GIBBS-MARKOV-YOUNG STRUCTURES WITH (STRETCHED) EXPONENTIAL TAIL FOR PARTIALLY HYPERBOLIC ATTRACTORS

JOSÉ F. ALVES AND XIN LI

Abstract. In this work we extend the results obtained by Gouëzel in [14] to partially hyperbolic attractors. We study a forward invariant set K on a Riemannian manifold M whose tangent space splits as dominated decomposition $T_K M = E^{cu} \oplus E^s$, for which the center-unstable direction E^{cu} is non-uniformly expanding on some local unstable disk. We prove that the (stretched) exponential decay of recurrence times for an induced scheme can be deduced under the assumption of (stretched) exponential decay of the time that typical points need to achieve some uniform expanding in the center-unstable direction. As an application of our results we obtain exponential decay of correlations and exponential large deviations for a class of partially hyperbolic diffeomorphisms considered in [2].

Contents

Date: December 18, 2015.

2000 *Mathematics Subject Classification.* 37A05, 37C40, 37D25.

Key words and phrases. Partially hyperbolic attractor, Gibbs-Markov-Young structure, recurrence times, decay of correlations, large deviations.

JFA was partially supported by Fundação Calouste Gulbenkian, by CMUP, by the European Regional Development Fund through the Programme COMPETE and by FCT under the projects PTDC/MAT/099493/2008 and PEst-C/MAT/UI0144/2011. XL was supported by FCT.

1. Introduction

In the late 60's and beginning of 70's, Sinai, Ruelle and Bowen brought Markov partitions and symbolic dynamics into the theory of uniformly hyperbolic dynamics to prove the existence of the so-called *Sinai-Ruelle-Bowen (SRB) measures* for these systems; see [19, 8, 18]. According to Ruelle [9, Preface], "this allowed the powerful techniques and results of statistical mechanics to be applied into smooth dynamics".

To study systems beyond those uniformly hyperbolic, in [21, 22] Young used some type of Markov partitions with infinitely many symbols to build towers for systems with nonuniform hyperbolic behavior, including Axiom A attractors, piecewise hyperbolic maps, billiards with convex scatterers, logistic maps, intermittent maps and Hénon-type attractors. Using towers, Young studied some statistical properties of these nonuniformly hyperbolic systems, including the existence of SRB measures, exponential decay of correlations and the validity of the Central Limit Theorem for the SRB measure. Roughly speaking, Young towers are characterized by some region of the phase space partitioned into an at most countable number of subsets with associated *recurrence times*. Young called it a *horseshoe with infinitely many branches*. These structures have some properties which address to Gibbs states and for that reason they are nowadays commonly referred to as *Gibbs-Markov-Young (GMY) structures*.

In [10], Bonatti and Viana considered partially hyperbolic attractors with mostly contracting central direction, meaning that the tangent bundle splits as $E^{cs} \oplus E^u$, with the E^u direction being uniformly expanding and the E^{cs} direction having negative Lyapunov exponents. They gave sufficient conditions for the existence of SRB measures under those conditions. In [12], Castro showed the existence of GMY structures for such systems, thus obtaining statistical properties like exponential decay of correlations and the validity of the Central Limit Theorem. The Central Limit Theorem had also been obtained by Dolgopyat in [13].

However, as most of the richness of the dynamics in partially hyperbolic attractors appears in the unstable direction, the case $E^{cu} \oplus E^s$ (now with the stable direction being uniform and the unstable one being nonuniform) comprises more difficulties than the case $E^{cs} \oplus E^u$. The existence of SRB's for some classes of non-uniformly hyperbolic systems has been proved in [2] by Alves, Bonatti and Viana, both for non-uniformly expanding maps, in the non-invertible case, and for partially hyperbolic attractors of the type $E^{cu} \oplus E^s$, in the invertible case. In the non-invertible case, Alves, Luzzatto and Pinheiro proved in [5] the existence of GMY structures of non-uniformly expanding maps. Their approach, originated from [21] for Axiom A attractors, has shown to be not efficient enough to estimate the tail of recurrence times for non-uniformly hyperbolic systems with exponential or stretched exponential tail of hyperbolic times. This is due to the fact that at each step of their algorithmic construction just a definite fraction of hyperbolic times is used to construct new elements in the partition. In the invertible case, using arguments similar to those in [5], Alves and Pinheiro in [7] obtained GMY structures for partially hyperbolic attractors. Again, they only managed to prove the polynomial case: if the lack of expansion of the system at time n in the center-unstable direction is polynomially small, then the system has some GMY structure with polynomial decay of recurrence times.

Gouëzel developed a new construction in [14] with more efficient control for the tail of the recurrence times in the non-invertible setting. As a starting point, Gouëzel used the fact that the attractor could be partitioned into a finite number of sets with small size. That gave rise to more precise estimates than those in [5], yielding also the (streched) exponential case for non-uniformly expanding maps. However, for important combinatorial reasons, Gouëzel strategy could not be generalized to the partially hyperbolic setting $E^{cu} \oplus E^s$, in particular because the attractor is typically made of unstable leaves, which are not bounded in their intrinsic distance. Partly inspired by [14, 17], Alves, Dias and Luzzatto gave in [3] an improved *local* GMY structure, more efficient than [5] in the use of hyperbolic times, which made it possible to prove the integrability of recurrence times under very general conditions.

The main goal of this work is to fill a gap in the theory of partially hyperbolic diffeomorphisms of the type $E^{cu} \oplus E^s$, where, after [7], GMY structures are only known with polynomial tail of recurrence times. From these structures we get (stretched) exponential Decay of Correlations and exponential Large Deviations for the systems under consideration, by related results in [21, 6, 16]. Our strategy is based in a mixture of techniques from [3] and [14] and we construct a GMY structure by a method similar to [3], where recurrence times were only proved to be integrable. To improve the efficiency of the algorithm in [7], our method has a main difference, namely, we keep track of all points with hyperbolic times at a given iterate and not just of a proportion of those points.

1.1. **Gibbs-Markov-Young structures.** Here we recall the structures which have been introduced in [21]. Let $f : M \to M$ be a C^{1+} diffeomorphism of a finite dimensional Riemannian manifold M, Leb (Lebesgue measure) the normalized Riemannian volume on the Borel sets of M. Given a submanifold $\gamma \subset M$, we use Leb_γ to denote the Lebesgue measure on γ, induced by the restriction of the Riemannian structure to γ.

Definition 1.1. An embedded disk $\gamma \subset M$ is called a local *unstable manifold* if for all $x, y \in \gamma$

$$\mathrm{dist}(f^{-n}(x), f^{-n}(y)) \to 0, \quad \text{as } n \to \infty.$$

Similarly, γ is called a *local stable manifold* if for all $x, y \in \gamma$

$$\mathrm{dist}(f^{n}(x), f^{n}(y)) \to 0, \quad \text{as } n \to \infty.$$

Definition 1.2. Given $n \geq 1$, let D^u be a unit disk in $\mathbb{R}^n$ and let $\mathrm{Emb}^1(D^u, M)$ be the space of C^1 embeddings from D^u into M. A *continuous family of C^1 unstable manifolds* is a set Γ^u of unstable disks γ^u satisfying the following properties: there is a compact set K^s and a map $\Phi^u \colon K^s \times D^u \to M$ such that

(1) $\gamma^u = \Phi^u(\{x\} \times D^u)$ is a local unstable manifold;
(2) Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
(3) $x \mapsto \Phi^u|(\{x\} \times D^u)$ is a continuous map from K^s to $\mathrm{Emb}^1(D^u, M)$.

Continuous families of C^1 stable manifolds are defined analogously.

Definition 1.3. A subset $\Lambda \subset M$ has a *product structure* if, for some $n \geq 1$, there exist a continuous family of n-dimensional unstable manifolds $\Gamma^u = \{\gamma^u\}$ and a continuous family of $(\dim(M) - n)$-dimensional stable manifolds $\Gamma^s = \{\gamma^s\}$ such that

(1) $\Lambda = (\cup \gamma^u) \cap (\cup \gamma^s)$;

(2) each γ^s meets each γ^u in exactly one point, with the angle of γ^s and γ^u uniformly bounded away from zero.

Definition 1.4. Let $\Lambda \subset M$ have a product structure defined by families Γ^s and Γ^u. A subset $\Lambda_0 \subset \Lambda$ is an *s-subset* if Λ_0 has a hyperbolic product structure defined by families $\Gamma_0^s \subset \Gamma^s$ and $\Gamma_0^u = \Gamma^u$; *u-subsets* are defined similarly.

For $* = u, s$, given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, and let f^* denote the restriction of the map f to γ^*-disks and $|\det Df^*|$ denote the Jacobian of Df^*.

Definition 1.5. A set Λ with a product structure for which properties $(\mathbf{P_0})$-$(\mathbf{P_4})$ below hold will be called a *Gibbs-Markov-Young (GMY) structure*. From here on we assume that $C > 0$, $0 < \beta < 1$ and $0 < \zeta \leq 1$ are constants depending only on f and Λ.

($\mathbf{P_0}$) *Lebesgue detectable*: for every $\gamma \in \Gamma^u$, we have $\mathrm{Leb}_\gamma(\Lambda \cap \gamma) > 0$;

($\mathbf{P_1}$) *Markov partition and recurrence times*: there are finitely or countably many pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that

(a) for each $\gamma \in \Gamma^u$, $\mathrm{Leb}_\gamma\big((\Lambda \setminus \cup\Lambda_i) \cap \gamma\big) = 0$;

(b) for each $i \in \mathbb{N}$ there is integer $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is u-subset, and for all $x \in \Lambda_i$
$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$
We define the *recurrence time* function $R\colon \cup_i \Lambda_i \to \mathbb{N}$ as $R|_{\Lambda_i} = R_i$. We call $f^{R_i} : \Lambda_i \to \Lambda$ the *induced map.*

($\mathbf{P_2}$) *Uniform contraction on stable leaves*: for each $x \in \Lambda$, $y \in \gamma^s(x)$ and $n \geq 1$
$$\mathrm{dist}(f^n(y), f^n(x)) \leq C\beta^n.$$

($\mathbf{P_3}$) *Backward contraction and bounded distortion on unstable leaves*: for all $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$, and $0 \leq n < R_i$

(a) $$\mathrm{dist}(f^n(y), f^n(x)) \leq C\beta^{R_i - n}\,\mathrm{dist}(f^{R_i}(x), f^{R_i}(y));$$

(b) $$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \leq C\,\mathrm{dist}(f^{R_i}(x), f^{R_i}(y))^\zeta.$$

($\mathbf{P_4}$) *Regularity of the foliations*:

(a) *Convergence of $D(f^i|\gamma^u)$*: for all $y \in \gamma^s(x)$ and $n \geq 0$
$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C\beta^n;$$

(b) *Absolutely continuity of the stable foliation*: given $\gamma, \gamma' \in \Gamma^u$, define the holonomy map $\phi\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ as $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous with
$$\frac{d(\phi_* \mathrm{Leb}_\gamma)}{d\,\mathrm{Leb}_{\gamma'}}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

The notion of absolute continuity is precisely stated in Section 5.3. Under these conditions we say that $F = f^R : \Lambda \to \Lambda$ is an *induced GMY map.*

1.2. **Partially hyperbolic attractors.** Here we recall the definition of partially hyperbolic attractors with mostly expanding center-unstable direction and then we state our main theorem, Theorem A. This result extends the polynomial estimates in [7, Theorem A] to the (stretched) exponential case.

Let $f : M \to M$ be a C^{1+} diffeomorphism of a finite dimensional Riemannian manifold M. We say that f is C^{1+} if f is C^1 and Df is Hölder continuous. A set $K \subset M$ is said to be invariant if $f(K) = K$.

Definition 1.6. A compact invariant subset $K \subset M$ has a *dominated splitting*, if there exists a continuous Df-invariant splitting $T_K M = E^{cs} \oplus E^{cu}$ and $0 < \lambda < 1$ such that (for some choice of Riemannian metric on M)

$$\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for all } x \in K. \tag{1}$$

We call E^{cs} the *center-stable bundle* and E^{cu} the *center-unstable bundle*.

Definition 1.7. A compact invariant set $K \subset M$ is called *partially hyperbolic*, if it has a dominated splitting $T_K M = E^{cs} \oplus E^{cu}$ for which E^{cs} is *uniformly contracting* or E^{cu} is *uniformly expanding*, i.e. there is $0 < \lambda < 1$ such that (for some choice of a Riemannian metric on M)

$$\|Df \mid E^{cs}_x\| \leq \lambda \quad \text{or} \quad \|Df^{-1} \mid E^{cu}_{f(x)}\|^{-1} \leq \lambda, \quad \text{for all } x \in K.$$

In this work we consider partially hyperbolic sets of the same type of those considered in [2], for which the center-stable direction is uniformly contracting and the central-unstable direction is non-uniformly expanding. To emphasize that, we shall write E^s instead of E^{cs}.

Definition 1.8. Given $b > 0$, we say that f is *non-uniformly expanding* at a point $x \in K$ in the central-unstable direction, if

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -b. \tag{NUE}$$

If f satisfies (NUE) at some point $x \in K$, then the *expansion time* function at x

$$\mathcal{E}(x) = \min \left\{ N \geq 1 \colon \frac{1}{n} \sum_{i=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^i(x)}\| < -b, \quad \forall\, n \geq N \right\} \tag{2}$$

is defined and finite. We call $\{\mathcal{E} > n\}$ *the tail of hyperbolic times* (at time n).

We remark that if condition (NUE) holds for every point in a subset with positive Lebesgue measure of a forward invariant set $\tilde{K} \subset M$, then $K = \cap_{n\geq 0} f^n(\tilde{K})$ contains some local unstable disk D for which condition (NUE) is satisfied Leb_D almost everywhere; see [7, Theorem A].

Theorem A. *Let $f : M \to M$ be a C^{1+} diffeomorphism with $K \subset M$ an invariant transitive partially hyperbolic set. Assume that there are a local unstable disk $D \subset K$ and constants $0 < \tau \leq 1$, $c > 0$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau})$. Then there exists $\Lambda \subset K$ with a GMY structure. Moreover, there exists $d > 0$ such that $\mathrm{Leb}_\gamma\{R > n\} = \mathcal{O}(e^{-dn^\tau})$ for any $\gamma \in \Gamma^u$.*

The proof of this result will be given in Section 3. Under the assumptions of Theorem A, the set Λ coincides with Γ^u, but there are other possibilities, e.g. in [11] Λ is a Cantor set for the Hénon attractors.

In Section 6 we present an open class of diffeomorphisms for which $K = M$ is partially hyperbolic and satisfies the assumptions of Theorem A. The transitivity of the diffeomorphisms in that class was proved in [20].

1.3. **Statistical properties.** A good way of describing the dynamical behavior of chaotic dynamical systems is through invariant probability measures; in our setting, a special role is played by SRB measures.

Definition 1.9. An f-invariant probability measure μ on the Borel sets of M is called a *Sinai-Ruelle-Bowen (SRB) measure* if f has no zero Lypaunov exponents μ almost everywhere and the conditional measures of μ on local unstable manifolds are absolutely continuous with respect to the Lebesgue measure on these manifolds.

It is well known that SRB measures are *physical measures*: for a positive Lebesgue measure set of points $x \in M$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi\, d\mu, \quad \text{for any continuous } \varphi : M \to \mathbb{R}. \tag{3}$$

SRB measures for partially hyperbolic diffeomorphisms whose central direction is non-uniformly expanding were already obtained in [2]. Under the assumptions of Theorem A, we also get the existence of such measures by means of [21, Theorem 1].

Definition 1.10. Given observables $\varphi, \psi \colon M \to \mathbb{R}$, we define the *correlation function* with respect to a measure μ as

$$\mathcal{C}_\mu(\varphi, \psi\circ f^n) = \left| \int \varphi(\psi\circ f^n)\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu \right|, \quad n \geq 0.$$

Sometimes it is possible to obtain specific rates for which $\mathcal{C}_\mu(\varphi,\psi)$ decays to 0 as $n \to \infty$, at least for certain classes of observables with some regularity. See that if we take the observables as characteristic functions of Borel sets, we get the classical definition of *mixing*.

The next corollary follows from Theorem A together with [6, Theorem B]; see also [6, Remark 2.4]. Though in [6] the decay of correlations depends on some backward decay rates in the unstable direction, in our case we clearly have exponential backward contraction along that direction. So the next result is indeed an extension of [7, Corollary B] to the (stretched) exponential case.

Corollary B (Decay of Correlations). *Let $f : M \to M$ be a C^{1+} diffeomorphism with an invariant transitive partially hyperbolic set $K \subset M$. Assume that there are a local unstable disk $D \subset K$ and constants $0 < \tau \leq 1$, $c > 0$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau})$. Then some power f^k has an SRB measure μ and there is $d > 0$ such that $\mathcal{C}_\mu(\varphi, \psi\circ f^{kn}) = \mathcal{O}(e^{-dn^\tau})$ for Hölder continuous $\varphi\colon M \to \mathbb{R}$, and $\psi \in L^\infty(\mu)$.*

If the recurrence times associated to the elements of the GMY structure given by Theorem A are relatively prime, i.e. $\gcd\{R_i\} = 1$, then the same conclusion holds with respect to f, i.e. for $k = 1$. Using Theorem A and [16, Theorem 4.1], we also deduce a large deviations result for f.

Corollary C (Large Deviations). *Let $f : M \to M$ be a C^{1+} diffeomorphism with an invariant transitive partially hyperbolic set $K \subset M$. Assume that there are a local unstable disk $D \subset K$ and $c > 0$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn})$. Given any Hölder continuous*

$\varphi : M \to \mathbb{R}$, the limit

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{j=0}^{n-1} \varphi \circ f^j - n \int \varphi \, d\mu \right)^2 d\mu$$

exists. Moreover, if $\sigma^2 > 0$, then there is a rate function $c(\epsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu \left(\left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j - \int \varphi \, d\mu \right| > \epsilon \right) = -c(\epsilon).$$

We observe that in this last result we do not need to take any power of f; see the considerations in [16, Section 2.2]. It remains an interesting open question to know whether we have a similar result in the stretched exponential case; this depends only on a stretched exponential version of [16, Theorem 4.1]. Further statistical properties, as the Central Limit Theorem or an Almost Sure Invariant Principle, which have already been obtained in [7], could still be deduced from Theorem A.

2. Preliminary results

In this section we make a revision of some concepts and results from [2] that will be useful for the proof of Theorem A. In particular, we state a bounded distortion property at hyperbolic times for iterations of f over center-unstable disks with a Hölder control on the tangent direction.

First we give the definition of the center-unstable cone field. We consider continuous extensions of E^s and E^{cu} to some neighborhood U of K that we denote by $\tilde{E}^s$ and $\tilde{E}^{cu}$, respectively. These extensions are not necessarily invariant under Df.

Definition 2.1. Given $0 < a < 1$, the *center-unstable cone field $C_a^{cu} = (C_a^{cu}(x))_{x\in U}$ of width a* is defined by

$$C_a^{cu}(x) = \left\{ v_1 + v_2 \in \tilde{E}_x^s \oplus \tilde{E}_x^{cu} \text{ such that } \|v_1\| \le a\|v_2\| \right\};$$

the *stable cone field $C_a^s = (C_a^s(x))_{x\in U}$ of width a* is defined similarly,

$$C_a^s(x) = \left\{ v_1 + v_2 \in \tilde{E}_x^s \oplus \tilde{E}_x^{cu} \text{ such that } \|v_2\| \le a\|v_1\| \right\}.$$

Notice that the dominated splitting property still holds for the extensions $\tilde{E}^s$ and $\tilde{E}^{cu}$, provided U is taken sufficiently small. Up to slightly increasing $\lambda < 1$, we fix $a > 0$ and U small enough so that the domination condition (1) still holds for any point $x \in U \cap f^{-1}(U)$ and every $v^s \in C_a^s(x)$, $v^{cu} \in C_a^{cu}(f(x))$:

$$\|Df(x)v^s\| \cdot \|Df^{-1}(f(x))v^{cu}\| \le \lambda \|v^s\| \, \|v^{cu}\|.$$

The center-unstable cone field is forward invariant

$$Df(x)C_a^{cu}(x) \subset C_a^{cu}(f(x)), \quad \text{any } x \in K,$$

and this holds for any $x \in U \cap f^{-1}(U)$ by continuity.

We say that an embedded C^1 submanifold $N \subset U$ is *tangent to the center-unstable cone field* or a *center-unstable (cu) disk* if the tangent subspace to N at each point $x \in N$ is contained in the corresponding cone $C_a^{cu}(x)$.

2.1. **Hölder control of the tangent direction.** The goal of this subsection is to introduce $\kappa(N)$ in (5) for a submanifold $N \subset U$ and the constant C_1 in Proposition 2.2. This will be useful for the statement of Proposition 2.7.

The notion of Hölder variation of the tangent bundle for a centre-unstable manifold $N \subset U$ is introduced in local coordinates as follows. First we take $\delta_0 > 0$ small enough so that the inverse of the exponential map $\exp_x$ is defined on the δ_0 neighborhood of every point $x \in U$. Then we identify this neighborhood of x with the corresponding neighborhood V_x of the origin in T_xN, through the local chart defined by $\exp_x^{-1}$. Accordingly, we identify x with $0 \in T_xN$. Reducing δ_0, if necessary, we may suppose that $\tilde{E}^{cs}_x$ is contained in the center-stable cone $C^{cs}_a(y)$ of every $y \in V_x$. In particular, the intersection of $C^{cu}_a(y)$ with $\tilde{E}^{cs}_x$ reduces to the zero vector. Then, the tangent space to T_yN is parallel to the graph of a unique linear map $A_x(y) : T_xN \to \tilde{E}^{cs}_x$. Given constants $C > 0$ and $0 < \zeta \leq 1$, we say that *the tangent bundle to N is (C, ζ)-Hölder* if

$$\|A_x(y)\| \leq C d_x(y)^\zeta \quad \text{for every } y \in N \cap V_x \text{ and } x \in U,$$

where $d_x(y)$ denotes the distance from x to y measured along $N \cap V_x$, defined as the length of the shortest curve in $N \cap V_x$ joining x to y.

Recall that we have taken the neighborhood U and the cone width a sufficiently small so that the domination property remains valid for vectors in the cones $C^{cs}_a(z)$, $C^{cu}_a(z)$, and for any point $z \in U$. Hence, there are $\lambda_1 \in (\lambda, 1)$ and $\zeta \in (0, 1]$ such that

$$\|Df(z)v^{cs}\| \cdot \|Df^{-1}(f(z))v^{cu}\|^{1+\zeta} \leq \lambda_1 < 1 \tag{4}$$

for all unit vectors $v^{cs} \in C^{cs}_a(z)$ and $v^{cu} \in C^{cu}_a(z)$, with $z \in U$. Then, up to reducing $\delta_0 > 0$ and slightly increasing $\lambda_1 < 1$, inequality (4) still holds if we replace z by any $y \in V_x$, $x \in U$ (where $\| \cdot \|$ means the Riemannian metric in the corresponding local chart).

From here on we fix $\lambda_1 \in (\lambda, 1)$ and $\zeta \in (0, 1]$ as above. Given a C^1 submanifold $N \subset U$, we define

$$\kappa(N) = \inf\{C > 0 : \text{the tangent bundle of } N \text{ is } (C, \zeta)\text{-Hölder}\}. \tag{5}$$

The proof of the next result is given in [2, Corollary 2.4].

Proposition 2.2. *There exists $C_1 > 0$ such that for any C^1 submanifold $N \subset U$ tangent to the center-unstable cone field*

1. *there exists $n_0 \geq 1$ such that $\kappa(f^n(N)) \leq C_1$ for every $n \geq n_0$ such that $f^k(N) \subset U$ for all $0 \leq k \leq n$;*
2. *if $\kappa(N) \leq C_1$, then $\kappa(f^n(N)) \leq C_1$ for each $n \geq 1$ such that $f^k(N) \subset U$ for all $0 \leq k \leq n$;*
3. *if N and n are as in the previous item, then the functions*

$$J_k : f^k(N) \ni x \longmapsto \log |\det \left(Df \mid T_x f^k(N)\right)|, \quad 0 \leq k \leq n,$$

are (L_1, ζ)-Hölder continuous with $L_1 > 0$ depending only on C_1 and f.

2.2. **Hyperbolic times and bounded distortion.** We can derive uniform expansion and bounded distortion from NUE assumption in the center-unstable direction, with the definition below. Here we do not need the full strength of partially hyperbolic, we only consider the cu-direction has condition (NUE).

Definition 2.3. Given $0 < \sigma < 1$, we say that n is a σ-*hyperbolic time* for $x \in K$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^k, \qquad \text{for all } 1 \leq k \leq n.$$

For $n \geq 1$, we define

$$H_n(\sigma) = \{x \in K\colon\ n \text{ is a } \sigma\text{-hyperbolic time for } x\ \}.$$

Remark 2.4. Given $0 < \sigma < 1$ and $x \in H_n(\sigma)$, we obtain

$$\|Df^{-k} \mid E^{cu}_{f^n(x)}\| \leq \prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^k, \tag{6}$$

which means that $Df^{-k} \mid E^{cu}_{f^n(x)}$ is a contraction for $1 \leq k \leq n$.

The next result gives the existence of σ-hyperbolic times for almost all points in a center-unstable disk D as in Theorem A, and gives indeed the asymptotic positive frequency of σ-hyperbolic times for such points. Its proof can be found in [2, Lemma 3.1, Corollary 3.2].

Proposition 2.5. *There exist* $0 < \theta \leq 1$ *and* $0 < \sigma < 1$ *such that for every* $x \in D$ *with* $\mathcal{E}(x) \leq n$ *there exist* σ*-hyperbolic times* $1 \leq n_1 < \cdots < n_\ell \leq n$ *for* x *with* $\ell \geq \theta n$.

We remark that both θ and σ are uniform constants independent of the point x or the iterate n. In the sequel, we fix $0 < \sigma < 1$ as in the previous proposition and write simply H_n for $H_n(\sigma)$.

Remark 2.6. By continuity, we may choose $a > 0$ (recall the definition of the cone-fields) and $\delta_1 > 0$ sufficiently small (in particular, the δ_1-neighborhood of K must be contained in U) such that

$$\|Df^{-1}(f(y))v\| \leq \frac{1}{\sqrt{\sigma}}\|Df^{-1}|E^{cu}_{f(x)}\|\,\|v\|, \tag{7}$$

whenever $x \in K$, $\operatorname{dist}(y, x) \leq \delta_1$ and $v \in C^{cu}_a(y)$.

By the first item of Proposition 2.2 we may assume that the center-unstable disk $D \subset K$ in the statement of Theorem A satisfies $\kappa(D) \leq C_1$. The next result is then a consequence [2, Lemma 2.7 & Proposition 2.8].

Proposition 2.7. *There exists* $C_2 > 1$ *such that for any* $x \in D \cap H_n$ *at a positive distance from* ∂D*, for* n *sufficiently large there is a neighborhood* $V_n(x)$ *of* x *in* D *such that:*

(1) f^n *maps* $V_n(x)$ *diffeomorphically onto a center-unstable disk* $B^{cu}(f^n(x), \delta_1)$*;*
(2) *for every* $1 \leq k \leq n$ *and* $y, z \in V_n(x)$

$$\operatorname{dist}_{f^{n-k}(V_n(x))}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} \operatorname{dist}_{f^n(V_n(x))}(f^n(y), f^n(z));$$

(3) *for all* $y, z \in V_n(x)$

$$\log \frac{|\det Df^n \mid T_y\Delta|}{|\det Df^n \mid T_z\Delta|} \leq C_2 \operatorname{dist}_{f^n(D)}(f^n(y), f^n(z))^\zeta;$$

(4) *for any Borel sets* $Y, Z \subset V_n(x)$

$$\frac{1}{C_2}\frac{\mathrm{Leb}(Y)}{\mathrm{Leb}(Z)} \le \frac{\mathrm{Leb}(f^n(Y))}{\mathrm{Leb}(f^n(Z))} \le C_2\frac{\mathrm{Leb}(Y)}{\mathrm{Leb}(Z)}.$$

The sets $V_n(x)$ will be called *hyperbolic pre-balls*, and their images $B^{cu}(f^n(x), \delta_1)$ called *hyperbolic balls*. Item (3) gives the *bounded distortion* at hyperbolic times.

3. Partition on a reference disk

In this section we prove the existence of a set with product structure in K. We essentially describe the geometrical and dynamical nature. This process has three steps. Firstly we prove the existence of a center-unstable disk Δ_0 whose hyperbolic pre-disks contained in it return to a neighborhood of Δ_0 under forward iterations and the image projects along stable leaves covering Δ_0 completely. Secondly, we define a partition on Δ_0 whose construction is inspired essentially on [7, Section 3] and [3, Section 3 & 4]. That is, we improve the product structure construction performed in [3] for non-invertible NUE maps, extending it to the partially hyperbolic setting; see Subsections 3.2 and 5.1. Finally we show that the set with a product structure satisfies Definition 1.5.

3.1. **The reference disk.**

Let D be a local unstable disk as in Theorem A. Given $\delta_1 > 0$ as in Remark 2.6, we take $0 < \delta_s < \delta_1/2$ such that points in K have local stable manifolds of radius δ_s. In particular, these local stable leaves are contained in the neighborhood U of K; recall (7).

Definition 3.1. Given a disk $\Delta \subset D$, we define the *cylinder* over Δ

$$\mathcal{C}(\Delta) = \bigcup_{x\in\Delta} W^s_{\delta_s}(x)$$

and consider π the projection from $\mathcal{C}(\Delta)$ onto Δ along local stable leaves. We say that a center-unstable disk γ^u *u-crosses* $\mathcal{C}(\Delta)$ if $\pi(\gamma^u \cap \mathcal{C}(\Delta)) = \Delta$.

For technical reasons (see Lemma 3.10) we shall take the constant

$$\delta_1' = \frac{\delta_1}{12} > 0,$$

and consider $V_n'(x)$ the part of $V_n(x)$ which is sent by f^n onto $B^{cu}(f^n(x), \delta_1')$. These sets $V_n'(x)$ will also be called hyperbolic pre-balls. The next lemma is a consequence of [7, Lemma 3.1 & 3.2].

Lemma 3.2. *There are* $p \in D$ *and* $N_0 \ge 1$ *such that for all* $\delta_0 > 0$ *sufficiently small and each hyperbolic pre-disk* $V_n'(x) \subseteq D$ *there is* $0 \le m \le N_0$ *such that* $f^{n+m}(V_n'(x))$ *u-crosses* $\mathcal{C}(\Delta_0)$*, where* $\Delta_0 = B^{cu}(p, \delta_0)$ *is the subdisk in* D *of radius* δ_0 *centerd at* p*.*

Now we fix $p \in D$, $N_0 \ge 1$ and $\delta_0 > 0$ small enough such that the conclusions of Lemma 3.2 hold and define

$$\Delta_0 = \Delta_0^0 = B^{cu}(p, \delta_0) \quad \text{and} \quad \Delta_0^1 = B^{cu}(p, 2\delta_0).$$

We consider the corresponding cylinders

$$\mathcal{C}_0^i = \bigcup_{x\in\Delta_0^i} W^s_{\delta_s}(x), \qquad \text{for } i = 0, 1. \tag{8}$$

Denoting π the projection along stable leaves, we have

$$\pi(\mathcal{C}_0^i) = \Delta_0^i, \quad \text{for } i = 0, 1.$$

Remark 3.3. We assume that each disk γ^u u-crossing $\mathcal{C}_0^i$ $(i = 0, 1)$ is a disk centered at a point of $W^s_{\delta_s}(p)$ and with the same radius of Δ_0^i. We ignore the difference of radius caused by the height of the cylinder and the angles of the two dominated splitting bundles. Let the top and bottom components of $\partial\mathcal{C}_0^1$ be denoted by $\partial^u\mathcal{C}_0^1$, i.e. the set of points $z \in \partial\mathcal{C}_0^1$ such that $z \in \partial W^s_{\delta_s}(x)$ for some $x \in \Delta_0$. By the domination property, we may take $\delta_0 > 0$ small enough so that any center-unstable disk γ^u which is contained in $\mathcal{C}_0^1$ and intersecting $W^s_{\delta_s/2}(p)$ does not reach $\partial^u\mathcal{C}_0^1$.

Given a hyperbolic pre-ball $V_n'(x)$, there is $0 \le m \le N_0$ as in the conclusion of Lemma 3.2, and for each $i = 0, 1$ there is a center-unstable disk $\omega^{i,x}_{n,m} \subset V_n'(x)$ such that

$$\pi\left(f^{n+m}(\omega^{i,x}_{n,m})\right) = \Delta_0^i. \tag{9}$$

As condition (9) may in principle hold for several values of m, for definiteness we shall always assume that m takes the smallest possible value. Observe that the center-unstable disk $\omega^{i,x}_{n,m}$ is associated to x, by construction, but does not necessarily contain x.

The sets of the type $\omega^{0,x}_{n,m}$, with $x \in H_n \cap \Delta_0$, are the natural candidates to be in the partition $\mathcal{P}$. For $k \ge n$, set the *annulus* around $\omega^{0,x}_{n,m}$

$$A_k(\omega^{0,x}_{n,m}) = \left\{ y \in \omega^{1,x}_{n,m} : 0 < \mathrm{dist}_D\left((\pi \circ f^{n+m})(y), \Delta_0\right) \le \delta_0\sigma^{\frac{k-n}{2}} \right\}. \tag{10}$$

Obviously

$$A_n(\omega^{0,x}_{n,m}) \cup \omega^{0,x}_{n,m} = \omega^{1,x}_{n,m}.$$

In the sequel, we shall frequently omit the symbols m, 0, x or n in the notation and simply use ω_n^x, ω_n or even ω to denote an element $\omega^{0,x}_{n,m}$.

3.2. **The partition.** In this subsection we describe an algorithm to construct a countable (Leb_D mod 0) partition $\mathcal{P}$ of Δ_0. The algorithm is similar to the one in [3], but in the present context of a diffeomorphism, each element of the partition will return to another center-unstable disk which *u-crosses* $\mathcal{C}_0^0$. Along the process we shall introduce inductively sequences of objects $(\Delta_n)_n$, $(\Omega_n)_n$, $(A_n)_n$ and $(S_n)_n$. For each n, Δ_n is the set of points which does not belong to any element of the partition constructed up to time n, Ω_n is the union of elements of the partition constructed at step n and A_n is the union of rings around the chosen elements at time n. The set S_n (called the union of *satellites*) contains the components which could have been chosen for the partition but intersect already chosen elements. A key point in our argument is property (13) below, which says that every point having a hyperbolic time at a given time n will belong to either to an element of the partition or to some satellite. All these and some other auxiliary objects will be defined inductively in the remaining part of this subsection.

First step of induction. Fixing some large $n_0 \in \mathbb{N}$, we only consider the dynamics after time n_0. Notice that, by boundedness on the derivative, there is a minimum radius $r_{n_0} > 0$ such that each hyperbolic pre-disk $V'_{n_0}(x)$ with $x \in H_{n_0}$ contains a center-unstable disk of radius r_{n_0}. Hence, there is a finite set $I_{n_0} = \{z_1, \dots, z_{N_{n_0}}\} \in H_{n_0} \cap \Delta_0$ such that

$$H_{n_0} \cap \Delta_0 \subset V'_{n_0}(z_1) \cup \cdots \cup V'_{n_0}(z_{N_{n_0}}).$$

Consider a maximal family

$$\Omega_{n_0} = \left\{ \omega^{0,x_0}_{n_0,m_0}, \omega^{0,x_1}_{n_0,m_1}, \dots, \omega^{0,x_{k_{n_0}}}_{n_0,m_{k_{n_0}}} \right\}.$$

of pairwise disjoint sets of the type (9) contained in Δ_0 with $\{x_0, \cdots, x_{k_{n_0}}\} \subset I_{n_0}$, and let

$$\tilde{I}_{n_0} = I_{n_0} \setminus \{x_0, \cdots, x_{k_{n_0}}\}.$$

The sets in Ω_{n_0} are precisely the elements of the partition $\mathcal{P}$ constructed in the n_0-step of the algorithm (our first step of induction). We define the *recurrence time* $R(x) = n_0 + m_i$ for each $x \in \omega^{0,x_i}_{n_0,m_i}$ with $0 \le i \le k_{n_0}$. We need to keep track of the sets $\{\omega^{1,z}_{n_0,m} : z \in \tilde{I}_{n_0}, 0 \le m \le N_0\}$ which, for some $\omega \in \Omega_{n_0}$, overlap $\omega \cup A_{n_0}(\omega)$ or $\Delta_0^c = D \setminus \Delta_0$. Given $\omega \in \Omega_{n_0}$, we define for each $0 \le m \le N_0$

$$I^m_{n_0}(\omega) = \left\{ x \in \tilde{I}_{n_0} : \omega^{1,x}_{n_0,m} \cap (\omega \cup A_{n_0}(\omega)) \neq \emptyset \right\},$$

(recall (10)) and the n_0-*satellite* around ω

$$S_{n_0}(\omega) = \bigcup_{m=0}^{N_0} \bigcup_{x \in I^m_{n_0}(\omega)} V'_{n_0}(x) \cap (\Delta_0 \setminus \omega). \tag{11}$$

We define

$$S_{n_0}(\Omega_{n_0}) = \bigcup_{\omega \in \Omega_{n_0}} S_{n_0}(\omega)$$

and

$$S_{n_0}(\Delta_0) = S_{n_0}(\Omega_{n_0}).$$

We also define the n_0-satellite associated to $\Delta_0^c = D \setminus \Delta_0$

$$S_{n_0}(\Delta_0^c) = \bigcup_{m=0}^{N_0} \bigcup_{\substack{x \in \tilde{I}_{n_0} \\ \omega^{1,x}_{n_0,m} \cap \Delta_0^c \neq \emptyset}} V'_{n_0}(x) \cap \Delta_0, .$$

We will show in the general step that the Lebesgue measure of $S_{n_0}(\Delta_0^c)$ is exponentially small. The *global* n_0-*satellite* is

$$S_{n_0} = S_{n_0}(\Delta_0) \cup S_{n_0}(\Delta_0^c),$$

The remaining points at step n_0 are

$$\Delta_{n_0} = \Delta_0 \setminus \cup_{\omega \in \Omega_{n_0}} \omega.$$

We clearly have for this first step of induction the key property

$$H_{n_0} \cap \Delta_0 \,\subset\, S_{n_0} \cup \bigcup_{\omega \in \Omega_{n_0}} \omega.$$

General step of induction. The general step of the construction follows the ideas of the first step with minor modifications. Given $n > n_0$, assume that the sets Ω_ℓ, Δ_ℓ, A_ℓ and S_ℓ are defined for each $n_0 \le \ell \le n-1$. As in the first step, there is a finite set of points $I_n = \{z_1, \ldots, z_{N_n}\} \in H_n \cap \Delta_{n-1}$ such that

$$H_n \cap \Delta_{n-1} \subset V_n'(z_1) \cup \cdots \cup V_n'(z_{N_n}).$$

Now we consider a maximal family

$$\Omega_n = \{\omega_{n,m_0}^{0,x_0}, \omega_{n,m_1}^{0,x_1}, \ldots, \omega_{n,m_{k_n}}^{0,x_{k_n}}\}$$

of pairwise disjoint sets of type (9) with $\{x_0, \cdots, x_{k_n}\} \subset I_n$ contained in Δ_{n-1} and satisfying

$$\omega_{n,m}^{1,x_i} \cap \left(\cup_{\ell=n_0}^{n-1} \cup_{\omega\in\Omega_\ell} (\omega \cup A_n(\omega))\right) = \emptyset, \qquad \text{for } i = 1, \ldots, k_n.$$

The sets in Ω_n are the elements of the partition $\mathcal{P}$ constructed in the n-step of the algorithm. We set the *recurrence time* $R(x) = n + m_i$ for each $x \in \omega_{n,m_i}^{0,x_i}$ with $0 \le i \le \ell_n$. Let

$$\tilde{I}_n = I_n \setminus \{x_0, \cdots, x_{k_n}\}.$$

Given $\omega \in \Omega_{n_0} \cup \cdots \cup \Omega_n$ and $0 \le m \le N_0$, set

$$I_n^m(\omega) = \left\{x \in \tilde{I}_n : \omega_{n,m}^{1,x} \cap (\omega \cup A_n(\omega)) \neq \emptyset\right\}.$$

(recall (10)) and the *n-satellite* around ω

$$S_n(\omega) = \bigcup_{m=0}^{N_0} \bigcup_{x\in I_n^m(\omega)} V_n'(x) \cap (\Delta_0 \setminus \omega). \tag{12}$$

We define for $n_0 \le i \le n$

$$S_n(\Omega_i) = \bigcup_{\omega\in\Omega_i} S_n(\omega)$$

and

$$S_n(\Delta_0) = \bigcup_{i=n_0}^{n} S_n(\Omega_i).$$

Similarly, the *n-satellite* associated to Δ_0^c is

$$S_n(\Delta_0^c) = \bigcup_{m=0}^{N_0} \bigcup_{\substack{x\in\tilde{I}_n \\ \omega_{n,m}^{1,x}\cap\Delta_0^c\neq\emptyset}} V_n'(x) \cap \Delta_0.$$

Remark 3.4. Observe that the volume of $S_n(\Delta_0^c)$ decays exponentially fast. In fact, it follows from the definition of $S_n(\Delta_0^c)$ and Proposition 2.7 that

$$S_n(\Delta_0^c) \subset \{x \in \Delta_0 : \operatorname{dist}_D(x, \partial\Delta_0) \le 2\delta_0\sigma^{n/2}\}.$$

Thus, there exists $\rho > 0$ such that $\operatorname{Leb}_D(S_n(\Delta_0^c)) \le \rho\sigma^{n/2}$.

Finally we define the *global n-satellite*

$$S_n = S_n(\Delta_0) \cup S_n(\Delta_0^c),$$

and

$$\Delta_n = \Delta_0 \setminus \bigcup_{i=n_0}^{n} \bigcup_{\omega\in\Omega_i} \omega.$$

We clearly have by construction

$$H_n \cap \Delta_0 \subset S_n \cup \bigcup_{i=n_0}^{n} \bigcup_{\omega\in\Omega_i} \omega. \tag{13}$$

3.3. **Estimates on the satellites.** For the sake of notational simplicity, we shall avoid the superscript 0 in the sets $\omega_{n,m}^{0,x}$. The next lemma shows that, given n and m, the conditional volume of the union of $\omega_{n,m}^{x}$ which is not far from one chosen element is proportional to the conditional volume of this element. The proportion constant is uniformly summable with respect to n.

Though we consider here the case of partially hyperbolic attractor and the construction has naturally obvious modifications, the proofs of the two items in the next lemma are essentially the same of [3, Lemmas 4.4 & 4.5].

Lemma 3.5. (1) *There exists $C_3 > 0$ such that, for any $n \geq n_0$, $0 \leq m \leq N_0$, and finitely many points $\{x_1, \ldots, x_N\} \in I_n$ satisfying $\omega_{n,m}^{x_i} = \omega_{n,m}^{x_1}$ ($1 \leq i \leq N$), we have*

$$\mathrm{Leb}_D\left(\bigcup_{i=1}^{N} V_n'(x_i)\right) \leq C_3 \,\mathrm{Leb}_D(\omega_{n,m}^{x_1}).$$

(2) *There exists $C_4 > 0$ such that for $k \geq n_0$, $\omega \in \Omega_k$ and $0 \leq m \leq N_0$, given any $n \geq k$, we obtain*

$$\mathrm{Leb}_D\left(\bigcup_{x\in I_n^m(\omega)} \omega_{n,m}^{x}\right) \leq C_4 \sigma^{\frac{n-k}{2}} \,\mathrm{Leb}_D(\omega).$$

Proposition 3.6. *There exists $C_5 > 0$ such that for any $\omega \in \Omega_k$ and $n \geq k$, we have*

$$\mathrm{Leb}_D(S_n(\omega)) \leq C_5 \sigma^{\frac{n-k}{2}} \,\mathrm{Leb}_D(\omega).$$

Proof. Consider now $k \geq n_0$ and $n \geq k$. Fix $\omega \in \Omega_k$ and consider $S_n(\omega)$ the n-satellite associated to it. By definition of $S_n(\omega)$ and the first item of Lemma 3.5 we have

$$\begin{aligned}
\mathrm{Leb}_D(S_n(\omega)) &\leq \sum_{m=0}^{N_0} \sum_{x\in I_n^m(\omega)} \mathrm{Leb}_D\left(V_n'(x) \cap (\Delta_0 \setminus \omega)\right) + \mathrm{Leb}_D(V_k'(\omega) \setminus \omega) \\
&\leq C_3 \sum_{m=0}^{N_0} \mathrm{Leb}_D\left(\bigcup_{x\in I_n^m(\omega)} \omega_{n,m}^{x}\right) + C_3 \,\mathrm{Leb}_D(\omega).
\end{aligned}$$

In this last step we have used the obvious fact that for fixed n, m the sets of the form $\omega^x_{n,m}$ with $x \in I^m_n(\omega)$ are pairwise disjoint. Thus, by the second item of Lemma 3.5,

$$\mathrm{Leb}_D(S_n(\omega)) \le C_3(C_4(N_0+1)+1)\sigma^{\frac{n-k}{2}} \mathrm{Leb}_D(\omega).$$

Take $C_5 = C_3(C_4(N_0+1)+1)$. □

Definition 3.7. Given $k \ge n_0$ and $\omega^x_{k,m} \in \Omega_k$ we define for $n \ge k$

$$B^k_n(x) = S_n(\omega^x_{k,m}) \cup \omega^x_{k,m} \quad \text{and} \quad t(B^k_n(x)) = k.$$

The set $\omega^x_{k,m}$ will be called the *core* of $B^k_n(x)$ and denoted $C(B^k_n(x))$.

Notice that given a set $B^k_n(x)$ as above, k is a hyperbolic time for x and n is a hyperbolic time for some point in the n-satellite involved in the definition of $B^k_n(x)$; recall the definition of the satellites.

Remark 3.8. From the construction of the partition $\mathcal{P}$ performed in Section 3.2, we easily deduce that, given any two cores $C(B^{k_1}_{n_1}(x_1))$ and $C(B^{k_2}_{n_2}(x_2))$, we have either $x_1 = x_2$ and $C(B^{k_1}_{n_1}(x_1)) = C(B^{k_2}_{n_2}(x_2))$, or $C(B^{k_1}_{n_1}(x_1)) \cap C(B^{k_2}_{n_2}(x_2)) = \emptyset$.

In the sequel we prove the main features of these sets $B^k_n(x)$. The next result follows immediately from Proposition 3.6.

Corollary 3.9. *For all $n \ge k$ we have*

$$\mathrm{Leb}_D(B^k_n(x)) \le (C_5+1)\,\mathrm{Leb}_D(C(B^k_n(x))).$$

The dependence of δ'_1 on δ_1 becomes clear in the proof of the next lemma.

Lemma 3.10. *If $\delta'_1 > 0$ is sufficiently small (only depending on δ_1), then for all $k' \ge k \ge n_0$, $n \ge k$, $n' \ge k'$ and $B^k_n(x) \cap B^{k'}_{n'}(y) \ne \emptyset$ we have*

$$C(B^k_n(x)) \cup C(B^{k'}_{n'}(y)) \subset V_k(x).$$

Proof. First of all observe that k is a hyperbolic time for x and $f^k(C(B^k_n(x)))$ is contained in a cu-disk of radius δ'_1 centered at $f^k(x)$. On the other hand, by definition each point in $B^k_n(x)$ which does not belong to $C(B^k_n(x))$ must necessarily be in some hyperbolic pre-disk (with hyperbolic time n) intersecting $C(B^k_n(x))$. Then, the second assertion of Proposition 2.7 yields

$$\mathrm{diam}_{f^k(D)}\left(f^k(B^k_n(x))\right) \le 2\delta'_1 + 4\delta'_1\sigma^{\frac{n-k}{2}} \le 6\delta'_1.$$

Similarly,

$$\mathrm{diam}_{f^{k'}(D)}\left(f^{k'}(B^{k'}_{n'}(y))\right) \le 6\delta'_1.$$

Using this and the second assertion of Proposition 2.7, we also have

$$\mathrm{diam}_{f^k(D)}\left(f^k(B^{k'}_{n'}(y))\right) \le 6\delta'_1\sigma^{\frac{k'-k}{2}} \le 6\delta'_1.$$

Hence, as we are taking $\delta'_1 = \delta_1/12$, we have $f^k(B^k_n(x)) \cup f^k(B^{k'}_{n'}(y))$ contained in the center-unstable disk of radius δ_1 centered at $f^k(x)$. This clearly gives the result. □

Notice that the proof of the previous lemma gives in fact $B_n^k(x) \cup B_{n'}^{k'}(y) \subset V_k(x)$, but the conclusion with the cores is enough for our purposes. The sets A_n play a key role in the proof of the next result and have been introduced exclusively to make this proof work.

Lemma 3.11. *There exists $P \geq N_0$ such that for all $t_2 > t_1 \geq n_0$, $x \in H_{t_1}$ and $y \in H_{t_2} \setminus B_{t_2}^{t_1}(x)$ we have*

$$B_{t_2+P}^{t_1}(x) \cap B_{t_2+P}^{t_2}(y) = \emptyset.$$

Proof. Notice that by definition we have t_1 a hyperbolic time for x and t_2 a hyperbolic time for y. Suppose, by contradiction, that we have $B_{t_2+P}^{t_2}(y) \cap B_{t_2+P}^{t_1}(x) \neq \emptyset$ for all $P \geq N_0$. Take a point $z \in B_{t_2+P}^{t_2}(y) \cap B_{t_2+P}^{t_1}(x)$. Observing that t_2+P is a hyperbolic time associated to hyperbolic pre-disks intersecting $C(B_{t_2+P}^{t_1}(x))$ and $C(B_{t_2+P}^{t_2}(y))$, by the second assertion of Proposition 2.7 we obtain for $R_1 = R(C(B_{t_2+P}^{t_1}(x)))$

$$\operatorname{dist}_{f^{R_1}(D)}\left(f^{R_1}(z), f^{R_1}(C(B_{t_2+P}^{t_1}(x)))\right) \leq 2\delta_1' \sigma^{\frac{t_2+P-R_1}{2}};$$

and also

$$\operatorname{dist}_{f^{R_1}(D)}\left(f^{R_1}(z), f^{R_1}(C(B_{t_2+P}^{t_2}(y)))\right) \leq 2\delta_1' \sigma^{\frac{t_2+P-R_1}{2}}.$$

Hence,

$$\operatorname{dist}_{f^{R_1}(D)}\left(f^{R_1}(C(B_{t_2+P}^{t_1}(x))), f^{R_1}(C(B_{t_2+P}^{t_2}(y)))\right) \leq 4\delta_1' \sigma^{\frac{t_2+P-R_1}{2}}.$$

Recall that, by definition, $R_1 = t_1 + m_1$ for some $0 \leq m_1 \leq N_0$, and so

$$\frac{t_2+P-R_1}{2} \geq \frac{t_2-t_1+P-N_0}{2}.$$

Thus, taking P large enough such that $4\delta_1'\sigma^{P/2} < \delta_0 \sigma^{N_0/2}$, we have

$$\operatorname{dist}_{f^{R_1}(D)}\left(f^{R_1}(C(B_{t_2+P}^{t_1}(x))), f^{R_1}(C(B_{t_2+P}^{t_2}(y)))\right) \leq \delta_0 \sigma^{\frac{t_2-t_1}{2}}$$

which means $C(B_{t_2+P}^{t_2}(y)) \subset A_{t_2}(C(B_{t_1+P}^{t_1}(x)))$. This gives a contradiction. □

4. Tail of recurrence times

Though our construction of the objects in the previous section is significantly different from [14], our approach on the estimates below is inspired in [14, Section 3.2]. Our goal in this section is to prove that if the Lebesgue measure of $\{\mathcal{E} > n\}$ decays (stretched) exponentially fast, then the tail of the recurrence times also decays (stretched) exponentially fast. More precisely, we shall see that given a local unstable disk $D \subset K$ and constants $c > 0$ and $0 < \tau \leq 1$, there is $d > 0$ such that

$$\operatorname{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau}) \quad \Rightarrow \quad \operatorname{Leb}_D\{R > n\} \leq \mathcal{O}(e^{-dn^\tau}). \tag{14}$$

First of all observe that if there exists $d, \tau > 0$ such that

$$\operatorname{Leb}_D(\Delta_n) \leq \mathcal{O}(e^{-dn^\tau}), \tag{15}$$

then we have $\{R > n\} \subset \Delta_{n-N_0}$, and so

$$\operatorname{Leb}_D\{R > n\} \leq \operatorname{Leb}_D(\Delta_{n-N_0}) = \mathcal{O}(e^{-d(n-N_0)^\tau}) = \mathcal{O}(e^{-dn^\tau}).$$

Hence, for proving (14), it is enough to see that (15) holds.

By Remark 3.4 there exists a constant $\rho > 0$ such that for all $n \in \mathbb{N}$

$$\mathrm{Leb}_D\{x \in D \mid \mathrm{dist}_D(x, \partial\Delta_0) \le 2\delta_0\sigma^{\theta n/4}\} \le \rho\sigma^{\theta n/4}, \tag{16}$$

where θ is given in Proposition 2.5. Take $x \in \Delta_n$ and assume it belongs neither to $\{\mathcal{E} > n\}$ nor to $\{x \mid \mathrm{dist}_D(x, \partial\Delta_0) \le 2\delta_0\sigma^{\theta n/4}\}$. Since $n \ge \mathcal{E}(x)$, by Proposition 2.5 the point x has at least $[\theta n]$ hyperbolic times between 1 and n, and so it has at least $[\theta n/2]$ hyperbolic times between $\theta n/2$ and n. Ordering them as $\theta n/2 \le t_1 < \cdots < t_k \le n$, then $x \in H_{t_i} \cap \Delta_0$ for $1 \le i \le k$. From (13) we know that

$$H_{t_i} \cap \Delta_0 \subset S_{t_i} \cup \bigcup_{j=n_0}^{t_i} \bigcup_{\omega \in \Omega_j} \omega, \quad \text{for } 1 \le i \le k.$$

If $x \notin S_{t_i}$, then $x \in \cup_{j=n_0}^{t_i} \Omega_j$ and this means that $x \notin \Delta_n$, which gives a contradiction. Hence $x \in S_{t_i}$. As $x \in \{x \in \Delta_0 \mid \mathrm{dist}_D(x, \partial\Delta_0) > 2\delta_0\sigma^{\theta n/4}\}$, we have

$$x \in H_{t_i} \cap \{x \in \Delta_0 \mid \mathrm{dist}_D(x, \partial\Delta_0) > 2\delta_0\sigma^{t_i/2}\}, \quad \text{for } 1 \le i \le k.$$

Recalling Remark 3.4, we obtain $x \notin S_{t_i}(\Delta_0^c)$, and so

$$x \in S_{t_i}(\Delta_0), \quad \text{for } i = 1, \dots, k.$$

Thus, we have seen that x belongs to the set $X([\theta n/2], n)$, where we define

$$X(k, N) = \left\{ x \mid \exists t_1 < \ldots < t_k \le n \text{ such that } x \in \bigcap_{i=1}^{k} S_{t_i}(\Delta_0) \right\}$$

for integers k and N. Hence

$$\Delta_n \subset \{x \in \Delta_0 \mid \mathcal{E}(x) > n\} \cup \left\{ x \in \Delta_0 \mid \mathrm{dist}_D(x, \partial\Delta_0) \le 2\delta_0\sigma^{\frac{\theta n}{4}} \right\} \cup X([\theta n/2], n).$$

Since the middle set in the union above has exponentially small measure by (16), it remains to see that the measure of $X([\theta n/2], n)$ decays exponentially fast in n. This follows from the next result, whose proof will be given in the remaining of this section.

Proposition 4.1. *There exist $D_0 > 0$ and $0 < \lambda_0 < 1$ such that for all k and N*

$$\mathrm{Leb}_D(X(k, N)) \le D_0\lambda_0^k \,\mathrm{Leb}_D(\Delta_0).$$

We start by fixing some integer $P' \ge P$ (recall Lemma 3.11) whose value will be made precise later. Given $x \in X(k, N)$, consider all the instants $u_1, \dots, u_p$ for which x belongs to some $S_{u_i+n_i}(\omega_{u_i}^{x_i})$ with $n_i \ge P'$, ordered so that $u_1 < \ldots < u_p$. Defining for $B_1 = B_{u_1}^{u_1}(x_1)$

$$Y(n_1, \dots, n_p, B_1) = \left\{ x \mid \exists\, t_1 < \ldots < t_p \text{ and } x_2, \dots, x_p \text{ s.t. } x \in \bigcap_{i=1}^{p} S_{t_i+n_i}(\omega_{t_i}^{x_i}) \right\}, \tag{17}$$

we then have $x \in Y(n_1, \dots, n_p, B_1)$. Assume that $\sum_{i=1}^{p} n_i < k/2$. As we are considering $n_1, \dots, n_k \ge P'$, we must have $p < k/(2P')$. Let $v_1 < \ldots < v_q$ be the other instants for which $x \in S_{v_i+m_i}(\omega_{v_i}^{z_i})$, for times $m_1, \dots, m_q < P'$. As $p + q = k$, for $P' > 1$ we have

$$q \ge \frac{(2P'-1)k}{2P'} \ge \frac{k}{2P'} \ge \left[\frac{k}{2P'}\right].$$

Thus, considering

$$Z(q,N)=\left\{x\in\Delta_N \mid \exists t_1<\ldots<t_q\le N \text{ and } m_1,\ldots,m_q<P' \text{ s.t. } x\in\bigcap_{i=1}^{q}S_{t_i+m_i}(\Omega_{t_i})\right\},$$

we have shown that assuming $\sum_{i=1}^{p} n_i < k/2$ we necessarily have $x \in Z\left([k/(2P')],N\right)$. Hence,

$$X(k,N)\ \subset \bigcup_{B_1}\ \bigcup_{\substack{n_1,\ldots,n_p\ge P',\\ \sum n_i\ge \frac{k}{2}}} Y(n_1,\ldots,n_p,B_1)\cup Z\left(\left[\frac{k}{2P'}\right],N\right), \tag{18}$$

where the first union is taken over all possible sets $B_1=B^{t_1}_{n_1}(x_1)$.

Our goal now is to obtain estimates on the Leb_D measure of the sets involved in (18). We start with $Z(k,N)$. For the sake of notational simplicity, in the sequel we shall denote for $i\ge 0$

$$B_i=B^{t_i}_{t_i+m_i}(x_i)\quad\text{and}\quad B'_i=B^{t'_i}_{t'_i+m'_i}(x'_i).$$

We introduce below auxiliary sets $Z_1(k,B_0)$ and $Z_2(k,N)$ which will be useful to estimate the Leb_D measure of $Z(k,N)$.

Given $E\in\mathbb{N}$, define for a positive integer k

$$Z_1(k,B_0)=\Bigg\{x\mid \exists B'_1,B_1,\ldots,B'_r,B_r \text{ so that } B_i\not\subseteq B'_i \text{ and } t_{i-1}\le t'_i\le t_i-E,\ \forall 1\le i\le r,$$
$$\sum_{i=1}^{r}\left[\frac{t_i-t'_i}{E}\right]\ge k \text{ and } x\in\bigcap_{i=0}^{r}B_i\cap\bigcap_{i=1}^{r}B'_i\Bigg\}.$$

Lemma 4.2. *There is $D_1>0$ (independent of E) such that for all k and B_0*

$$\mathrm{Leb}_D(Z_1(k,B_0))\le D_1(D_1\sigma^{E/2})^k\,\mathrm{Leb}_D(C(B_0)).$$

Proof. We shall prove the result by induction on $k\ge 0$. For $k=0$, we have by Corollary 3.9

$$\mathrm{Leb}_D(Z_1(0,B_0))\le \mathrm{Leb}_D(B_0)\le (C_5+1)\,\mathrm{Leb}_D(C(B_0)),$$

In this case it is enough to take

$$D_1\ge C_5+1 \tag{19}$$

For $k\ge 1$, we have

$$Z_1(k,B_0)\subset\bigcup_{t=1}^{k}\ \bigcup_{B'_1\cap B_0\neq\emptyset}\ \bigcup_{\substack{B'_1\cap B_1\neq\emptyset,\,B_1\not\subseteq B'_1,\\ \left[\frac{t_1-t'_1}{E}\right]\ge t}} Z_1(k-t,B_1).$$

Let $n=t_1-t'_1$. Fix some B'_1, and take one from all the possible B_1's intersecting B'_1. Setting $p=t'_1$ and $Q'_1=f^p(B'_1)$, we have

$$f^p(B_1)\subset\mathcal{C}:=\left\{y\mid \mathrm{dist}_{f^p(D)}(y,\partial Q'_1)\le 6\delta'_1\sigma^{n/2}\right\}. \tag{20}$$

Indeed, as B_1 contains a point of $\partial B'_1$, then $f^p(B_1)$ contains a point of $\partial Q'_1$. We obtain

$$\mathrm{diam}_{f^p(D)}\,f^p(B_1)\le\sigma^{n/2}\mathrm{diam}_{f^{p+n}(D)}\,f^{p+n}(B_1)\le 6\delta'_1\sigma^{n/2},$$

and so we get (20). Similarly to (16), we have for some uniform constant $\rho_1 > 0$

$$\mathrm{Leb}_{f^p(D)}(\mathcal{C}) \le \rho_1 \sigma^{n/2}\,\mathrm{Leb}_{f^p(D)}(Q_1'). \tag{21}$$

By Remark 3.8 the cores $C(B_1)$ of all those possible B_1's are pairwise disjoint. Moreover, by Lemma 3.10 these cores $C(B_1)$ must be all contained in $V_p(x_1')$, where x_1' is the point such that $C(B_1') = \omega_{t_1'}^{x_1'}$. Then, using (20), (21), Corollary 3.9 and the bounded distortion we obtain

$$\sum_{\substack{B_1'\cap B_1\neq\emptyset,\, B_1\not\subseteq B_1',\\ \left[\frac{t_1-t_1'}{E}\right]\ge t}} \mathrm{Leb}_D(C(B_1)) \le C_2(C_5+1)\rho_1\sigma^{Et/2}\,\mathrm{Leb}_D(C(B_1')). \tag{22}$$

Let now $q = t_0$ and $C(B_0) = \omega_{q,m}^x$. Once more by Remark 3.8 and Lemma 3.10, the possible sets $C(B_1')$'s are pairwise disjoint and are all included in $V_q(x)$. Moreover, as $f^q(V_q(x)) = B^{cu}(f^q(x),\delta_1)$ and $f^{q+m}(C(B_0))$ is a cu-disk of radius $\delta_0 > 0$, there is some uniform constant $\rho_2 > 0$ such that

$$\mathrm{Leb}_{f^q(D)}(B^{cu}(f^q(x),\delta_1)) \le \rho_2\,\mathrm{Leb}_{f^q(D)}(f^q(C(B_0)));$$

recall that $0 \le m \le N_0$. Using bounded distortion we get

$$\sum_{B_1'\cap B_0\neq\emptyset} \mathrm{Leb}_D(C(B_1')) \le \mathrm{Leb}_D(V_q(x)) \le \rho_2 C_2\,\mathrm{Leb}_D(C(B_0)). \tag{23}$$

Finally, using (22), (23) and the inductive hypothesis, we deduce

$$\begin{aligned}
\mathrm{Leb}_D(Z_1(k,B_0)) &\le \sum_{t=1}^{k}\sum_{B_1'\cap B_0\neq\emptyset}\sum_{\substack{B_1'\cap B_1\neq\emptyset,\, B_1\not\subseteq B_1',\\ \left[\frac{t_1-t_1'}{E}\right]\ge t}} \mathrm{Leb}_D(Z_1(k-t,B_1))\\
&\le \sum_{t=1}^{k}\sum_{B_1'\cap B_0\neq\emptyset}\sum_{\substack{B_1'\cap B_1\neq\emptyset,\, B_1\not\subseteq B_1',\\ \left[\frac{t_1-t_1'}{E}\right]\ge t}} D_1(D_1\sigma^{E/2})^{k-t}\,\mathrm{Leb}_D(C(B_1))\\
&\le \sum_{t=1}^{k}\sum_{B_1'\cap B_0\neq\emptyset} D_1(D_1\sigma^{E/2})^{k-t} C_2(C_5+1)\rho_1\sigma^{Et/2}\,\mathrm{Leb}_D(C(B_1'))\\
&\le D_1(D_1\sigma^{E/2})^{k} C_2^2(C_5+1)\rho_1\rho_2 \sum_{t=1}^{k} D_1^{-t}\,\mathrm{Leb}_D(C(B_0)).
\end{aligned}$$

Thus, taking $D_1 > 0$ large enough so that

$$\frac{C_2^2(C_5+1)\rho_1\rho_2 D_1^{-1}}{1-D_1^{-1}} \le 1$$

we finish the proof. ∎

Now set for positive integers k and N

$$Z_2(k,N) = \left\{ x \in \Delta_N \mid \exists B_1 \supsetneq B_2 \ldots \supsetneq B_k \text{ with } t_1 < \cdots < t_k \le N \text{ and } x \in \bigcap_{i=1}^{k} B_i \right\}. \quad (24)$$

Lemma 4.3. *There exists $\lambda_2 < 1$ such that for all $N \ge 1$ and $1 \le k \le N$*

$$\mathrm{Leb}_D(Z_2(k,N)) \le \lambda_2^k \,\mathrm{Leb}_D(\Delta_0).$$

Proof. We assume N is fixed in this proof and simply write $Z_2(k) = Z_2(k,N)$. We shall prove that the conclusion of the lemma holds with $\lambda_2 = \frac{D_1}{D_1+1}$. Using that $C_5 + 1 \le D_1$, by Corollary 3.9 we have for each possible B

$$\mathrm{Leb}_D(B) \le D_1 \,\mathrm{Leb}_D(C(B)). \quad (25)$$

We define $\mathcal{Q}_1$ as the class of sets B with $t(B) \le N$ and not contained in any other B's. Consider $\mathcal{Q}_2$ as the class of sets $B \notin \mathcal{Q}_1$ with $t(B) \le N$ which are included in elements of $\mathcal{Q}_1$ and not contained in any other B's. We proceed inductively. Notice that this process must stop in a finite number of steps because we always take $t(B) \le N$. We say that an element in $\mathcal{Q}_i$ has *rank* i.

Let now

$$G_k = \bigcup_{i=1}^{k} \bigcup_{B \in \mathcal{Q}_k} C(B),$$

and

$$\tilde{Z}_2(k) = \left(\bigcup_{B \in \mathcal{Q}_k} B \right) \setminus G_k.$$

Now we prove that $Z_2(k) \subset \tilde{Z}_2(k)$. Given $x \in Z_2(k)$, we have $x \in B_1 \cap \ldots \cap B_k \cap \Delta_N$ with $B_1 \supsetneq B_2 \ldots \supsetneq B_k$ and $t(B_k) \le N$. We clearly have that B_k is of rank $r \ge k$. Take $B'_1 \supsetneq B'_2 \ldots \supsetneq B'_{r-1} \supsetneq B'_r$ a sequence with $B'_i \in \mathcal{Q}_i$ and $B'_r = B_k$. In particular, $x \in B'_i$ for $i = 1, \ldots, k$, and so $x \in \bigcup_{B \in \mathcal{Q}_k} B$. On the other hand, since $x \in \Delta_N$ and $G_k \cap \Delta_N = \emptyset$, we get $x \notin G_k$. So $x \in \tilde{Z}_2(k)$.

Now we deduce the relation between $\mathrm{Leb}_D(\tilde{Z}_2(k+1))$ and $\mathrm{Leb}_D(\tilde{Z}_2(k))$, in such a way that we may estimate $\mathrm{Leb}_D(\tilde{Z}_2(k))$. Take $B \in \mathcal{Q}_{k+1}$. Let B' be an element of rank k containing B. As the cores are pairwise disjoint by nature, $C(B) \cap G_k = \emptyset$. We obtain $C(B) \subset B' \setminus G_k \subset \tilde{Z}_2(k)$. By definition $C(B) \subset G_{k+1}$, thus $C(B) \cap \tilde{Z}_2(k+1) = \emptyset$. This means that $C(B) \subset \tilde{Z}_2(k) \setminus \tilde{Z}_2(k+1)$. Finally, by (25),

$$\begin{aligned}
\mathrm{Leb}_D(\tilde{Z}_2(k+1)) &\le \sum_{B \in \mathcal{Q}_{k+1}} \mathrm{Leb}_D(B) \\
&\le D_1 \sum_{B \in \mathcal{Q}_{k+1}} \mathrm{Leb}_D(C(B)) \\
&\le D_1 \,\mathrm{Leb}_D(\tilde{Z}_2(k) \setminus \tilde{Z}_2(k+1))
\end{aligned}$$

since the $C(B)$ are pairwise disjoint; recall Remark 3.8. Then, we obtain

$$(D_1+1)\,\mathrm{Leb}_D(\tilde{Z}_2(k+1)) \le D_1 \,\mathrm{Leb}_D(\tilde{Z}_2(k+1)) + D_1 \,\mathrm{Leb}_D(\tilde{Z}_2(k) \setminus \tilde{Z}_2(k+1))$$

$$= D_1 \operatorname{Leb}_D(\tilde{Z}_2(k)).$$

It yields $\operatorname{Leb}_D(\tilde{Z}_2(k)) \leq \left(\frac{D_1}{D_1+1}\right)^k \operatorname{Leb}_D(\Delta_0)$ by induction. Since $Z_2(k) \subset \tilde{Z}_2(k)$, the same inequality holds for $Z_2(k)$. □

The previous two lemmas are enough for us to establish the desired estimate on the Leb_D measure of $Z(k,N)$ in the next lemma. The idea is to divide N into blocks of length E and choosing properly certain instants from each block. Depending on the number of such instants, we shall apply either Lemma 4.2 or Lemma 4.3 to yield the conclusion.

Lemma 4.4. *There are $D_3 > 0$ and $0 < \lambda_3 < 1$ such that for all $1 \leq k \leq N$,*

$$\operatorname{Leb}_D(Z(k,N)) \leq D_3 \lambda_3^k \operatorname{Leb}_D(\Delta_0).$$

Proof. Choose E large enough such that $D_1 \sigma^{E/2} < 1$; recall D_1 in Lemma 4.2. We write $N = rE + s$ with $s < E$. Given any $x \in Z(k,N)$, choose the instants $t_1 < \ldots < t_k$ as in the definition of $Z(k,N)$. For $0 \leq u < r$, take from each interval $[uE, (u+1)E)$ the first $t_i \in \{t_1, \ldots, t_k\}$ (if there is at least one). Denote that subsequence of t_i's by $t_{1'} < \cdots < t_{k'}$. Since $t_1 < \cdots < t_k \leq N$, we have $k' \geq [\frac{k}{E}]$, which means that $Ek' + E \geq k$. Keeping only the instants with odd indexes, we get a sequence of instants $u_1 < \ldots < u_\ell$ with $2\ell \geq k'$, and necessarily $\ell \geq \frac{k-E}{2E}$. Moreover, we have $u_{i+1} - u_i \geq E$ for $1 \leq i \leq \ell$, by construction. According to our construction process, we know that associated to the instant u_i there must be some set B_i such that $x \in B_i$, for $1 \leq i \leq \ell$. Define

$$I = \{1 \leq i \leq \ell, B_i \subset B_1 \cap \cdots \cap B_{i-1}\} \quad \text{and} \quad J = [1, \ell] \setminus I.$$

Now we split the proof according to the following possible cases:

If $\#I \geq \ell/2$, we keep only the elements with indexes in I. We necessarily have $x \in Z_2(\ell/2, N)$; recall (24). Then, using Lemma 4.3 we see that $Z_2(\ell/2, N)$ has Leb_D measure exponentially small in ℓ (hence in k), which gives the result in this case.

If $\#I \leq \ell/2$, then $\#J \geq \ell/2$. Set $j_0 = \sup J$ and $i_0 = \inf\{i < j_0, B_{j_0} \not\subseteq B_i\}$. Next set $j_1 = \sup\{j \leq i_0, j \in J\}$ and $i_1 = \inf\{i < j_1, B_{j_1} \not\subseteq B_i\}$. Proceeding inductively, the process must necessarily stop at some step i_n. Then $J \subset \cup_{s=0}^n (i_s, j_s]$, by construction. We obtain $\sum_{s=0}^n (j_s - i_s) \geq \#J \geq \ell/2$, which shows that

$$\sum_{s=0}^n \left[\frac{t(B_{j_s}) - t(B_{i_s})}{E}\right] = \sum_{s=0}^n \left[\frac{u_{j_s} - u_{i_s}}{E}\right] \geq \ell/2,$$

since $|u_j - u_i| \geq E(j-i)$ by the constructing process. Hence $x \in Z_1(\ell/2, B_{i_n})$ with the sequence $B_{i_n}, B_{i_n}, B_{j_n}, \ldots, B_{i_0}, B_{j_0}$. As the cores of these sets are pairwise disjoint, we use Lemma 4.2 and, summing over all the possible B'_{i_n}s, we get the result also in this case. □

Finally, we deduce an estimate on the Leb_D measure of the sets $Y(n_1, \ldots, n_p, B_1)$ as in (17). Recall that P has been introduce in Lemma 3.11.

Lemma 4.5. *There is $D_4 > 0$ such that for all $n_1, \ldots, n_p > P$ and B_1*

$$\operatorname{Leb}_D(Y(n_1, \ldots, n_p, B_1)) \leq D_4 (D_4 \sigma^{n_1/2}) \cdots (D_4 \sigma^{n_p/2}) \operatorname{Leb}_D(C(B_1)).$$

Proof. The proof is by induction on p. Taking $D_4 > {C_5}^{1/2}$ (recall the constant C_5 in Proposition 3.6), we immediately get the result for $p = 1$. Now suppose $p > 1$ and let $x \in Y(n_1, \dots, n_p, B_1)$. Then there exists $B_2 = B^{t_2}_{t_2}(x_2)$ constructed at an instant $t_2 > t_1$ such that $x \in Y(n_2, \dots, n_p, B_2)$. By Lemma 3.11 we have $B^{t_2}_{t_2+P}(x_2) \cap B^{t_1}_{t_2+P}(x_1) = \emptyset$. But for all $1 \le i \le s$, we have $x \in B^{t_i}_{t_i+n_i}(x_i)$. So, $t_1 + n_1 < t_2 + P$, i.e. $t_2 - t_1 > n_1 - P$. By the uniform expansion at hyperbolic times, we get

$$\mathrm{diam}_{f^{t_1}(D)}(f^{t_1}(B_2)) \le \sigma^{\frac{t_2-t_1}{2}} \mathrm{diam}_{f^{t_2}(D)}(f^{t_2}(B_2)) \le 6\delta_1' \sigma^{\frac{n_1-P}{2}}.$$

On the other hand, setting $Q = f^{t_1}(C(B_1))$, we have $\mathrm{dist}_{f^{t_1}(D)}(f^{t_1}(x), \partial Q) \le 2\delta_1' \sigma^{\frac{n_1}{2}}$ when $x \in B^{t_1}_{t_1+n_1}(x_1) \cap B_2$. Thus, there is some constant $D_5 > 0$ such that

$$f^{t_1}(B_2) \subset \mathcal{C} := \left\{ y \mid \mathrm{dist}_{f^{t_1}(D)}(y, \partial Q) \le D_5 \sigma^{\frac{n_1}{2}} \right\}.$$

By the induction hypothesis we have

$$\mathrm{Leb}_D(Y(n_2, \dots, n_p, B_2)) \le D_4(D_4\sigma^{n_2/2}) \dots (D_4\sigma^{n_p/2}) \,\mathrm{Leb}_D(C(B_2)),$$

which together with bounded distortion yields

$$\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Y(n_2, \dots, n_p, B_2))) \le C_2 D_4(D_4\sigma^{n_2/2}) \dots (D_4\sigma^{n_p/2}) \,\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(C(B_2))).$$

By Remark 3.8 and Lemma 3.10, the possible cores $C(B_2)$'s are pairwise disjoint and all contained in $V_{t_1}(x_1)$. So, the sets $f^{t_1}(C(B_2))$ are still pairwise disjoint and all contained in the annulus $\mathcal{C}$. Hence

$$\begin{aligned}
\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Y(n_1, \dots, n_p, B_1))) &\le \sum_{B_2} \mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Y(n_2, \dots, n_p, B_2))) \\
&\le C_2 D_4(D_4\sigma^{n_2/2}) \dots (D_4\sigma^{n_p/2}) \sum_{B_2} \mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(C(B_2))) \\
&\le C_2 D_4(D_4\sigma^{n_2/2}) \dots (D_4\sigma^{n_p/2}) \,\mathrm{Leb}_{f^{t_1}(D)}(\mathcal{C}).
\end{aligned}$$

Similarly to (16), we have for some $\rho > 0$

$$\mathrm{Leb}_{f^{t_1}(D)}(\mathcal{C}) \le \rho\sigma^{n_1/2} \,\mathrm{Leb}_{f^{t_1}(D)}(Q),$$

where $Q = f^{t_1}(C(B_1))$. Then,

$$\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Y(n_1, \dots, n_p, B_1))) \le C_2\rho\sigma^{n_1/2}(D_4\sigma^{n_2/2}) \dots (D_4\sigma^{n_p/2}) \,\mathrm{Leb}_{f^{t_1}(D)}(Q),$$

which together with bounded distortion yields

$$\mathrm{Leb}_D(Y(n_1, \dots, n_p, B_1)) \le C_2^2\rho(D_4\sigma^{n_1/2})(D_4\sigma^{n_2/2}) \dots (D_4\sigma^{n_p/2}) \,\mathrm{Leb}_D(C(B_1)).$$

Taking $D_4 \ge C_2^2\rho$, we finish the proof. □

Now we are ready to complete the proof of Proposition 4.1. Take $P' \ge P$ (recall P in Lemma 3.11) so that

$$\sigma^{1/2} + D_4\sigma^{P'/2} < 1. \tag{26}$$

As shown in (18), we have

$$X(k,N) \subset \bigcup_{B_1} \bigcup_{\substack{n_1,\dots,n_p \geq P', \\ \sum n_i \geq \frac{k}{2}}} Y(n_1,\dots,n_p,B_1) \cup Z\left(\left[\frac{k}{2P'}\right],N\right).$$

It follows from Lemma 4.4 and Lemma 4.5 that

$$\mathrm{Leb}_D(X(k,N)) \leq \sum_{B_1} \sum_{\substack{n_1,\dots,n_p \geq P', \\ \sum n_i \geq \frac{k}{2}}} D_4(D_4\sigma^{n_1/2})\dots(D_4\sigma^{n_p/2})\,\mathrm{Leb}_D(C(B_1)) + D_3\lambda_3^{\frac{k}{2P'}}\,\mathrm{Leb}_D(\Delta_0).$$

We have $\sum_{B_1}\mathrm{Leb}_D(C(B_1)) \leq \mathrm{Leb}_D(\Delta_0) < \infty$, because the cores $C(B_1)$ are pairwise disjoint. We are left to show that the sum

$$\sum_{\substack{n_1,\dots,n_p \geq P', \\ \sum n_i \geq \frac{k}{2}}} (D_4\sigma^{n_1/2})\dots(D_4\sigma^{n_p/2})$$

is exponentially small in k. We use the generating series

$$\sum_n \sum_{\substack{n_1,\dots,n_p \geq P', \\ \sum n_i = n}} (D_4\sigma^{n_1/2})\dots(D_4\sigma^{n_p/2})z^n = \sum_{p=1}^{\infty}\left(D_4 \sum_{n=P'}^{\infty} \sigma^{n/2}z^n\right)^p = \frac{D_4\sigma^{P'/2}z^{P'}}{1-\sigma^{1/2}z - D_4\sigma^{P'/2}z^{P'}}.$$

Under condition (26), the function above has no pole in a neighborhood of the unit disk in $\mathbb{C}$. Thus, its coefficients decay exponentially fast: there are constants $D_6 > 0$ and $\lambda_6 < 1$ such that

$$\sum_{\substack{n_1,\dots,n_p \geq P', \\ \sum n_i = n}} (D_4\sigma^{n_1/2})\dots(D_4\sigma^{n_p/2}) \leq D_6\lambda_6^n.$$

Then we sum over $n \geq k/2$ and B_1 to obtain constants $D_0 > 0$ and $0 < \lambda_0 < 1$ such that

$$\mathrm{Leb}_D(X(k,N)) \leq D_0\lambda_0^k\,\mathrm{Leb}_D(\Delta_0),$$

which gives Proposition 4.1.

5. Gibbs-Markov-Young structure

In this section we construct the GMY structure given by Theorem A.

5.1. **Product structure.** Consider the center-unstable disk $\Delta_0 \subset D$ of Section 3.2 and the (Leb_D mod 0) partition $\mathcal{P}$ of Δ_0. We define

$$\Gamma^s = \left\{W^s_{\delta_s}(x) : x \in \Delta_0\right\}$$

and the family of unstable leaves Γ^u as the set of all local unstable leaves u-crossing $\mathcal{C}^0$; recall Definition 3.1 and (8). Clearly Γ^u is nonempty because $\Delta_0 \in \Gamma^u$. We need to see that $\cup\gamma^u$ is compact. By the domination property and Ascoli-Arzela Theorem, any limit leaf Δ_∞ of leaves in Γ^u is a center-unstable disk u-crossing $\mathcal{C}^0$. Hence $\Delta_\infty \in \Gamma^u$, by definition of Γ^u, and so $\cup\gamma^u$ is compact.

The *s-subsets* are defined in the following way: given $\omega \in \mathcal{P}$, consider

$$\mathcal{C}(\omega) = \bigcup_{x\in\omega} W^s_{\delta_s}(x).$$

The pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \dots$ are precisely the sets

$$\{\mathcal{C}(\omega) \cap (\cup\gamma^u) : \omega \in \mathcal{P}\}.$$

Then we should check that $f^{R_i}(\Lambda_i)$ is *u-subset*. Given an element $\omega \in \mathcal{P}$, by construction there is some $R(\omega) \in \mathbb{N}$ such that $f^{R(\omega)}(\omega)$ is a center-unstable disk u-crossing $\mathcal{C}^0$. Since by construction $f^{R(\omega)}(\omega)$ intersects $W^s_{\delta_s/4}(p)$, then according to the choice of δ_0 and the invariance of the stable foliation, we have that each element of $f^{R(\omega)}(\mathcal{C}(\omega) \cap \Gamma^u)$ must u-cross $\mathcal{C}^0$ and is contained in the $\lambda^{R(\omega)}\delta_s$ height neighborhood of $f^{R(\omega)}(\omega)$. Ignore the difference caused by the angle. We can say it is contained in $\mathcal{C}^0$. So, that is a *u-subset.*

In the sequel we prove that the set $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$ with a *product structure* has indeed a *GMY structure*. Observe that Λ coincides with the union of the leaves in Γ^u and properties $(\mathbf{P_0})$–$(\mathbf{P_2})$ are naturally satisfied. In the next two subsections we prove properties $(\mathbf{P_3})$ and $(\mathbf{P_4})$.

5.2. **Uniform expansion and bounded distortion.** Property $(\mathbf{P_3})$(a) follows from the next result.

Lemma 5.1. *There is $C > 0$ such that, given $\omega \in \mathcal{P}$ and $\gamma \in \Gamma^u$, we have for all $1 \le k \le R(\omega)$ and all $x, y \in \mathcal{C}(\omega) \cap \gamma$*

$$\mathrm{dist}_{f^{R(\omega)-k}(\mathcal{C}(\omega)\cap\gamma)}(f^{R(\omega)-k}(x), f^{R(\omega)-k}(y)) \le C\sigma^{k/2}\, \mathrm{dist}_{f^{R(\omega)}(\mathcal{C}(\omega)\cap\gamma)}(f^{R(\omega)}(x), f^{R(\omega)}(y)).$$

Proof. Let ω be an element of the partition $\mathcal{P}$ constructed in Section 3.2. There is necessarily a point $x \in D$ with σ-hyperbolic time $n(\omega)$ satisfying $R(\omega) - N_0 \le n(\omega) \le R(\omega)$. Since we take $\delta_s, \delta_0 < \delta_1/2$, by (7), $n(\omega)$ is a $\sqrt{\sigma}$-hyperbolic time for every point in $\mathcal{C}(\omega)\cap\gamma$. Recalling (6), we obtain that for all $1 \le k \le n(\omega)$ and all $x, y \in \mathcal{C}(\omega) \cap \gamma$

$$\mathrm{dist}_{f^{n(\omega)-k}(\mathcal{C}(\omega)\cap\gamma)}(f^{n(\omega)-k}(x), f^{n(\omega)-k}(y)) \le \sigma^{k/2}\, \mathrm{dist}_{f^{n(\omega)}(\mathcal{C}(\omega)\cap\gamma)}(f^{n(\omega)}(x), f^{n(\omega)}(y)).$$

Considering $R(\omega) - n(\omega) \le N_0$, we take C depending only on N_0 and the derivative of f, then we get the result. □

Property $(\mathbf{P_3})$(b) follows from Proposition 2.2 together with Lemma 5.1 as in [2, Proposition 2.8]. We prove it here for the sake completeness.

Lemma 5.2. *There is $\bar{C} > 0$ such that, for all $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$, we have*

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \le \bar{C}\, \mathrm{dist}(f^{R_i}(x), f^{R_i}(y))^\zeta.$$

Proof. For $0 \le k < R_i$ and $y \in \gamma^u(x) \in \Gamma^u$, we set $J_k(y) = \log|\det Df^u(f^k(y))|$ as in the last item of Proposition 2.2. Then,

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} = \sum_{k=0}^{R_i-1} (J_k(x) - J_k(y)) \le \sum_{k=0}^{R_i-1} L_1\, \mathrm{dist}_D(f^k(x), f^k(y))^\zeta.$$

By Proposition 2.2, the sum of $\mathrm{dist}_D(f^k(x), f^k(y))^\zeta$ over $0 \le k \le R_i$ is bounded by

$$\mathrm{dist}_D(f^{R_i}(x), f^{R_i}(y))^\zeta/(1 - \sigma^{\zeta/2}).$$

Take $\bar{C} = L_1(1 - \sigma^{\zeta/2})$ to get the result. □

5.3. **Regularity of the foliations.** Property ($\mathbf{P_4}$) has already been proved in [7]. This is standard for uniformly hyperbolic attractors and the ideas can be adapted to the partially hyperbolic setting. Property ($\mathbf{P_4}$)(a) follows from the next result whose proof may be found in [7, Corollary 3.8].

Proposition 5.3. *There are $C > 0$ and $0 < \beta < 1$ such that for all $y \in \gamma^s(x)$ and $n \geq 0$*

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C\beta^n.$$

For ($\mathbf{P_4}$)(b) we need to introduce some useful notion.

Definition 5.4. Given N and G submanifolds of M, we say that $\phi : N \to G$ is *absolutely continuous* if it is an injective map for which there exists $J : N \to \mathbb{R}$, called the *Jacobian* of ϕ, such that

$$\mathrm{Leb}_G(\phi(A)) = \int_A J d\,\mathrm{Leb}_N\,.$$

Property ($\mathbf{P_4}$)(b) follows from the next result whose proof is given in [7, Proposition 3.9].

Proposition 5.5. *Given $\gamma, \gamma' \in \Gamma^u$, define $\phi\colon \gamma' \to \gamma$ by $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous and the Jacobian of ϕ is given by*

$$J(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

We have from Proposition 5.3 that this infinite product converges uniformly.

6. Application

Here we present a open class of partially hyperbolic diffeomorphisms whose center-unstable direction is non-uniformly expanding at Lebesgue almost everywhere in M and, for any center-unstable disk D, we have $\mathrm{Leb}_D\{\mathcal{E} > n\}$ is exponentially small. This example was introduced in [2, Appendix] and we sketch below the main steps of its description.

We consider a linear Anosov diffeomorphism f_0 on the d-dimensional torus $M = T^d$, $d \geq 2$, with a hyperbolic splitting $TM = E^u \oplus E^s$. Let $V \subset M$ be some small compact domain, such that for $\pi : \mathbb{R}^d \to T^d$ the canonical projection, there exist unit open cubes K^0, K^1 in $\mathbb{R}^d$ such that $V \subset \pi(K^0)$ and $f_0(V) \subset \pi(K^1)$. Let f be a diffeomorphism on T^d such that:

(1) f has invariant cone fields C^{cu} and C^s which are with small width $\alpha > 0$ and contain, respectively, the unstable bundle E^u and the stable bundle E^s of the Anosov diffeomorphism f_0;
(2) f^{cu} is *volume expanding everywhere*: there is $\sigma_1 > 0$ such that $|\det(Df|T_xD^{cu})| > \sigma_1$ for any $x \in M$ and any disk D^{cu} through x tangent to the center-unstable cone field C^{cu};
(3) f is C^1-close to f_0 in the compliment of V, so that f^{cu} is *expanding outside* V: there is $\sigma_2 < 1$ satisfying $\|(Df|T_xD^{cu})^{-1}\| < \sigma_2$ for $x \in M \setminus V$ and any disks D^{cu} tangent to C^{cu};

(4) f^{cu} is *not too contracting* on V: there is small $\delta_0 > 0$ satisfying $\|(Df|T_xD^{cu})^{-1}\| < 1 + \delta_0$ for any $x \in V$ and any disks D^{cu} tangent to C^{cu}.

For example, if $f_1 : T^d \to T^d$ is a diffeomorphism satisfying itens (1), (2) and (4) above and coinciding with f_0 outside V, then any f in a C^1 neighborhood of f_1 satisfies all the conditions (1)-(4). The C^1 open classes of transitive non-Anosov diffeomorphisms given in [10, Section 6], and also other robust examples from [15], are constructed in this way and they satisfy: both these diffeomorphisms and their inverse satisfy conditions (1)-(4) above.

Next we show that any f satisfying (1)-(4) is non-uniformly expanding along the cu-direction on a full Lebesgue set of points in M. Let $B_1, \dots, B_p, B_{p+1} = V$ be any partition of T^d into small subsets such that there exist open cubes K_i^0 and K_i^1 in $\mathbb{R}^d$ for which

$$B_i \subset \pi(K_i^0) \quad \text{and} \quad f(B_i) \subset \pi(K_i^1).$$

Let $\mathcal{F}_0^u$ be the unstable foliation of f_0 and let us fix any small disk D contained in a leaf of $\mathcal{F}_0^u$. Using the same arguments in the proof of [2, Lemma A.1] we deduce the next result.

Lemma 6.1. *There exist $\theta > 0$ such that the orbit of Lebesgue almost every $x \in D$ spends a fraction θ of the time in $B_1 \cup \ldots \cup B_p$:*

$$\#\{0 \leq j < n : f^j(x) \in B_1 \cup \ldots \cup B_p\} \geq \theta n$$

for every large n.

Hence, Leb_D-almost every point $x \in D$ spends a positive fraction θ of time outside the domain V. Then by itens (3) and (4) above, there exists $c_0 > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df \mid E^{cu}_{f^j(x)})^{-1}\| \leq -c_0$$

for Leb_D-almost every point $x \in D$. Moreover, there exists a constant $c > 0$ such that

$$\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn});$$

see the Claim in [2, p.396]. Furthermore, as D is an arbitrary cu-disk, f is non-uniformly expanding along the cu-direction on a full Lebesgue set of points in M.

References

[1] J. F. Alves, *SRB measures for non-hyperbolic systems with multidimensional expansion*, Ann. Scient. Éc. Norm. Sup. (4) **33** (2000), 1-32.

[2] J. F. Alves, C. Bonatti, M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding*, Invent. Math. **140** (2000), 351-398.

[3] J. F. Alves, C. L. Dias, S. Luzzatto, *Geometry of expanding absolutely continuous invariant measures and the liftability problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013) 101-120.

[4] J. F. Alves, J. M. Freitas, S. Luzzatto, S. Vaienti, *From rates of mixing to recurrence times via large deviations*, Adv. Math. **228**, n.2 (2011), 1203-1236.

[5] J. F. Alves, S. Luzzatto, V. Pinheiro, *Markov structures and decay of correlations for non-uniformly expanding dynamical systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, n.6 (2005) 817-839.

[6] J. F. Alves, V. Pinheiro, *Slow rates of mixing for dynamical systems with hyperbolic structures*, J. Stat. Phys. **131**, n.3 (2008) 505-534.

[7] J. F. Alves, V. Pinheiro, *Gibbs-Markov structures and limit laws for partially hyperbolic attractors with mostly expanding central direction*, Adv. Math. 223 (2010) 1706-1730.

[8] R. Bowen, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math. (1970) vol. 92 pp. 725-747.

[9] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lect. Notes in Math. Springer Verlag, (1975) vol. 470.

[10] C. Bonatti, M. Viana, *SRB measures for partially hyperbolic systems with mostly contracting central direction*, Israel J. Math. **115** (2000), 157-193.

[11] M. Benedicks, L.-S. Young, *Markov extensions and decay of correlations for certain Hénon maps*, Astérisque **261** (2000), 13-56.

[12] A. A. de Castro Júnior, *Backward inducing and exponential decay of correlations for partially hyperbolic attractors with mostly contracting central direction*, Israel J. of Math. **130** (2001), 29-75.

[13] D. Dolgopyat, *Limit theorems for partially hyperbolic systems*, Trans. Amer. Math. Soc. **356**, no. 4 (2004), 1637–1689.

[14] S. Gouëzel, *Decay of correlations for nonuniformly expanding systems*, Bull. Soc. Math. France **134**, n.1 (2006), 1-31.

[15] R. Mañé, *Contributions to the stability conjecture*, Topology **17** (1978), no. 4, 383-396.

[16] I. Melbourne, M. Nicol, *Large deviations for nonuniformly hyperbolic systems*, Trans. Amer. Math. Soc. **360** (2008) 6661-6676.

[17] V. Pinheiro, *Sinai-Ruelle-Bowen measures for weakly expanding maps*, Nonlinearity **19** (2006) 1185-1200

[18] D. Ruelle, *Ergodic theory of differentiable dynamical systems*, Inst. Hautes Études Sci. Publ. Math. **50** (1979) 27-58.

[19] Y. Sinai. *Markov partitions and C-diffeomorphisms.* Funkcional. Anal. i Prilozen (1968) vol. 2 (1) pp. 64-89.

[20] A. Tahzibi, *Stably ergodic diffeomorphisms which are not partially hyperbolic*, Israel J. Math. **142** (2004), 315–344.

[21] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. Math. **147** (1998), 585-650.

[22] L.-S. Young, *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153-188.

José F. Alves, Departamento de Matemática, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: jfalves@fc.up.pt

URL: http://www.fc.up.pt/cmup/jfalves

Xin Li, Department of Mathematics, Soochow University, Suzhou 215006, Jiangsu, PR China; and Departamento de Matemática, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: lxinporto@gmail.com