Classification of factorial generalized down-up algebras

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Abstract

We determine when a generalized down-up algebra is a Noetherian unique factorisation domain or a Noetherian unique factorisation ring.

Keywords: generalized down-up algebra; Noetherian unique factorisation domain; Noetherian unique factorisation ring.

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Introduction

Down-up algebras were introduced by Benkart and Roby in [6], motivated by the study of certain "down" and "up" operators on posets. In this seminal paper, the highest weight theory for a down-up algebra was developed and a parallel was drawn between down-up algebras and enveloping algebras of Lie algebras, based on the apparent similarity between their respective representation theories and structural properties. Later, in [11], Cassidy and Shelton introduced a larger class of algebras which, when defined over an algebraically closed field, contains all down-up algebras.

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Fix $r, s, \gamma \in \mathbb{K}$ and $f \in \mathbb{K}[x]$. The *generalized down-up algebra* $L = L(f, r, s, \gamma)$ is the unital associative $\mathbb{K}$-algebra generated by d, u and h, subject to the relations:

$$[d, h]_r + \gamma d = 0, \quad [h, u]_r + \gamma u = 0 \quad \text{and} \quad [d, u]_s + f(h) = 0,$$

where $[a, b]_\lambda := ab - \lambda ba$. A down-up algebra can be seen as a generalized down-up algebra, as above, with $\deg(f) = 1$.

Noteworthy examples of generalized down-up algebras are the enveloping algebra of the semisimple Lie algebra $\mathfrak{sl}_2$, of traceless matrices of size 2, which is isomorphic to $L(x, 1, 1, 1)$, and the enveloping algebra of the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}$, which is isomorphic to $L(x, 1, 1, 0)$. Another example is the quantum Heisenberg Lie algebra $U_q(\mathfrak{sl}_3^+)$, where $q \in \mathbb{K}^*$, which can be seen as $L(x, q, q^{-1}, 0)$. Under a mild restriction on the parameters, the algebra of regular functions on quantum affine 3-space, $\mathcal{O}_Q(\mathbb{K}^*)$, is a generalized down-up algebra of the form $L(0, r, s, 0)$, with $rs \neq 0$. In [34], Smith defined a class of algebras *similar to the enveloping algebra of* $\mathfrak{sl}_2$. Subsequently, Rueda considered in [33] a larger family of algebras, including Smith's algebras. The algebras in Rueda's family are generalized down-up algebras of the form $L(f, 1, s, 1)$, and by setting $s = 1$ we retrieve Smith's algebras. Other examples of generalized down-up algebras can be found in [4, Secs. 5 and 6].

Like down-up algebras, generalized down-up algebras display several features of the structure and representation theory of a semisimple Lie algebra, but their defining parameters allow enough freedom to obtain a variety of different behaviours. An example of this is the global dimension of a generalized down-up algebra, which can be 1, 2 or 3, by [11, Thm. 3.1] (for a down-up algebra,

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the global dimension is always 3). Similarly, in some cases the centre is reduced to the scalars, but in others it can be large, and there are cases in which the generalized down-up algebra is finite over its centre. Other examples of properties which hold in some generalized down-up algebras and do not in others are: being Noetherian, being primitive, having all finite-dimensional modules semisimple, and having a Hopf algebra structure.

Generalized down-up algebras have been studied mostly from the point of view of representation theory (see [6], [10], [24], [19], [11] and [29]); their primitive ideals have been determined in [21], [28], [31], [30] and [32]. In this paper we study generalized down-up algebras from the point of view of noncommutative algebraic geometry, namely, we provide a complete classification of those generalized down-up algebras which are (noncommutative) Noetherian unique factorisation rings (resp. domains), as defined by Chatters and Jordan in [12] and [13].

An element p of a Noetherian domain R is *normal* if $pR = Rp$. In our case, a Noetherian domain R is said to be a *unique factorisation ring*, Noetherian UFR for short, if R has at least one height one prime ideal, and every height one prime ideal is generated by a normal element. If, in addition, every height one prime factor of R is a domain, then R is called a *unique factorisation domain*, Noetherian UFD for short. As well as the usual commutative Noetherian UFDs, examples of Noetherian UFDs include certain group algebras of polycyclic-by-finite groups [8] and various quantum algebras [26, 25] such as quantised coordinate rings of semisimple groups. Unfortunately, the notion of a Noetherian UFD is not closed under polynomial extensions. To the opposite, the notion of a Noetherian UFR is closed under polynomial extensions. Moreover, Chatters and Jordan proved general results for skew polynomial extensions of the type $R[x;\sigma]$ and $R[x;\delta]$. The general case of skew polynomial extensions of type $R[x;\sigma,\delta]$ is more intricate and only partial results have been obtained for a class of "quantum" algebras called *CGL extensions* [26], which includes (generic) quantum matrices, positive parts of quantum enveloping algebras of semisimple Lie algebras, etc.

Going back to enveloping algebras, it follows from results of Conze in [14] that, over the complex numbers, the universal enveloping algebra of a finite-dimensional semisimple Lie algebra is a Noetherian UFD, and an analogous result holds for a finite-dimensional solvable Lie algebra, by [12]. It is thus natural to investigate the factorial properties of generalized down-up algebras. Moreover, by considering cases in which the parameters r and s are roots of unity, we hope to get some insight into the behaviour of enveloping algebras over fields of finite characteristic (see [7] and references therein). Indeed, our analysis yields the following result, which shows that, for generalized down-up algebras, the distinction between a Noetherian UFR and a Noetherian UFD depends only on the existence of torsion in the multiplicative subgroup of $\mathbb{K}^*$ generated by r and s.

Theorem A. *Let $L = L(f, r, s, \gamma)$ be a generalized down-up algebra with $rs \neq 0$. Then L is a Noetherian UFD if and only if L is a Noetherian UFR and $\langle r, s\rangle$ is torsionfree.*

Noetherian generalized down-up algebras can be viewed as iterated skew polynomial rings as well as generalized Weyl algebras (see [22] and [11]). They also can be described as ambiskew polynomial rings (see [21]). In his paper [20], Jordan determined the height one prime ideals of ambiskew polynomial rings under two additional conditions:

- *conformality*; recall that f is *conformal* in $L(f, r, s, \gamma)$ if there exists $g \in \mathbb{K}[h]$ such that $f(h) = sg(h) - g(rh - \gamma)$;
- *σ-simplicity* (see below for the definition of σ-simplicity).

He then applied these results in [21, Sec. 6] to determine the height one prime ideals of down-up algebras, under certain technical restrictions arising from [20]. Here we consider any Noetherian generalized down-up algebra and obtain the following classification:

Theorem B. *Let $L = L(f, r, s, \gamma)$ be a generalized down-up algebra with $rs \neq 0$. Then L is a Noetherian UFR except if $f \neq 0$ and one of the following conditions is satisfied:*

(a) *f is not conformal, r is not a root of unity, and there exists $\zeta \neq \gamma/(r-1)$ such that $f(\zeta) = 0$;*

(b) *f is conformal, $\langle r,s\rangle$ is a free abelian group of rank 2, and there exists $\zeta \neq \gamma/(r-1)$ such that $f(\zeta)=0$;*

(c) *$\gamma \neq 0$, $r=1$, s is not a root of unity, and $f \notin \mathbb{K}$.*

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1 Generalized down-up algebras and Factoriality

Throughout this paper, $\mathbb{N}$ is the set of nonnegative integers, $\mathbb{K}$ denotes an algebraically closed field of characteristic 0 and $\mathbb{K}^*$ is the multiplicative group of units of $\mathbb{K}$. If X is a subset of the ring L then the two-sided ideal of L generated by X is denoted by $\langle X\rangle$; we also write $\langle x_1,\dots,x_n\rangle$ in place of $\langle\{x_1,\dots,x_n\}\rangle$. Moreover, we denote by $\mathcal{Z}(L)$ the centre of L.

Given a polynomial $f=a_0+a_1x+\cdots+a_nx^n\in\mathbb{K}[x]$, with all $a_i\in\mathbb{K}$, we define the support of f to be the set $\operatorname{supp}(f)=\{i\mid a_i\neq 0\}$ and the degree of f, denoted $\deg(f)$, as the supremum of $\operatorname{supp}(f)$. In particular, the zero polynomial has degree $-\infty$. In the context of this paper, a monomial in the variable x is a (nonzero) polynomial of the form λx^k, for some $\lambda\in\mathbb{K}^*$ and some $k\geq 0$.

1.1 Noetherian generalized down-up algebras

Let $f\in\mathbb{K}[x]$ be a polynomial and fix scalars $r,s,\gamma\in\mathbb{K}$. The *generalized down-up algebra* $L=L(f,r,s,\gamma)$ was defined in [11] as the unital associative $\mathbb{K}$-algebra generated by d, u and h, subject to the relations:

$$dh-rhd+\gamma d=0, \tag{1.1}$$
$$hu-ruh+\gamma u=0, \tag{1.2}$$
$$du-sud+f(h)=0. \tag{1.3}$$

When f has degree one, we retrieve all down-up algebras $A(\alpha,\beta,\gamma)$, α, β, $\gamma\in\mathbb{K}$, for suitable choices of the parameters of L.

It is well known that L is Noetherian $\iff$ L is a domain $\iff$ $rs\neq 0$. Thus, from now on, we will always assume $rs\neq 0$. Moreover we can view L as an iterated skew polynomial ring,

$$L=\mathbb{K}[h][d;\sigma][u;\sigma^{-1},\delta], \tag{1.4}$$

where $\sigma(h)=rh-\gamma$, $\sigma(d)=sd$, $\delta(h)=0$, $\delta(d)=s^{-1}f(h)$. (See [11] for more details.)

To finish this section, we describe the $\mathbb{Z}$-graduation of L obtained by assigning to the generators the following degrees (see [11, Sec. 4]):

$$\deg(u)=1,\ \ \deg(d)=-1,\ \ \deg(h)=0. \tag{1.5}$$

The decomposition $L=\oplus_{i\in\mathbb{Z}}L_i$ of L into homogeneous components has been described in [11, Prop. 4.1]:

$$L_0=\mathbb{K}[h,ud] \text{ is the commutative polynomial algebra generated by } h \text{ and } ud,$$

and

$$L_{-i}=L_0d^i=d^iL_0,\quad L_i=L_0u^i=u^iL_0,\quad \text{for } i>0. \tag{1.6}$$

1.2 Conformality and isomorphisms

When we consider two generalized down-up algebras, say $L = L(f, r, s, \gamma)$ and $\tilde{L} = L(\tilde{f}, \tilde{r}, \tilde{s}, \tilde{\gamma})$, we may denote their canonical generators by d, u, h and $\tilde{d}$, $\tilde{u}$, $\tilde{h}$, respectively, if any confusion could arise regarding which algebra we are referring to.

Lemma 1.1. *The sets $\left\{d^i\right\}_{i\geq 0}$ and $\left\{u^i\right\}_{i\geq 0}$ are right and left denominator sets in L.*

Proof. See [20, 1.5]. It follows from [16, Lem. 1.4] that $\left\{d^i\right\}_{i\geq 0}$ is a right and left denominator set in L. Using the anti-automorphism that fixes h and interchanges d and u we obtain the corresponding statement for $\left\{u^i\right\}_{i\geq 0}$. □

Fix the parameters $r, s \in \mathbb{K}^*$, $\gamma \in \mathbb{K}$, and consider the linear transformation $s \cdot 1 - \sigma$ of $\mathbb{K}[h]$. We denote the image of $p \in \mathbb{K}[h]$ under this transformation by p^*. Specifically, $p^*(h) = sp(h) - p(rh - \gamma)$.

Lemma 1.2. *Let $L = L(f, r, s, \gamma)$, $p \in \mathbb{K}[h]$ and $\tilde{L} = L(f - p^*, r, s, \gamma)$. Consider the denominator sets $D = \left\{d^i\right\}_{i\geq 0}$ in L, $\tilde{D} = \{\tilde{d}^i\}_{i\geq 0}$ in $\tilde{L}$ and the corresponding localisations LD^{-1} and $\tilde{L}\tilde{D}^{-1}$.*

There is an isomorphism $\phi : LD^{-1} \to \tilde{L}\tilde{D}^{-1}$, determined by $\phi(d) = \tilde{d}$, $\phi(h) = \tilde{h}$, $\phi(u) = \tilde{u} + p(\tilde{h})\tilde{d}^{-1}$.

Proof. To show the existence of an algebra endomorphism $\phi : L \to \tilde{L}\tilde{D}^{-1}$ as stated, the following relations need to be checked in $L(f - p^*, r, s, \gamma)\tilde{D}^{-1}$:

$$\tilde{d}\tilde{h} - r\tilde{h}\tilde{d} + \gamma\tilde{d} = 0; \tag{1.7}$$

$$\tilde{h}\left(\tilde{u} + p(\tilde{h})\tilde{d}^{-1}\right) - r\left(\tilde{u} + p(\tilde{h})\tilde{d}^{-1}\right)\tilde{h} + \gamma\left(\tilde{u} + p(\tilde{h})\tilde{d}^{-1}\right) = 0; \tag{1.8}$$

$$\tilde{d}\left(\tilde{u} + p(\tilde{h})\tilde{d}^{-1}\right) - s\left(\tilde{u} + p(\tilde{h})\tilde{d}^{-1}\right)\tilde{d} + f(\tilde{h}) = 0. \tag{1.9}$$

As the first two of these relations are immediately checked, we show only (1.9):

$$\begin{aligned}
\tilde{d}\left(\tilde{u} + p(\tilde{h})\tilde{d}^{-1}\right) &= \tilde{d}\tilde{u} + \tilde{d}p(\tilde{h})\tilde{d}^{-1} \\
&= \tilde{d}\tilde{u} + p(r\tilde{h} - \gamma) \\
&= s\tilde{u}\tilde{d} - (f - p^*)(\tilde{h}) + \left(sp(\tilde{h}) - p^*(\tilde{h})\right) \\
&= s\left(\tilde{u} + p(\tilde{h})\tilde{d}^{-1}\right)\tilde{d} - f(\tilde{h}).
\end{aligned}$$

As $\phi(d)$ is a unit in $\tilde{L}\tilde{D}^{-1}$, the map ϕ above extends (uniquely) to a map $\phi : LD^{-1} \to \tilde{L}\tilde{D}^{-1}$. Now, similar considerations show the existence of an inverse map $\psi : \tilde{L}\tilde{D}^{-1} \to LD^{-1}$, such that $\psi(\tilde{d}) = d$, $\psi(\tilde{h}) = h$, $\psi(\tilde{u}) = u - p(h)d^{-1}$. Hence, ϕ is bijective.

□

Given $r, s, \gamma \in \mathbb{K}$, we say that $f \in \mathbb{K}[h]$ is *conformal* if there is g such that $f = g^*$. We also say, somewhat abusively, that f is conformal in $L(f, r, s, \gamma)$. Thus, if f is conformal, then LD^{-1} is isomorphic to $L(0, r, s, \gamma)\tilde{D}^{-1}$. In this case, in particular, the nonzero element $z := ud - g(h)$ is normal and satisfies the relations $zh = hz$, $dz = szd$ and $zu = suz$.

The following results from [9] determine when a polynomial f is conformal in $L(f, r, s, \gamma)$.

Lemma 1.3 ([9, Lem. 1.6])**.** *Let $f = \sum a_i h^i$. Then f is conformal in $L(f, r, s, 0)$ if and only if $s \neq r^i$ for all $i \in \operatorname{supp}(f)$. In that case, a polynomial g satisfying $f(h) = sg(h) - g(rh)$ exists and is unique if we impose the additional condition that* $\operatorname{supp}(f) = \operatorname{supp}(g)$*; in particular, g can be chosen so that* $\deg(g) = \deg(f)$.

Proposition 1.4 ([9, Prop. 1.7])**.** *If $r \neq 1$ then $L(f, r, s, \gamma) \simeq L(\tilde{f}, r, s, 0)$ for some polynomial $\tilde{f}$ of the same degree as f. Furthermore, f is conformal in $L(f, r, s, \gamma)$ if and only if $\tilde{f}$ is conformal in $L(\tilde{f}, r, s, 0)$.*

Proposition 1.5 ([9, Prop. 1.8])**.** *f is conformal in $L(f, 1, s, \gamma)$ except if $s = 1$, $\gamma = 0$ and $f \neq 0$.*

1.3 Noetherian unique factorisation rings and domains

In this section, we recall the notions of Noetherian unique factorisation rings and Noetherian unique factorisation domains introduced by Chatters and Jordan (see [12, 13]).

An ideal I in a ring L is called *principal* if there exists a normal element x in L such that $I = \langle x \rangle = xL = Lx$.

Definition 1.6. A ring L is called a *Noetherian unique factorisation ring* (Noetherian UFR for short) if the following two conditions are satisfied:

(a) L is a prime Noetherian ring;

(b) Any nonzero prime ideal in L contains a nonzero principal prime ideal.

Definition 1.7. A Noetherian UFR L is said to be a *Noetherian unique factorisation domain* (Noetherian UFD for short) if L is a domain and each height one prime ideal P of L is completely prime; that is, L/P is a domain for each height one prime ideal P of L.

Note that the generalized down-up algebra $L = L(f, r, s, \gamma)$, with $rs \neq 0$, is Noetherian and has finite Gelfand-Kirillov dimension; so, it satisfies the descending chain condition for prime ideals, see for example, [23, Cor. 3.16]. As, moreover, L is a prime Noetherian ring, we deduce from [13] the following result.

Proposition 1.8. *Let $L = L(f, r, s, \gamma)$ be a generalized down-up algebra with $rs \neq 0$. Then L is a Noetherian UFR if and only if all of its height one prime ideals are principal.*

To end this section, we recall a noncommutative analogue of Nagata's Lemma (in the commutative case, see [15, 19.20 p. 487]) that allows one to prove that a ring is a Noetherian UFR or a Noetherian UFD by proving this property for certain localisations of the ring under consideration.

If L is a prime Noetherian ring and x is a nonzero normal element of L, we denote by L_x the (right) localisation of L with respect to the powers of x.

Lemma 1.9 ([26, Lem. 1.4])**.** *Let L be a prime Noetherian ring and x a nonzero, nonunit, normal element of L such that $\langle x \rangle$ is a completely prime ideal of L.*

(a) *If P is a prime ideal of L not containing x and such that the prime ideal PL_x of L_x is principal, then P is principal.*

(b) *If L_x is a Noetherian UFR, then so is L.*

(c) *If L_x is a Noetherian UFD, then so is L.*

1.4 Some prime ideals of L

In [20, 2.10], Jordan defines prime ideals $Q(P)$ which depend on certain prime ideals P of a subalgebra which, in our setting, is $\mathbb{K}[h]$. It is easy to generalize that construction so as to include the case when f is not conformal in L, which we will do below. We give the details only for the prime ideals of $\mathbb{K}[h]$ of the form $\langle h - \lambda \rangle$, with $\lambda \in \mathbb{K}$. The only case remaining concerns $\langle 0 \rangle$, the zero ideal of $\mathbb{K}[h]$, which will not be necessary for our discussion and carries additional technical issues.

Lemma 1.10. *Let $L = L(f, r, s, 0)$ and write $f = \sum a_i h^i$. Then, for every $k \geq 0$,*

$$du^k = s^k u^k d - P_k(h) u^{k-1}, \tag{1.10}$$

where:

(a) $P_k(h) = \sum_{i=0}^{k-1} s^i f(r^{-i} h)$;

(b) *If $f = g^*$ then $P_k(h) = s^k g(r^{1-k}h) - g(rh)$;*

(c) *The coefficient of h^m in $P_k(h)$ is $a_m k$, if $s = r^m$, and $a_m \frac{(sr^{-m})^k - 1}{sr^{-m}-1}$ if $s \neq r^m$.*

In particular, if f is not conformal then $P_k \neq 0$ for all $k > 0$.

Proof. Equation (1.10) along with parts (a) and (b) follow readily by induction on $k \geq 0$. Part (c) follows from (a). Finally, if f is not conformal, recall from [9, Lem. 1.6] that there is m such that $a_m \neq 0$ and $s = r^m$. Thus, the coefficient of h^m in $P_k(h)$ is nonzero, for $k > 0$. □

Let $L = L(f, r, s, 0)$. Fix $\lambda \in \mathbb{K}$ and define the L-module V_λ as follows. As a $\mathbb{K}$-vector space,

$$V_\lambda = \bigoplus_{i \geq 0} \mathbb{K} v_i$$

and the L-action is given by:

$$\begin{aligned} h.v_k &= r^k \lambda v_k \\ u.v_k &= v_{k+1} \\ d.v_k &= -P_k(r^{k-1}\lambda) v_{k-1}, \text{ for } k \geq 1, \text{ and } \quad d.v_0 = 0. \end{aligned}$$

Assume there is $k > 0$ such that $P_k(r^{k-1}\lambda) = 0$. Then $\bigoplus_{i \geq k} \mathbb{K} v_i$ is a proper submodule of V_λ. Let $k > 0$ be minimal with this property, and define $M_\lambda = \bigoplus_{i \geq k} \mathbb{K} v_i$. Thus, $L_\lambda := V_\lambda / M_\lambda$ is a finite-dimensional representation of L. Let $Q_\lambda := \operatorname{ann}_L L_\lambda$. By the minimality of k it is straightforward to see that L_λ is simple. Thus, Q_λ is a primitive ideal; in particular, it is prime.

Remark 1.11.

(a) M_λ, L_λ and Q_λ are defined only if there exists $k > 0$ such that $P_k(r^{k-1}\lambda) = 0$.

(b) When f is conformal, this construction is a special case of the construction in [20, 2.10], where P is the ideal of $\mathbb{K}[h]$ generated by $h - \lambda$ and $Q_\lambda = Q(\langle h - \lambda \rangle)$.

Theorem 1.12. *Let $L = L(f, r, s, 0)$. Suppose $\lambda \in \mathbb{K}$ is such that $P_k(r^{k-1}\lambda) = 0$ for some $k > 0$. Then Q_λ is a non-principal maximal ideal of L containing d^k and u^k.*

Furthermore, if $P_k \neq 0$ for all $k > 0$ (e.g., if f is not conformal) and Q is any prime ideal of L containing d^k and u^k for some $k > 0$, then there exists $\lambda \in \mathbb{K}$ such that Q_λ is defined and $Q = Q_\lambda$.

Proof. This follows essentially as in [20, Thm. 2.12]. We give details for completeness.

Let $\rho : L \to \operatorname{End}_{\mathbb{K}}(L_\lambda)$ be the map which defines the representation. Since L_λ is finite-dimensional and simple, and $\mathbb{K}$ is algebraically closed, Schur's Lemma implies that $\operatorname{End}_L(L_\lambda)$, the centraliser algebra of L_λ, is just $\mathbb{K}$. Thus, by the Jacobson Density Theorem, ρ is onto and induces an algebra isomorphism $L/Q_\lambda \simeq \operatorname{End}_{\mathbb{K}}(L_\lambda)$. As $\operatorname{End}_{\mathbb{K}}(L_\lambda)$ is simple, the ideal Q_λ is maximal.

If Q is any prime ideal of L containing d^k and u^k for some $k > 0$, then the proof of [20, Thm. 2.12] shows that there is $k > 0$ and a prime ideal P of $\mathbb{K}[h]$ such that $P_k(r^{k-1}h) \in P$. As we are assuming $P_k \neq 0$ for all $k > 0$, and $\mathbb{K}$ is algebraically closed, it follows that there is $\lambda \in \mathbb{K}$ such that $P_k(r^{k-1}\lambda) = 0$. Then, as in the proof of [20, Thm. 2.12], we have $Q_\lambda \subseteq Q$, and hence, by the maximality of Q_λ, we obtain $Q = Q_\lambda$.

Finally, Q_λ is not principal because, by the definition of L_λ, we have $d^k, u^k \in Q_\lambda$. This is a general fact concerning any generalized Weyl algebra $D(\phi, a)$ over a commutative domain D such that $0 \neq a \in D$ is not a unit. (Recall, e.g. [11, Lem. 2.7], that L is a generalized Weyl algebra, where D is the polynomial algebra in the variables h and $a = ud$.) Nevertheless, we give the specific details for L.

Assume ξL is a principal ideal of L containing u^k, for some $k > 0$. Then, the equation $\xi x = u^k$, for $x \in L$, implies that both ξ and x must be homogeneous, with respect to the $\mathbb{Z}$-grading defined in (1.5). Assume ξ has degree $n < 0$. Then we can write $\xi = td^{-n}$ and $x = t'u^{k-n}$, for some $t, t' \in \mathbb{K}[h, ud]$. We have:

$$u^k = (td^{-n})(t'u^{k-n}) = t\phi^{-n}(t')d^{-n}u^{-n}u^k = t\phi^{-n}(t')\left(\prod_{i=1}^{-n}\phi^i(ud)\right)u^k,$$

where ϕ is the automorphism of $\mathbb{K}[h, ud]$ defined by $\phi(h) = rh$ and $\phi(ud) = sud - f(h)$. The above equation implies that ud is a unit in $\mathbb{K}[h, ud]$, which is a contradiction. Hence, ξ has degree $n \geq 0$. Similarly, assuming that $d^k \in \xi L$, we conclude that ξ has degree $n \leq 0$. It follows that , if ξL contains both u^k and d^k, then $\xi \in \mathbb{K}[h, ud]$. But then the equation $\xi(tu^k) = u^k$, for $t \in \mathbb{K}[h, ud]$, implies that ξ is a unit and $\xi L = L$. Thus, no proper ideal of L containing u^k and d^k can be principal. □

We end this section by pointing out some principal height one prime ideals which will also be of interest later.

Lemma 1.13. *Let $L = L(f, r, s, 0)$. Then the normal element h generates a height one, completely prime ideal of L. Furthermore, if r is a primitive root of unity of order $l \geq 1$ then, for any $\lambda \in \mathbb{K}^*$, the central element $h^l - \lambda$ generates a height one prime ideal of L which is completely prime if and only if $r = 1$.*

Proof. First, notice that h is normal, as $\gamma = 0$, and generates a completely prime ideal, as the factor algebra $L/\langle h\rangle$ is either a quantum plane or a quantum Weyl algebra, or one of their classical analogues, in case $s = 1$. By the Principal Ideal Theorem (see [27, 4.1.11]), $\langle h\rangle$ has height one.

If r is a primitive root of unity of order $l \geq 1$ then h^l is central. Consider the presentation $L = \mathbb{K}[h][d; \sigma][u; \sigma^{-1}, \delta]$ of L as an iterated skew polynomial ring, as given in (1.4) above, with $\sigma(h) = rh$ and $\delta(h) = 0$. It is easy to see that $(h^l - \lambda)\mathbb{K}[h]$ is a σ-prime ideal of $\mathbb{K}[h]$ (i.e., it is a prime ideal in the lattice of σ-stable ideals of $\mathbb{K}[h]$). It follows, e.g. by [5, Prop. 2.1], that $h^l - \lambda$ generates a prime ideal of $\mathbb{K}[h][d; \sigma]$. In particular, this ideal is σ-prime and δ-stable, so it follows by [5, Prop. 2.1] that $\langle h^l - \lambda\rangle$ is a prime ideal of $L = \mathbb{K}[h][d; \sigma][u; \sigma^{-1}, \delta]$. Again by the Principal Ideal Theorem, this ideal has height one.

If $l \geq 2$, then $h^l - \lambda$ factors nontrivially, as $\mathbb{K}$ is algebraically closed, so $\langle h^l - \lambda\rangle$ is not completely prime, by simple degree arguments. Otherwise, if $l = 1$ then $r = 1$ and the factor algebra $L/\langle h - \lambda\rangle$ is again a quantum plane or a quantum Weyl algebra, or one of their classical analogues, so in this case the ideal $\langle h - \lambda\rangle$ is completely prime. □

2 The case f not conformal

Assume $f = \sum a_i h^i$ is not conformal. Then, by Propositions 1.4 and 1.5, we can assume $\gamma = 0$. By Lemma 1.3, we can write $f = f_c + f_{nc}$, where $f_c = g^*$ is conformal and f_{nc} is such that $s = r^i$ for all $i \in \operatorname{supp}(f_{nc})$. Such a decomposition $f = f_c + f_{nc}$ is unique, and $f_{nc} \neq 0$, as f is not conformal.

Lemma 2.1. *Let $L = L(f, r, s, 0)$ with f not conformal. There is $j \in \operatorname{supp}(f)$ such that $s = r^j$ and $f_{nc} = h^j F$, where $0 \neq F \in \mathbb{K}[h] \cap \mathcal{Z}(L)$. Furthermore:*

(a) *If r is not a root of unity, then $F \in \mathbb{K}^*$;*

(b) *If r is a root of unity of order $l \geq 1$, then $F(h) = G(h^l)$ is a polynomial in the central indeterminate h^l.*

Proof. Let us write $f_{nc} = \sum_{i \in \operatorname{supp}(f_{nc})} a_i h^i$. Let $j = \min \operatorname{supp}(f_{nc})$. Thus, $s = r^j$ and we can write $f_{nc} = h^j F(h)$, where $F(h) = \sum_{i \in \operatorname{supp}(f_{nc})} a_i h^{i-j}$. Given $i \in \operatorname{supp}(f_{nc})$, we have $i - j \geq 0$ and $r^i = s = r^j$, so $r^{i-j} = 1$.

If r is not a root of unity, then $i = j$ and $F(h) \in \mathbb{K}^*$. Otherwise, if r is a primitive l-th root of unity, then l divides $i - j$ and $F(h)$ is a polynomial in h^l, which is thus central, as $\gamma = 0$. □

Proposition 2.2. *Let $L = L(f, r, s, 0)$, with f not conformal, and consider the localisation LD^{-1}, where $D = \{d^i\}_{i \geq 0}$. Then $\{h^i\}_{i \geq 0}$ is a right and left denominator set in LD^{-1} and the localisation at this set is isomorphic to $\hat{L} = L(F, r, 1, 0)$ localised at the multiplicative set generated by the corresponding elements $\hat{d}$ and $\hat{h}$ in $\hat{L}$, where $f = f_c + f_{nc}$ and $f_{nc} = h^j F$, as in the previous lemma.*

Proof. The first statement follows from the normality of h in L. We have already seen in Lemma 1.2 that LD^{-1} is isomorphic to $\tilde{L}\tilde{D}^{-1}$, where $\tilde{L} = L(f_{nc}, r, s, 0)$, under an isomorphism that maps h to $\tilde{h}$, d to $\tilde{d}$ and u to $\tilde{u} + g(\tilde{h})\tilde{d}^{-1}$, where $f_c = g^*$. So it suffices to show that $\tilde{L}$ localised at the multiplicative set generated by $\tilde{d}$ and $\tilde{h}$ is isomorphic to the corresponding localisation of $\hat{L} = L(F, r, 1, 0)$ at the multiplicative set generated by $\hat{d}$ and $\hat{h}$.

It is easy to see that there is an algebra homomorphism $\Phi : L(f_{nc}, r, s, 0) \longrightarrow L(F, r, 1, 0)$ such that $\Phi(\tilde{d}) = \hat{d}$, $\Phi(\tilde{h}) = \hat{h}$ and $\Phi(\tilde{u}) = \hat{u}\hat{h}^j$. This homomorphism clearly extends to an isomorphism when we pass to the localisation under consideration. □

Next, we define a (left and right) denominator set X in $L(f, r, s, 0)$, which depends on r:

(a) If r is not a root of unity, then X is the multiplicative set generated by d and h.

(b) If r is a root of unity of order $l \geq 1$, then X is the multiplicative set generated by d, h and the central elements of the form $h^l - \lambda$, for $\lambda \in \mathbb{K}^*$.

Proposition 2.3. *Let $L = L(f, r, s, 0)$ and assume f is not conformal. Then the localisation of L at the denominator set X defined above is a simple algebra.*

Proof. By the previous result, it is enough to assume $L = L(F, r, 1, 0)$, where $F \neq 0$ is either a scalar (if r is not a root of unity) or a polynomial in the central indeterminate h^l (if r has order $l \geq 1$). Furthermore, since F is central and invertible in the localisation under consideration ($\mathbb{K}$ is algebraically closed), we can assume $F = 1$, on replacing u by uF^{-1}. The result then follows from the description below of the prime ideals of $L(1, r, 1, 0)$. □

Theorem 2.4. *Let $L = L(1, r, 1, 0)$.*

(a) *Assume that r is not a root of unity. Then* $\mathrm{Spec}(L) = \{\langle 0 \rangle, \langle h \rangle\}$.

(b) *Assume that r is a primitive l-th root of unity, for $l \geq 1$. Then*

$$\mathrm{Spec}(L) = \{\langle 0 \rangle, \langle h \rangle\} \cup \{\langle h^l - \lambda \rangle \mid \lambda \in \mathbb{K}^*\}.$$

Proof. This follows from the isomorphism $L(1, r, 1, 0) \simeq \mathbb{A}_1(\mathbb{K})[h; \phi]$, where $\mathbb{A}_1(\mathbb{K})$ denotes the first Weyl algebra over $\mathbb{K}$, generated by d and u, subject to the relation $ud - du = 1$, and ϕ is the automorphism of $\mathbb{A}_1(\mathbb{K})$ defined by $\phi(d) = r^{-1}d$ and $\phi(u) = ru$. Thus, we identify the algebras L and $\mathbb{A}_1(\mathbb{K})[h; \phi]$. Below we sketch the proof, which relies on the simplicity of $\mathbb{A}_1(\mathbb{K})$ (recall that $\mathbb{K}$ has characteristic 0).

Firstly, all ideals listed in the statement are prime, e.g. by Lemma 1.13. We will show that there are no other prime ideals. There is an $\mathbb{N}$-grading on $\mathbb{A}_1(\mathbb{K})[h; \phi]$ such that the homogenous component of degree $n \geq 0$ is $\mathbb{A}_1(\mathbb{K})h^n = h^n\mathbb{A}_1(\mathbb{K})$. This grading is, of course, different from the usual $\mathbb{Z}$-grading of L we consider in the paper, but for the remainder of this proof, this is the grading we will consider. As usual, the degree of an element in $\mathbb{A}_1(\mathbb{K})[h; \phi]$ is the maximum of the degrees of its nonzero homogeneous components, i.e., its degree as a polynomial in h.

Assume $P \neq \langle 0 \rangle$ is a prime ideal of $\mathbb{A}_1(\mathbb{K})[h; \phi]$. Let $0 \neq \xi \in P$ be a (not necessarily homogeneous) element of minimum degree, say $n \geq 0$. Then the set of leading coefficients of nonzero elements of P of degree n, adjoined with 0, is easily seen to be an ideal of $\mathbb{A}_1(\mathbb{K})$. As the latter is simple and $\xi \neq 0$, it follows that this ideal contains 1. Therefore, we can assume that ξ is monic. By the minimality of the degree of ξ and the fact that its leading coefficient is a unit, we can use right and left division algorithms to conclude that P is principal and generated by ξ, both on the right and on the left. In particular, ξ is normal.

Since $\mathbb{A}_1(\mathbb{K})[h;\phi]$ is $\mathbb{N}$-graded, every homogeneous constituent of ξ is normal, so we will first determine the homogeneous elements of $\mathbb{A}_1(\mathbb{K})[h;\phi]$ which are normal. Assume ah^i is normal, where $a \in \mathbb{A}_1(\mathbb{K})$ and $i \geq 0$. Then, as h^i is itself normal, it follows that a is normal in $\mathbb{A}_1(\mathbb{K})$. Thus, $a \in \mathbb{K}$. This shows, in particular, that the normal elements of $\mathbb{A}_1(\mathbb{K})[h;\phi]$ are polynomials in h with coefficients in $\mathbb{K}$, but not every such polynomial is normal, except if $r = 1$. Indeed, suppose $0 \neq \xi = \sum_{i\geq 0} \lambda_i h^i$ is normal, where $\lambda_i \in \mathbb{K}$. Then there is a such that $d\xi = \xi a$. It must then be that $a \in \mathbb{A}_1(\mathbb{K})$, by degree considerations, and $\lambda_i d = \lambda_i \phi^i(a)$, for all i. If λ_i and λ_j are nonzero, then $r^i d = \phi^{-i}(d) = a = \phi^{-j}(d) = r^j d$, so $r^{i-j} = 1$. This implies that we can write $\xi = h^k G$, where $k \geq 0$ and either G is a (nonzero) scalar, if r is not a root of unity, or G is a polynomial in h^l with scalar coefficients and nonzero constant term, if r is a primitive l-th root of unity.

As h and G are normal, and $P = \langle \xi \rangle$ is prime, either $h \in P$ or $G \in P$. If the former occurs, then $P = \langle h \rangle$. Otherwise, $k = 0$, r is a primitive l-th root of unity, and $P = \langle h^l - \lambda \rangle$, for some $\lambda \in \mathbb{K}^*$, as $\mathbb{K}$ is algebraically closed and, up to a scalar, G can be factored into central polynomials of the form $h^l - \mu$, for $\mu \in \mathbb{K}^*$. This establishes the claim. $\square$

Similarly, we can define a (left and right) denominator set Y in $L(f, r, s, 0)$, which can be obtained from X by replacing d by u. Specifically:

(a) If r is not a root of unity, then Y is the multiplicative set generated by u and h.

(b) If r is a root of unity of order $l \geq 1$ then Y is the multiplicative set generated by u, h and the central elements of the form $h^l - \lambda$, for $\lambda \in \mathbb{K}^*$.

Proposition 2.5. *Let $L = L(f, r, s, 0)$ and assume f is not conformal. Then the localisation of L at the denominator set Y defined above is a simple algebra.*

Proof. Consider the isomorphism $L(f, r, s, 0) \longrightarrow L(f, r^{-1}, s^{-1}, 0)$, defined by the correspondence $d \mapsto -s^{-1}u$, $u \mapsto d$, $h \mapsto h$. Notice that f is conformal in $L(f, r, s, 0)$ if and only if f is conformal in $L(f, r^{-1}, s^{-1}, 0)$ (if $f(h) = sg(h) - g(rh)$ then $f(h) = s^{-1}G(h) - G(r^{-1}h)$, for $G(h) = -sg(rh)$). Thus, our claim follows from applying our previous result to $L(f, r^{-1}, s^{-1}, 0)$ and the denominator set X, and using this isomorphism. $\square$

We can now determine when $L(f, r, s, 0)$ is a Noetherian UFR or a Noetherian UFD, assuming f is not conformal.

Theorem 2.6. *Let $L = L(f, r, s, 0)$ and assume f is not conformal. Then L is a Noetherian UFR, except in the case that f is not a monomial and r is not a root of unity. Moreover, L is a Noetherian UFD if and only if either $r = 1$ or if r is not a root of unity and f is a monomial.*

Proof. Let us first identify the possible height one primes.

Let P be a height one prime ideal of L. If P does not contain any power of d or if P does not contain any power of u then, by Propositions 2.3 and 2.5, P must contain either h or $h^l - \lambda$, for some $\lambda \in \mathbb{K}^*$ (the latter can occur only when r is a primitive l-th root of unity, for some $l \geq 1$), as these elements are normal. But both h and $h^l - \lambda$ generate prime ideals, by Lemma 1.13, so it follows that either $P = \langle h \rangle$ or $P = \langle h^l - \lambda \rangle$.

Otherwise, P must contain both a power of d and a power of u. Thus, $P = Q_\lambda$, for some $\lambda \in \mathbb{K}$, by Theorem 1.12. In particular, $P_k(r^{k-1}\lambda) = 0$ for some $k > 0$ (with the notation of Lemma 1.10). Assume that $\lambda = 0$. Then $h \in Q_0 = P$ and $P = \langle h \rangle$, which is a contradiction, as $\langle h \rangle$ does not contain any power of d. So $P = Q_\lambda$ for some $\lambda \in \mathbb{K}^*$.

To summarise, the possible height one primes of L are: $\langle h \rangle$; $\langle h^l - \lambda \rangle$ with $\lambda \in \mathbb{K}^*$ and Q_λ for some $\lambda \in \mathbb{K}^*$ such that $P_k(r^{k-1}\lambda) = 0$, for some $k > 0$.

We now distinguish between three different cases.

Let us first consider the case that f is a monomial. Then $P_k \neq 0$ is also a monomial and hence the only possibility for λ to satisfy $P_k(r^{k-1}\lambda) = 0$ is $\lambda = 0$, which is a contradiction. So the only possible height one primes of L are $\langle h \rangle$ and $\langle h^l - \lambda \rangle$ with $\lambda \in \mathbb{K}^*$. They are all principal so that

L is a Noetherian UFR. Furthermore, it follows from Lemma 1.13 that L is a Noetherian UFD except if r is a primitive root of unity of order $l \geq 2$.

If f is not a monomial, then we consider two cases:

Case 1: r is a primitive l-th root of unity. In this case, $h^l - \lambda^l$ annihilates L_λ, so $h^l - \lambda^l \in Q_\lambda$. It follows that Q_λ is not a height one prime. So, as above, the only height one primes of L are $\langle h \rangle$ and $\langle h^l - \lambda \rangle$ with $\lambda \in \mathbb{K}^*$, whence the final statement follows.

Case 2: r is not a root of unity. As f is not a monomial, there is $\eta \in \mathbb{K}^*$ such that $P_1(\eta) = f(\eta) = 0$. Assume Q_η does not have height 1. Then it properly contains a nonzero prime ideal Q. This ideal Q cannot be of the form Q_λ, as these ideals are maximal, hence either Q does not contain a power of d or Q does not contain a power of u. By the first part of our argument, as r is not a root of unity, Q must contain h. In particular, h annihilates the module L_η, which is a contradiction, as $\eta \neq 0$. Thus, Q_η indeed has height one and is not principal, so L is not a Noetherian UFR in this case. $\square$

3 The case $f = 0$

We will consider separately the cases $\gamma = 0$ and $\gamma \neq 0$.

3.1 The case $f = 0$ and $\gamma = 0$

In this case, the defining relations of $L = L(0, r, s, \gamma)$ are:

$$dh = rhd, \quad hu = ruh, \quad du = sud,$$

and L is the so-called *quantum coordinate ring of affine* 3*-space* over $\mathbb{K}$. The normal elements d, u and h generate pairwise distinct completely prime ideals so, by [26, Prop. 1.6], it is enough to show that the localisation T of L with respect to the Ore set generated by these three elements is a Noetherian UFR. Well, by [17, 1.3(i) and Cor. 1.5], the height one prime ideals of T are generated by a single central element, so T is a Noetherian UFR. Thus, L is a Noetherian UFR. We record this result below.

Proposition 3.1. *Assume $f = 0$ and $\gamma = 0$. Then $L(0, r, s, 0)$ is a Noetherian UFR.*

We conclude this section by studying for which values of r and s the generalized down-up algebra $L(0, r, s, 0)$ is a Noetherian UFD.

Proposition 3.2. *Assume $f = 0$ and $\gamma = 0$. Then $L = L(0, r, s, 0)$ is a Noetherian UFD if and only if $\langle r, s \rangle$ is torsionfree.*

Proof. As was observed above, L is just the quantum coordinate ring of an affine 3-space. So we deduce from [18, Thm. 2.1] that, if $\langle r, s \rangle$ is torsionfree, then all prime ideals of L are completely prime. Thus, the result is proved in this case.

Now assume that $\langle r, s \rangle$ is not torsionfree. First, if r is a root of unity of order $l \geq 2$, then the result follows from Lemma 1.13. So we are left with the cases where r is either 1 or not a root of unity. Before distinguishing between different cases, let us describe our strategy to prove that L is not a Noetherian UFD in these cases.

If L were a Noetherian UFD, then so would be the localisation T of L at the Ore set generated by the normal elements h, d, u. (Note that this is due to the fact that we are localising at elements that are "q-central" - see also Proposition 4.1.) This localised algebra T is a quantum torus. More precisely, it is the quantum torus generated by the three indeterminates h, d and u, and their inverses h^{-1}, d^{-1} and u^{-1}, subject to the relations

$$dh = rhd, \; hu = ruh, \; du = sud.$$

Now it follows from [18] that extension and contraction provide mutually inverse bijections between the prime spectrum of T and the prime spectrum of the centre $\mathcal{Z}(T)$ of T, and that $\mathcal{Z}(T)$ is a (commutative) Laurent polynomial algebra over $\mathbb{K}$. Moreover, we can compute this centre explicitly, using [18, 1.3]. So, to prove that L is not a Noetherian UFD in the remaining cases, we will construct a height one prime ideal of T which is not completely prime. This is achieved by computing the centre of T in each case.

We distinguish between three cases:

Case 1: $r = 1$ and s is a root of unity of order $\beta \geq 2$. In this case, we get

$$\mathcal{Z}(T) = \mathbb{K}[u^{\pm\beta}, h^{\pm 1}, d^{\pm\beta}].$$

Hence, $u^\beta - 1$ generates a height one prime ideal in T which is not completely prime.

Case 2: r is not a root of unity and s is a root of unity of order $\beta \geq 2$. In this case, we get

$$\mathcal{Z}(T) = \mathbb{K}[(ud)^{\pm\beta}].$$

Hence, $(ud)^\beta - 1$ generates a height one prime ideal in T which is not completely prime.

Case 3: r and s are not roots of unity. Hence, there exists (α_0, β_0) with $\beta_0 > 0$ minimal such that $r^{\alpha_0} s^{\beta_0} = 1$. In this case, we deduce from [18, 1.3] that

$$\mathcal{Z}(T) = \mathbb{K}[(u^{\beta_0} h^{\alpha_0} d^{\beta_0})^{\pm 1}].$$

Now observe that the fact that $\langle r, s \rangle$ is not torsionfree imposes that $\gcd(\alpha_0, \beta_0) > 1$. Hence, $u^{\beta_0} h^{\alpha_0} d^{\beta_0} - 1$ generates a height one prime ideal in T which is not completely prime. $\square$

3.2 The case $f = 0$ and $\gamma \neq 0$

If $r \neq 1$, then on replacing h by $\tilde{h} = h + \gamma/(1-r)$ we can reduce to the case $\gamma = 0$ studied above. So we can assume $r = 1$. We can further assume $\gamma = 1$, by replacing the generator h by $\gamma^{-1}h$. Let Q be the subalgebra of L generated by d and u. Then Q is the quantum plane with relation $du = sud$ and $L = Q[h; \partial]$, where ∂ is the derivation of Q determined by $\partial(d) = d$, $\partial(u) = -u$. By the arguments of Section 3.1, Q is a Noetherian UFR.

Proposition 3.3. *Assume $f = 0$, $r = 1$ and $\gamma \neq 0$. Then $L = L(0, 1, s, \gamma)$ is a Noetherian UFR.*

Proof. Without loss of generality, we assume $\gamma = 1$. By [13, Thm. 5.5], it is enough to show that every non-zero ∂-prime ideal of Q contains a non-zero principal ∂-ideal, for the derivation ∂ of Q defined above.

Let $0 \neq I \leq Q$ be a ∂-prime ideal of Q. Choose $p = p(d, u) \in I \setminus \{0\}$ with minimal support, i.e., $0 \neq p = \sum a_{ij} d^i u^j \in I$ such that $a_{ij} \in \mathbb{K}$ and the set $\{(i,j) \mid a_{ij} \neq 0\}$ has minimal cardinality. Fix (α, β) such that $a_{\alpha\beta} \neq 0$. It is straightforward to check that $\partial(d^i u^j) = (i-j) d^i u^j$, for all nonnegative integers i, j. Then,

$$\begin{aligned} I \ni (\alpha - \beta)p - \partial(p) &= \sum a_{ij}(\alpha - \beta) d^i u^j - \sum a_{ij}(i-j) d^i u^j \\ &= \sum a_{ij}(\alpha + j - \beta - i) d^i u^j. \end{aligned}$$

Furthermore, $(\alpha - \beta)p - \partial(p)$ has a smaller support than p, as its coefficient of $d^\alpha u^\beta$ is 0. By the minimality assumption, it must be that $(\alpha - \beta)p - \partial(p) = 0$. Thus, $\partial(p) = (\alpha - \beta)p$. In particular, $i - j$ is constant for all (i,j) such that $a_{ij} \neq 0$, and we can write $p = \sum_{i \geq 0} a_i d^i u^{j(i)}$.

Choose α such that $a_\alpha \neq 0$. Then,

$$up - s^{-\alpha} pu = \sum_{i \geq 0} a_i (s^{-i} - s^{-\alpha}) d^i u^{j(i)+1}$$

and this is still an element of I, with smaller support than p. Thus, $up - s^{-\alpha}pu = 0$ and $up = s^{-\alpha}pu$. Similarly, $dp = s^\beta pd$ for some $\beta \in \mathbb{Z}$, which shows that p is normal. In particular, I contains the nonzero principal ideal generated by p, which is a ∂-ideal, as $\partial(p) = \lambda p$ for some integer λ. This proves our claim. $\square$

We now inquire when L is a Noetherian UFD. For that purpose, we will determine the height one prime ideals of L explicitly and check which are completely prime. Since L is a Noetherian UFR, we know that all height one prime ideals are principal. So we start out by determining the normal elements of L.

Recall the $\mathbb{Z}$-graduation of L described in (1.5).

Lemma 3.4. *Assume $f = 0$, $r = 1$ and $\gamma \neq 0$. Let $L = L(0, 1, s, \gamma)$ and consider the normal element $z = ud$. The normal elements of L are homogeneous and of the form $p(z)u^i$ or $p(z)d^i$, for $i \in \mathbb{N}$ and $p(z) \in \mathbb{K}[z]$. Furthermore:*

(a) *If s is not a root of unity, then $p(z) = \lambda z^c$ for some $\lambda \in \mathbb{K}$ and some integer $c \geq 0$;*

(b) *If s is a primitive l-th root of unity ($l \geq 1$), then $p(z) = z^c \tilde{p}(z^l)$ for some integer $0 \leq c < l$ and some polynomial $\tilde{p}(z^l)$ in the central element z^l.*

Proof. We again assume $\gamma = 1$. Let $\nu = \sum_{i \in \mathbb{Z}} \nu_i$ be a nonzero normal element of L, with ν_i homogeneous of degree i. Notice that $hu^i d^j = u^i d^j (h + j - i)$. In particular, $h\nu_i = \nu_i (h - i)$ for all $i \in \mathbb{Z}$. By degree considerations, there is $\xi(h, z) \in \mathbb{K}[h, z]$ such that $h\nu = \nu\xi(h, z)$. Thus,

$$\sum \nu_i \xi(h, z) = h \sum \nu_i = \sum \nu_i (h - i).$$

Therefore, equating homogeneous components and factoring out the nonzero ν_i, we get that $h - i = \xi(h, z)$ for all i such that $\nu_i \neq 0$, which proves that ν is homogeneous.

Assume $\nu \neq 0$ has degree $i \geq 0$. We can write $\nu = p(h, z)u^i$ (see (1.6)). As $u^i d^j$ is clearly normal, for all $i, j \in \mathbb{N}$, and L is a domain, $p(h, z)u^i$ is normal if and only if $p(h, z)$ is normal. Write $p(h, z) = \sum_{j \geq 0} p_j(h) z^j$, with $p_j(h) \in \mathbb{K}[h]$. As before, there must exist $\xi(h, z) \in \mathbb{K}[h, z]$ such that $up(h, z) = p(h, z)\xi(h, z)u$. Using the commutation relations $up_j(h) = p_j(h + 1)u$ and $uz^j = s^{-j} z^j u$, and factoring out u on the right from both terms of that equation, we obtain

$$\sum_{j \geq 0} p_j(h + 1) s^{-j} z^j = \xi(h, z) \sum_{j \geq 0} p_j(h) z^j.$$

From the above equation we readily conclude that $\xi(h, z) = \xi \in \mathbb{K}$, as we are assuming $p(h, z) \neq 0$. Next, equating coefficients of z^j, we get $p_j(h + 1)s^{-j} = \xi p_j(h)$ for all j. This implies that $p_j(h)$ is a constant polynomial, for all j. Thus, we conclude that $\nu = p(z)u^i$, for some $p(z) \in \mathbb{K}[z]$. The case $i \leq 0$ is symmetric.

It remains to determine when a nonzero element $p(z) \in \mathbb{K}[z]$ is normal. Write $p(z) = \sum_i a_i z^i$, with $a_i \in \mathbb{K}$. Since $up(z) = \sum_i s^{-i} a_i z^i u$, it is easy to conclude that $p(z)$ is normal if and only if there is $\lambda \in \mathbb{K}$ such that $s^{-i} = \lambda$ for all i such that $a_i \neq 0$. Let $c \geq 0$ be the first index for which $a_c \neq 0$. It follows that $p(z) = a_c z^c$ if s is not a root of unity. In case s is a primitive l-th root of unity, with $l \geq 1$, then $p(z) = z^c p'(z^l)$, where $p'(z^l)$ is a polynomial in z^l, and the result follows. $\square$

We can now list all height one prime ideals of L and check when L is a Noetherian UFD.

Theorem 3.5. *Assume $f = 0$, $r = 1$ and $\gamma \neq 0$. Let $L = L(0, 1, s, \gamma)$ and $z = ud$. Then L is a Noetherian UFD if and only if either s is not a root of unity or $s = 1$. The height one prime ideals of L are:*

(a) *$\langle d \rangle$ and $\langle u \rangle$, if s is not a root of unity. These ideals are completely prime.*

(b) *$\langle d \rangle$, $\langle u \rangle$ and $\langle z - \lambda \rangle$, for $\lambda \in \mathbb{K}^*$, if $s = 1$. These ideals are completely prime.*

(c) *$\langle d \rangle$, $\langle u \rangle$ and $\langle z^l - \lambda \rangle$, for $\lambda \in \mathbb{K}^*$, if s is a primitive l-th root of unity, with $l > 1$. The ideals $\langle d \rangle$ and $\langle u \rangle$ are completely prime but those of the form $\langle z^l - \lambda \rangle$ are not.*

Proof. Once more, we assume $\gamma = 1$. Let P be a height one prime ideal of L. By Proposition 3.3, there exists a normal element ν such that $P = \langle \nu \rangle$. Furthermore, as P is prime, ν cannot be the product of two nonzero, nonunit normal elements. Thus, by Lemma 3.4, the only possibilities are, up to nonzero scalars, $\nu = u$, $\nu = d$ or $\nu = p(z)$. The ideals generated by either u or d are indeed completely prime, as the corresponding factor algebra is isomorphic to the enveloping algebra of the two-dimensional nonabelian Lie algebra. By the Principal Ideal Theorem, they have height one. So it remains to consider the case $\nu = p(z)$. Note first that $z = ud$, and since u and d are nonzero nonunit normal elements, the ideal generated by z is not prime.

Assume first that s is not a root of unity. Then, by Lemma 3.4(a) and the above, there is no other possibility for P. This proves part (a). Now assume s is a primitive l-th root of unity, with $l \geq 1$. Since $\mathbb{K}$ is algebraically closed, the only other possibility for the generator ν of P is $\nu = z^l - \lambda$, for some $\lambda \in \mathbb{K}^*$. If $l = 1$, i.e., in case $s = 1$, then $\langle z - \lambda \rangle$ is completely prime, for $\lambda \neq 0$, as the factor algebra is isomorphic to the differential operator ring $\mathbb{K}[u^{\pm 1}][h; \partial]$, where $\partial(u^i) = -iu^i$, for all $i \in \mathbb{Z}$. This proves part (b).

Finally, assume $l > 1$. Recall that L can be presented as the differential operator ring $Q[h; \partial]$, where Q is the quantum plane with relation $du = sud$ and ∂ is the derivation of Q determined by $\partial(d) = d$, $\partial(u) = -u$. The centre of Q is the polynomial algebra $\mathbb{K}[d^l, u^l]$, by [17, 1.3(i)], and the element $z^l - \lambda$ is irreducible in this polynomial algebra. Thus, $z^l - \lambda$ generates a prime ideal of $\mathbb{K}[d^l, u^l]$. By [17, Cor. 1.5], $z^l - \lambda$ also generates a prime ideal of Q. Furthermore, $\partial(z^l - \lambda) = 0$, as $\partial(z) = 0$. Hence, by [27, Prop. 14.2.5], $z^l - \lambda$ generates a prime ideal of L. This ideal has height one, by the Principal Ideal Theorem. Since $l > 1$ and $\mathbb{K}$ is algebraically closed, $z^l - \lambda$ factors nontrivially as a polynomial in z, so the ideal $\langle z^l - \lambda \rangle$ is not completely prime, which proves part (c). $\square$

4 The case f conformal and r not a root of unity

In this case, as $r \neq 1$, we can and will assume that $\gamma = 0$. Thus, the defining relations for $L = L(f, r, s, 0)$ are:

$$dh - rhd = 0, \tag{4.11}$$

$$hu - ruh = 0, \tag{4.12}$$

$$du - sud + f(h) = 0. \tag{4.13}$$

In particular, h is a nonzero, nonunit normal element of L which generates a completely prime ideal of L.

Let L_h be the localisation of L with respect to the powers of h. It is clear that

$$L_h = \mathbb{K}[h^{\pm 1}][d; \sigma][u; \sigma^{-1}, \delta],$$

where σ and δ are extended by setting $\sigma(h^{-1}) = r^{-1}h^{-1}$ and $\delta(h^{-1}) = 0$.

Proposition 4.1. *Assume $\gamma = 0$. Then L_h is a Noetherian UFR (resp. UFD) if and only if L is a Noetherian UFR (resp. UFD).*

Proof. The direct implication follows from Lemma 1.9. The converse follows from standard arguments in localisation theory, provided we can show that any normal element of L is still normal in L_h.

Let $\nu \in L$ be normal. We can assume $\nu \neq 0$. Write $\nu = \sum_{i \in \mathbb{Z}} \nu_i$ with ν_i homogeneous of degree i. As $h\nu_i = \nu_i \sigma^i(h) = r^i \nu_i h$, it follows that $h\nu = \sum_{i \in \mathbb{Z}} r^i \nu_i h$. On the other hand, by the normality of ν, there is $h' \in L$ such that $h\nu = \nu h'$. The $\mathbb{Z}$-grading implies that h' has degree 0. Hence, the degree i component of $\nu h'$ is $\nu_i h'$. Equating elements of the same degree we conclude that $r^i \nu_i h = \nu_i h'$, for all i. Choose α with $\nu_\alpha \neq 0$. We must have $h' = r^\alpha h$ and thus $h\nu = r^\alpha \nu h$. Hence, in L_h, we have $\nu h^{-1} = r^\alpha h^{-1} \nu$, which is enough to show that ν is normal in L_h. This concludes the proof. $\square$

Let $g \in \mathbb{K}[h]$ be such that $f = g^*$, i.e., $f(h) = sg(h) - g(rh)$. We will assume $f \neq 0$, as the case $f = 0$ has already been dealt with in Section 3. In particular, $g \neq 0$.

The algebra L_h is in the scope of the algebras studied by Jordan in [20], where our polynomial g plays the role of the element u of [20]. We will start out with a few technical observations which will allow us to apply the results of [20] to L_h. We will remind the reader of the necessary definitions as they are needed.

Lemma 4.2. *Assume $\gamma = 0$ and r is not a root of unity. Then the Laurent polynomial algebra $\mathbb{K}[h^{\pm 1}]$ is σ-simple, i.e., its only σ-invariant ideals are itself and the zero ideal.*

Proof. This is worked out in Example 1.2.(i) of [20]. $\square$

We recall Definition 1.7 of [20], applied in our context. Suppose there exists $0 \neq p \in \mathbb{K}[h^{\pm 1}]$ such that $\sigma(p) = s^{-n}p$ for some positive integer n. Let $n \geq 1$ be minimal with respect to the existence of such an element p. Then, any $0 \neq p \in \mathbb{K}[h^{\pm 1}]$ satisfying $\sigma(p) = s^{-n}p$ will be called a *principal eigenvector*, and n will be its *degree*.

In order to discuss the existence of principal eigenvectors, we will make use of two integers $\epsilon \in \mathbb{Z}$ and $\tau \in \mathbb{N}$, which have been defined in [9], as follows:

$$\tau = \min\{i > 0 \mid s^i = r^j \quad \text{for some } j \in \mathbb{Z}\} \qquad \text{and} \qquad r^\epsilon = s^\tau,$$

if $\{i > 0 \mid s^i = r^j \quad \text{for some } j \in \mathbb{Z}\} \neq \emptyset$, and $\tau = 0 = \epsilon$, otherwise. As long as r is not a root of unity, ϵ is uniquely defined. Furthermore, by [9, Lem. 2.1], if $\delta, \eta \in \mathbb{Z}$ then $r^\delta s^\eta = 1$ if and only if there is $\lambda \in \mathbb{Z}$ such that $(\delta, \eta) = \lambda(-\epsilon, \tau)$.

Lemma 4.3. *Assume $\gamma = 0$. There exist principal eigenvectors in L_h if and only if $\tau > 0$, i.e., if and only if there are integers α, β, with $\alpha \neq 0$, such that $s^\alpha r^\beta = 1$.*

Proof. Assume $s^\alpha r^\beta = 1$, with $\alpha \neq 0$. We can thus assume $\alpha \geq 1$. Then $\sigma(h^\beta) = r^\beta h^\beta = s^{-\alpha} h^\beta$, so there are principal eigenvectors.

Conversely, assume $\sigma(p) = s^{-n}p$ for some $n \geq 1$ and some $p = \sum_{i=\alpha}^{\beta} a_i h^i$, with $\alpha \leq \beta$ and $a_\alpha a_\beta \neq 0$. Then, $0 = \sigma(p) - s^{-n}p = \sum_{i=\alpha}^{\beta} a_i(r^i - s^{-n})h^i$. In particular, $r^\beta - s^{-n} = 0$ and $s^n r^\beta = 1$. $\square$

Theorem 4.4. *Assume $\gamma = 0$, r is not a root of unity and $f \neq 0$ is conformal. Then L is a Noetherian UFR if and only if either $\tau > 0$ or f is a monomial.*

Proof. By Proposition 4.1, we can work over the localisation L_h. Then, the result for L_h follows from Example 2.21 of [20]. $\square$

It remains to establish when L is a Noetherian UFD, which we do next.

Theorem 4.5. *Assume $\gamma = 0$, r is not a root of unity and $f \neq 0$ is conformal. Then L is a Noetherian UFD if and only if either one of the following two conditions holds:*

(a) *$\langle r, s\rangle$ is a free abelian group of rank 2 and f is a monomial, or*

(b) *$\langle r, s\rangle$ is a free abelian group of rank 1.*

Proof. Once again, we can work over the localisation L_h, by virtue of Proposition 4.1. Notice that, since r is not a root of unity, $\langle r, s\rangle$ is a free abelian group of rank 2 if and only if $\tau = 0$, and $\langle r, s\rangle$ is a free abelian group of rank 1 if and only if $\tau \geq 1$ and $\gcd(\tau, \epsilon) = 1$.

Assume first that $\tau = 0$. Then, by the above, L_h is a Noetherian UFR if and only if f is a monomial. When this is the case, $\langle z := ud - g(h)\rangle$ is the unique height one prime ideal of L_h, and it is clearly completely prime, as the factor algebra is a quantum torus in two variables (namely, the cosets of u and h).

Let us now suppose $\tau \geq 1$. Then, by the proof of Lemma 4.3, there is a principal eigenvector, $h^{-\epsilon}$, and it has degree τ. Furthermore, this principal eigenvector is unique, up to nonzero scalar multiples. It follows that $\langle h^{-\epsilon}z^{\tau}-\lambda\rangle$ is a height one prime ideal of L_h, for all $\lambda\in\mathbb{K}^*$ such that $\lambda h^{\epsilon}\neq(-g(h))^{\tau}$, by [20, Cor. 2.9.(ii)]. Note that $\lambda h^{\epsilon}=(-g(h))^{\tau}$ can occur for at most one value of $\lambda\in\mathbb{K}^*$. By [20, Thm. 2.24], it is easy to conclude that, for $\lambda\in\mathbb{K}^*$, $\langle h^{-\epsilon}z^{\tau}-\lambda\rangle$ is completely prime if and only if $\gcd(\tau,\epsilon)=1$. In particular, under the current hypotheses, if L_h is a Noetherian UFD, then $\langle r,s\rangle$ is free abelian of rank 1.

Conversely, assume $\gcd(\tau,\epsilon)=1$, with $\tau\geq 1$. Then, by [20, 2.17 and Prop. 2.18], the height one prime ideals of L_h include $\langle z\rangle$, which is completely prime, and the ideals of the form $\langle h^{-\epsilon}z^{\tau}-\lambda\rangle$, for $\lambda\in\mathbb{K}^*$ such that $\lambda h^{\epsilon}\neq(-g(h))^{\tau}$, which are all completely prime, as $\gcd(\tau,\epsilon)=1$. In case $h^{-\epsilon}(-g(h))^{\tau}\notin\mathbb{K}$, then this is the complete list of height one prime ideals of L_h, and it follows that L_h is a Noetherian UFD. Suppose that $h^{-\epsilon}(-g(h))^{\tau}\in\mathbb{K}$. Then $g(h)$ is a unit, say $g(h)=\mu h^{a}$, and it follows that $\epsilon=\tau a$. As we are assuming τ and ϵ to be coprime, it must be that $\tau=1$ and $\epsilon=a$. Thus, $f(h)=sg(h)-g(rh)=\mu(s-r^{a})h^{a}=\mu(s^{\tau}-r^{\epsilon})h^{a}=0$, which contradicts our hypothesis on f. Therefore, it is always the case that $h^{-\epsilon}(-g(h))^{\tau}\notin\mathbb{K}$ and the proof is complete. □

5 The case f conformal and $r=1$

When $r=1$, we cannot assume that $\gamma=0$, so we will consider separately the cases $\gamma\neq 0$ and $\gamma=0$. The defining relations of $L=L(f,1,s,\gamma)$ are:

$$dh-hd+\gamma d=0, \tag{5.14}$$
$$hu-uh+\gamma u=0, \tag{5.15}$$
$$du-sud+f(h)=0. \tag{5.16}$$

Note that if $r=1$ and $\gamma\neq 0$ we retrieve the algebras studied by Rueda in [33]. The latter include Smith's algebras [34], which occur as generalized down-up algebras when $r=s=1$ and $\gamma\neq 0$. We assume throughout that $f\neq 0$.

5.1 The case f conformal, $r=1$ and $\gamma\neq 0$

Let g be such that $f(h)=sg(h)-g(h-\gamma)$. In particular, $g\neq 0$. Recall, from Section 4, the definition of a principal eigenvector.

Lemma 5.1. *Assume $r=1$ and $\gamma\neq 0$. If $p\in\mathbb{K}[h]$ is such that $\sigma(p)=\mu p$ for some $\mu\in\mathbb{K}$ then $p\in\mathbb{K}$. In particular, the only nonzero σ-invariant ideal of $\mathbb{K}[h]$ is $\mathbb{K}[h]$ and there are principal eigenvectors if and only if s is a root of unity.*

Proof. Suppose that $\sigma(p)=\mu p$ for some $p\in\mathbb{K}[h]\setminus\mathbb{K}$. Then, since $\mathbb{K}$ is algebraically closed, there is $\alpha\in\mathbb{K}$ such that $p(\alpha)=0$. It follows that $0=\mu p(\alpha)=\sigma(p)(\alpha)=p(\alpha-\gamma)$, and hence $\alpha-\gamma$ is also a root of p. Since α was an arbitrary root of p and $\gamma\neq 0$, this is impossible. Thus, $p\in\mathbb{K}$.

Let I be a σ-invariant ideal of $\mathbb{K}[h]$. Then $I=\langle p\rangle$, for some $p\in\mathbb{K}[h]$, and $\sigma(p)\in\mathbb{K}^*p$, so either $I=\langle 0\rangle$ or $I=\mathbb{K}[h]$.

Finally, assume there is a principal eigenvector $0\neq p\in\mathbb{K}[h]$. Then there is $n\geq 1$ so that $\sigma(p)=s^{-n}p$. In particular, by the above, it follows that $p\in\mathbb{K}^*$ and $s^n=1$. Conversely, if s is a primitive n-th root of unity, then 1 is a principal eigenvector of degree n. □

Proposition 5.2. *Assume s is not a root of unity and $\gamma\neq 0$. Take $p\in\mathbb{K}[h]$. Then p satisfies*

$$\forall\lambda\in\mathbb{K}\quad\forall n\geq 1\quad s^{n}p(\lambda)=p(\lambda-n\gamma)\implies p(\lambda)=0 \tag{5.17}$$

if and only if $p\in\mathbb{K}$.

Proof. Let $p\in\mathbb{K}[h]$ and assume, by way of contradiction, that p is not constant. Then the set of roots of p is finite and nonempty. Let $\Delta=\{\alpha-\beta\mid\alpha\text{ and }\beta\text{ are roots of }p\}$ be the set of differences of (not necessarily distinct) roots of p. Since Δ is finite, there exists an integer $n\geq 1$ such that $n\gamma\notin\Delta$. Consider the polynomial $p_n(h)=s^np(h)-p(h-n\gamma)$. Since s is not a root of unity, p_n

has the same degree as p. In particular, p_n has some root, say $\alpha \in \mathbb{K}$. By (5.17), α is a root of p, which in turn implies that $\alpha - n\gamma$ is a root of p, as well. Hence, $n\gamma = \alpha - (\alpha - n\gamma) \in \Delta$, which contradicts our choice of n. So indeed, $p \in \mathbb{K}$. The converse implication is trivial. $\square$

We are now ready to say when L is a Noetherian UFR.

Theorem 5.3. *Assume f and γ are nonzero and $r = 1$. Then $L = L(f, 1, s, \gamma)$ is a Noetherian UFR if and only if one of the following conditions hold:*

(a) *s is a root of unity, or*

(b) *s is not a root of unity and $f \in \mathbb{K}$.*

Proof. The first and second parts follow from [20, Prop. 2.18] and [20, Prop. 2.20], respectively, and the results in this section. Note that, since $s \neq 1$, then $f \in \mathbb{K} \iff g \in \mathbb{K}$. $\square$

Next, we deduce from [20, 2.22 and Remark 2.25] the cases where L is a Noetherian UFD.

Theorem 5.4. *Assume f and γ are nonzero and $r = 1$. Then $L = L(f, 1, s, \gamma)$ is a Noetherian UFD if and only if one of the following conditions hold:*

(a) *$s = 1$, or*

(b) *s is not a root of unity and $f \in \mathbb{K}$.*

5.2 The case f conformal, $r = 1$ and $\gamma = 0$

In this case, h is central in L. Note also that since f is conformal in L, then $s \neq 1$. Nevertheless, the conformality condition will not be used in this section, as we will not refer to [20]. We will consider separately the cases s not a root of unity and s a root of unity of order $l \geq 2$.

5.2.1 The case f conformal, $r = 1$, $\gamma = 0$ and s not a root of unity

Theorem 5.5. *Assume f is nonzero, $r = 1$, $\gamma = 0$ and s is not a root of unity. Then $L = L(f, 1, s, 0)$ is a Noetherian UFD. In fact, the height one prime ideals of L are $\langle h - \lambda \rangle$, for $\lambda \in \mathbb{K}$, and $\langle du - ud \rangle$.*

Proof. Let $\lambda \in \mathbb{K}$. Then $h - \lambda$ is central and the factor algebra $L/\langle h - \lambda \rangle$ is either the quantum plane or the quantum Weyl algebra, depending on whether λ is a root of f or not. In either case, $L/\langle h - \lambda \rangle$ is a domain, and thus the ideal $\langle h - \lambda \rangle$ is completely prime.

The element $du - ud = (s - 1)z$ is normal and the factor algebra $L/\langle du - ud \rangle$ is a commutative algebra generated by h, d and u, subject to the relation $ud - \frac{1}{s-1} f(h) = 0$. It is easy to see that the element $ud - \frac{1}{s-1} f(h)$, viewed as an element of the polynomial algebra in the 3 commuting variables h, d and u, is irreducible, provided that $f \neq 0$. This shows that $L/\langle du - ud \rangle$ is a domain. Another way of reaching this conclusion is by realising this factor algebra as the generalized Weyl algebra $\mathbb{K}[h] \left(\mathrm{id}_{\mathbb{K}[h]}, \frac{1}{s-1} f(h) \right)$. (The reader is referred to [3] for more details on generalized Weyl algebras.)

The above shows that all ideals of the form $\langle h - \lambda \rangle$, for $\lambda \in \mathbb{K}$, and $\langle du - ud \rangle$ are completely prime and principal. By the Principal Ideal Theorem, they have height one. To finish the proof, we need only show that any nonzero prime ideal of L must contain one of these ideals. We do so in the next proposition. $\square$

Proposition 5.6. *Assume f is nonzero, $r = 1$, $\gamma = 0$ and s is not a root of unity. Then any nonzero prime ideal of $L = L(f, 1, s, 0)$ must contain either $du - ud$ or $h - \lambda$, for some $\lambda \in \mathbb{K}$.*

Proof. Let P be a nonzero prime ideal of L and assume $h-\lambda$ is not in P, for any scalar λ. We will show that $du-ud \in P$.

Since $\mathbb{K}$ is algebraically closed and $\mathbb{K}[h]$ is a central subalgebra, it follows that $P \cap \mathbb{K}[h] = \langle 0 \rangle$. Let $\widetilde{L}$ be the localisation of L at the central multiplicative set of nonzero elements of $\mathbb{K}[h]$. Let $\mathbb{F} = \mathbb{K}(h)$ be the field of fractions of $\mathbb{K}[h]$. Then $\widetilde{L}$ can be seen as the first quantised Weyl algebra $\mathbb{A}_1^s(\mathbb{F})$, generated over $\mathbb{F}$ by X and Y, and subject to the relation $XY - sYX = 1$. In fact, it is easy to check that there are mutually inverse $\mathbb{F}$-algebra maps, $\Phi : \widetilde{L} \to \mathbb{A}_1^s(\mathbb{F})$ and $\Psi : \mathbb{A}_1^s(\mathbb{F}) \to \widetilde{L}$, such that $\Phi(d) = X$, $\Phi(u) = -f(h)Y$, $\Psi(X) = d$ and $\Psi(Y) = -u\,(f(h))^{-1}$.

Thus, P extends to a nonzero prime ideal $\widetilde{P}$ of $\widetilde{L}$, which we identify, via the map Φ above, with a prime ideal of $\mathbb{A}_1^s(\mathbb{F})$. The element $Z := XY - YX$ of $\mathbb{A}_1^s(\mathbb{F})$ is normal, nonzero and not a unit. In fact, Z corresponds, via Ψ, to the element $(ud-du)\,(f(h))^{-1}$. Let $\mathbb{B}_1^s(\mathbb{F})$ be the localisation of $\mathbb{A}_1^s(\mathbb{F})$ at the powers of Z. Since s is not a root of unity, $\mathbb{B}_1^s(\mathbb{F})$ is simple, by [2, Lem. 2.2] (note that this result does not depend on the base field being algebraically closed). Since Z is normal, this means that every nonzero prime ideal of $\mathbb{A}_1^s(\mathbb{F})$ contains Z. In particular, $Z \in \Phi(\widetilde{P})$, i.e., $(ud-du)\,(f(h))^{-1} \in \widetilde{P}$. Thus, $du - ud \in P = \widetilde{P} \cap L$. □

5.2.2 The case f conformal, $r=1$, $\gamma=0$ and $s \neq 1$ a root of unity

We finally tackle the case in which s is a primitive l-th root of unity, for some $l \geq 2$. It is straightforward to see that, in this case, both d^l and u^l are central. Our aim is to prove that, in this case, L is a Noetherian UFR.

Let $\widetilde{L}$ be the localisation of L with respect to the multiplicative set generated by the central elements of the form $h-\lambda$, where λ runs through the roots of f. In case f is a (nonzero) constant polynomial, we have $\widetilde{L} = L$.

Since, for λ a root of f, $L/\langle h-\lambda \rangle$ is a quantum plane, the ideals of the form $\langle h-\lambda \rangle$, with $f(\lambda) = 0$, are completely prime as well as pairwise distinct. Thus, by [26, Prop. 1.6], it will be enough to show that $\widetilde{L}$ is a Noetherian UFR.

Let S be the localisation of $\mathbb{K}[h]$ at the multiplicative set generated by the $h-\lambda$, with λ running through the roots of f. Since $\mathbb{K}$ is algebraically closed, f is a product of linear factors, and thus is invertible in S. The localised algebra $\widetilde{L}$ can be seen as the algebra over S, generated by elements D and U, subject to the relation

$$DU - sUD = 1,$$

where $D = d$ and $U = -u\,(f(h))^{-1}$. Consider the nonzero normal element $Z = DU - UD$, of $\widetilde{L}$. It satisfies $ZU = sUZ$ and $ZD = s^{-1}DZ$. In particular, Z^l is central in $\widetilde{L}$. The algebra $\widetilde{L}/\langle Z \rangle$ is isomorphic to the commutative Laurent polynomial algebra $S[U^{\pm 1}]$, and hence Z generates a completely prime ideal of $\widetilde{L}$. Therefore, it will suffice to show that the localisation $\widehat{L}$ of $\widetilde{L}$ at the multiplicative set generated by Z is a Noetherian UFR, by [26, Prop. 1.6]. The latter is a consequence of the result that follows.

Proposition 5.7. *Under the above assumptions, $\widehat{L}$ is an Azumaya algebra over its centre $\mathcal{Z}(\widehat{L})$, with $[\widehat{L} : \mathcal{Z}(\widehat{L})] = l^2$. Moreover, the centre $\mathcal{Z}(\widehat{L})$ of $\widehat{L}$ is the localisation of $S[U^l, D^l]$ at the powers of Z^l.*

Proof. The proof is entirely analogous to that of [1, Prop. 1.3]. We give details for completeness.

First, it is easy to see that the centre of $\widetilde{L}$ is $S[U^l, D^l]$, and it must contain Z^l, as this element commutes with D and U.

Let $b = aZ^n$ be an element of $\widehat{L}$ with $a \in \widetilde{L}$ and $n \in \mathbb{Z}$. Take $q \in \mathbb{Z}$ and $0 \leq r < l$ such that $n = ql + r$. As Z^{ql} is central in $\widehat{L}$, we get that $b = aZ^{ql}Z^r$ is central in $\widehat{L}$ if and only if aZ^r is central in $\widetilde{L}$. Hence, $\mathcal{Z}(\widehat{L}) = \{cZ^{ql} \mid c \in S[U^l, D^l],\ q \in \mathbb{Z}\}$ is the localisation of $S[U^l, D^l]$ at the powers of Z^l.

By [1, Lem. 1.2], $\{U^iD^j\}_{0 \leq i,j \leq l-1}$ is a basis for $\widehat{L}$ over its centre. So $[\widehat{L} : \mathcal{Z}(\widehat{L})] = l^2$.

To conclude, it is enough to show that the irreducible finite dimensional representations of $\widehat{L}$ over $\mathbb{K}$ all have dimension l, by the Artin-Procesi Theorem. Let $\rho : \widehat{L} \to \mathrm{End}_{\mathbb{K}}(V)$ be such an irreducible representation, with $\dim_{\mathbb{K}} V = m$. Since $\mathbb{K}$ is algebraically closed, and V is finite-dimensional, it follows by Schur's Lemma that the centre of $\widehat{L}$ acts on V as scalars. Thus, $\dim_{\mathbb{K}} \rho(\widehat{L}) \leq l^2$. By the Jacobson Density Theorem, ρ is surjective. Therefore,

$$m^2 = \dim_{\mathbb{K}} \mathrm{End}_{\mathbb{K}}(V) = \dim_{\mathbb{K}} \rho(\widehat{L}) \leq l^2,$$

and $m \leq l$.

On the other hand, let $X = \rho(D)$, $Y = \rho(U)$. Then, $XY - sYX = \rho(DU - sUD) = \rho(1) = 1 \in \mathrm{End}_{\mathbb{K}}(V)$. Furthermore, as Z is invertible in $\widehat{L}$, the same is true of $\rho(Z) = XY - YX \in \mathrm{End}_{\mathbb{K}}(V)$. Thus, $\dim_{\mathbb{K}} \mathrm{End}_{\mathbb{K}}(V) \geq l^2$, again by [1, Lem. 1.2]. So $m \geq l$ and $m = l$. □

Theorem 5.8. *Assume f is nonzero, $\gamma = 0$, $r = 1$ and s is a primitive l-th root of unity, for some $l \geq 2$. Then $L = L(f, 1, s, 0)$ is a Noetherian UFR, but not a Noetherian UFD.*

Proof. Since the algebra $\widehat{L}$ is Azumaya over its centre, it follows that all ideals of $\widehat{L}$ are centrally generated. Hence, as $\mathcal{Z}(\widehat{L})$ is a (commutative) UFR, by Proposition 5.7, we deduce from the Principal Ideal Theorem that $\widehat{L}$ is a Noetherian UFR. We can thus conclude that L is a Noetherian UFR, by [26, Prop. 1.6].

We will now observe that the ideal $\langle d^l - 1\rangle$ of L is prime. To see this, notice that L is an Ore extension over the commutative polynomial algebra $\mathbb{K}[h, d]$. So, by [5, Prop. 2.1], it will be enough to prove that $d^l - 1$ generates a δ-stable, σ-prime ideal of this polynomial algebra, where σ and δ are as in (1.4). In particular, $\sigma(d) = sd$. This ideal is stable under δ and σ because $d^l - 1$ is central in $L = \mathbb{K}[h, d][u; \sigma^{-1}, \delta]$. Consider the prime ideal I of $\mathbb{K}[h, d]$ generated by $d - 1$. Since s is a primitive root of unity of order l, it follows that

$$\bigcap_{i\in\mathbb{Z}} \sigma^i(I) = \prod_{0\leq i\leq l-1} (d - s^i)\mathbb{K}[h, d] = (d^l - 1)\mathbb{K}[h, d],$$

so $(d^l - 1)\mathbb{K}[h, d]$ is indeed a σ-prime ideal of $\mathbb{K}[h, d]$, as it is the intersection of a σ-orbit of a prime ideal. Thus, $\langle d^l - 1\rangle$ is a prime ideal of L.

By the Principal Ideal Theorem, $\langle d^l - 1\rangle$ has height one. Yet, it is not completely prime, as $l \geq 2$ and hence the central element $d^l - 1$ factors non-trivially. So L is not a Noetherian UFD. □

6 The case f conformal and $r \neq 1$ a root of unity

The final part of our discussion concerns the case when f is conformal and r is a primitive root of unity of order $l \geq 2$. Since $r \neq 1$ we will assume, without loss of generality, that $\gamma = 0$, by Proposition 1.4.

We start with a negative result, which follows immediately from Lemma 1.13.

Corollary 6.1. *Let $L = L(f, r, s, 0)$ and assume $r \neq 1$ is a root of unity. Then L is not a Noetherian UFD.*

The remainder of this section is devoted to establishing that, under the current assumptions, $L = L(f, r, s, 0)$ is a Noetherian UFR. The following general result will play, in this section, the role of Propositions 2.3 and 2.5.

We consider a Noetherian ring R, with a subring A, which is a domain, and such that R is free both as a left and as a right A-module, with basis $S := \{X^i \mid i \geq 0\}$. Assume the multiplicative system S satisfies the Ore condition on both sides, and let $\widehat{R} := RS^{-1}$ be the corresponding localisation.

Lemma 6.2. *Let P be a nonzero prime ideal of R such that $P \cap S = \emptyset$, and assume that there exists $b \in \widehat{R}$ such that:*

(a) $PS^{-1} = \widehat{R}b = b\widehat{R}$;

(b) $Xb = \eta bX$, *for some central unit* η *of* A.

Then $P = xR = Rx$*, where* $e \in \mathbb{Z}$ *is minimal such that* $bX^e \in R$*, and* $x = bX^e$.

Proof. Observe that, since $b \neq 0$, a minimal $e \in \mathbb{Z}$ such that $bX^e \in R$ exists; also, $Xx = \eta xX$. We will prove that $P = Rx$. As $Xb = \eta bX$, e is also minimal such that $X^e b \in R$, and $X^e b = \eta^e x$, so a similar argument will show that $P = xR$, using the fact that $\{X^i \mid i \geq 0\}$ is a free basis for R as a right A-module.

By construction, it is clear that $Rx \subseteq P$, as $x \in PS^{-1} \cap R = P$. Let $y \in P$. Then $y \in PS^{-1} = \widehat{R}b = \widehat{R}xX^{-e} = \widehat{R}x$, as x and X η-commute. Hence, there exists $u \in \widehat{R}$ such that $y = ux$. Moreover, there exists $t \geq 0$ such that $uX^t \in R$. Therefore, $yX^t = uxX^t = \eta^{-t}uX^t x$, i.e., there exist $t \geq 0$ and $r \in R$ such that $yX^t = rx$. We choose a minimal such t.

Assume that $t \geq 1$. Write

$$r = \sum_{i=0}^{k} r_i X^i, \quad y = \sum_{i=0}^{k} y_i X^i, \quad x = \sum_{i=0}^{k} x_i X^i,$$

where $r_i, y_i, x_i \in A$. Note that $x_0 \neq 0$, as otherwise $xX^{-1} \in R$, so that $bX^{e-1} \in R$, contradicting the minimality of e.

On the other hand, as $Xx = \eta xX$, the equality $yX^t = rx$ can be written as follows:

$$\sum_{i=0}^{k} y_i X^{i+t} = \sum_{i=0}^{k} r_i X^i bX^e = \sum_{i=0}^{k} r_i \eta^i bX^{e+i} = \sum_{i=0}^{k} r_i \eta^i xX^i = \sum_{i,j=0}^{k} r_i \eta^i x_j X^{i+j}.$$

As $t \geq 1$, identifying the degree 0 coefficients yields $0 = r_0 x_0$. As $x_0 \neq 0$ and A is a domain, this forces $r_0 = 0$. Hence, $rX^{-1} \in R$ and $yX^{t-1} = rxX^{-1} = \eta rX^{-1}x$. This contradicts the minimality of t. Thus, $t = 0$ and $y = rx \in Rx$, as desired. □

Proposition 6.3. *Let* $L = L(f, r, s, 0)$*, with* f *conformal. If* P *is a prime ideal of* L *of height one, which either does not contain any power of* d *or does not contain any power of* u*, then* P *is a principal ideal, generated by a normal element of* L.

Proof. By Lemma 1.2, the localisation $\widehat{L}$ of L at the denominator set $D = \{d^i\}_{i \geq 0}$ is isomorphic to a quantum coordinate ring of affine 3-space over $\mathbb{K}$, localised at the powers of one of its canonical generators. As in Section 3.1, it follows that $\widehat{L}$ is a Noetherian UFR.

If P is a height one prime ideal of L which is disjoint from D, then PD^{-1} is a height one prime ideal of $\widehat{L}$, so it is generated by a normal element $b \in \widehat{L}$. It is easy to see that in a quantum coordinate ring the normal elements are q-central, so there is $\eta \in \mathbb{K}^*$ such that $db = \eta bd$. Thus, by Lemma 6.2, P is a principal ideal, generated by some normal element $x \in L$.

The statement regarding u follows similarly. □

So it remains to consider the prime ideals that contain both a power of d and a power of u. We start by discussing the simpler case where s is not a root of unity.

Proposition 6.4. *Let* $L = L(f, r, s, 0)$*, with* $f \neq 0$ *conformal and* $r \neq 1$ *a root of unity. If* s *is not a root of unity, then* L *is a Noetherian UFR, but not a Noetherian UFD.*

Proof. In view of Corollary 6.1 and Proposition 6.3, it is enough to show that the height one prime ideals of L either do not contain any power of d or do not contain any power of u.

Let P be a prime ideal of L which contains a power of d and a power of u. Since r is a root of unity and s is not, it follows that $(s/r^m)^k \neq 1$, for all $k > 0$. Thus, by Lemma 1.10, the polynomials P_k are all nonzero, for $k > 0$. Hence, $P = Q_\lambda$, for some $\lambda \in \mathbb{K}$, by Theorem 1.12.

If $\lambda = 0$, then $h \in P$; otherwise $h^l - \lambda^l \in P$, where $l \geq 2$ is the order of r. Therefore, either $\langle h \rangle \subsetneq P$ or $\langle h^l - \lambda^l \rangle \subsetneq P$, as $P = Q_\lambda$ is not principal, so P has height at least two, by Lemma 1.13, thus proving our claim. □

In the next lemma we deal with the case in which s is a root of unity. Note that if r and s are roots of unity and f is conformal, then Lemma 1.10 guarantees the existence of a positive integer k such that $P_k = 0$. For any such k, the elements u^k and d^k are normal.

Lemma 6.5. *Let $L = L(f, r, s, 0)$, with $f \neq 0$ conformal, and assume r and s are roots of unity. Take $k > 0$ minimal such that $P_k = 0$. Then, u^k and d^k are normal and each generates a height one prime ideal of L.*

Proof. We will prove the statement for u^k; the result for d^k will thus follow, by symmetry.

Consider the Ore set $D = \left\{d^i\right\}_{i\geq 0}$ in L and the localisation $\widehat{L} = LD^{-1}$. Recall that $z := ud - g(h)$ is normal and satisfies $zh = hz$, $dz = szd$ and $zu = suz$ (see Section 1.2). It is easy to see that h and z generate a (commutative) polynomial algebra in two variables, $\mathbb{K}[h, z]$, and $\widehat{L} = \mathbb{K}[h, z][d^{\pm 1}; \tau]$, where $\tau(h) = rh$, $\tau(z) = sz$, with $u = (z + g(h))d^{-1}$.

Let $\xi = z + g(h) \in \mathbb{K}[h, z]$. This is an irreducible polynomial in the polynomial algebra $\mathbb{K}[h, z]$, hence it generates a prime ideal $P = \xi\mathbb{K}[h, z]$. Furthermore, $\tau^i(\xi)$ and $\tau^j(\xi)$ are associated irreducible polynomials if and only if k divides $i - j$. The latter follows from the minimality of k, as $P_i(h) = 0 \iff s^i g(h) = g(r^i h) \iff k$ divides i.

Thus,

$$I := \bigcap_{i\in\mathbb{Z}} \tau^i(P) = \bigcap_{i\in\mathbb{Z}} \tau^i(\xi)\mathbb{K}[h, z] = \bigcap_{1-k\leq i\leq 0} \tau^i(\xi)\mathbb{K}[h, z] = \prod_{1-k\leq i\leq 0} \tau^i(\xi)\mathbb{K}[h, z]$$

is a τ-prime ideal of $\mathbb{K}[h, z]$. It follows (e.g. by [5, Prop. 2.1]) that $Q := I\widehat{L}$ is a prime ideal of $\widehat{L}$.

Claim: $\displaystyle\prod_{1-n\leq i\leq 0} \tau^i(\xi) = u^n d^n$, for all $n \geq 0$.

The claim above can be readily established by induction. In particular, $Q = u^k d^k \widehat{L} = u^k \widehat{L}$.

It remains to show that the prime ideal that Q contracts to in L is generated by u^k. This follows by applying Lemma 6.2 to the contraction of Q to L, and noting that:

- $du^k = s^k u^k d$, and
- for $n \in \mathbb{Z}$, $u^k d^n \in L \iff n \geq 0$.

Finally, the height of $\langle u^k \rangle$ is one, by the Principal Ideal Theorem. □

Our final result finishes the classification of which generalized down-up algebras are Noetherian UFR's.

Theorem 6.6. *Let $L = L(f, r, s, 0)$, with $f \neq 0$ conformal and $r \neq 1$ a root of unity. Then L is a Noetherian UFR but not a Noetherian UFD.*

Proof. By Proposition 6.4, it remains to consider the case where s is a root of unity (possibly equal to 1), and by Corollary 6.1 and Proposition 6.3, it will be enough to show that there are no height one prime ideals of L which contain both a power of d and a power of u.

Let P be a prime ideal of L which contains a power of d and a power of u. Let $k > 0$ be minimal such that $P_k = 0$. Since u^k is normal, we must have $u^k \in P$, so P contains the height one prime ideal $\langle u^k \rangle$, by Lemma 6.5. So P does not have height one, as $\langle u^k \rangle$ contains no power of d. □

7 Proofs of Theorems A and B

In this final section, we start by proving Theorem B, which gives a complete classification of the generalized down-up algebras which are a Noetherian UFR, and then we prove Theorem A. We also specialise our results to down-up algebras, as introduced by Benkart and Roby in [6].

Proof of Theorem B. Assume first that $\gamma = 0$. Then the condition *there exists* $\zeta \neq \gamma/(r-1)$ *such that* $f(\zeta) = 0$ is equivalent to the condition f *is not a monomial*, and the condition $\langle r, s\rangle$ *is a free abelian group of rank* 2 is equivalent to the condition r *is not a root of unity and* $\tau = 0$. Thus, in this case, the result follows from Theorem 2.6, Proposition 3.1, Theorem 4.4, Theorem 5.5, Theorem 5.8 and Theorem 6.6.

Now assume that $\gamma \neq 0$ and $r = 1$. Then, by Proposition 1.5, f is conformal, and the result follows from Proposition 3.3 and Theorem 5.3.

Finally, if $\gamma \neq 0$ and $r \neq 1$, then Proposition 1.4 asserts that L is isomorphic to a generalized down-up algebra $L(\tilde{f}, r, s, 0)$, such that f is conformal in $L(f, r, s, \gamma)$ if and only if $\tilde{f}$ is conformal in $L(\tilde{f}, r, s, 0)$. Furthermore, by the proof of this result (see [9, Prop. 1.7]), we can take $\tilde{f}(h) = f(\frac{h+\gamma}{r-1})$. Hence, in this case, the result follows from applying our previously established criteria to $L(\tilde{f}, r, s, 0)$. $\square$

To finish the classification, we just need to determine the generalized down-up algebras which are a Noetherian UFD, and prove Theorem A.

Proof of Theorem A. It will be enough to establish this result in the case $\gamma = 0$, and the case $\gamma \neq 0$, $r = 1$, by Proposition 1.4, as the statement does not involve f or γ. So we assume that either $\gamma = 0$ or $r = 1$.

- If f is not conformal then $\gamma = 0$, by Proposition 1.5, and thus, by Lemma 1.3, $\langle r, s\rangle = \langle r\rangle$. Then Theorem 2.6 establishes the result.
- If $f = 0$ and $\gamma = 0$, then the result follows from Propositions 3.1 and 3.2.
- If $f = 0$ and $\gamma \neq 0$, then we assume $r = 1$ and the result follows from Proposition 3.3 and Theorem 3.5.
- If $f \neq 0$ is conformal and r is not a root of unity, then we assume $\gamma = 0$ and the result follows from Theorems 4.4 and 4.5.
- If $f \neq 0$ is conformal, $r = 1$ and $\gamma \neq 0$, then Theorems 5.3 and 5.4 establish the result.
- If $f \neq 0$ is conformal, $r = 1$ and $\gamma = 0$, then Proposition 1.5 implies that $s \neq 1$. Thus, Theorems 5.5 and 5.8 imply the result.
- If $r \neq 1$ is a root of unity, then we can assume that $\gamma = 0$, and the result follows directly from Corollary 6.1.

$\square$

We note that the hypothesis that L be a Noetherian UFR, in Theorem A, is essential, as the following example illustrates. Let $r \in \mathbb{K}^*$ be a non-root of unity, $s \in \{1, r\}$ and $f = h \in \mathbb{K}[h]$. Then $L = L(h, r, s, 1)$ is not a Noetherian UFD, by Proposition 1.4 and Theorem B(a). Yet, $\langle r, s\rangle \simeq \mathbb{Z}$ is torsionfree. Notice that $L(h, r, s, 1)$ is isomorphic to the down-up algebra $A(r+s, -rs, 1)$.

In general, the down-up algebra $A(\alpha, \beta, \gamma)$, as defined in [6], can be viewed as the generalized down-up algebra $L(h, r, s, \gamma)$, where $\alpha = r + s$ and $\beta = -rs$ (see [9, Lem. 1.1] for more details). So we have:

Corollary 7.1. *Let* $A = A(\alpha, \beta, \gamma)$ *be a down-up algebra over* $\mathbb{K}$ *with* $\beta \neq 0$. *Let* $r, s \in \mathbb{K}$ *be the roots of* $h^2 - \alpha h - \beta$. *Then* A *is a Noetherian UFR except if* $\gamma \neq 0$, β *is not a root of unity and one of the following conditions is satisfied:*

(a) $\alpha + \beta = 1$;

(b) $\alpha^2 + 4\beta = 0$;

(c) $\langle r, s\rangle$ *is a free abelian group of rank* 2.

Furthermore, A is a Noetherian UFD if and only if A is a Noetherian UFR and $\langle r, s\rangle$ is torsionfree.

Proof. We use the isomorphism $A(\alpha, \beta, \gamma) \simeq L(h, r, s, \gamma)$. First, by Proposition 1.4, Lemma 1.3 and Proposition 1.5, we conclude that $f(h) = h$ is conformal in $L(h, r, s, \gamma)$ if and only if one of the following conditions holds:

- $\gamma = 0$ and $r \neq s$;
- $\gamma \neq 0$, $r \neq 1$, $s \neq 1$ and $r \neq s$;
- $\gamma \neq 0$ and $r = 1$.

Thus, we can apply Theorem B to conclude that A is a Noetherian UFR except in the cases listed below:

- $\gamma \neq 0$, $s = 1$ and r is not a root of unity;
- $\gamma \neq 0$, $s = r$ and r is not a root of unity;
- $\gamma \neq 0$ and $\langle r, s\rangle$ is a free abelian group of rank 2;
- $\gamma \neq 0$, $r = 1$ and s is not a root of unity.

Notice that, in all of these cases, $\gamma \neq 0$ and $\beta = -rs$ is not a root of unity. Also, $\alpha + \beta = 1 \iff r = 1$ or $s = 1$, and $\alpha^2 + 4\beta = 0 \iff r = s$. The first part of the theorem thus follows. The second part is a direct consequence of Theorem A. □

Two down-up algebras of particular interest are the enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ and the enveloping algebra of the 3-dimensional Heisenberg Lie algebra, which occur as $A(2, -1, 1)$ and $A(2, -1, 0)$, respectively. Using Corollary 7.1, we retrieve the well-known fact that each of these two algebras is a Noetherian UFD (see [14] and [12, Prop. 3.1]).

Generalized down-up algebras also include other classes of algebras, such as Smith's algebras [34] and Rueda's algebras [33]. In the case of Smith's algebras, the result is quite straightforward. Let $f \in \mathbb{K}[H]$. Recall that the Smith algebra $S(f)$ is the $\mathbb{K}$-algebra generated by A, B, H with relations:

$$[H, A] = A, \ [H, B] = -B \text{ and } [A, B] = f(H).$$

It is well known that $S(f) \simeq L(f, 1, 1, 1)$. Hence, we deduce from Theorems A and B the following result.

Corollary 7.2. *Let $S(f)$ be a Smith algebra with $f \in \mathbb{K}[H]$. Then, $S(f)$ is a Noetherian UFD.*

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