ_i) is a smooth point in the moduli space of G_i -Higgs bundles, where G_i is the corresponding real group $\mathrm{Sp}(2n_i, \mathbb{R})$, $\mathrm{U}(p_i, q_i)$, $\mathrm{U}(n_i)$ or $\mathrm{GL}(n_i, \mathbb{R})$.

Proof. This follows from Propositions 3.18, 3.38 and 3.39 and Theorems 3.15 and 3.27 \square

4. MAXIMAL DEGREE $\mathrm{Sp}(2n, \mathbb{R})$ -HIGGS BUNDLES AND THE CAYLEY CORRESPONDENCE

4.1. Cayley correspondence. In this section we shall describe the $\mathrm{Sp}(2n, \mathbb{R})$ moduli space for the extreme value $|d| = n(g - 1)$. In fact, for the rest of this section we shall assume that $d = n(g - 1)$. This involves no loss of generality, since, by Proposition 3.36, $(V, \varphi) \mapsto (V^*, \varphi^t)$ gives an isomorphism between the $\mathrm{Sp}(2n, \mathbb{R})$ moduli spaces for d and $-d$. The main result is Theorem 4.3, which we refer to as the *Cayley correspondence*. This is stated as Theorem 1.3 in the Introduction, where the reason for the name is also explained.

Let (V, β, γ) be an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle with $d = n(g - 1)$ such that $\gamma \in H^0(K \otimes S^2 V^*)$ is an isomorphism. Let $L_0 = K^{-1/2}$ be a fixed square root of K^{-1} , and define $W := V \otimes L_0$. Then $q := \gamma \otimes I_{L_0} : W \rightarrow W^*$ is a symmetric isomorphism defining a non-degenerate symmetric bilinear form Q on W , in other words, (W, Q) is an $\mathrm{O}(n, \mathbb{C})$ -holomorphic bundle. The K^2 -twisted endomorphism $\psi : W \rightarrow W \otimes K^2$ defined by $\psi := \beta \otimes I_{L_0^{-1}} \circ (\gamma \otimes I_{L_0})$ is Q -symmetric and hence (W, Q, ψ) defines a K^2 -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair (in the sense of Section 3.4), from which we can recover the original $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle.

We shall need the following Lemma. In the statement we use the notions of β - and γ -invariance of filtrations introduced in Definition 3.5.

Lemma 4.1. *Let (V, β, γ) be a $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle with $d = n(g - 1)$ such that γ is an isomorphism. Let (W, Q, ψ) be the corresponding K^2 -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair. Let*

$$(4.27) \quad 0 \subset V_1 \subset V_2 \subset V$$

be a filtration and let $W_1 \subset W$ be the subbundle defined by $W_1 = V_1 \otimes L_0$. Then the following statements hold:

- (1) *The filtration (4.27) is γ -invariant if and only if $W_1 \subset W$ is isotropic.*
- (2) *Assume that the filtration (4.27) is (β, γ) -invariant. Then $W_1 \subset W$ is ψ -invariant.*

- (3) Assume that $V_2 = V_1^{\perp\gamma}$. Then the filtration (4.27) is (β, γ) -invariant if and only if $W_1 \subset W$ is isotropic and ψ -invariant.

Proof. Statement (1) is immediate using that γ -invariance of the filtration is equivalent to $V_1^{\perp\gamma} \subset V_2$ (see Remark 3.7).

For Statement (2), note that $\psi = (\beta \otimes \text{Id}_{L_0^{-1}}) \circ (\gamma \otimes \text{Id}_{L_0})$. Hence, using the conditions for (β, γ) -invariance given in Remark 3.7, we obtain

$$\psi(W_1) = (\beta \otimes \text{Id}_{L_0^{-1}})(\gamma(V_1) \otimes L_0) \subset (\beta \otimes \text{Id}_{L_0^{-1}})(V_2^{\perp} \otimes L_0^{-1}) \subset V_1 \otimes L_0 K^2 = W_1 \otimes K^2.$$

In order to prove Statement (3), in view of (1) and (2), we can use (3.11) of Remark 3.7, ψ -invariance of W_1 and $V_2^{\perp\gamma} = V_1 = W_1 \otimes L_0^{-1}$ to deduce that

$$\beta(V_2^{\perp}) = \beta(\gamma \otimes \text{Id}_{K^{-1}}(V_2^{\perp\gamma} \otimes K^{-1})) = (\psi \otimes \text{Id}_{L_0})(W_1 \otimes L_0) \subset W_1 \otimes L_0 K^2 = V_1 \otimes K.$$

Thus the result follows from the β -invariance condition given in Remark 3.6. \square

Theorem 4.2. *Let (V, β, γ) be a $\text{Sp}(2n, \mathbb{R})$ -Higgs bundle with $d = n(g - 1)$ such that γ is an isomorphism. Let (W, Q, ψ) be the corresponding K^2 -twisted $\text{GL}(n, \mathbb{R})$ -Higgs pair. Then (V, β, γ) is semistable (respectively stable, polystable) if and only if (W, Q, ψ) is semistable (respectively stable, polystable).*

Proof. We start by proving the equivalence of the semistability conditions.

Assume that (V, β, γ) is a semistable $\text{Sp}(2n, \mathbb{R})$ -Higgs bundle with $d = n(g - 1)$, so that γ is an isomorphism. Let $W_1 \subset W$ be a ψ -invariant Q -isotropic subbundle, i.e.

$$(4.28) \quad W_1 \subset W_1^{\perp Q}.$$

We shall prove that $\deg(W_1) \leq 0$. Define

$$V_1 := W_1 \otimes L_0^{-1} \quad \text{and} \quad V_2 := W_1^{\perp Q} \otimes L_0^{-1},$$

and note that

$$(4.29) \quad \deg(W_1) = \deg(V_1) - \text{rk}(V_1)(g - 1).$$

By (4.28) we have a filtration $0 \subset V_1 \subset V_2 \subset V$, which moreover satisfies $V_2 = V_1^{\perp\gamma}$. Thus, by (3) of Lemma 4.1, it is (β, γ) -invariant and so, by semistability of (V, β, γ) , we have

$$(4.30) \quad \deg(V) - \deg(V_1) - \deg(V_2) \geq 0.$$

Now, we have an exact sequence

$$(4.31) \quad 0 \rightarrow V_1^{\perp\gamma} \rightarrow V \rightarrow V_1^* \otimes K \rightarrow 0,$$

defined by composing the isomorphism γ with the projection $V^* \otimes K \rightarrow V_1^* \otimes K$. Thus

$$(4.32) \quad \deg(V_1^{\perp\gamma}) = \deg(V) + \deg(V_1) - (2g - 2) \text{rk}(V_1)$$

and, since $V_2 = V_1^{\perp\gamma}$, it follows that

$$(4.33) \quad \deg(V) - \deg(V_1) - \deg(V_2) = -2 \deg(V_1) + (2g - 2) \text{rk}(V_1) = -2 \deg(W_1),$$

where we have used (4.29) in the second identity. Hence (4.30) implies that $\deg(W_1) \leq 0$ and therefore, according to Definition 3.3, (W, Q, ψ) is semistable.

To prove the converse, assume now that (W, Q, ψ) is semistable. Consider a (β, γ) -invariant filtration

$$0 \subset V_1 \subset V_2 \subset V.$$

Let

$$W_1 := V_1 \otimes L_0.$$

Then (1) and (2) of Lemma 4.1 imply that W_1 is isotropic and ψ -invariant. Hence, by semistability of (W, Q, ψ) , we have $\deg(W_1) \leq 0$, i.e.,

$$(4.34) \quad \deg(V_1) \leq \text{rk}(V_1)(g-1).$$

In order to complete the proof of semistability of (V, β, γ) , we shall consider a second ψ -invariant Q -isotropic subbundle $W_2 \subset W$. For this, we view $V_2 \cap V_2^{\perp\gamma} \subset V$ as a subbundle (i.e., define it in sheaf theoretic terms and take its saturation) and let

$$W_2 = V_2 \cap V_2^{\perp\gamma} \otimes L_0 \subset W.$$

Similarly, we consider the subbundle $V_2 + V_2^{\perp\gamma} \subset V$. Define

$$V'_1 = V_2 \cap V_2^{\perp\gamma} \quad \text{and} \quad V'_2 = V_2 + V_2^{\perp\gamma}.$$

Then we have a filtration

$$(4.35) \quad 0 \subset V'_1 \subset V'_2 \subset V.$$

We shall see that this filtration is (β, γ) -invariant and that $V'_2 \subset V_1^{\perp\gamma}$. Once this is established, it will follow from (3) of Lemma 4.1 that W_2 is isotropic and ψ -invariant.

We start by showing γ -invariance. There is an inclusion

$$(4.36) \quad V_2 + V_2^{\perp\gamma} \subset (V_2 \cap V_2^{\perp\gamma})^{\perp\gamma},$$

which, in turn follows from $V_2 \subset (V_2 \cap V_2^{\perp\gamma})^{\perp\gamma}$ and $V_2^{\perp\gamma} \subset (V_2 \cap V_2^{\perp\gamma})^{\perp\gamma}$, which are obvious. Hence $V'_2 \subset V_1^{\perp\gamma}$, from which γ -invariance of (4.35) follows (cf. Remark 3.7). To show β -invariance, note that we have inclusions

$$V_1 \subset V'_1 = V_2 \cap V_2^{\perp\gamma}, \quad V_2 \subset V'_2 = V_2 + V_2^{\perp\gamma}.$$

The first one follows from $V_1 \subset V_2$ and from applying $^{\perp\gamma}$ to the inclusion $V_2 \subset V_1^{\perp\gamma}$, which gives $V_1 \subset V_2^{\perp\gamma}$. The second inclusion is obvious. From the first inclusion we have $V_1^{\perp\perp} \subset V_1^{\perp}$ and hence, using the second inclusion and β -invariance of the original filtration $0 \subset V_1 \subset V_2 \subset V$ we have

$$\beta(V_1^{\perp\perp}) \subset \beta(V_1^{\perp}) \subset V_2 \otimes K \subset V'_2 \otimes K.$$

Thus (4.35) is also β -invariant (using Remark 3.7) and we have established (β, γ) -invariance of (4.35). Note that we did not use that γ is non-degenerate for this.

Now, since γ is non-degenerate, the sequence

$$(4.37) \quad 0 \rightarrow V_2 \cap V_2^{\perp\gamma} \rightarrow V_2 \oplus V_2^{\perp\gamma} \rightarrow V_2 + V_2^{\perp\gamma} \rightarrow 0$$

is generically short exact. Hence both bundles in (4.36) have the same rank and, therefore,

$$V_2 + V_2^{\perp\gamma} = (V_2 \cap V_2^{\perp\gamma})^{\perp\gamma}.$$

In other words, $V'_2 = V_1'^{\perp\gamma}$ and, as noted above, it follows from (3) of Lemma 4.1 that $W_2 \subset W$ is isotropic and ψ -invariant. Thus, by semistability of (W, Q, ψ) we have that $\deg(W_2) \leq 0$. This, arguing as in (4.33), is equivalent to

$$(4.38) \quad \deg(V) - \deg(V'_1) - \deg(V'_2) \geq 0.$$

But, using the generically short exact sequence (4.37), we have that

$$\deg(V'_1) + \deg(V'_2) \geq \deg(V_2) + \deg(V_2^{\perp\gamma})$$

Hence (4.38) implies that

$$(4.39) \quad \deg(V) - \deg(V_2) - \deg(V_2^{\perp\gamma}) \geq 0$$

and so, using (4.32) for $V_2^{\perp\gamma}$, we obtain

$$\deg(V_2) \leq (g-1) \operatorname{rk}(V_2).$$

Adding this to (4.34) we get

$$\deg(V_1) + \deg(V_2) \leq (g-1) \operatorname{rk}(V_1) + (g-1) \operatorname{rk}(V_2).$$

Now we can estimate, using the inclusion $V_2 \subset V_1^{\perp\gamma}$,

$$(4.40) \quad \operatorname{rk}(V_2) \leq \operatorname{rk}(V_1)^{\perp\gamma} = \operatorname{rk}(V) - \operatorname{rk}(V_1),$$

where $\operatorname{rk}(V_1^{\perp\gamma}) = \operatorname{rk}(V) - \operatorname{rk}(V_1)$ follows from the fact that γ is nondegenerate. Consequently,

$$\deg(V_1) + \deg(V_2) \leq (g-1) \operatorname{rk}(V).$$

On the other hand, the nondegeneracy of γ implies that $V \simeq V^* \otimes K$, which gives $\deg(V) = (g-1) \operatorname{rk}(V)$. Summing up, we finally get

$$\deg(V_1) + \deg(V_2) \leq \deg(V),$$

concluding the proof of semistability of (V, β, γ)

The proof of the statement for stability is essentially the same, observing that the trivial filtration $0 = V_1 \subset V_2 = V$ corresponds to the trivial subbundle $0 \subset W$.

We turn now to the proof of the equivalence of the polystability conditions. We will first prove that if (V, β, γ) is polystable then (W, Q, ψ) is also polystable. For this, suppose that $W' \subset W$ is a ψ -invariant Q -isotropic non-trivial subbundle such that $\deg(W') = 0$. According to Definition 3.21, we have to show that there exists a Q -coisotropic ψ -invariant subbundle $W'' \subset W$ such that $W = W' \oplus W''$. We proceed as in the proof that semistability of (V, β, γ) implies semistability of (W, Q, ψ) . So, let $V_1 := W' \otimes L_0^{-1}$ and $V_2 := W'^{\perp Q} \otimes L_0^{-1}$. Our previous arguments now show that the filtration $0 \subset V_1 \subset V_2 \subset V$ is a non-trivial (β, γ) -invariant filtration, and that $\deg(W') = 0$ is equivalent to

$$\deg(V) - \deg(V_1) - \deg(V_2) = 0.$$

Moreover, since (V, β, γ) is polystable, from Definition 3.16 and Remark 3.17, there are subbundles F_1, F_2 and F_3 of V , such that $V = F_1 \oplus F_2 \oplus F_3$, with $V_1 = F_1$, $V_2 = F_1 \oplus F_2$, and $\beta(F_i^*) \subset F_{4-i} \otimes K$ and $\gamma(F_i) \subset F_{4-i}^* \otimes K$ for $i = 1, 2, 3$. Let $W'' := (F_2 \oplus F_3) \otimes L_0$. The ψ -invariance of W'' follows clearly from the (β, γ) -invariance of the F_i . Moreover, the γ -invariance of the F_i implies that $(F_2 \oplus F_3)^{\perp\gamma} = F_3$, i.e. $W''^{\perp Q} \subset W''$, which is the coisotropic condition.

To prove the converse, suppose that (W, Q, ψ) is polystable. Consider a general (β, γ) -invariant filtration $0 \subset V_1 \subset V_2 \subset V$ with

$$(4.41) \quad \deg(V) - \deg(V_1) - \deg(V_2) = 0.$$

The arguments of the proof that semistability of (W, Q, ψ) implies semistability of (V, β, γ) of course continue to hold and, moreover, the condition (4.41) forces all inequalities in that proof to be equalities. In particular, equality must hold in (4.40) and so we conclude that in fact

$$V_2 = V_1^{\perp\gamma}.$$

Therefore, it suffices to consider (β, γ) -invariant filtrations of the form $0 \subset V_1 \subset V_1^{\perp\gamma} \subset V$ with

$$\deg(V) - \deg(V_1) - \deg(V_1^{\perp\gamma}) = 0.$$

Now, running backwards the argument given above that polystability of (V, β, γ) implies polystability of (W, Q, ψ) , we see that there is a decomposition $V = F_1 \oplus F_2 \oplus F_3$ with $V_1 = F_1$, $V_1^{\perp\gamma} = F_1 \oplus F_2$ and satisfying the (β, γ) -invariance conditions.

This concludes the proof. □

Theorem 4.3. *Let \mathcal{M}_{\max} be the moduli space of polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles with $d = n(g-1)$ and let \mathcal{M}' be the moduli space of polystable K^2 -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pairs. The map $(V, \beta, \gamma) \mapsto (W, Q, \psi)$ defines an isomorphism of complex algebraic varieties*

$$\mathcal{M}_{\max} \simeq \mathcal{M}'.$$

Proof. Let (V, β, γ) be a semistable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle with $d = n(g-1)$. By Proposition 3.34, γ is an isomorphism and hence the map $(V, \beta, \gamma) \mapsto (W, Q, \psi)$ is well defined. The result follows now from Theorem 4.2 and the existence of local universal families (see [41]). □

4.2. Invariants of $\mathrm{GL}(n, \mathbb{R})$ -Higgs pairs. To a K^2 -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair (W, Q, ψ) one can attach topological invariants corresponding to the first and second Stiefel-Whitney classes of a reduction to $\mathrm{O}(n)$ of the $\mathrm{O}(n, \mathbb{C})$ bundle defined by (W, Q) . The first class $w_1 \in H^1(X, \mathbb{Z}_2) \simeq \mathbb{Z}_2^{2g}$ measures the obstruction for the $\mathrm{O}(n)$ -bundle to have an orientation, i.e. to the existence of a reduction to a $\mathrm{SO}(n)$ bundle, while the second one $w_2 \in H^2(X, \mathbb{Z}_2) \simeq \mathbb{Z}_2$ measures the obstruction to lifting the $\mathrm{O}(n)$ -bundle to a $\mathrm{Pin}(n)$ -bundle, where

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \mathrm{Pin}(n) \rightarrow \mathrm{O}(n) \rightarrow 1.$$

If we define

$$\mathcal{M}'_{w_1, w_2} := \{(W, Q, \psi) \in \mathcal{M}' \text{ such that } w_1(W) = w_1 \text{ and } w_2(W) = w_2\},$$

we have that

$$(4.42) \quad \mathcal{M}' = \bigcup_{w_1, w_2} \mathcal{M}'_{w_1, w_2}.$$

We thus have, via the isomorphism given by Theorem 4.3, that the moduli space \mathcal{M}_{\max} is partitioned in disjoint closed subvarieties corresponding to fixing (w_1, w_2) .

5. THE HITCHIN FUNCTIONAL

5.1. The Hitchin functional. In order to define this functional, we consider the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles (V, φ) from the gauge theory point of view, i.e., we use the identification of \mathcal{M}_d with the moduli space $\mathcal{M}_d^{\mathrm{gauge}}$ of solutions (A, φ) to the Hitchin equations given by Theorem 2.6. There is an action of S^1 on \mathcal{M}_d via multiplication of φ by scalars: $(A, \varphi) \mapsto (A, e^{i\theta}\varphi)$. Restricted to the smooth locus \mathcal{M}_d^s this action is Hamiltonian with symplectic moment map $-f$, where the *Hitchin functional* f is defined by

$$(5.43) \quad \begin{aligned} f: \mathcal{M}_d &\rightarrow \mathbb{R}, \\ (A, \varphi) &\mapsto \frac{1}{2}\|\varphi\|^2 = \frac{1}{2}\|\beta\|^2 + \frac{1}{2}\|\gamma\|^2. \end{aligned}$$

Here $\|\cdot\|$ is the L^2 -norm obtained by using the Hermitian metric in V and integrating over X . The function f is well defined on the whole moduli space (not just on the smooth locus). It was proved by Hitchin [27, 28] that f is proper and therefore it has a minimum on every closed subspace of $\mathcal{M} = \bigcup_d \mathcal{M}_d$. Thus we have the following result.

Proposition 5.1. *Let $\mathcal{M}' \subseteq \mathcal{M}$ be any closed subspace and let $\mathcal{N}' \subseteq \mathcal{M}'$ be the subspace of local minima of f on \mathcal{M}' . If \mathcal{N}' is connected then so is \mathcal{M}' . \square*

The following observation was also made by Hitchin [28].

Proposition 5.2. *The Hitchin functional is additive with respect to direct sum of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles, in other words,*

$$f\left(\bigoplus (V_i, \varphi_i)\right) = \sum f(V_i, \varphi_i).$$

Let (V, φ) represent a smooth point of \mathcal{M}_d . Then the moment map condition shows that the critical points of f are exactly the fixed points of the circle action. These can be identified as follows (cf. [27, 28, 43]).

Proposition 5.3. *An $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) represents a fixed point of the circle action on \mathcal{M}_d if and only if it is a complex variation of Hodge structure (also called a Hodge bundle): this means that there is a decomposition in holomorphic subbundles*

$$V = \bigoplus F_i$$

for real indices, or weights, i such that, attributing weight $-i$ to F_i^* , $\varphi = (\beta, \gamma)$ has weight one with respect to this decomposition; more explicitly this means that

$$\gamma: F_i \rightarrow F_{-i-1}^* \otimes K \quad \text{and} \quad \beta: F_i^* \rightarrow F_{-i+1} \otimes K.$$

The decomposition $V = \bigoplus F_i$ of Proposition 5.3 gives rise to corresponding decompositions

$$(5.44) \quad \mathrm{End}(V)_k = \bigoplus_{i-j=k} F_i \otimes F_j^*,$$

$$(5.45) \quad (S^2V \otimes K)_{k+1} = \bigoplus_{\substack{i+j=k+1 \\ i < j}} F_i \otimes F_j \otimes K \oplus S^2F_{\frac{k+1}{2}} \otimes K,$$

$$(5.46) \quad (S^2V^* \otimes K)_{k+1} = \bigoplus_{\substack{-i-j=k+1 \\ i < j}} F_i^* \otimes F_j^* \otimes K \oplus S^2F_{-\frac{k+1}{2}}^* \otimes K.$$

The map $\text{ad}(\varphi)$ in the deformation complex (2.5) has weight 1 with respect to these decompositions, so that we can define complexes

$$(5.47) \quad C_k^\bullet(V, \varphi): \text{End}(V)_k \xrightarrow{\text{ad}(\varphi)} (S^2V \otimes K \oplus S^2V^* \otimes K)_{k+1},$$

for any k . The deformation complex (2.5) decomposes accordingly as

$$C^\bullet(V, \varphi) = \bigoplus C_k^\bullet(V, \varphi).$$

We shall also need the positive weight subcomplex

$$(5.48) \quad C_-^\bullet(V, \varphi) = \bigoplus_{k>0} C_k^\bullet(V, \varphi).$$

It can be shown (see, e.g., [20, §3.2]) that $\mathbb{H}^1(C_k^\bullet(V, \varphi))$ is the weight $-k$ -subspace of $\mathbb{H}^1(C^\bullet(V, \varphi))$ for the infinitesimal circle action. Thus $\mathbb{H}^1(C_-^\bullet(V, \varphi))$ is the positive weight space for the infinitesimal circle action.

Proposition 5.4. *Let (V, φ) be a polystable $\text{Sp}(2n, \mathbb{R})$ -Higgs bundle whose isomorphism class is fixed under the circle action.*

- (1) *Assume that (V, φ) is simple and stable as an $\text{Sp}(2n, \mathbb{C})$ -Higgs bundle. Then (V, φ) represents a local minimum of f if and only if $\mathbb{H}^1(C_-^\bullet(V, \varphi)) = 0$.*
- (2) *Suppose that there is a family (V_t, φ_t) of polystable $\text{Sp}(2n, \mathbb{R})$ -Higgs bundles, parametrized by t in the open unit disk D , deforming (V, φ) (i.e., such that $(V_0, \varphi_0) = (V, \varphi)$) and that the corresponding infinitesimal deformation is a non-zero element of $\mathbb{H}^1(C_-^\bullet(V, \varphi))$. Then (V, φ) is not a local minimum of f on \mathcal{M}_d .*

Proof. (1) From Proposition 2.18, when the hypotheses are satisfied, (V, φ) represents a smooth point of the moduli space. Then one can use the moment map condition on f to show that $\mathbb{H}^1(C_k^\bullet(V, \varphi))$ is the eigenvalue $-k$ subspace of the Hessian of f (cf. [20, §3.2]; this goes back to Frankel [18], at least). This proves (1).

(2) Take a corresponding family of solutions to Hitchin's equations. One can then prove that the second variation of f along this family is negative in certain directions (see Hitchin [28, § 8]). \square

5.2. A cohomological criterion for minima. The following result was proved in [2, Proposition 4.14¹ and Remark 4.16]. It is the key to obtaining the characterization of the minima of the Hitchin functional f .

Proposition 5.5. *Let (V, φ) be a polystable $\text{Sp}(2n, \mathbb{R})$ -Higgs bundle whose isomorphism class is fixed under the circle action. Then for any k we have $\chi(C_k^\bullet(V, \varphi)) \leq 0$ and equality holds if and only if*

$$\text{ad}(\varphi): \text{End}(V)_k \rightarrow (S^2V \otimes K \oplus S^2V^* \otimes K)_{k+1}$$

is an isomorphism.

Corollary 5.6. *Let (V, φ) be a simple $\text{Sp}(2n, \mathbb{R})$ -Higgs bundle which is stable as an $\text{Sp}(2n, \mathbb{C})$ -Higgs bundle. If (V, φ) is fixed under the circle action then it represents a local minimum of f if and only if the map*

$$\text{ad}(\varphi): \text{End}(V)_k \rightarrow (S^2V \otimes K \oplus S^2V^* \otimes K)_{k+1}$$

is an isomorphism for all $k > 0$.

¹a corrected proof can be found in [5, Lemma 3.11]

Proof. We have the vanishing $\mathbb{H}^0(C_k^\bullet(V, \varphi)) = \mathbb{H}^2(C_k^\bullet(V, \varphi)) = 0$ for all $k > 0$ from Proposition 2.17. Hence $\dim \mathbb{H}^1(C_-^\bullet(V, \varphi)) = -\chi(C_-^\bullet(V, \varphi))$. Now the result is immediate from Proposition 5.5 and (1) of Proposition 5.4. \square

5.3. Minima of the Hitchin functional. In order to describe the minima, it is convenient to define the following subspaces of \mathcal{M}_d .

Definition 5.7. For each d , define the following subspace of \mathcal{M}_d .

$$\mathcal{N}_d = \{(V, \beta, \gamma) \in \mathcal{M}_d \mid \beta = 0 \text{ or } \gamma = 0\}.$$

Remark 5.8. It is easy to see that polystability of (V, φ) implies that, in fact,

$$\begin{aligned} \mathcal{N}_d &= \{(V, \beta, \gamma) \mid \beta = 0\} && \text{for } d > 0, \\ \mathcal{N}_d &= \{(V, \beta, \gamma) \mid \gamma = 0\} && \text{for } d < 0, \\ \mathcal{N}_d &= \{(V, \beta, \gamma) \mid \beta = \gamma = 0\} && \text{for } d = 0. \end{aligned}$$

Note, in particular, that for $d = 0$ the vanishing of one of the sections β or γ implies the vanishing of the other one.

Proposition 5.9. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle with $\beta = 0$ or $\gamma = 0$. Then (V, φ) represents the absolute minimum of f on \mathcal{M}_d . Thus \mathcal{N}_d is contained in the subspace of local minima of f on \mathcal{M}_d .*

Proof. This can be seen in a way similar to the proof of [2, Proposition 4.5]. \square

Theorem 5.10. *Let (V, β, γ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle and assume that $n \geq 3$. Then (V, β, γ) represents a minimum of the Hitchin functional if and only if one of the following situations occurs:*

- (1) (V, β, γ) belongs to \mathcal{N}_d .
- (2) The degree $d = -n(g-1)$ with $n = 2q+1$ odd, and there exists a square root L of K such that the bundle V is of the form

$$V = \bigoplus_{\lambda=-q}^q L^{-1}K^{-2\lambda}.$$

With respect to this decomposition of V and the corresponding decomposition of V^* , the maps β and γ are of the form:

$$\beta = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}$$

where, in the matrix for β , we denote by 1 the canonical section of

$$\mathrm{Hom}((L^{-1}K^{-2\lambda})^*, L^{-1}K^{2\lambda}) \otimes K \simeq \mathcal{O}$$

and analogously for γ .

- (3) The degree $d = -n(g-1)$ with $n = 2q+2$ even, and there exists a square root L of K such that the bundle V is of the form

$$V = \bigoplus_{\lambda=-q}^{q+1} LK^{-2\lambda}.$$

With respect to this decomposition of V and the corresponding decomposition of V^* , the maps β and γ are of the form given above.

- (4) The degree $d = n(g-1)$ and the dual $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle $(V', \beta', \gamma') = (V^*, \gamma^t, \beta^t)$ is of the form given in (2) or (3) above.

Definition 5.11. If (V, β, γ) is a minimum which does not belong to \mathcal{N}_d we say that it is a **quiver type** minimum.

Remark 5.12. The cases $n = 1$ and $n = 2$ are special and were treated in [27] and [26], respectively (cf. (1) of Corollary 6.5 and Remark 6.6).

Proof of Theorem 5.10. This proof relies on the results of Sections 6 and 7 below.

Consider first the case of simple $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles (V, φ) which are stable as $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundles. In this case, the analysis of the minima is based on Corollary 5.6 and is carried out in Section 6 below. The main result is Theorem 6.7, which says that Theorem 5.10 holds for such (V, φ) .

Next, consider a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, φ) which is not simple and stable as an $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle. Then the decomposition $(V, \varphi) = \bigoplus (V_i, \varphi_i)$ given in the structure Theorem 3.40 is non-trivial. The main result of Section 7, Proposition 7.1, says that if such a (V, φ) is a local minimum then it belongs to \mathcal{N}_d , i.e., $\beta = 0$ or $\gamma = 0$. This concludes the proof. \square

6. MINIMA IN THE SMOOTH LOCUS OF THE MODULI SPACE

In this section we consider simple $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles (V, ϕ) which are stable as $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundles. Thus, by Proposition 2.18, they belong to the smooth locus of the moduli space \mathcal{M}_d . In Theorem 6.7 below we prove that the statement of Theorem 5.10 holds in this case.

Our results are based on a careful analysis of the structure of $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles (V, φ) satisfying the criterion of Corollary 5.6.

6.1. Hodge bundles. In this subsection we give a description of simple $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles which are complex variations of Hodge structure (cf. Proposition 5.3). Assume that the $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle $(V, \varphi) = (V, \beta, \gamma)$ is a Hodge bundle, so that there is a splitting $V = \bigoplus_{i \in \mathbb{R}} F_i$ and

$$(6.49) \quad \beta \in H^0\left(\bigoplus_{i+j=1} F_i \otimes F_j \otimes K\right), \quad \gamma \in H^0\left(\bigoplus_{-i-j=1} F_i^* \otimes F_j^* \otimes K\right),$$

as described in Proposition 5.3 (these tensor products should be interpreted as subbundles of $S^2V \otimes K$ and $S^2V^* \otimes K$, so for example when $i = j = \frac{1}{2}$ the summand $F_i \otimes F_j \otimes K$ is to be thought of as the symmetric product $S^2F_{\frac{1}{2}} \otimes K$). It is important to bear in mind that the indices i of the summands F_i are in general real numbers, not necessarily integers (in fact, we will deduce below from the condition that V is simple that F_i is zero unless i belongs to $\frac{1}{2} + \mathbb{Z}$).

The following definitions will be useful in the subsequent arguments. Let Γ be the group of maps from \mathbb{R} to itself generated by the functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 1 - x$ and $g(x) = -1 - x$. Let $\mathcal{O} \subset \mathbb{R}$ be an orbit of the action of Γ . A parametrization of \mathcal{O} is a surjective map $r : \mathbb{Z} \rightarrow \mathcal{O}$ which satisfies $r(2k + 1) = f(r(2k))$ and $r(2k + 2) =$

$g(r(2k+1))$ for each integer k . Since the maps f, g are involutions, any orbit of Γ admits a parametrization. We now have:

Lemma 6.1. *Let $\mathcal{O} \subset \mathbb{R}$ be any orbit of the action of Γ . Then \mathcal{O} belongs to one of the following sets of orbits:*

- (1) \mathbb{Z} ,
- (2) $\frac{1}{2} + 2\mathbb{Z}$,
- (3) $-\frac{1}{2} + 2\mathbb{Z}$,
- (4) $(\alpha + 2\mathbb{Z}) \cup ((1 - \alpha) + 2\mathbb{Z})$, where $0 < \alpha < \frac{1}{2}$ is a real number,
- (5) $(-\alpha + 2\mathbb{Z}) \cup ((\alpha - 1) + 2\mathbb{Z})$, where $0 < \alpha < \frac{1}{2}$ is a real number.

Furthermore, any parametrization $r : \mathbb{Z} \rightarrow \mathcal{O}$ is bijective unless \mathcal{O} is either $\frac{1}{2} + 2\mathbb{Z}$ or $-\frac{1}{2} + 2\mathbb{Z}$.

Proof. If two real numbers $x, y \in \mathbb{R}$ satisfy $x - y \in 2\mathbb{Z}$ then $f(x) - f(y) \in 2\mathbb{Z}$ and $g(x) - g(y) \in 2\mathbb{Z}$, so the action of Γ on \mathbb{R} descends to any action on $\mathbb{R}/2\mathbb{Z}$. Since $f(g(x)) = 2 + x$, for any Γ -orbit $\mathcal{O} \subset \mathbb{R}$ and any $x \in \mathcal{O}$ we have $x + 2\mathbb{Z} \subset \Gamma$. It follows that the quotient map $\mathbb{R} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ gives a bijection between Γ -orbits. Consequently, to classify the orbits of Γ acting on \mathbb{R} is equivalent to classify the orbits on $\mathbb{R}/2\mathbb{Z}$. Such classification can be easily made by hand, so the first statement of the lemma follows. The second statement can also be checked directly in a straightforward way. \square

Lemma 6.2. *Assume that (V, β, γ) is simple. Then there exists a unique Γ -orbit $\mathcal{O} \subset \mathbb{R}$, which is either $\frac{1}{2} + 2\mathbb{Z}$ or $-\frac{1}{2} + 2\mathbb{Z}$, such that*

$$V = \bigoplus_{i \in \mathcal{O}} F_i.$$

In other words, $F_i = 0$ unless $i \in \mathcal{O}$.

Proof. For any two reals $i, j \in \mathbb{R}$ let β_{ij} be the piece of β contained in $H^0(F_i \otimes F_j \otimes K)$, and define similarly $\gamma_{ij} \in H^0(F_i^* \otimes F_j^* \otimes K)$. It follows from (6.49) that both β_{ij} and γ_{ij} vanish unless i, j belong to the same Γ -orbit. We now prove that there is a unique Γ -orbit \mathcal{O} such that $F_i \neq 0 \Rightarrow i \in \mathcal{O}$. Suppose that this is not the case. Then there exists a Γ -orbit \mathcal{O} such that both bundles

$$V' = \bigoplus_{i \in \mathcal{O}} F_i \quad \text{and} \quad V'' = \bigoplus_{i \notin \mathcal{O}} F_i$$

are nonzero. Clearly, $V = V' \oplus V''$. Furthermore, by the previous observation, defining

$$\beta' = \bigoplus_{i, j \in \mathcal{O}} \beta_{ij}, \quad \beta'' = \bigoplus_{i, j \notin \mathcal{O}} \beta_{ij}, \quad \gamma' = \bigoplus_{i, j \in \mathcal{O}} \gamma_{ij}, \quad \gamma'' = \bigoplus_{i, j \notin \mathcal{O}} \gamma_{ij},$$

we have $\beta = \beta' + \beta''$ and $\gamma = \gamma' + \gamma''$. It follows that the automorphism of V defined as $\sigma = \text{Id}_{V'} - \text{Id}_{V''}$ fixes both β and γ , so (V, β, γ) is not simple, contradicting our hypothesis. Now let \mathcal{O} be the Γ -orbit satisfying $V = \bigoplus_{i \in \mathcal{O}} F_i$, and let $r : \mathbb{Z} \rightarrow \mathcal{O}$ be a parametrization. Assume that \mathcal{O} is not of the form $\frac{1}{2} + 2\mathbb{Z}$ nor of the form $-\frac{1}{2} + 2\mathbb{Z}$. Then, by Lemma 6.1, the map r is a bijection. Define then

$$V' = \bigoplus_{k \in \mathbb{Z}} F_{r(2k)}, \quad \text{and} \quad V'' = \bigoplus_{k \in \mathbb{Z}} F_{r(2k+1)}.$$

Then we have

$$\beta \in H^0(V' \otimes V'' \otimes K), \quad \gamma \in H^0((V')^* \otimes (V'')^* \otimes K).$$

Hence, any automorphism of V of the form $\sigma = \theta \text{Id}_{V'} + \theta^{-1} \text{Id}_{V''}$, with $\theta \in \mathbb{C}^*$, fixes both β and γ , contradicting the assumption that (V, β, γ) is simple. It follows that \mathcal{O} is equal either to $\frac{1}{2} + 2\mathbb{Z}$ or to $-\frac{1}{2} + 2\mathbb{Z}$, so the lemma is proved. \square

6.2. Simple minima with $\beta \neq 0$ and $\gamma \neq 0$. Assume, as in the previous subsection, that (V, β, γ) is simple and a Hodge bundle. Assume additionally that $\beta \neq 0$ and $\gamma \neq 0$.

Denote as before by $\mathcal{O} \subset \mathbb{R}$ the Γ -orbit satisfying $V = \bigoplus_{i \in \mathcal{O}} F_i$. We claim that there are at least two nonzero summands in the previous decomposition. Indeed, if there is a unique nonzero summand F_i , then $\beta \neq 0$ implies $2i = 1$, whereas $\gamma \neq 0$ implies $2i = -1$. Since these assumptions are mutually contradictory, the claim follows.

Now define $M_+ = \sup\{i \mid F_i \neq 0\}$ and $M_- = \inf\{i \mid F_i \neq 0\}$. We claim that $|M_+| \neq |M_-|$. Indeed, by Lemma 6.2 we have either $\mathcal{O} = \frac{1}{2} + 2\mathbb{Z}$ or $\mathcal{O} = -\frac{1}{2} + 2\mathbb{Z}$. Suppose we are in the first case. Then we can write $M_+ = \frac{1}{2} + 2k$, $M_- = \frac{1}{2} + 2l$ for some integers k, l . The equality $|M_+| = |M_-|$ implies that $M_+ = M_-$, so we conclude that there is a unique nonzero F_i , contradicting our previous observation. The case $\mathcal{O} = -\frac{1}{2} + 2\mathbb{Z}$ is completely analogous.

In view of the preceding observation, we may distinguish two cases: either $|M_+| > |M_-|$ or $|M_+| < |M_-|$. Henceforth we shall assume, for definiteness, that we are in the situation $|M_+| > |M_-|$.

Remark 6.3. Recall from Proposition 3.36 that, for each d , there is an isomorphism $\mathcal{M}_d \xrightarrow{\cong} \mathcal{M}_{-d}$, given by the duality $(V, \beta, \gamma) \mapsto (V^*, \gamma^t, \beta^t)$. Under this duality the two cases $|M_+| > |M_-|$ and $|M_+| < |M_-|$ get interchanged (in fact, as we shall see, the former situation corresponds to $d < 0$, whereas the latter corresponds to $d > 0$).

Let $M = M_+$. We have $M = p + \frac{1}{2}$ for some integer p . Define $m = -p + \frac{1}{2}$. We can write

$$(6.50) \quad V = \bigoplus_{\lambda=0}^p F_{M-2\lambda}.$$

A priori, in this decomposition there might be some summands which are zero. Nevertheless, we will see below that this is not the case.

Theorem 6.4. *Let (V, β, γ) be simple and a Hodge bundle with $\beta \neq 0$ and $\gamma \neq 0$. Assume additionally that $|M_+| > |M_-|$ so that (V, β, γ) is of the form (6.50). Then the map*

$$\text{ad}(\varphi): \text{End}(V)_k \rightarrow (S^2V \otimes K \oplus S^2V^* \otimes K)_{k+1}$$

is an isomorphism for all $k > 0$ if and only if the following holds:

- (i) *For any $0 \leq \lambda \leq p$ the rank of $F_{M-2\lambda}$ is 1 (in particular, it is nonzero);*
- (ii) *for any $0 \leq \lambda \leq p - 1$ the piece of β in*

$$F_{M-2\lambda} \otimes F_{m+2\lambda} \otimes K \subset S^2V \otimes K$$

never vanishes;

(iii) for any $1 \leq \lambda \leq p-1$ the piece of γ in

$$F_{M-2\lambda}^* \otimes F_{m+2\lambda-2}^* \otimes K \subset S^2V^* \otimes K$$

never vanishes.

Analogous statements hold in the case $|M_+| < |M_-|$ (cf. Remark 6.3).

Proof. We already proved that the assumption $\beta \neq 0$ and $\gamma \neq 0$ implies that $p \geq 1$ (for otherwise in the decomposition (6.50) we would only have one summand). If we take the piece in degree $k = 2p$ of the map $\text{ad}(\varphi)$, we get

$$A := \text{ad}(\varphi)_{2p} : F_M \otimes F_m^* \rightarrow S^2F_M \otimes K,$$

which by assumption is an isomorphism. Computing the ranks $r_i = \text{rk}(F_i)$, we deduce

$$r_M r_m = \frac{r_M(r_M + 1)}{2}.$$

To prove that $r_M = r_m = 1$, we assume the contrary and show that this leads to a contradiction. If $r_M > 1$ then by the formula above we must have $r_m < r_M$. Let b be the piece of β in $F_M \otimes F_m \otimes K \subset (S^2V \otimes K)_{2p}$. Then the map A sends any $e \in F_M \otimes F_m^*$ to

$$A(e) = eb + be^*.$$

The first summand denotes the composition of maps

$$F_M^* \xrightarrow{b} F_m \xrightarrow{e} F_M$$

and the second summand

$$F_M^* \xrightarrow{e^*} F_m^* \xrightarrow{b} F_M.$$

Take a basis u_1, \dots, u_{r_M} of F_M whose first r_m elements are a basis of $b(F_m^*)$, and take on F_M^* the dual basis. If we write the matrices of eb and be^* with respect to these basis, one readily checks that the $(r_M - r_m) \times (r_M - r_m)$ block in the bottom left of both matrices vanishes. Consequently, an element in S^2F_M represented by a symmetric matrix whose entry at the bottom left corner is nonzero cannot belong to the image of A . Hence A is not an isomorphism, in contradiction to our assumption, so we deduce that

$$r_M = r_m = 1.$$

One also deduces that the section $b \in H^0(F_M \otimes F_m \otimes K)$ never vanishes. This proves statements (i) and (ii) when $\lambda = 0$ or p .

Observation. The following observation will be useful: if $e \in F_i \otimes F_j^* \subset \text{End}(V)$, then any nonzero piece of $\text{ad}(\varphi)(e)$ in the decomposition (5.45) belongs to a summand of the form $F_i \otimes F_u \otimes K$, and any nonzero piece in (5.46) belongs to a summand of the form $F_j^* \otimes F_v^* \otimes K$ (in both cases the symmetrization should be understood if the two indices coincide). This follows from the fact that $\text{ad}(\varphi)(e)$ is the sum of compositions of e with another map (either on the right and on the left). Hence each summand in $\text{ad}(\varphi)(e)$ must share with e at least the domain or the target.

Now let us take any $k = 2p - 2\lambda \geq 1$, such that $\lambda \geq 1$, so that $1 \leq \lambda \leq p-1$. Then we have

$$(6.51) \quad \text{End}(V)_{2p-2\lambda} = F_M \otimes F_{m+2\lambda}^* \oplus F_{M-2} \otimes F_{m+2\lambda-2}^* \oplus \cdots \oplus F_{M-2\lambda} \otimes F_m^*.$$

We claim that there is no nonzero block in $(S^2V^* \otimes K)_{2p-2\lambda+1}$ of the form $F_{m+2\lambda}^* \otimes F_v^* \otimes K$. Indeed, for that one should take $v = -(2p - 2\lambda + 1) - (m + 2\lambda) = -M - 1$, but $F_{-M-1} =$

0, because $-M - 1 < m$. On the other hand, $(S^2V^* \otimes K)_{2p-2\lambda+1}$ contains the block $F_M \otimes F_{M-2\lambda} \otimes K$ and no other block involving F_M . Hence we must have

$$\text{ad}(\varphi)_k(F_M \otimes F_{m+2\lambda}^*) \subset F_M \otimes F_{M-2\lambda} \otimes K.$$

Taking ranks and using the fact that $\text{ad}(\varphi)_k$ is injective, we deduce that

$$r_{m+2\lambda} \leq r_{M-2\lambda}.$$

Since $1 \leq \lambda \leq p-1 \iff 1 \leq p-\lambda \leq p-1$, we automatically deduce that

$$r_{m+2p-2\lambda} \leq r_{M-2p+2\lambda}.$$

But $m+2p=M$, so we conclude that

$$(6.52) \quad r_{m+2\lambda} = r_{M-2\lambda}.$$

Let us distinguish two possibilities.

Case (1). Suppose that $\lambda = 2l + 1$ is odd. Then we have

$$S^2F_{m+\lambda-1}^* \otimes K \subset (S^2V^* \otimes K)_{2p-2\lambda+1},$$

and the observation above implies that

$$\text{ad}(\varphi)_{2p-2\lambda}^{-1}(S^2F_{m+\lambda-1}^* \otimes K) \subset F_{M-\lambda-1} \otimes F_{m+\lambda-1}^*.$$

The argument given above for $\lambda = 0$ proves now that the piece of γ in

$$F_{M-\lambda-1}^* \otimes F_{m+\lambda-1}^* \otimes K$$

never vanishes.

Case (2). Suppose that $\lambda = 2l$ is even. Then we have

$$S^2F_{M-\lambda} \otimes K \subset (S^2V \otimes K)_{2p-2\lambda+1},$$

and the observation above implies that

$$\text{ad}(\varphi)_{2p-2\lambda}^{-1}(S^2F_{M-\lambda} \otimes K) \subset F_{M-\lambda} \otimes F_{m+\lambda}^*.$$

The argument given above for $\lambda = 0$ proves now that the piece of β in

$$F_{M-\lambda} \otimes F_{m+\lambda} \otimes K$$

never vanishes.

These arguments prove statements (ii) and (iii).

We are now going to prove that for any $1 \leq \lambda \leq p/2$ the ranks $r_{M-2\lambda} = r_{m+2\lambda} = 1$ using induction. Fix such a λ and assume that for any $0 \leq l < \lambda$ we have $r_{M-2l} = r_{m+2l} = 1$ (when $l = 0$ we already know this is true). Since $2p - 2\lambda \geq 1$ we must have

$$(6.53) \quad \text{rk End}(V)_{2p-2\lambda} = \text{rk}(S^2V \otimes K \oplus S^2V^* \otimes K)_{2p-2\lambda+1}.$$

Using induction we can compute the left hand side:

$$\begin{aligned} \text{rk End}(V)_{2p-2\lambda} &= r_M r_{m+2\lambda} + r_{M-2} r_{m+2\lambda-2} + \cdots + r_{M-2\lambda+2} r_{m+2} + r_{M-2\lambda} r_m \\ &= r_{m+2\lambda} + r_{M-2\lambda} + (\lambda - 1). \end{aligned}$$

We now distinguish again two cases.

Case (1). Suppose that $\lambda = 2l + 1$ is odd. Then we compute

$$\begin{aligned} \text{rk}(S^2V)_{2p-2\lambda+1} &= r_M r_{M-2\lambda} + r_{M-2} r_{M-2\lambda+2} + \cdots + r_{M-\lambda+1} r_{M-\lambda-1} \\ &= r_{M-2\lambda} + l \end{aligned}$$

and

$$\begin{aligned} \mathrm{rk}(S^2V^*)_{2p-2\lambda+1} &= r_m r_{m+2\lambda-2} + r_{m+2} r_{m+2\lambda-4} + \cdots + r_{m+\lambda-3} r_{m+\lambda+1} \\ &\quad + \binom{r_{m+\lambda-1} + 1}{2} = l + 1. \end{aligned}$$

Comparing the two computations it follows from (6.53) that

$$r_{m+2\lambda} = 1,$$

and using (6.52) we deduce that

$$r_{M-2\lambda} = 1.$$

Case (2). Now suppose that $\lambda = 2l$ is even. Then we have

$$\begin{aligned} \mathrm{rk}(S^2V)_{2p-2\lambda+1} &= r_M r_{M-2\lambda} + r_{M-2} r_{M-2\lambda+2} + \cdots + r_{M-\lambda+2} r_{M-\lambda-2} \\ &\quad + \binom{r_{M-\lambda}}{2} = r_{M-2\lambda} + l \end{aligned}$$

and

$$\begin{aligned} \mathrm{rk}(S^2V^*)_{2p-2\lambda+1} &= r_m r_{m+2\lambda-2} + r_{m+2} r_{m+2\lambda-4} + \cdots + r_{m+\lambda-2} r_{m+\lambda} \\ &= l. \end{aligned}$$

Comparing again the two computations we deduce that

$$r_{m+2\lambda} = r_{M-2\lambda} = 1.$$

This finishes the proof of statement (i) and thus the proof of the Theorem in the case $|M_+| > |M_-|$.

Finally, in the case $|M_+| < |M_-|$ the analysis is completely analogous. \square

Corollary 6.5. *Let (V, β, γ) be simple and a Hodge bundle with $\beta \neq 0$ and $\gamma \neq 0$. Assume additionally that $|M_+| > |M_-|$ so that (V, β, γ) is of the form (6.50). Assume that the map*

$$\mathrm{ad}(\varphi): \mathrm{End}(V)_k \rightarrow (S^2V \otimes K \oplus S^2V^* \otimes K)_{k+1}$$

is an isomorphism for all $k > 0$. Then the following holds.

- (1) *If $n = 2$ then $F_{\frac{3}{2}} \otimes F_{-\frac{1}{2}} \otimes K \simeq \mathcal{O}$.*
- (2) *If $n = 2q + 1 \geq 3$ is odd then $\beta: F_{\frac{1}{2}-2\lambda}^* \xrightarrow{\simeq} F_{\frac{1}{2}+2\lambda} K$ for any integer $-q \leq \lambda \leq q$. In particular, there exists a square root L of K such that for any integer $-q \leq \lambda \leq q$ we have*

$$F_{M-2(q-\lambda)} \simeq F_{m+2(\lambda+q)} \simeq F_{\frac{1}{2}+2\lambda} \simeq L^{-1} \otimes K^{-2\lambda},$$

and the bundle V is of the form

$$V = \bigoplus_{\lambda=-q}^q L^{-1} K^{-2\lambda}.$$

- (3) *If $n = 2q + 2 \geq 4$ then $\gamma: F_{-\frac{1}{2}} \xrightarrow{\simeq} F_{-\frac{1}{2}}^* K$ and $\beta: F_{-\frac{1}{2}-2\lambda}^* \xrightarrow{\simeq} F_{-\frac{1}{2}+2\lambda} K$ for any integer $-q \leq \lambda \leq q + 1$. In particular, there exists a square root L of K such that for any integer $-q \leq \lambda \leq q + 1$ we have*

$$F_{-\frac{1}{2}+2\lambda} \simeq L \otimes K^{-2\lambda} \simeq F_{M-2(q+1-\lambda)} \simeq F_{m+2(\lambda+q)},$$

and the bundle V is of the form

$$V = \bigoplus_{\lambda=-q}^{q+1} LK^{-2\lambda}.$$

- (4) For any $n \geq 2$, the degree of V is $\deg(V) = n(1 - g)$.
 (5) For any $n \geq 2$, an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle of the form described in (1)–(3) above is stable as an $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundle, and thus also as an $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle.

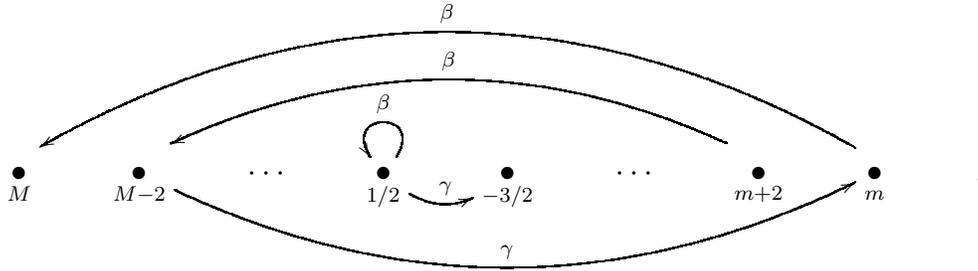
Analogous statements hold in the case $|M_+| < |M_-|$. In particular, in this case the degree of V is $\deg(V) = n(g - 1)$ (cf. Remark 6.3).

Remark 6.6. In the case $n = 1$ it is not possible for (V, φ) to be a Hodge bundle with $\beta \neq 0$ and $\gamma \neq 0$.

Proof of Corollary 6.5. First we observe that, since the F_i are all line bundles, we have $n = p + 1$, $M = p + \frac{1}{2}$ and $m = -p + \frac{1}{2}$.

(1) In this case we have $n = 2$, $p = 1$, $M = 3/2$, $m = -1/2$. Then, taking $\lambda = 0$ in (ii) of Theorem 6.4 we get $F_{\frac{3}{2}} \otimes F_{-\frac{1}{2}} \otimes K \simeq \mathcal{O}$.

(2) In this case we have $n = p + 1 = 2q + 1$ so that $M = 2q + 1/2$ and $m = -2q + 1/2$. Hence, using (ii) and (iii) of Theorem 6.4, we can describe the structure of the maps β and γ in the following diagram:

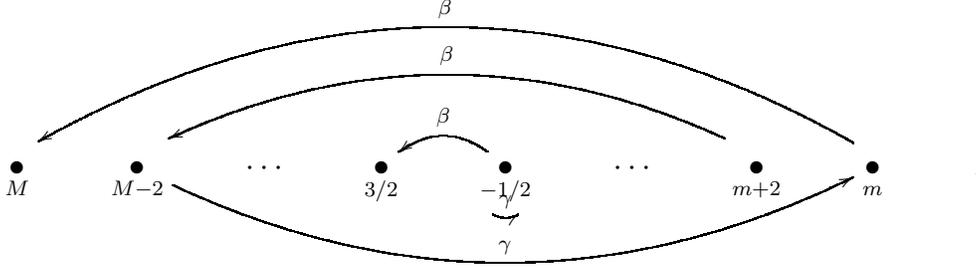


where an arrow $\bullet_i \xrightarrow{\beta} \bullet_j$ means that there is an isomorphism $\beta: F_i^* \rightarrow F_j \otimes K$ (and

thus $j = -i + 1$); similarly, an arrow $\bullet_i \xrightarrow{\gamma} \bullet_j$ means that there is an isomorphism

$\gamma: F_i \rightarrow F_j^* \otimes K$. In particular, we see that the isomorphism $\beta: F_{\frac{1}{2}}^* \xrightarrow{\simeq} F_{\frac{1}{2}} \otimes K$ means that $F_{\frac{1}{2}} \simeq L^{-1}$ for a square root L of K . This proves the case $\lambda = 0$ of (2). Now repeated application of (ii) and (iii) of Theorem 6.4 proves the general case. Note that this argument can be phrased as saying that the graph above is connected and its only closed loop is the one at $1/2$: thus the remaining F_i are uniquely determined by $F_{\frac{1}{2}}$.

(3) In this case we have $n = p + 1 = 2q + 2$ so that $M = 2q + 3/2$ and $m = -2q - 1/2$ and, as above, we have a diagram



The argument is now analogous to the previous case.

(4) Easy from the formulas for V given in (2) and (3).

(5) Let (V, φ) be of the kind described in (1)–(3), and consider the associated $\mathrm{SL}(2n, \mathbb{C})$ -Higgs bundle $(V \oplus V^*, \Phi) = H(V, \varphi)$. The Φ -invariant subbundles of $V \oplus V^*$ are of the form $\bigoplus_{i \geq i_0} (F_i \oplus F_{-i}^*)$. From the given description, it is easy to check that such a subbundle, when proper and non-zero, has degree strictly negative.

Finally, in the case $|M_+| < |M_-|$ the analysis is completely analogous. \square

6.3. Simple minima: final characterization. Finally, we use the analysis carried out so far to determine the minima of the Hitchin functional on the locus of the moduli space corresponding to simple $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles which are stable as $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundles.

Theorem 6.7. *Let (V, β, γ) be a simple $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle which is stable as an $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle.*

- (1) *If $|d| < n(g - 1)$ then (V, β, γ) represents a minimum of the Hitchin functional if and only if it belongs to \mathcal{N}_d .*
- (2) *If $|d| = n(g - 1)$ and $n \geq 3$ then (V, β, γ) represents a minimum of the Hitchin functional if and only if one of the following situations occurs:*
 - (i) *the $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, β, γ) belongs to \mathcal{N}_d ;*
 - (ii) *the $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V, β, γ) is of the type described in (2) or (3) of Corollary 6.5.*
 - (iii) *the dual $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V^*, γ^t, β^t) is of the type described in (2) or (3) of Corollary 6.5 (cf. Remark 6.3).*

In cases (ii) and (iii) we say that (V, β, γ) is a quiver type minimum.

Proof. If (V, β, γ) belongs to \mathcal{N}_d then we know from Proposition 5.9 that it represents a minimum. And, if (V, β, γ) (or the dual (V^*, γ^t, β^t)) is of the type described in (2) or (3) of Corollary 6.5, then Corollary 5.6 and Theorem 6.4 show that it represents a minimum.

On the other hand, if (V, β, γ) is a minimum which does not belong to \mathcal{N}_d , then Corollary 5.6, Theorem 6.4 and Corollary 6.5 show that it (or the dual (V^*, γ^t, β^t)) is of the type described in (2) or (3) of Corollary 6.5. \square

7. MINIMA ON THE ENTIRE MODULI SPACE

7.1. Main result and strategy of proof. In Section 6 we characterized the minima of the Hitchin functional on the locus of \mathcal{M}_d corresponding to simple $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs

bundles (V, φ) which are stable as $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundles. In this section we provide the remaining results required to extend this characterization to the whole moduli space, thus completing the proof of Theorem 5.10. As explained in the proof of that Theorem, what is required is to rule out certain type of potential minima of the Hitchin functional. In each case this is done by using (2) of Proposition 5.4. The main result of this Section is the following.

Proposition 7.1. *Let $(V, \varphi = \beta + \gamma)$ be a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle and assume that the decomposition $(V, \varphi) = (V_1, \varphi_1) \oplus \cdots \oplus (V_k, \varphi_k)$ of Theorem 3.40 is non-trivial. If (V, φ) is a local minimum of the Hitchin functional then either $\beta = 0$ or $\gamma = 0$.*

Proof. The starting point is the structure Theorem 3.40. Recall that this describes a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle as a direct sum

$$(7.54) \quad (V, \varphi) = \bigoplus (V_i, \varphi_i),$$

where each $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V_i, φ_i) comes from a G_i -Higgs bundle which is a smooth point in its respective moduli space. If (V, φ) is a minimum, then Proposition 5.2 implies that each (V_i, φ_i) is a minimum on the corresponding moduli space of G_i -Higgs bundles. Consider each of the possible G_i 's in turn.

The case $G_i = \mathrm{Sp}(2n_i, \mathbb{R})$. This is the case covered by Theorem 6.7. (Except for the case $n_i = 2$, which will require special attention.)

The case $G_i = \mathrm{U}(n_i)$. In this case $\varphi_i = 0$ for any G_i -Higgs bundle, as we have already seen.

The case $G_i = \mathrm{U}(p_i, q_i)$. In this case, the minima of the Hitchin functional were determined in [2, Theorem 4.6]. There it is shown that a $\mathrm{U}(p_i, q_i)$ -Higgs bundle $(\tilde{V}_i, \tilde{W}_i, \tilde{\beta} + \tilde{\gamma})$ is a minimum if and only if $\tilde{\beta} = 0$ or $\tilde{\gamma} = 0$. Hence $(V_i, \varphi_i) = v_*^{\mathrm{U}(p_i, q_i)}(\tilde{V}_i, \tilde{W}_i, \tilde{\beta} + \tilde{\gamma})$ (cf. (3.14)) is a minimum if and only if $\beta_i = 0$ or $\gamma_i = 0$.

The case $G_i = \mathrm{GL}(n_i, \mathbb{R})$. The moduli space of such Higgs bundles was studied in [3]. Using the results of that paper we show in Lemma 7.8 below that a $\mathrm{Sp}(2n_i, \mathbb{R})$ -Higgs bundle (V_i, φ_i) coming from a $\mathrm{GL}(n_i, \mathbb{R})$ -Higgs bundle is a minimum if and only if $\varphi_i = 0$.

A quiver type minimum (V, φ) is simple and stable as a $\mathrm{Sp}(2n, \mathbb{C})$ -Higgs bundle by (5) of Corollary 6.5. Thus, to conclude the proof of the Proposition, it remains to show that if (V, φ) is a minimum and the decomposition (7.54) is non-trivial, then it belongs to \mathcal{N}_d , i.e., $\beta = 0$ or $\gamma = 0$. By the above analysis of the minima coming from G_i -Higgs bundles, it therefore suffices to show that (V, φ) is not a minimum when the decomposition (7.54) falls in one of the following cases:

- (1) There is a (V_i, φ_i) in \mathcal{N}_{d_i} with $\beta_i \neq 0$ and a (V_j, φ_j) in \mathcal{N}_{d_j} with $\gamma_j \neq 0$.
- (2) There is a (V_i, φ_i) which is a quiver type minimum and a (V_j, φ_j) which lies in \mathcal{N}_{d_i} .
- (3) There are (distinct) (V_i, φ_i) and (V_j, φ_j) which are quiver type minima.

In order to accommodate the possibility $n_i = 2$, the quiver type minima must here be understood to include all minima with $\beta \neq 0$ and $\gamma \neq 0$ (cf. (1) of Corollary 6.5). The case $n_i = 1$ is included since such minima must have $\beta = 0$ or $\gamma = 0$ (cf. Remark 6.6).

Note that, by Proposition 5.2, in fact it suffices to consider the case when $k = 2$ in (7.54). With this in mind, the results of Lemmas 7.2, 7.4 and 7.6 below conclude the proof. \square

7.2. Deforming a sum of minima in \mathcal{N}_d .

Lemma 7.2. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle which decomposes as a direct sum $(V, \varphi) = (V', \varphi') \oplus (V'', \varphi'')$ with $\varphi' = (\beta', \gamma')$ and $\varphi'' = (\beta'', \gamma'')$. Suppose that $\beta' = 0$, $\gamma' \neq 0$, $\beta'' \neq 0$ and $\gamma'' = 0$. Suppose additionally that (V', φ') and (V'', φ'') are stable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles or stable $\mathrm{U}(p, q)$ -Higgs bundles. Then (V, φ) is not a minimum of f on \mathcal{M}_d .*

Proof. We prove the Lemma by applying the criterion in (2) of Proposition 5.4. As a first step, we identify the complex C_-^\bullet defined in (5.48), and for that we need to know the weights of each piece V', V'' . Recall that the weight of φ', φ'' is always 1.

- (1) Since $\gamma': V' \rightarrow V'^*K$, the weight on V'^* is $1 + \lambda' = -\lambda'$, where λ' is the weight on V' . Thus $\lambda' = -1/2$.
- (2) Similarly, the weight on V'' is $\lambda'' = 1/2$.

From this it follows immediately that the complex C_-^\bullet is given by

$$C_-^\bullet: \mathrm{Hom}(V', V'') \rightarrow 0,$$

so that

$$\mathbb{H}^1(C_-^\bullet) = H^1(\mathrm{Hom}(V', V'')).$$

Recall from Remark 5.8 that $d' = \deg(V') \geq 0$ and $d'' \leq 0$ so, by Riemann–Roch,

$$H^1(\mathrm{Hom}(V', V'')) \neq 0.$$

This proves that C_-^\bullet has nonzero first hypercohomology. To finish the argument we need to integrate any element of $\mathbb{H}^1(C_-^\bullet)$ to a deformation of (V, φ) through polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles.

Chose any² nonzero element $a \in H^1(\mathrm{Hom}(V', V''))$. Denote by D the open unit disk. Define $\mathbb{V}' = D \times V'$ and $\mathbb{V}'' = D \times V''$, which we view as vector bundles over $X \times D$. We denote by $\gamma'_D: \mathbb{V}' \rightarrow \mathbb{V}'^* \otimes K$ (here K denotes the pullback to $X \times D$) the extension of γ' which is constant on the D direction, and we define similarly $\beta''_D: \mathbb{V}''^* \rightarrow \mathbb{V}'' \otimes K$. Take the extension

$$0 \rightarrow \mathbb{V}'' \rightarrow \mathbb{V} \rightarrow \mathbb{V}' \rightarrow 0$$

classified by

$$a \otimes 1 \in H^1(\mathrm{Hom}(\mathbb{V}', \mathbb{V}'')) = H^1(X; \mathrm{Hom}(V', V'')) \otimes H^0(D; \mathbb{C}).$$

The restriction of this to $X \times \{t\}$ is the extension

$$(7.55) \quad 0 \rightarrow V'' \rightarrow V_t \rightarrow V' \rightarrow 0$$

classified by $ta \in H^1(\mathrm{Hom}(V', V''))$. Define $\gamma_D: \mathbb{V} \rightarrow \mathbb{V}^* \otimes K$ as the composition

$$\mathbb{V} \longrightarrow \mathbb{V}' \xrightarrow{\gamma'_D} \mathbb{V}'^* \otimes K \rightarrow \mathbb{V}^* \otimes K,$$

where the first arrow comes from the exact sequence defining \mathbb{V} and the third one comes from dualizing the same exact sequence and tensoring by the pullback of K . Similarly, define $\beta_D: \mathbb{V}^* \rightarrow \mathbb{V} \otimes K$ as the composition

$$\mathbb{V}^* \longrightarrow \mathbb{V}''^* \xrightarrow{\beta''_D} \mathbb{V}'' \otimes K \rightarrow \mathbb{V} \otimes K.$$

²when one of (V', φ') and (V'', φ'') is a $\mathrm{U}(p, q)$ -Higgs bundle, this choice is not completely arbitrary, cf. the proof of Lemma 7.3 below.

The resulting triple $(\mathbb{V}, \beta_D, \gamma_D)$ is a family of symplectic Higgs bundles parametrized by the disk, whose restriction to the origin coincides with (V, φ) , and which integrates the element a in the deformation complex.

It remains to show that each member of the family $(\mathbb{V}, \beta_D, \gamma_D)$ is a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle. This is done in Lemma 7.3 below. We have thus proved that (V, φ) is not a local minimum. \square

Lemma 7.3. *The $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle $(V_t, \varphi_t = \beta_t + \gamma_t)$ on X , obtained by restricting to $X \times \{t\}$ the family $(\mathbb{V}, \beta_D, \gamma_D)$ constructed in the proof of Lemma 7.2, is polystable.*

Proof. It will be convenient to use the stability condition for $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles as given in Lemma 3.29. Thus, if (V_t, φ_t) is not stable, there are subbundles $A \subset V_t$ and $B \subset V_t^*$ such that $\gamma_t(A) \subset B \otimes K$ and $\beta_t(B) \subset A \otimes K$, and with $\deg(A \oplus B) = 0$. Since X is a Riemann surface, the kernel of the restriction to A of the sheaf map $V_t \rightarrow V''$ is locally free and corresponds to a subbundle $A' \subset A$. The quotient $A'' := A/A'$ then gives a subbundle $A'' \subset V''$ so that we have a commutative diagram with exact rows and columns:

$$(7.56) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A'' & \longrightarrow & A & \longrightarrow & A' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & V'' & \longrightarrow & V_t & \longrightarrow & V' \longrightarrow 0. \end{array}$$

Similarly, we obtain subbundles $B'' \subset V''^*$ and $B' \subset V'^*$ and a diagram:

$$(7.57) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & B' & \longleftarrow & B & \longleftarrow & B'' \longleftarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & V'^* & \longleftarrow & V_t & \longleftarrow & V''^* \longleftarrow 0. \end{array}$$

One easily checks that $B'^{\perp} \subset A'$ and $B''^{\perp} \subset A''$. By definition of γ_t , the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & V' & \longrightarrow & V_t & \longrightarrow & V'' \longrightarrow 0 \\ & & \downarrow \gamma' & & \downarrow \gamma_t & & \\ 0 & \longleftarrow & V'^* & \longleftarrow & V_t & \longleftarrow & V''^* \longleftarrow 0. \end{array}$$

commutes. Thus, since $\gamma_t(A) \subset B \otimes K$, we have that $\gamma'(A') \subset B' \otimes K$. Similarly, $\beta''(B'') \subset A'' \otimes K$. It follows that the pair of subbundles $A' \subset V'$ and $B' \subset V'^*$ destabilizes (V', φ') and that the pair of subbundles $A'' \subset V''$ and $B'' \subset V''^*$ destabilizes (V'', φ'') .

Consider now the case in which both (V', φ') and (V'', φ'') are stable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. Then we must have $A' \oplus B' = V' \oplus V'^*$ or $A' \oplus B' = 0$ and similarly for $A'' \oplus B''$. The only case in which the original destabilizing subbundle $A \oplus B \subset V_t \oplus V_t^*$ is non-trivial is when $A' \oplus B' = V' \oplus V'^*$ and $A'' \oplus B'' = 0$ (or vice-versa). But, in this case, $V' \simeq A' \simeq A$ and hence (7.56) shows that the non-trivial extension (7.55) splits, which is a contradiction. Hence there is no non-trivial destabilizing pair of subbundles of (V_t, φ_t) , which is therefore stable.

It remains to deal with case in which one, or both, of (V', φ') and (V'', φ'') are stable $U(p, q)$ -Higgs bundles. The remaining cases being similar, for definiteness we consider the case in which (V'', φ'') is a stable $\mathrm{Sp}(2n'', \mathbb{R})$ -Higgs bundle and (V', φ') is a stable $U(n'_1, n'_2)$ -Higgs bundle, i.e.,

$$V' = V'_1 \oplus V'_2, \quad \varphi' = \gamma' \in H^0(V'_1 \otimes V'_2 \otimes K).$$

In addition to the cases considered above, we now also need to consider the case when $A' \oplus B'$ is non-trivial, say $A' \oplus B' = V'_1 \oplus V'^*_2$. There are now two possibilities for $A'' \oplus B''$: either it is zero or it equals $V'' \oplus V''^*$; we leave the first (simpler) case to the reader and consider the second one. In this case, the element

$$a = a_1 + a_2 \in H^1(\mathrm{Hom}(V', V'') = H^1(\mathrm{Hom}(V'_1, V'')) \oplus H^1(\mathrm{Hom}(V'_2, V''))$$

chosen in the proof of Lemma 7.2 above must be taken such that both a_1 and a_2 are non-zero (this is possible by Riemann–Roch). Thus, for $i = 1, 2$ we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & V'' & \longrightarrow & V_{t_i} & \longrightarrow & V'_i & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V'' & \longrightarrow & V_t & \longrightarrow & V'_1 \oplus V'_2 & \longrightarrow & 0 \end{array}$$

of non-trivial extensions, where the two vertical maps on the right are inclusions. This, together with (7.57) for $B' = V'^*_2$ and $B'' = V''^*$, gives rise to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & V'^*_2 & \longrightarrow & B & \longrightarrow & V''^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & V'^*_1 \oplus V'^*_2 & \longrightarrow & V_t^* & \longrightarrow & V''^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & V'^*_2 & \longrightarrow & V_{t_2}^* & \longrightarrow & V''^* & \longrightarrow & 0. \end{array}$$

The composites of the vertical maps on the left and on the right are isomorphisms. Hence the composite of the middle vertical maps is also an isomorphism and this provides a splitting of the extension

$$0 \rightarrow V'^*_1 \rightarrow V_t^* \rightarrow V_{t_2}^* \rightarrow 0.$$

Denote the splitting maps in the dual split extension by

$$i: V'_1 \rightarrow V_t \quad \text{and} \quad p: V_t \rightarrow V_{t_2}.$$

We now have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & V'' & \longrightarrow & V_{t_1} & \longrightarrow & V'_1 & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V'' & \longrightarrow & V_t & \longrightarrow & V'_1 \oplus V'_2 & \longrightarrow & 0 \\ & & \parallel & & \downarrow p & & \downarrow & & \\ 0 & \longrightarrow & V'' & \longrightarrow & V_{t_2} & \longrightarrow & V'_2 & \longrightarrow & 0, \end{array}$$

where the vertical maps on the right are the natural inclusion and projection, respectively. Using the existence of the splitting map $i: V'_1 \rightarrow V_t$ and the inclusion $V_{t_2} \rightarrow V_t$ one readily

sees that this diagram commutes. This finally gives us the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & V_t/V_{t_1} & \xrightarrow{\simeq} & V'_2 & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & V'' & \longrightarrow & V_{t_2} & \longrightarrow & V'_2 & \longrightarrow & 0, \end{array}$$

which shows that the sequence at the bottom is split, a contradiction. \square

7.3. Deforming a sum of a quiver type minimum and a minimum in \mathcal{N}_d .

Lemma 7.4. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle which decomposes as a direct sum $(V, \varphi) = (V', \varphi') \oplus (V'', \varphi'')$ with $\varphi' = (\beta', \gamma')$ and $\varphi'' = (\beta'', \gamma'')$. Suppose that*

- (1) (V', φ') is a quiver type minimum,
- (2) (V'', φ'') is a minimum with $\beta'' = 0$ or $\gamma'' = 0$ which is a stable G'' -Higgs bundle for G'' one of the following groups: $\mathrm{Sp}(2n'', \mathbb{R})$, $\mathrm{U}(p'', q'')$, $\mathrm{U}(n'')$ or $\mathrm{GL}(n'', \mathbb{R})$.

Then (V, φ) is not a minimum of f on \mathcal{M}_d .

Proof. Consider for definiteness the case in which (V', φ') is a quiver type minimum with $\deg(V') = n'(1 - g)$ and (V'', φ'') has $\gamma'' = 0$ and $\beta'' \neq 0$. The case in which $\beta'' = 0$ and $\gamma'' \neq 0$ can be treated along the same lines as the present case, so we will not give the details. The case in which (V', φ') is a quiver type minimum with $\deg(V') = n'(g - 1)$ is obtained by symmetry. Note that some degenerate cases can occur, namely:

- (1) (V', φ') is a quiver type minimum with $\mathrm{rk}(V') = 2$ (cf. (1) of Corollary 6.5).
- (2) (V'', φ'') has $\beta'' = \gamma'' = 0$.

With respect to Case (1), all we need for the arguments below is that $\beta: F_{\frac{3}{2}}^* \xrightarrow{\simeq} F_{-\frac{1}{2}} \otimes K$ is an isomorphism, which is guaranteed by (1) of Corollary 6.5. In what concerns Case (2), slight modifications are required in the arguments given below; we leave these to the reader.

With these introductory remarks out of the way, Corollary 6.5 tells us that V' decomposes as a direct sum of line bundles $V' = F_m \oplus \cdots \oplus F_M$ and that restricting β' we get an isomorphism

$$\beta' : F_m^* \xrightarrow{\simeq} F_M \otimes K.$$

Our first task is to identify nonzero elements in the first hypercohomology of C_\bullet . A good place to look for them is in the hypercohomology of the piece of highest weight in the deformation complex, which is

$$(7.58) \quad V''^* \otimes F_M \oplus V'' \otimes F_m^* \rightarrow V'' \otimes F_M \otimes K.$$

This morphism cannot be an isomorphism, because the ranks do not match. Thus Proposition 5.5 implies that \mathbb{H}^1 of this complex is non-vanishing.

In the hypercohomology long exact sequence (cf. (2.6)) of the complex (7.58), the map

$$H^0(V''^* \otimes F_M \oplus V'' \otimes F_m^*) = H^0(V''^* \otimes F_M) \oplus H^0(V'' \otimes F_m^*) \rightarrow H^0(V'' \otimes F_M \otimes K)$$

is always onto because the map $f: H^0(V'' \otimes F_m^*) \rightarrow H^0(V'' \otimes F_M \otimes K)$ is induced by tensoring $\beta': F_m^* \rightarrow F_M \otimes K$ (which is an isomorphism) with the identity map $V'' \rightarrow V''$, so f is also an isomorphism. Hence the image of $H^0(V'' \otimes F_M \otimes K) \rightarrow \mathbb{H}^1$ is zero, and this by exactness implies that $\mathbb{H}^1 \rightarrow H^1(V''^* \otimes F_M \oplus V'' \otimes F_m^*)$ is injective. We now want

to characterize the image of this inclusion. Tensoring the Higgs fields β'' and β' with the identity on F_M and V'' respectively, we get maps

$$\beta'' : V''^* \otimes F_M \rightarrow V'' \otimes F_M \otimes K,$$

and

$$\beta' : V'' \otimes F_m^* \xrightarrow{\cong} V'' \otimes F_M \otimes K.$$

Now the map ζ in the long exact sequence

$$\mathbb{H}^1 \rightarrow H^1(V''^* \otimes F_M \oplus V'' \otimes F_m^*) \xrightarrow{\zeta} H^1(V'' \otimes F_M \otimes K) \rightarrow \mathbb{H}^2$$

can be interpreted as follows: given elements $(\delta, \epsilon) \in H^1(V''^* \otimes F_M) \oplus H^1(V'' \otimes F_m^*)$,

$$\zeta(\delta, \epsilon) = -\beta''(\delta) - \beta'(\epsilon) \in H^1(V'' \otimes F_M \otimes K).$$

Hence we may take a nonzero pair (δ, η) satisfying $\beta''(\delta) + \beta'(\epsilon) = 0$ and corresponding to a nonzero element in the hypercohomology of the complex (7.58). We next prove that the deformation along (δ, η) is unobstructed, by giving an explicit construction of a family of Higgs bundles (V_t, β_t, γ_t) parametrized by $t \in \mathbb{C}$ and restricting to $(V' \oplus V'', \varphi' + \varphi'')$ at $t = 0$.

Pick Dolbeault representatives $a_\delta \in \Omega^{0,1}(V''^* \otimes F_M)$ and $a_\epsilon \in \Omega^{0,1}(F_m^* \otimes V'')$ of δ and ϵ . We are going to construct a pair (W_t, ν_t) satisfying the following.

- There is a C^∞ isomorphism of vector bundles $W_t \simeq F_M \oplus V'' \oplus F_m$ with respect to which the $\bar{\partial}$ operator of W_t can be written as

$$\bar{\partial}_{W_t} = \begin{pmatrix} \bar{\partial}_{F_M} & ta_\delta & t^2\gamma \\ 0 & \bar{\partial}_{V''} & ta_\epsilon \\ 0 & 0 & \bar{\partial}_{F_m} \end{pmatrix} = \bar{\partial}_0 + ta_1 + t^2a_2,$$

where $\gamma \in \Omega^{0,1}(F_m^* \otimes F_M)$ will be specified later,

- ν_t is a holomorphic section of $H^0(S^2W_t \otimes K)$ of the form

$$\nu_t = \beta' + \beta'' + t\nu_1.$$

Now the condition $\bar{\partial}_{W_t}\nu_t = 0$ translates into

$$\begin{aligned} \bar{\partial}_0(\beta' + \beta'') &= 0, \\ \bar{\partial}_1\nu_1 + a_1(\beta' + \beta'') &= 0, \\ a_1\nu_1 + a_2(\beta' + \beta'') &= 0. \end{aligned}$$

The first equation is automatically satisfied. As for the second equation note that

$$a_1(\beta' + \beta'') = \beta''(a_\delta) + \beta'(a_\epsilon) \in \Omega^{1,1}(V'' \otimes_S F_M).$$

Since by hypothesis the Dolbeault cohomology class represented by $\beta''(a_\delta) + \beta'(a_\epsilon)$ is equal to zero, we may choose a value of $\nu_1 \in \Omega^{0,1}(V'' \otimes_S F_M)$ solving the second equation. It remains to consider the third equation. Note that $a_2\beta'' = 0$ and that $a_2\beta' = \gamma(\beta') \in \Omega^{1,1}(F_M \otimes F_M)$. Since β' is an isomorphism, for any $\eta \in \Omega^{1,1}(F_M \otimes F_M)$ there exist some γ such that $\gamma(\beta') = \eta$. Taking $\eta = -a_1\nu_1$, we obtain a value of γ solving the third equation above.

It follows from the construction that there are short exact sequences of holomorphic bundles

$$0 \rightarrow F_M \rightarrow W_t \rightarrow Z_t \rightarrow 0, \quad 0 \rightarrow V'' \rightarrow Z_t \rightarrow F_m \rightarrow 0.$$

Dualizing both sequences we have inclusions $F_m^* \rightarrow Z_t^*$ and $Z_t^* \rightarrow W_t^*$ which can be composed to get an inclusion

$$(7.59) \quad F_m^* \rightarrow W_t^*.$$

Now let

$$V_t = W_t \oplus \bigoplus_{m < \lambda < M} F_\lambda.$$

To finish the construction of the family of Higgs bundles we have to define holomorphic maps

$$\beta_t : V_t^* \rightarrow V_t \otimes K, \quad \gamma_t : V_t \rightarrow V_t^* \otimes K$$

defining sections in $H^0(S^2V_t \otimes K)$ and $H^0(S^2V_t^* \otimes K)$ respectively. The following conditions are in fact satisfied by a unique choice of maps (β_t, γ_t) :

- the restriction of β_t to W_t is equal to ν_t ,
- the restriction of β_t to $\bigoplus_{m < \lambda < M} F_\lambda$ is equal to β' ,
- the restriction of γ_t to W_t is equal to 0,
- the restriction of γ_t to $F_M \subset V_t$ is 0,
- the restriction of γ_t to $F_{M-2} \subset V_t$ is the composition of $\gamma' : F_{M-2} \rightarrow F_m^* \otimes K$ with the inclusion (7.59) tensored by the identity on K ,
- the restriction of γ_t to $\bigoplus_{m < \lambda < M-2} F_\lambda$ is equal to γ' .

The proof of the lemma is completed by using Lemma 7.5. □

Lemma 7.5. *The $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V_t, φ_t) , obtained by restricting the family constructed in the proof of Lemma 7.4 to $X \times \{t\}$, is polystable.*

Proof. Analogous to the proof of Lemma 7.3. □

7.4. Deforming a sum of two quiver type minima.

Lemma 7.6. *Let (V, φ) be a polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle which decomposes as a direct sum $(V, \varphi) = (V', \varphi') \oplus (V'', \varphi'')$ with $\varphi' = (\beta', \gamma')$ and $\varphi'' = (\beta'', \gamma'')$. Suppose that both (V', φ') and (V'', φ'') are quiver type minima. Then (V, φ) is not a minimum of f on \mathcal{M}_d .*

Proof. Suppose we have two minima which are quiver pairs (minimal degree)

$$V' = F'_{m'} \oplus \cdots \oplus F'_{M'} = \bigoplus F'_\lambda \quad \text{and} \quad V'' = F''_{m''} \oplus \cdots \oplus F''_{M''} = \bigoplus F''_\mu.$$

All morphisms $\beta', \beta'', \gamma', \gamma''$ are isomorphisms. We want to deform $V' \oplus V''$.

The same ideas as before tell us (looking at the negative deformation complex) that we should look at the piece of the exact sequence of maximal weight, which is

$$C^\bullet : F'^*_{m'} \otimes F''_{M''} \oplus F''^*_{m''} \otimes F'_{M'} \rightarrow F'_{M'} \otimes F''_{M''} \otimes K.$$

Define $V''_0 := F''_{m''} \oplus F''_{M''}$. The restriction of the β'' to V''_0 defines an isomorphism

$$\beta''_0 : V''_0 \rightarrow V''_0 \otimes K,$$

so we can apply exactly the same construction as before, replacing V'' by V''_0 , and obtain a deformation $W_{t\delta, t\epsilon}$ of the bundle

$$F'_{m'} \oplus F'_{M'} \oplus V''_0 = F'_{m'} \oplus F'_{M'} \oplus F''_{m''} \oplus F''_{M''}.$$

A very important point, however, is that now the extension classes of the bundles W_δ and W_ϵ are more restricted, since they belong respectively to the groups $H^1(F''_{m''} \otimes F'_{M'})$ and $H^1(F'_{m'} \otimes F''_{M''})$. In particular, to define $W_{t\epsilon}$ the line bundle $F'_{m'}$ only merges with $F''_{M''}$, and not with $F''_{m''}$. This implies that there is a map

$$(7.60) \quad W_{t\epsilon} \rightarrow F''_{m''}$$

which deforms the projection $V_0'' \rightarrow F''_{m''}$.

We leave all the remaining F'_λ and F''_μ untouched. There are only two maps which have to be deformed (apart from the β 's which are internal in $W_{\delta,\epsilon}$). These are

$$\gamma' : F'_{m'} \rightarrow F'_{M'-2} \otimes K \quad \text{and} \quad \gamma'' : F''_{m''} \rightarrow F''_{M''-2} \otimes K.$$

The first one can be deformed to a map

$$\gamma'_{\delta,\epsilon} : W_{t\delta,t\epsilon} \rightarrow F'_{M'-2} \otimes K$$

exactly as in the previous section. As for γ'' , we combine the projection $W_{t\delta,t\epsilon} \rightarrow W_{t\epsilon}$ with the map in (7.60) and with γ'' to obtain the desired deformation

$$W_{t\delta,t\epsilon} \rightarrow F''_{M''-2} \otimes K.$$

Lemma 7.7 below completes the proof. \square

Lemma 7.7. *The $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V_t, φ_t) , obtained by restricting the family constructed in the proof of Lemma 7.6 to $X \times \{t\}$, is polystable.*

Proof. Analogous to the proof of Lemma 7.3. \square

7.5. $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles. In this section, we will assume that

$$(V, \varphi) = v_*^{\mathrm{GL}(n, \mathbb{C})}((W, Q), \psi)$$

is an $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle associated to a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle $((W, Q), \psi)$. Recall that $d = \deg(V) = 0$ in this case.

Lemma 7.8. *Let (V, φ) be the $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle associated to a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle $((W, Q), \psi)$ as in (3.26). If (V, φ) is a minimum of f on \mathcal{M}_0 then $\varphi = 0$.*

Proof. In [3] it is shown that there are two types of minima on the moduli space $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles $((W, Q), \psi)$. The first type has $\psi = 0$. The second type corresponds to the minimum on the Hitchin–Teichmüller component and has non-vanishing Higgs field. They are of the form:

$$W = F_{-m} \oplus \cdots \oplus F_m$$

for line bundles F_i , indexed by integers for $n = 2m + 1$ odd and half-integers for $n = 2m + 1$ even. More precisely, $F_i \simeq K^{-i}$ so that, in particular, $F_i \simeq F_{-i}^*$. With respect to this decomposition of W ,

$$Q = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & \ddots & 0 \\ \vdots & & 1 & & \vdots \\ 0 & \ddots & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

We shall apply the criterion in (2) of Proposition 5.4 to show that $v_*^{\mathrm{GL}(n, \mathbb{C})}((W, Q), \psi)$ is not a minimum of the Hitchin functional for such $((W, Q), \psi)$.

Recall that $V = W$, $\beta = \psi f^{-1}$ and $\gamma = f\psi$, where $f: V \rightarrow V^*$ is the symmetric isomorphism associated to Q . Hence the components of β and γ are the canonical sections

$$\beta: F_i^* \rightarrow F_{-i+1} \otimes K \quad \text{and} \quad \gamma: F_i \rightarrow F_{-i-1}^* \otimes K.$$

Since φ has weight one, the weight of F_i is i (cf. Proposition 5.3). It follows that the highest weight piece of the complex C_\bullet defined in (5.48) is

$$C_{2m}^\bullet: \mathrm{Hom}(F_{-m}, F_m) \rightarrow 0.$$

Hence

$$\mathbb{H}^1(C_{2m}^\bullet) = H^1(\mathrm{Hom}(F_{-m}, F_m)) = H^1(K^{-2m}),$$

which is non-vanishing. Take a non-zero $a \in H^1(\mathrm{Hom}(F_{-m}, F_m))$. Let D be the open unit disk and let \mathbb{F}_j be the pull-back of F_j to $X \times D$. Let

$$(7.61) \quad 0 \rightarrow \mathbb{F}_m \rightarrow \mathbb{W}_a \rightarrow \mathbb{F}_{-m} \rightarrow 0$$

be the extension with class

$$a \otimes 1 \in H^1(\mathrm{Hom}(\mathbb{F}_{-m}, \mathbb{F}_m)) \simeq H^1(X; \mathrm{Hom}(F_{-m}, F_m)) \otimes H^0(D; \mathbb{C}).$$

Then $\mathbb{V}_a = \mathbb{W}_a \oplus \bigoplus_{i < m} \mathbb{F}_i$ is a family deforming V which is tangent to a at $t = 0 \in D$. To obtain the required deformation of (V, φ) it thus remains to define the Higgs field $\varphi_D \in H^0(S^2 \mathbb{V}_a \otimes K)$ deforming φ . The only pieces of φ which do not automatically lift are the ones involving F_{-m} and F_m , i.e., $\beta \in H^0(\mathrm{Hom}(F_{-m+1}^*, F_m) \otimes K)$ and $\gamma \in H^0(\mathrm{Hom}(F_{-m}, F_{m-1}^*) \otimes K)$. In order to lift β , clearly we should define β_D to be the composition

$$\mathbb{F}_{-m+1}^* \xrightarrow{\beta} \mathbb{F}_m \rightarrow \mathbb{W}_a,$$

where the last map is induced from the injection in (7.61). A similar construction gives the lift γ_D of γ . We have thus constructed a family $(\mathbb{V}_a, \beta_D, \gamma_D)$ which is tangent to $a \in H^1(C_{2m}^\bullet(V, \varphi))$ for $t = 0 \in D$. Hence Lemma 7.9 below completes the proof. \square

Lemma 7.9. *The $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle (V_t, φ_t) , obtained by restricting $(\mathbb{V}_a, \beta_D, \gamma_D)$ constructed in the proof of Lemma 7.8 above to $X \times \{t\}$, is polystable.*

Proof. Analogous to the proof of Lemma 7.3. \square

8. COUNTING COMPONENTS: MAIN RESULTS

8.1. Connected components of \mathcal{M}_d for $d = 0$ and $|d| = n(g-1)$. With the description of the minima of the Hitchin functional given in Theorem 5.10 at our disposal we are now in a position to complete the count of connected components of the moduli space in the situation of $d = 0$ and $|d| = n(g-1)$.

Proposition 8.1. *The quiver type minima belong to a Hitchin–Teichmüller component of the moduli space. In particular, they are stable and simple and correspond to smooth points of the moduli space.*

Proof. This is immediate from the description of the $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles of the Hitchin–Teichmüller component given in [28]. \square

Proposition 8.2. *Assume that $d = -n(g - 1)$ and let (V, β, γ) be a quiver type minimum for the Hitchin functional. Let L_0 be a fixed square root of the canonical bundle, giving rise to the Cayley correspondence isomorphism $\mathcal{M}_{-n(g-1)} \xrightarrow{\cong} \mathcal{M}'$ of Theorem 4.3, via $V \mapsto W \otimes L_0$. Then the following holds.*

- (1) *The second Stiefel–Whitney class $w_2(W) \in H^2(X, \mathbb{Z}_2)$ vanishes.*
- (2) *If n is odd, the first Stiefel–Whitney class $w_1(W)$ corresponds to the two-torsion point $L^{-1}L_0$ in the Jacobian of X under the standard identification $J_2 \simeq H^1(X, \mathbb{Z}_2)$.*
- (3) *If n is even, the first Stiefel–Whitney class $w_1(W) \in H^1(X, \mathbb{Z}_2)$ vanishes.*

Proof. Easy (similar to the arguments given in [28] for $G = \mathrm{SL}(n, \mathbb{R})$). □

Theorem 8.3. *Let X be a compact Riemann surface of genus g . Let \mathcal{M}_d be the moduli space of polystable $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles of degree d . Let $n \geq 3$. Then*

- (1) \mathcal{M}_0 *is non-empty and connected;*
- (2) $\mathcal{M}_{\pm n(g-1)}$ *has $3 \cdot 2^{2g}$ non-empty connected components.*

Proof. (1) When $d = 0$, we have from Theorem 5.10 that the subspace of minima of the Hitchin functional on \mathcal{M}_0 is \mathcal{N}_0 . It is immediate from Theorem 3.26 that \mathcal{N}_0 is isomorphic to the moduli space of poly-stable vector bundles of degree zero. This moduli space is well known to be non-empty and connected and hence \mathcal{M}_0 is non-empty and connected.

(2) For definiteness assume that $d = -n(g - 1)$. The decomposition (4.42) given by the Cayley correspondence gives a decomposition

$$(8.62) \quad \mathcal{M}_{-n(g-1)} = \bigcup_{w_1, w_2} \mathcal{M}_{w_1, w_2},$$

where \mathcal{M}_{w_1, w_2} corresponds to \mathcal{M}'_{w_1, w_2} under the Cayley correspondence.

For each possible value of (w_1, w_2) , there may be one or more corresponding Hitchin–Teichmüller components contained in \mathcal{M}_{w_1, w_2} (cf. Proposition 8.2); denote by $\tilde{\mathcal{M}}_{w_1, w_2}$ the complement to these. Since minima in $\mathcal{N}_{-n(g-1)}$ (i.e. with $\gamma = 0$) clearly do not belong to Hitchin–Teichmüller components, we see that the subspace of minima of $\tilde{\mathcal{M}}_{w_1, w_2}$ consists of those (V, β, γ) which have $\gamma = 0$. Thus, under the Cayley correspondence, this subspace of minima is identified with the moduli space of poly-stable $\mathrm{O}(n, \mathbb{C})$ -bundles with the given Stiefel–Whitney classes (w_1, w_2) .

If G is a connected reductive algebraic group over \mathbb{C} , Ramanathan proves in [37, Proposition 4.2] that the moduli space of G -principal bundles of a fixed topological type is connected. However, the hypothesis that G is connected is not used in Ramanathan’s proof (all that is required for his argument is that, when considering a holomorphic family of G -principal bundles, semistability is an open condition in the analytic topology), so the statement extends immediately to $G = \mathrm{O}(n, \mathbb{C})$. It follows that the subspace of minima on $\tilde{\mathcal{M}}_{w_1, w_2}$ is connected and, hence, this space itself is connected by Proposition 5.1. Additionally, each $\tilde{\mathcal{M}}_{w_1, w_2}$ is non-empty (see, e.g., [36]). Therefore, there is one connected component $\tilde{\mathcal{M}}_{w_1, w_2}$ for each of the 2^{2g+1} possible values of (w_1, w_2) . Adding to this the 2^{2g} Hitchin–Teichmüller components gives a total of $3 \cdot 2^{2g}$ connected components, as stated.

This accounts for all the connected components of $\mathcal{M}_{-n(g-1)}$ since there are no other minima of the Hitchin functional. □

8.2. Representations and $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundles. Let $\mathcal{R} := \mathcal{R}(\mathrm{Sp}(2n, \mathbb{R}))$ be the moduli space of reductive representations of $\pi_1(X)$ in $\mathrm{Sp}(2n, \mathbb{R})$. Since $U(n) \subset \mathrm{Sp}(2n, \mathbb{R})$ is a maximal compact subgroup, we have

$$\pi_1(\mathrm{Sp}(2n, \mathbb{R})) \simeq \pi_1(U(n)) \simeq \mathbb{Z},$$

and the topological invariant attached to a representation $\rho \in \mathcal{R}$ is hence an element $d = d(\rho) \in \mathbb{Z}$. This integer is called the **Toledo invariant** and coincides with the first Chern class of a reduction to a $U(n)$ -bundle of the flat $\mathrm{Sp}(2n, \mathbb{R})$ -bundle associated to ρ .

Fixing the invariant $d \in \mathbb{Z}$ we consider, as in (2.3),

$$\mathcal{R}_d := \{\rho \in \mathcal{R} \text{ such that } d(\rho) = d\}.$$

Proposition 8.4. *The transformation $\rho \mapsto (\rho^t)^{-1}$ in \mathcal{R} induces an isomorphism of the moduli spaces \mathcal{R}_d and \mathcal{R}_{-d} .*

As shown in Turaev [46] (cf. also Domic–Toledo [14], the Toledo invariant d of a representation satisfies the Milnor–Wood type inequality

$$(8.63) \quad |d| \leq n(g-1).$$

As a consequence we have the following.

Proposition 8.5. *The moduli space \mathcal{R}_d is empty unless*

$$|d| \leq n(g-1).$$

As a special case of Theorem 2.5 we have the following.

Proposition 8.6. *The moduli spaces \mathcal{R}_d and \mathcal{M}_d are homeomorphic.*

From Proposition 8.6 and Theorem 8.3 we have the main result of this paper regarding the connectedness properties of \mathcal{R} given by the following.

Theorem 8.7. *Let X be a compact oriented surface of genus g . Let \mathcal{R}_d be the moduli space of reductive representations of $\pi_1(X)$ in $\mathrm{Sp}(2n, \mathbb{R})$. Let $n \geq 3$. Then*

- (1) \mathcal{R}_0 is non-empty and connected;
- (2) $\mathcal{R}_{\pm n(g-1)}$ has $3 \cdot 2^{2g}$ non-empty connected components.

REFERENCES

- [1] I. Biswas and S. Ramanan, *An infinitesimal study of the moduli of Hitchin pairs*, J. London Math. Soc. (2) **49** (1994), 219–231.
- [2] S. B. Bradlow, O. García-Prada, and P. B. Gothen, *Surface group representations and $U(p, q)$ -Higgs bundles*, J. Differential Geom. **64** (2003), 111–170.
- [3] ———, *Representations of surface groups in the general linear group*, Proceedings of the XII Fall Workshop on Geometry and Physics (H. Albuquerque, R. Caseiro, J. Clemente-Gallardo, J. M. Nunes da Costa, and J. Teles, eds.), Publicaciones de la RSME, vol. 7, 2004, pp. 83–94.
- [4] ———, *Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces*, Geometriae Dedicata **122** (2006), 185–213.
- [5] ———, *Homotopy groups of moduli spaces of representations*, Topology **47** (2008), 203–224.
- [6] S. B. Bradlow, O. García-Prada, and I. Mundet i Riera, *Relative Hitchin-Kobayashi correspondences for principal pairs*, Quart. J. Math. **54** (2003), 171–208.
- [7] M. Burger and A. Iozzi, *Bounded Kähler class rigidity of actions on Hermitian symmetric spaces*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 77–103.

- [8] ———, *Bounded differential forms, generalized Milnor-Wood inequality and an application to deformation rigidity*, *Geom. Dedicata* **125** (2007), 1–23.
- [9] M. Burger, A. Iozzi, F. Labourie, and A. Wienhard, *Maximal representations of surface groups: symplectic Anosov structures*, *Pure Appl. Math. Q.* **1** (2005), no. 3, 543–590.
- [10] M. Burger, A. Iozzi, and A. Wienhard, *Surface group representations with maximal Toledo invariant*, *C. R. Math. Acad. Sci. Paris* **336** (2003), no. 5, 387–390.
- [11] ———, *Surface group representations with maximal Toledo invariant*, *Ann. of Math. (2)* **172** (2010), no. 1, 517–566.
- [12] ———, *Hermitian symmetric spaces and Kähler rigidity*, *Transform. Groups* **12** (2007), no. 1, 5–32.
- [13] K. Corlette, *Flat G -bundles with canonical metrics*, *J. Differential Geom.* **28** (1988), 361–382.
- [14] A. Domic and D. Toledo, *The Gromov norm of the Kaehler class of symmetric domains*, *Math. Ann.* **276** (1987), 425–432.
- [15] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, *Proc. London Math. Soc. (3)* **55** (1987), 127–131.
- [16] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1990.
- [17] V. V. Fock and A. B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, *Publ. Math. Inst. Hautes Études Sci.* **103** (2006), 1–211.
- [18] T. Frankel, *Fixed points and torsion on Kähler manifolds*, *Ann. of Math. (2)* **70** (1959), 1–8.
- [19] O. García-Prada, P. B. Gothen, and I. Mundet i Riera, *The Hitchin–Kobayashi correspondence, Higgs pairs and surface group representations*, v3: August 2012, [arXiv:0909.4487 \[math.AG\]](https://arxiv.org/abs/0909.4487).
- [20] O. García-Prada, P. B. Gothen, and V. Muñoz, *Betti numbers of the moduli space of rank 3 parabolic Higgs bundles*, *Mem. Amer. Math. Soc.* **187** (2007), no. 879, viii+80.
- [21] O. García-Prada and I. Mundet i Riera, *Representations of the fundamental group of a closed oriented surface in $\mathrm{Sp}(4, \mathbb{R})$* , *Topology* **43** (2004), 831–855.
- [22] W. M. Goldman, *Discontinuous groups and the Euler class*, Ph.D. thesis, University of California, Berkeley, 1980.
- [23] ———, *The symplectic nature of fundamental groups of surfaces*, *Adv. Math.* **54** (1984), No. 2, 200–225.
- [24] ———, *Representations of fundamental groups of surfaces*, Springer LNM 1167, 1985, pp. 95–117.
- [25] ———, *Topological components of spaces of representations*, *Invent. Math.* **93** (1988), 557–607.
- [26] P. B. Gothen, *Components of spaces of representations and stable triples*, *Topology* **40** (2001), 823–850.
- [27] N. J. Hitchin, *The self-duality equations on a Riemann surface*, *Proc. London Math. Soc. (3)* **55** (1987), 59–126.
- [28] ———, *Lie groups and Teichmüller space*, *Topology* **31** (1992), 449–473.
- [29] D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford University Press, 2006.
- [30] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Princeton University Press, 1987.
- [31] F. Labourie, *Cross ratios, Anosov representations and the energy functional on Teichmüller space*, *Ann. Sci. École Norm. Sup. (4)* **41** (2008), no. 3, 437–469.
- [32] ———, *Anosov flows, surface groups and curves in projective space*, *Invent. Math.* **165** (2006), no. 1, 51–114.
- [33] J. W. Milnor, *On the existence of a connection with curvature zero*, *Comm. Math. Helv.* **32** (1958), 215–223.
- [34] I. Mundet i Riera, *A Hitchin–Kobayashi correspondence for Kähler fibrations*, *J. Reine Angew. Math.* **528** (2000), 41–80.
- [35] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, *Ann. of Math. (2)* **82** (1965), 540–567.
- [36] A. G. Oliveira, *Higgs bundles, quadratic pairs and the topology of moduli spaces*, Ph.D. thesis, University of Porto, 2008.
- [37] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, *Math. Ann.* **213** (1975), 129–152.
- [38] ———, *Moduli for principal bundles over algebraic curves: I and II*, *Proc. Indian Acad. Sci. Math. Sci.* **106** (1996), 301–328 and 421–449.

- [39] R.W. Richardson, Conjugacy classes of n -tuples in Lie algebras and algebraic groups, *Duke Math. J.* **57** (1988) 1–35.
- [40] A. H. W. Schmitt, *Moduli for decorated tuples for sheaves and representation spaces for quivers*, Proc. Indian Acad. Sci. Math. Sci. **115** (2005), 15–49.
- [41] ———, *Geometric invariant theory and decorated principal bundles*, Zürich Lectures in Advanced Mathematics, European Mathematical Society, 2008.
- [42] C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), 867–918.
- [43] ———, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 5–95.
- [44] ———, *Moduli of representations of the fundamental group of a smooth projective variety I*, Publ. Math., Inst. Hautes Étud. Sci. **79** (1994), 47–129.
- [45] ———, *Moduli of representations of the fundamental group of a smooth projective variety II*, Publ. Math., Inst. Hautes Étud. Sci. **80** (1995), 5–79.
- [46] V. G. Turaev, *A cocycle of the symplectic first Chern class and the Maslov index*, Funct. Anal. Appl. **18** (1984), 35–39.
- [47] A. Wienhard, *The action of the mapping class group on maximal representations*, Geom. Dedicata **120** (2006), 179–191.
- [48] J. W. Wood, *Bundles with totally disconnected structure group*, Comment. Math. Helv. **46** (1971), 257–273.