

ON THE WEIGHT HIERARCHY OF CODES COMING FROM SEMIGROUPS WITH TWO GENERATORS

M. DELGADO, J. I. FARRÁN, P. A. GARCÍA-SÁNCHEZ, AND D. LLENA

ABSTRACT. The weight hierarchy of one-point algebraic geometry codes can be estimated by means of the generalized order bounds, which are described in terms of a certain Weierstrass semigroup. The asymptotical behaviour of such bounds for $r \geq 2$ differs from that of the classical Feng-Rao distance ($r = 1$) by the so-called Feng-Rao numbers. This paper is addressed to compute the Feng-Rao numbers for numerical semigroups of embedding dimension two (with two generators), obtaining a closed simple formula for the general case by using numerical semigroup techniques. These involve the computation of the Apéry set with respect to an integer of the semigroups under consideration. The formula obtained is applied to lower-bounding the generalized Hamming weights, improving the bound given by Kirfel and Pellikaan in terms of the classical Feng-Rao distance. We also compare our bound with a modification of the Griesmer bound, improving this one in many cases.

1. INTRODUCTION

Kirfel and Pellikaan introduced in [17] the concept of array of codes. More generally, the concepts of order function and weight function allows us to define arrays of codes with the same features (see [16]). For such an array there is a majority voting algorithm for decoding efficiently up to half the so-called Feng-Rao distance. This distance is obtained by numerical computations in a certain underlying numerical semigroup.

These results are actually a linear algebra formalization of the Feng-Rao decoding algorithm and the Feng-Rao distance $\delta_{RF}(m)$, introduced in [11] for one-point algebraic geometry codes (AG codes in short). The Feng-Rao distance (also known as *order bound* in the literature) becomes a lower bound for the minimum distance of the involved error-correcting codes.

In the case of one-point AG codes, the Feng-Rao distance improves the lower bound for the minimum distance given by the Riemann-Roch theorem, that is called the Goppa distance. This result has a translation in [17] to the case of arrays of codes, namely

$$\delta_{FR}(m+1) \geq m+2-2g$$

if $m > 2g - 2$, and the equality holds for $m \gg 0$. The number $m+2-2g$ corresponds to the Goppa bound.

Even though the Feng-Rao distance was introduced just for Weierstrass semigroups and with decoding purposes, it is just a combinatorial concept that makes sense for arbitrary numerical semigroups, so that it can be computed just with numerical semigroup techniques. The problem of computing Feng-Rao distances has been broadly studied in the literature for different types of numerical semigroups (see [2], [3] or [17]).

Later on, the concept of minimum distance for an error-correcting code has been generalized to the so-called *generalized Hamming weights* and the *weight hierarchy*. These concepts were independently

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introduced by Helleseth et al. in [14] and Wei in [22] for applications in coding theory and cryptography, respectively.

On the other hand, the Feng-Rao distance has been generalized in a natural way to higher weights (see [13]). The obtained generalized Feng-Rao distances (or *generalized order bounds*), defined on the underlying numerical semigroup for an array of codes (or a weight function, in a modern setting), become lower bounds for the corresponding generalized Hamming weights. However, the computation of these generalized Feng-Rao distances is a much more complicated problem than in the classical case. This means that very few results are known about this subject, and they are completely scattered in the literature (see for example [1], [6], [7], [10] or [13]).

This paper is addressed to the asymptotical behaviour of the generalized Feng-Rao distances, that is, $\delta_{FR}^r(m)$ for $r \geq 2$ and $m \gg 0$. In fact, it was proved in [10] that

$$\delta_{FR}^r(m) = m + 1 - 2g + E_r$$

for $m \gg 0$ (details are made precise in the next section). The number $E_r \equiv E(S, r)$ is called the r th Feng-Rao number of the semigroup S , and they are unknown but for very few semigroups and concrete r 's. For example, it was proved in [8] that

$$(1) \quad E(S, r) = \rho_r$$

for hyper-elliptic semigroups $S = \langle 2, 2g + 1 \rangle$, with multiplicity 2 and genus g , and for Hermitian-like semigroups $S = \langle a, a + 1 \rangle$, where $S = \{\rho_1 = 0 < \rho_2 < \dots\}$.

In [7] the authors compute the Feng-Rao numbers for numerical semigroups generated by intervals, generalizing the techniques for the Hermitian-like case, but not obtaining the same formula (note that such semigroups are not symmetric in general). In the present paper we generalize the results of [8], obtaining the above formula (1) for the general case of embedding dimension two numerical semigroups, that is, $S = \langle a, b \rangle$ generated by two elements. In particular, we get a lower bound for the generalized Hamming weights in an array of codes whose associated semigroup is such an S . This bound improves the one given in [17] in terms of the classical Feng-Rao distance. In fact, once the Feng-Rao number E_r is computed, we get a lower bound for the generalized Feng-Rao distance

$$\delta_{FR}^r(m) \geq m + 1 - 2g + E_r$$

for $m \geq c$ (see [10]).

The computation of δ_{FR}^r is related to divisors of multiple elements in a numerical semigroups, and we show that these can be calculated by using Apéry sets. These sets are a powerful tool in the study of numerical semigroups, basically because they provide fast computations, and they were known only when n equals to one of the generators. We obtain a general expression for the Apéry sets of a semigroup S with two generators, with respect to any integer n .

The paper is organized as follows. Section 2 sets the general definitions concerning numerical semigroups, Feng-Rao distances, Feng-Rao numbers and amenable sets. Computations on embedding dimension two numerical semigroups are introduced in main Section 3. More precisely, we revise the calculations on grounds and triangles in [7] for the case of semigroups with two generators, obtaining the desired formula (1) for the Feng-Rao numbers in Theorem 43 by working with amenable sets. Experimental results with the GAP [12] package `numericalsgps` [6] pointed precisely to this formula for this type of semigroups, and were actually the starting point and motivation to write this paper. The paper ends with some examples and conclusions in Section 4.

2. FENG-RAO DISTANCES ON NUMERICAL SEMIGROUPS

Let S be a numerical semigroup, that is, a submonoid of \mathbb{N} such that $\#(\mathbb{N} \setminus S) < \infty$ and $0 \in S$. Denote by $g := \#(\mathbb{N} \setminus S)$ the *genus* of S . Note that if S is the Weierstrass semigroup of a curve χ at a point P , g equals precisely to the geometric genus of χ , and the elements of $G(S) := \mathbb{N} \setminus S = \{\ell_1 < \dots < \ell_g\}$ are called the *gaps* of S (for the case S being a Weierstrass semigroup, they are also known as Weierstrass gaps of χ at P).

Let $c \in S$ be the *conductor* of S , that is the (unique) element in S such that $c - 1 \notin S$ and $c + l \in S$ for all $l \in \mathbb{N}$. We obviously have $c \leq 2g$, and thus the “largest gap” of S is $\ell_g \doteq c - 1 \leq 2g - 1$. The number

ℓ_g is precisely the *Frobenius number* of S , denoted by $F(S)$ in the literature. The semigroup S is called *symmetric* provided $r \in S$ if and only if $c - 1 - r \notin S$, for all $r \in \mathbb{Z}$. This is equivalent to say that $c = 2g$ or $F(S) = 2g - 1$.

The *multiplicity* of a numerical semigroup is the least positive integer belonging to it. Note that if S is the value semigroup of a curve χ at a point P , this number equals to the multiplicity of χ at the point P .

We say that a numerical semigroup S is generated by a set of elements $G \subseteq S$ if every element $m \in S$ can be written as a linear combination

$$m = \sum_{g \in G} \lambda_g g,$$

where finitely many $\lambda_g \in \mathbb{N}$ are non-zero. It is well-known that every numerical semigroup is finitely generated, that is, there exists a finite set G that is a generator set for S . Furthermore, every such generator set contains the set of irreducible elements, where $m \in S$ is *irreducible* if $m = a + b$ and $a, b \in S$ implies $a \cdot b = 0$. The set of irreducibles actually generates S , so that it is usually called “the” *generator set* of S (and thus its elements are sometimes referred as *generators*). The number of irreducibles is called *embedding dimension* of S (see [20] for further details). The smallest generator is precisely the *multiplicity*.

Finally, if we enumerate the elements of S in increasing order

$$S = \{\rho_1 = 0 < \rho_2 < \dots\},$$

we note that every $m \geq c$ is the $(m + 1 - g)$ th element of S , that is $m = \rho_{m+1-g}$.

Following [20], for $a, b \in \mathbb{Z}$ given, we say that a *divides* b , and write

$$a \leq_S b, \text{ if } b - a \in S.$$

This binary relation is an order relation.

The set $D(a)$ denotes the set of *divisors* of a in S , and for a given $M = \{m_1, \dots, m_r\} \subseteq S$, we write $D(M) = D(m_1, \dots, m_r) = \bigcup_{i=1}^r D(m_i)$. Thus, from now on, we will use the term *divisors* to refer to the elements in the sets $D(\cdot)$.

Note that $D(m_1) \subseteq [0, m_1]$, and $s \in D(m_1)$ implies $D(s) \subseteq D(m_1)$. The following result was proved in [7].

Lemma 1. $D(x) = S \cap (x - S)$.

Remark 2. As an immediate consequence we get $\#(D(m + \rho_n) \cap [m, \infty)) = n$ for $m \geq c$.

The above inclusion $D(m) \subseteq D(m+p)$, for all $p \in S$, is very useful for practical computations. Moreover, we easily get the following result (see [7]).

Proposition 3. *If $m \geq 2c - 1$, then $\#D(m) = \#(S \cap (m - S)) = m + 1 - 2g$.*

We now introduce the definitions of the generalized Feng-Rao distances and summarize known results about them.

Definition 4. *Let S be a numerical semigroup. The (classical) Feng-Rao distance of S is defined by the function*

$$\begin{aligned} \delta_{FR} : S &\longrightarrow \mathbb{N} \\ m &\mapsto \delta_{FR}(m) := \min\{\#D(m_1) \mid m_1 \geq m, m_1 \in S\}. \end{aligned}$$

There are some well-known facts about the δ_{FR} function for an arbitrary semigroup S (see [16], [17] or [2] for further details). The most important one for our purposes is that $\delta_{FR}(m) \geq m + 1 - 2g$ for all $m \in S$ with $m \geq c$, and that equality holds if moreover $m \geq 2c - 1$.

The classical Feng-Rao distance corresponds to $r = 1$ in the following definition.

Definition 5. *Let S be a numerical semigroup. For any integer $r \geq 1$, the r th Feng-Rao distance of S is defined by the function*

$$\begin{aligned} \delta_{FR}^r : S &\longrightarrow \mathbb{N} \\ m &\mapsto \delta_{FR}^r(m) := \min\{\#D(m_1, \dots, m_r) \mid m \leq m_1 < \dots < m_r, m_i \in S\}. \end{aligned}$$

Very few results are known for the numbers δ_{FR}^r , and their computation is very hard from both a theoretical and computational point of view. The main result we need describes the asymptotical behaviour for $m \gg 0$, and was proved in [10]. As we already mentioned in the introduction, this result tells us that there exists a certain constant $E_r = E(S, r)$, depending on r and S , such that

$$(2) \quad \delta_{FR}^r(m) = m + 1 - 2g + E_r$$

for $m \geq 2c - 1$. This constant is called the *r*th *Feng-Rao number* of the semigroup S . Furthermore, it is also true that $\delta_{FR}^r(m) \geq m + 1 - 2g + E(S, r)$ for $m \geq c$, and equality holds if S is symmetric and $m = 2g - 1 + \rho$ for some $\rho \in S \setminus \{0\}$ (see [10]). Somehow, this constant measures the difference between $\delta_{FR}^r(m)$ and $\delta_{FR}(m)$ for sufficiently large m , being $E(S, 1) = 0$. For the trivial semigroup with $g = 0$, it is easy to check that $E(S, r) = r - 1$.

We summarize some general properties of the Feng-Rao numbers, for $r \geq 2$ and S fixed, with $g \geq 1$ (see again [10] for the details):

1. The function $E(S, r)$ is non-decreasing in r .
2. $r \leq E(S, r) \leq \rho_r$. If furthermore $r \geq c$, then $E(S, r) = \rho_r = r + g - 1$.

The computation of the Feng-Rao numbers is a very hard task, even in very simple examples. So far, only the second Feng-Rao number ($r = 2$) is computed in the literature, with either a general algorithm based on Apéry systems, or concrete formulas for simple examples by counting deserts (see [10]). More precisely, the only exact formula, given in [10], is that

$$E(S, 2) = \rho_2$$

for S generated by two elements.

In a previous work [8], and by using different techniques, we have found two families of numerical semigroups with $E(S, r) = \rho_r$: that of numerical semigroups with multiplicity two (hyper-elliptic), and those embedding dimension two numerical semigroups generated by a positive integer a and $a + 1$ (hermitian-like). In this paper we generalize this result to the whole family of embedding dimension two numerical semigroups.

Note that in general this bound is not attained for other kinds of semigroups, not even for $r = 2$. For example, if we consider the semigroup $S = \langle 6, 13, 14, 15, 16, 17 \rangle$ then $E(S, 2) = 3 < \rho_2 = 6$.

The following definitions are addressed to find the minimum required by the definition of Feng-Rao distance.

Definition 6. *Let S be a numerical semigroup and let $m \in S$. A finite subset of $S \cap [m, \infty)$ is called a (S, m) -configuration, or simply a configuration. A configuration M of cardinality r is said to be optimal if $\delta_{FR}^r(m) = \#D(M)$.*

Motivated by Formula (2) and Proposition 3, in the sequel we denote by \mathfrak{m} any integer greater than or equal to $2c - 1$, where c is the conductor of the semigroup under consideration.

Definition 7. *Let S be a numerical semigroup with conductor c . Let $M = \{m_1, \dots, m_r\} \subseteq S$ with $\mathfrak{m} = m_1 < \dots < m_r$. We say that the set M is (S, \mathfrak{m}, r) -amenable if:*

$$(3) \quad \text{for all } i \in \{1, \dots, r\}, D(m_i) \cap [\mathfrak{m}, \infty) \subseteq M.$$

We will refer a set satisfying (3) as being *\mathfrak{m} -closed under division*. So, a subset of $S \cap [\mathfrak{m}, \infty)$ with cardinality r is (S, \mathfrak{m}, r) -amenable if and only if it contains \mathfrak{m} and is \mathfrak{m} -closed under division.

When no confusion arises or only the concept is important, we simply say *amenable set*.

The following is immediate from the definition (it has also been stated in [7, Lemma 40]).

Lemma 8. *Let $M \neq \{\mathfrak{m}\}$ be an amenable set. Then $M \setminus \{\max(M)\}$ is again an amenable set.*

The importance of amenable sets comes from the following result, which states that among the optimal configurations of cardinality r there is at least one (S, \mathfrak{m}, r) -amenable set.

Proposition 9. [7, Proposition 10] *Let S be a numerical semigroup with conductor c and let $\mathfrak{m} \geq 2c - 1$. Let r be a positive integer. Among the (S, \mathfrak{m}) -optimal configurations of cardinality r there is one (S, \mathfrak{m}, r) -amenable set.*

The following lemma is of extreme importance, since it will allow us to concentrate in computing amenable sets whose so-called grounds have as few divisors as possible.

Lemma 10. *Let S be a numerical semigroup minimally generated by $\{n_1 < \dots < n_e\}$ with conductor c . Let $\mathfrak{m} \geq 2c - 1$ and let $M = \{\mathfrak{m} = m_1 < \dots < m_r\}$ be an amenable set. Define $L = M \cap [\mathfrak{m}, \mathfrak{m} + n_e]$. Then L is an amenable set,*

$$D(L) \cap [0, \mathfrak{m}] = D(M) \cap [0, \mathfrak{m}]$$

and

$$\# D(M) = \#(D(L) \cap [0, \mathfrak{m}]) + \#M.$$

Proof. Clearly L is amenable.

By [7, Lemma 13], $D(M) = (M \setminus L) \cup D(L)$. Hence $D(M) \cap [0, \mathfrak{m}] = ((M \setminus L) \cup D(L)) \cap [0, \mathfrak{m}] = D(L) \cap [0, \mathfrak{m}]$.

Also $D(M) = (M \setminus L) \cup (D(L) \cap [\mathfrak{m}, \infty)) \cup (D(L) \cap [0, \mathfrak{m}])$. As L is amenable, $D(L) \cap [\mathfrak{m}, \infty) = L$. Hence $D(M) = (M \setminus L) \cup L \cup (D(L) \cap [0, \mathfrak{m}]) = M \cup (D(L) \cap [0, \mathfrak{m}])$, and this union is a disjoint union, whence $\# D(M) = \#(D(L) \cap [0, \mathfrak{m}]) + \#M$. \square

Let S be a numerical semigroup, n an integer and consider the following set:

$$\text{Ap}(S, n) = \{s \in S \mid s - n \notin S\}.$$

This set is said to be the *Apéry set of S with respect to n* . The Apéry set with respect to an element of S is one of the most powerful ingredients in the study of numerical semigroups, in part because it leads to fast computations, though in the literature usually n is chosen to be a nonzero element of S .

Next we present an useful relationship between Apéry sets and divisors.

Proposition 11. *The mapping $f : \text{Ap}(S, n) \rightarrow D(\mathfrak{m} + n) \setminus D(\mathfrak{m})$, $f(s) = \mathfrak{m} + n - s$ is a bijection.*

Proof. Let us see that this map is well defined. Let $s \in \text{Ap}(S, n)$. Then $\mathfrak{m} + n - s \in D(\mathfrak{m} + n)$. On the other hand, as $\mathfrak{m} + n - s = \mathfrak{m} - (s - n)$, $\mathfrak{m} + n - s \in S$ implies that $\mathfrak{m} + n - s \notin D(\mathfrak{m})$.

The fact that f is one to one is clear.

Let now $\mathfrak{m} + n - s \in S$ be a divisor of $\mathfrak{m} + n$ (which implies that $s \in S$) that is not a divisor of \mathfrak{m} . As $\mathfrak{m} + n - s = \mathfrak{m} - (s - n)$, the fact that $\mathfrak{m} + n - s$ belongs to S and is not a divisor of \mathfrak{m} implies that $s - n \notin S$. It follows that we can take s as a pre-image of $\mathfrak{m} + n - s$, concluding in this way that f is surjective. \square

For symmetric numerical semigroups we can get an alternative description.

Remark 12. Let S be a symmetric numerical semigroup. Then $D(\mathfrak{m} + n) \setminus D(\mathfrak{m}) = \text{Ap}(S, n) + \mathfrak{m} - F(S)$.

Proof. Let $t \in D(\mathfrak{m} + n) \setminus D(\mathfrak{m})$. Then $\mathfrak{m} + n - t \in S$ and $\mathfrak{m} - t \notin S$. As S is symmetric $F(S) - (\mathfrak{m} + n - t) \notin S$ and $F(S) - (\mathfrak{m} - t) \in S$. Set $s = F(S) - (\mathfrak{m} - t) \in S$. Then $s \in \text{Ap}(S, n)$, and $t = s + \mathfrak{m} - F(S) \in \text{Ap}(S, n) + \mathfrak{m} - F(S)$.

For the other inclusion, take $s \in \text{Ap}(S, n)$. Then $\mathfrak{m} + n - (s + \mathfrak{m} - F(S)) = F(S) - (s - n) \in S$ ($s - n \notin S$ and S is symmetric), and $\mathfrak{m} - (s + \mathfrak{m} - F(S)) = F(S) - s \notin S$ (since $s \in S$ and S is symmetric). Hence $\mathfrak{m} + s - F(S) \in D(\mathfrak{m} + n) \setminus D(\mathfrak{m})$. \square

3. FENG-RAO NUMBERS OF EMBEDDING DIMENSION TWO NUMERICAL SEMIGROUPS

Let $S = \langle a, b \rangle$, with $a < b$ coprime integers greater than two. Let c be the conductor of S . A well known result of Sylvester states that $c = ab - a - b + 1$. Let \mathfrak{m} be an integer greater than or equal to $2c - 1$.

Throughout this section, the letters a , b and \mathfrak{m} shall be used with the above meanings.

This section is composed of various subsections. The first one, recalls some known facts for Weierstrass semigroups with two generators. Then we introduce some technical results that will be used in the rest of the paper. It is worth to highlight that among these, we present an explicit description of the Apéry sets with respect to any positive integer. Later we introduce a way to draw sets of integers that may help to follow the text remaining. The pictures, which show results produced with the package [6], have been

created by using the GAP [12] package IntPic [5]. These type of images helped the authors to prove the results presented in this paper. Next we show how to organize sets of divisors in triangles, and finally we discuss how to arrange them to obtain optimal configurations.

3.1. Weierstrass semigroups with two generators. For a base field \mathbb{F} of characteristic zero, it is classically known that every numerical semigroup S generated by two elements is actually a Weierstrass semigroup, in the sense that there exists an irreducible smooth projective algebraic curve χ and a point $P \in \chi$ such that the Weierstrass semigroup of χ at P is precisely S (see [18]).

Unfortunately, this result is not proven to be true also in positive characteristic. Nevertheless, there are sufficiently many examples of embedding dimension two numerical semigroups that are actually Weierstrass. In fact, provided \mathbb{F} is a perfect field of positive characteristic (a finite field, in particular), one has that the plane curve given by the equation

$$\alpha x^a + \beta y^b = \gamma$$

has genus

$$g = \frac{1}{2}(a-1)(b-1)$$

where $\alpha, \beta, \gamma \in \mathbb{F} \setminus \{0\}$, $\gcd(a, b) = 1$ and $\text{char } \mathbb{F} \nmid a \cdot b$ (see [21]).

It is easy to check that the rational functions x and y have a unique pole at P , P being the only point at infinity, of order b and a respectively, so that the semigroup $S = \langle a, b \rangle$ is contained in the Weierstrass semigroup Γ of χ at P . But since both semigroups S and Γ have the same genus, one concludes that $S = \Gamma$.

The above example shows that, for a given characteristic p , infinitely many embedding dimension two numerical semigroups are Weierstrass (those whose generators are none of them multiple of p). Conversely, a given semigroup $S = \langle a, b \rangle$ is Weierstrass for every characteristic p but for a finite number of primes p (namely, those prime factors of a or b).

The above example does not work when the characteristic p divides one of the generators. For example, if $\mathbb{F} = \mathbb{F}_2$ and one considers the curve $x^4 + y^5 = 1$, the genus turns out to be 0 (use for example the library `brnoeth.lib` [9] of the computer algebra system SINGULAR [4]), so that the semigroup $S = \langle 4, 5 \rangle$ is not the Weierstrass semigroup of this curve at any of its points.

3.2. Technical results. We start by giving a procedure to decide when an integer belongs to $\langle a, b \rangle$. This criterion will be used many times in the rest of the paper. This is indeed a direct consequence of [20, Lemma 2.6], and its proof is included for the sake of completeness.

Lemma 13. *Let $u \in \{0, \dots, b-1\}$, and let v be an integer.*

1. *The integer $ua + vb \in S$ if and only if $v \geq 0$ (analogously, for $v \in \{0, \dots, a-1\}$ and $u \in \mathbb{Z}$, $ua + vb \in S$ if and only if $u \geq 0$).*
2. *If $ua + vb = u'a + v'b$ with $0 \leq u' \leq b-1$ and $v' \in \mathbb{Z}$, then $u' = u$ and $v' = v$ (the same for $v' \in \{0, \dots, a-1\}$, $u' \in \mathbb{Z}$)*

Proof. 1. Clearly, if $v \geq 0$, then $ua + vb \in S$. For the converse, if $ua + vb \in S$, then there exist $u', v' \in \mathbb{N}$ such that $ua + vb = u'a + v'b$. Assume to the contrary that $v < 0$. Then $ua = u'a + (v' - v)b$. As $b > a$, we get that $ua = u'a + (v' - v)b > (u' + v' - v)a$, and consequently $u > u'$. Hence $(u - u')a = (v' - v)b$. However, $\gcd(a, b) = 1$, which implies that $b \mid u - u'$. Since $u - u' \in \{0, \dots, b-1\}$, this forces $u = u'$, and then $v' - v = 0$, contradicting $v < 0, v' \in \mathbb{N}$.

2. If $ua + vb = u'a + v'b$, as $u'a + (v' - v)b = ua$, we have too that a divides $v' - v$ and we can write $v' - v = xa$, with $x \in \mathbb{Z}$, to obtain $(u' + xb)a = ua$. Since $0 \leq u = u' + xb \leq b-1$ and $0 \leq u' \leq b-1$, we deduce that $x = 0$ and $u = u'$.

□

We observe that for an integer n given, there exist integers u and v such that $n = ua + vb$, since a and b are coprime. Furthermore, u can be taken such that $0 \leq u < b$ (in fact, $u = na^{-1} \pmod{b}$) and, in this case, u and v are unique, by the above lemma.

Let n be a positive integer. Next we give a description of $\text{Ap}(S, n)$.

Theorem 14. *Let a and b be coprime positive integers, and let $S = \langle a, b \rangle$. Let n be an integer, and let u and v be integers with $0 \leq u < b$, such that $n = ua + vb$. Then*

$$\text{Ap}(S, n) = \begin{cases} \{\alpha a + \beta b \mid 0 \leq \alpha < u \text{ and } 0 \leq \beta < a + v\}, & \text{if } -a \leq v < 0, \\ \{\alpha a + \beta b \mid u \leq \alpha < b \text{ and } 0 \leq \beta < v\} \\ \cup \{\alpha a + \beta b \mid 0 \leq \alpha < u \text{ and } 0 \leq \beta < a + v\}, & \text{if } n \in S. \end{cases}$$

In particular,

$$\#\text{Ap}(S, n) = \begin{cases} 0 & \text{if } v < -a, \\ u(a + v) & \text{if } -a \leq v < 0, \\ n & \text{otherwise.} \end{cases}$$

Proof. Take $s = \alpha a + \beta b \in S$, with $0 \leq \alpha < b$, $\beta \geq 0$. Since $0 \leq u < b$ and $0 \leq \alpha < b$ implies that $-b < \alpha - u < b$, we have that $s - n = (\alpha - u)a + (\beta - v)b \notin S$ if and only if either $0 \leq \alpha - u < b$ and $\beta - v < 0$, or $-b < \alpha - u < 0$ and $\beta - v < a$. It follows that

$$\begin{aligned} \text{Ap}(S, n) &= \{\alpha a + \beta b \mid (0 \leq \alpha - u < b \text{ and } \beta - v < 0) \text{ and } (0 \leq \alpha < b \text{ and } \beta \geq 0)\} \\ &\cup \{\alpha a + \beta b \mid (-b < \alpha - u < 0 \text{ and } \beta - v < a) \text{ and } (0 \leq \alpha < b \text{ and } \beta \geq 0)\}. \end{aligned}$$

And the proof follows easily by studying the possible cases. \square

Observe that we recover the well known fact that the Apéry set of an element n in S has cardinality n .

In light of Proposition 11, the description of the Apéry sets for semigroups of embedding dimension two given in Theorem 14 yields a description of the set $D(\mathfrak{m} + n) \setminus D(\mathfrak{m})$, that is, of the new divisors that $\mathfrak{m} + n$ adds to those of \mathfrak{m} .

To better understand the result below, we refer the reader to the figures in Example 17.

Corollary 15. *Let $n \in \mathbb{N}$, $n = ua + vb$ with $u \in \{0, \dots, b - 1\}$ and $v \in \mathbb{Z}$. Then*

$$D(\mathfrak{m} + n) \setminus D(\mathfrak{m}) = \begin{cases} \{\mathfrak{m} + xa + yb \mid 0 < x \leq u \text{ and } -a < y \leq v < 0\}, & \text{if } v < 0, \\ \{\mathfrak{m} + xa + yb \mid u < x \leq b \text{ and } -a < y \leq v - a\} \\ \cup \{\mathfrak{m} + xa + yb \mid 0 < x \leq u \text{ and } -a < y \leq v\}, & \text{if } n \in S. \end{cases}$$

Proof. Let $s \in D(\mathfrak{m} + n) \setminus D(\mathfrak{m})$. By applying Proposition 11, there exists $\alpha a + \beta b \in \text{Ap}(S, n)$ such that

$$\begin{aligned} s &= f(\alpha a + \beta b) = \mathfrak{m} + ua + vb - (\alpha a + \beta b) = \mathfrak{m} + (u - \alpha)a + (v - \beta)b \\ &= \mathfrak{m} + (u - \alpha + b)a + (v - \beta - a)b. \end{aligned}$$

The inequalities in Theorem 14 involved in the description of $\text{Ap}(S, n)$ can be rewritten as follows.

- $0 \leq \alpha < u$ is equivalent to $0 < u - \alpha \leq u$.
- $0 \leq \beta < a + v$ if and only if $-a < v - \beta \leq v$.
- $u \leq \alpha < b$ is the same as $u < u - \alpha + b \leq b$.
- $0 \leq \beta < v$ is equivalent to $-a < v - \beta - a \leq v - a$.
- $0 \leq \beta < a + v$ corresponds to $-a < v - \beta \leq v$.

From the above inequalities and Theorem 14 we obtain

$$D(\mathfrak{m} + n) \setminus D(\mathfrak{m}) \subseteq \begin{cases} \{\mathfrak{m} + xa + yb \mid 0 < x \leq u \text{ and } -a < y \leq v < 0\}, & \text{if } v < 0, \\ \{\mathfrak{m} + xa + yb \mid u < x \leq b \text{ and } -a < y \leq v - a\} \\ \cup \{\mathfrak{m} + xa + yb \mid 0 < x \leq u \text{ and } -a < y \leq v\}, & \text{if } n \in S. \end{cases}$$

Now by Lemma 13, the set on the right hand side has the same cardinality as that of $\text{Ap}(S, n)$, which by Proposition 11, is the same as that of $D(\mathfrak{m} + n) \setminus D(\mathfrak{m})$. Hence the equality holds. \square

As a consequence of Corollary 15, we also obtain

$$(4) \quad D(\mathfrak{m}, \mathfrak{m} + n) = D(\mathfrak{m}) \cup \begin{cases} \{\mathfrak{m} + xa + yb \mid 0 < x \leq u \text{ and } -a < y \leq v < 0\}, & \text{if } v < 0, \\ \{\mathfrak{m} + xa + yb \mid u < x \leq b \text{ and } -a < y \leq v - a\} \\ \cup \{\mathfrak{m} + xa + yb \mid 0 < x \leq u \text{ and } -a < y \leq v\}, & \text{if } n \in S, \end{cases}$$

and this union is disjoint.

3.3. A way to visualize integers. Our purpose in this subsection is to construct a table where each integer appears exactly once and such that the way the integers are disposed helps the understanding of the problem treated in this paper, as well as many of the statements and proofs. Instead of the traditional arrangement of the integers in a straight line, we represent them in a an bi-infinite strip whose width depends on a given integer and the way the elements are presented depends on another integer which is smaller and coprime to the former one. The pictures, which show results produced with the package [6], have been produced by using the GAP [12] package `IntPic` [5].

Let $a, b \in \mathbb{N}$ be coprime integers such that $a < b$. We shall construct a bi-infinite table such that each row has length b . For this purpose, we choose an integer o which will work as the origin of a referential. Take the row $\{o, o + a, \dots, o + (b - 1)a\}$ (which will work as the x -axis.) The other rows are obtained by adding or subtracting multiples of b in such a way that the y -axis is $\{\dots, o - 2b, o - b, o, o + b, o + 2b, \dots\}$. Similarly, each of the other columns consist of o plus multiples of b plus a certain fixed multiple of a between 0 and $(b - 1)a$. It follows from Lemma 13 that each integer appears exactly once in the table and, if we write $n = ua + vb$, with $u \in \{1, \dots, b - 1\}$ and $v \in \mathbb{Z}$, u and v may be seen as the x -coordinate and the y -coordinate, respectively.

Throughout the paper, the integers a and b will be taken as the minimal generators of an embedding dimension two numerical semigroup and o is taken as m .

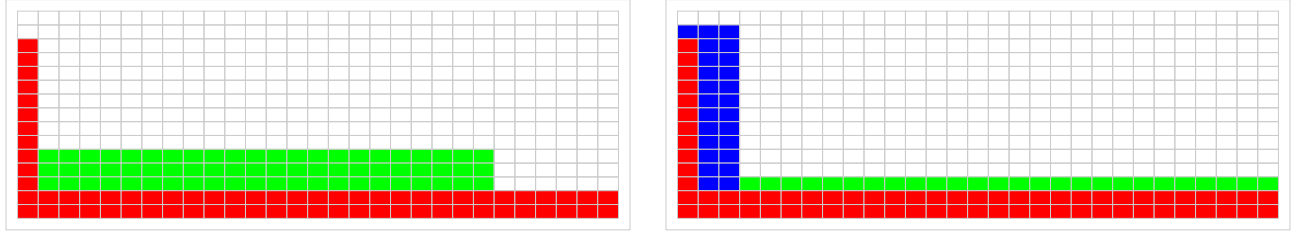
The examples in this subsection show the relevant parts of pictures that have been produced by taking $a = 11, b = 29$ and $o = m = 559$. In the next example the numbers are shown, so that the reader can easily verify which are the integers involved in the other examples. Colors (or gray tones) are used to highlight elements. For those elements belonging to more than one set whose elements are to be highlighted, we use gradients ranging through all the colors involved. The set $\{m, \dots, m + b - 1\}$ will be called the *ground* with respect to m , or simply ground when no possible confusion may arise.

Example 16. Let $a = 11, b = 29, o = m = 559$. The highlighted elements are those of the x and y axis and the ground. In this example, the steps of the ground have lengths 2 or 3.

617	628	639	650	661	672	683	694	705	716	727	738	749	760	771	782	793	804	815	826	837	848	859	870	881	892	903	914	925
588	599	610	621	632	643	654	665	676	687	698	709	720	731	742	753	764	775	786	797	808	819	830	841	852	863	874	885	896
559	570	581	592	603	614	625	636	647	658	669	680	691	702	713	724	735	746	757	768	779	790	801	812	823	834	845	856	867
530	541	552	563	574	585	596	607	618	629	640	651	662	673	684	695	706	717	728	739	750	761	772	783	794	805	816	827	838
501	512	523	534	545	556	567	578	589	600	611	622	633	644	655	666	677	688	699	710	721	732	743	754	765	776	787	798	809
472	483	494	505	516	527	538	549	560	571	582	593	604	615	626	637	648	659	670	681	692	703	714	725	736	747	758	769	780
443	454	465	476	487	498	509	520	531	542	553	564	575	586	597	608	619	630	641	652	663	674	685	696	707	718	729	740	751
414	425	436	447	458	469	480	491	502	513	524	535	546	557	568	579	590	601	612	623	634	645	656	667	678	689	700	711	722
385	396	407	418	429	440	451	462	473	484	495	506	517	528	539	550	561	572	583	594	605	616	627	638	649	660	671	682	693
356	367	378	389	400	411	422	433	444	455	466	477	488	499	510	521	532	543	554	565	576	587	598	609	620	631	642	653	664
327	338	349	360	371	382	393	404	415	426	437	448	459	470	481	492	503	514	525	536	547	558	569	580	591	602	613	624	635
298	309	320	331	342	353	364	375	386	397	408	419	430	441	452	463	474	485	496	507	518	529	540	551	562	573	584	595	606
269	280	291	302	313	324	335	346	357	368	379	390	401	412	423	434	445	456	467	478	489	500	511	522	533	544	555	566	577
240	251	262	273	284	295	306	317	328	339	350	361	372	383	394	405	416	427	438	449	460	471	482	493	504	515	526	537	548
211	222	233	244	255	266	277	288	299	310	321	332	343	354	365	376	387	398	409	420	431	442	453	464	475	486	497	508	519

The following example gives us the (correct) impression that the way the elements are sorted in the construction of the table leads to disposing the divisors in a way that makes easy to visualize and count them. The pictures are meant to illustrate the sets in Corollary 15.

Example 17. The highlighted cells in the leftmost picture, corresponding to the case $n \notin S$, are the divisors of $m (= 559)$, and the divisors of $m + n (= m + 22a - 8b = 569)$ that are not divisors of m . The highlighted cells in the picture on the right, correspond to the case $n \in S$, and are on the one hand the divisors of $m (= 559)$, and on the other hand, the divisors of $m + n (= m + 2a + b = 610)$ that are not divisors of m ; these consist of the union of two sets that are drawn by using different colors.



3.4. Ground, triangles and divisibility. Every $x \in \{0, \dots, b-1\}$ can be expressed as $ia \bmod b$ for a unique $i \in \{0, \dots, b-1\}$, because $\gcd(a, b) = 1$.

For $i \in \mathbb{N}$, we will write $\mathfrak{m} \oplus i$ for $\mathfrak{m} + (ia \bmod b)$, which is a rather convenient way to express uniquely the elements of the ground.

In order to avoid the unnecessary parentheses, we will assume that the precedence of \oplus is higher than the rest of binary operations. Thus, for instance, we will write $\mathfrak{m} \oplus i + ha$ to refer to $(\mathfrak{m} \oplus i) + ha$.

The divisors of an element in the ground, excluding the divisors of \mathfrak{m} are described in the following consequence of Corollary 15.

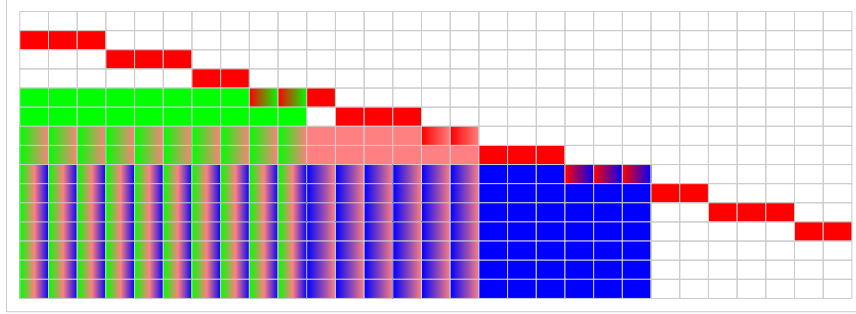
Corollary 18. *Let $i \in \{0, \dots, b-1\}$. Then*

$$D(\mathfrak{m} \oplus i) \setminus D(\mathfrak{m}) = \{\mathfrak{m} + xa + yb \mid 0 < x \leq i, -a < y \leq -\lfloor ia/b \rfloor\}.$$

Proof. Just use Corollary 15 with $ia \bmod b = ia - \lfloor ia/b \rfloor b$ ($u = i, v = -\lfloor ia/b \rfloor$). □

We are going to see that if an element divides two elements in the ground and does not divide \mathfrak{m} , then it divides all the elements between these two elements. First we give an example that may help to follow the proof.

Example 19. The divisors of $\mathfrak{m} \oplus 9$, $\mathfrak{m} \oplus 15$ and $\mathfrak{m} \oplus 21$ are represented in the following picture.



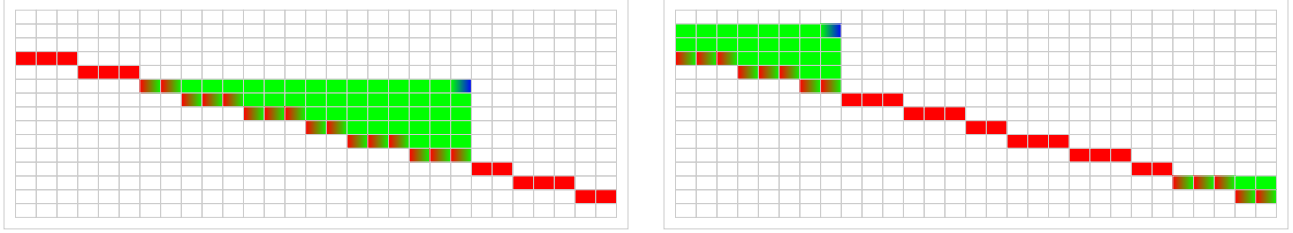
Corollary 20. *Let $i, j, k \in \{0, \dots, b-1\}$ with $i < j < k$. Then*

$$(D(\mathfrak{m} \oplus i) \cap D(\mathfrak{m} \oplus k)) \setminus D(\mathfrak{m}) \subseteq D(\mathfrak{m} \oplus j) \setminus D(\mathfrak{m}).$$

Proof. Following Corollary 18, if we take $s \in (D(\mathfrak{m} \oplus i) \cap D(\mathfrak{m} \oplus k)) \setminus D(\mathfrak{m})$, then $s = \mathfrak{m} + xa + yb$ with $0 < x \leq i$ and $-a < y \leq -\lfloor ka/b \rfloor$. Thus $0 < x \leq i < j$ and $-a < y \leq -\lfloor ka/b \rfloor \leq -\lfloor ja/b \rfloor$, and so by using again Corollary 18, we obtain that $s \in D(\mathfrak{m} \oplus j) \setminus D(\mathfrak{m})$. □

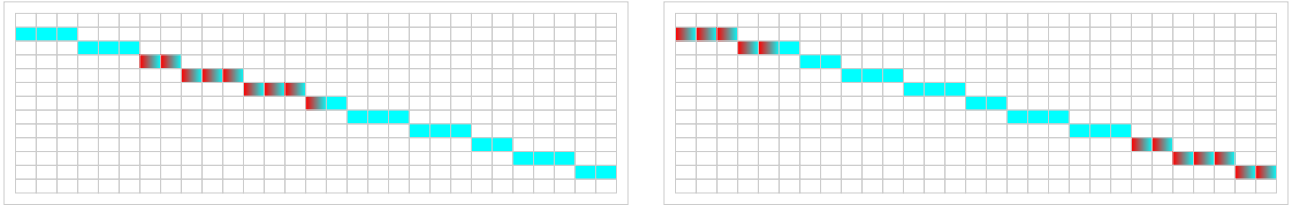
Given $n, n' \in \mathbb{N}$, $n' \leq_S n$ if and only if $D(\mathfrak{m} + n') \subseteq D(\mathfrak{m} + n)$, or equivalently, $D(\mathfrak{m} + n') \cap [\mathfrak{m}, \infty) \subseteq D(\mathfrak{m} + n) \cap [\mathfrak{m}, \infty)$. That is $n - n' \in S$ (n' divides n with respect to S) if and only if $D(\mathfrak{m} + n') \cap [\mathfrak{m}, \infty)$ is included in $D(\mathfrak{m} + n) \cap [\mathfrak{m}, \infty)$. We say that $D(\mathfrak{m} + n) \cap [\mathfrak{m}, \infty)$ is the *triangle associated* to n , and $D(\mathfrak{m} + n) \cap [\mathfrak{m}, \mathfrak{m} + b)$ is its *base*. Also we will refer to n as the *upper vertex* of the triangle. Thus we have shown that $n \leq_S n'$ if and only if the triangle associated to n is included in that associated to n' . We are going to see that we do not need to compare the whole triangles, but just the bases.

Example 21. The following pictures illustrate the two existing kinds of triangles (depending on having m in the base or not). The bases and upper vertices are highlighted.



An *interval* of the ground is a subset L of $[m, m + b)$ of the form $L = \{m \oplus i, \dots, m \oplus (i + h)\}$, with $i, h \in \{0, \dots, b-1\}$. Observe that if $ia \bmod b \geq a$, then $m + ia \bmod b - a = m \oplus (i-1) \in (D(L) \cap [m, m + b)) \setminus L$ and L is non amenable. An interval of the ground that happens to be an amenable set is said to be an *amenable interval*.

Example 22. The two existing types of amenable intervals are illustrated in the figures below. They have been obtained using, respectively, $i = 6, h = 8$ and $i = 22, h = 11$.



As we see next, every amenable interval is realizable as the base of a triangle.

Lemma 23. *Let $L = \{m \oplus i, \dots, m \oplus (i + h)\}$ be an amenable interval, with $0 \leq i, h < b$. Then $D(m \oplus i + ha) \cap [m, m + b) = L$.*

Proof. The divisors of $m \oplus i + ha$ can be expressed as $m \oplus i + ha - xa - yb$ with $x, y \in \mathbb{N}$.

Let $x, y \in \mathbb{N}$. Next we use the division algorithm to manipulate $m + ia \bmod b + ha - xa - yb$.

$$\begin{aligned} m + ia \bmod b + ha - xa - yb &= m + ia \bmod b + \lfloor (h-x)a/b \rfloor b + (h-x)a \bmod b - yb \\ &= m + ia \bmod b + (h-x)a \bmod b + (\lfloor (h-x)a/b \rfloor - y)b. \end{aligned}$$

Hence

$$(5) \quad m + ia \bmod b + ha - xa - yb = m + (i + h - x)a \bmod b + Yb,$$

where

$$Y = \begin{cases} \lfloor (h-x)a/b \rfloor - y & \text{if } 0 \leq ia \bmod b + (h-x)a \bmod b < b, \\ \lfloor (h-x)a/b \rfloor - y + 1 & \text{if } b \leq ia \bmod b + (h-x)a \bmod b < 2b. \end{cases}$$

Hence

$$m + ia \bmod b + ha - (m + (i + h - x)a \bmod b) = xa + (y + Y)b.$$

As the elements of L are in the ground and may be written as $m \oplus (i + h - x)$, with $x \in \{0, \dots, h\}$, the above equation proves that $L \subseteq D(m \oplus i + ha) \cap [m, m + b)$.

In order to prove the reverse inclusion recall that as L is amenable, we have that $ia \bmod b < a$. Thus

$$\begin{aligned} m \oplus i + ha - xa - yb &= m + (ia \bmod b) + ha - xa - yb \\ &< m + a + ha - xa - yb = m + (1 + h - x)a - yb. \end{aligned}$$

If $x > h$, then we get a divisor that is smaller than m , therefore we may assume that $x \in \{0, \dots, h\}$. Now it suffices to use again Equation (5), to see that $m \oplus i + ha - xa - yb \in [m, m + b)$ if and only if $Y = 0$, and then $m \oplus i + ha - xa - yb = m + (i + h - x)a \bmod b \in \{m \oplus i, \dots, m \oplus (i + h)\} = L$. \square

For $h > b - 1$, $D(m \oplus i + (b-1)a) \subseteq D(m \oplus i + ha)$. By Lemma 23, $D(m \oplus i + (b-1)a) \cap [m, m + b) = \{m, \dots, m + b - 1\}$, and consequently $D(m \oplus n) \cap [m, m + b) = \{m, \dots, m + b - 1\}$.

Remark 24. Let $n \in \mathbb{N}$. Then $n = ia \bmod b + ha$, with $i = (n \bmod a)a^{-1} \bmod b$ and $h = \lfloor \frac{n}{a} \rfloor$. This is because $n = \lfloor n/a \rfloor a + n \bmod a = ha + ia \bmod b$. Observe that

$$ia \bmod b = n - ha < a,$$

and thus $L = \{\mathfrak{m} \oplus i, \dots, \mathfrak{m} \oplus (i+h)\}$ is an amenable interval. Moreover, by Lemma 23, $L = D(\mathfrak{m} + n) \cap [\mathfrak{m}, \mathfrak{m} + b)$, and if $h < b$, $L \neq \{\mathfrak{m}, \dots, \mathfrak{m} + b - 1\}$.

- If $\mathfrak{m} \in L$, by Lemma 23, $\mathfrak{m} \in D(\mathfrak{m} \oplus i + ha) = D(\mathfrak{m} + n)$. Thus $\mathfrak{m} + n - \mathfrak{m} = n \in S$. The converse is trivially true. Hence $\mathfrak{m} \in L$ if and only if $n \in S$. In this setting, if $i \neq 0$, L can be written as

$$L = \{\mathfrak{m}, \dots, \mathfrak{m} \oplus (i+h-b)\} \cup \{\mathfrak{m} \oplus i, \dots, \mathfrak{m} \oplus (b-1)\}.$$

If $i = 0$, then $L = \{\mathfrak{m}, \dots, \mathfrak{m} \oplus h\}$.

- If $\mathfrak{m} \notin L$ ($n \notin S$), then $h+i < b$ (since if $i+h \geq b$, $\mathfrak{m} = \mathfrak{m} \oplus b \in L$), and $0 < ia \bmod b$ (since otherwise, $n = ha \in S$).

Observe that if M is an amenable set and $L = M \cap [\mathfrak{m}, \mathfrak{m} + b) = \{\mathfrak{m}, \dots, \mathfrak{m} + b - 1\}$, then according to Lemma 10, $\#D(M) = \#(D(\mathfrak{m}, \dots, \mathfrak{m} + b - 1) \cap [0, \mathfrak{m})) + \#M$. Hence whenever we add an element to M so that it remains amenable, the resulting number of divisors is increased just by one. Thus we are mainly interested in the case $M \cap [\mathfrak{m}, \mathfrak{m} + b) \subsetneq \{\mathfrak{m}, \dots, \mathfrak{m} + b - 1\}$.

Lemma 25. *Let $n, n' \in \mathbb{N}$ with $D(\mathfrak{m} + n) \cap [\mathfrak{m}, \mathfrak{m} + b) \neq \{\mathfrak{m}, \dots, \mathfrak{m} + b - 1\}$. Then $n' \leq_S n$ if and only if $D(\mathfrak{m} + n') \cap [\mathfrak{m}, \mathfrak{m} + b) \subseteq D(\mathfrak{m} + n) \cap [\mathfrak{m}, \mathfrak{m} + b)$.*

Proof. If $n' \leq_S n$, then as it was already mentioned above, trivially $D(\mathfrak{m} + n') \subseteq D(\mathfrak{m} + n)$.

For the converse, let i, h, i', h' be as in Remark 24, such that $n = ia \bmod b + ha$ and $n' = i'a \bmod b + h'a$. Notice that $n - n' = (h+i-h'-i')a + (\lfloor i'a/b \rfloor - \lfloor ia/b \rfloor)b = (h+i-h'-i'-b)a + (\lfloor i'a/b \rfloor - \lfloor ia/b \rfloor + a)b$.

In view of Remark 24, $D(\mathfrak{m} + n') \cap [\mathfrak{m}, \mathfrak{m} + b) = \{\mathfrak{m} \oplus i', \dots, \mathfrak{m} \oplus (i'+h')\}$, and $D(\mathfrak{m} + n) \cap [\mathfrak{m}, \mathfrak{m} + b) = \{\mathfrak{m} \oplus i, \dots, \mathfrak{m} \oplus (i+h)\}$.

As $D(\mathfrak{m} + n') \cap [\mathfrak{m}, \mathfrak{m} + b) \subseteq D(\mathfrak{m} + n) \cap [\mathfrak{m}, \mathfrak{m} + b) \neq \{\mathfrak{m}, \dots, \mathfrak{m} + b - 1\}$, we deduce $\{\mathfrak{m} \oplus i', \dots, \mathfrak{m} \oplus (i'+h')\} \subseteq \{\mathfrak{m} \oplus i, \dots, \mathfrak{m} \oplus (i+h)\}$ and $h < b - 1$. The following cases may occur.

1. If $i+h < b$, then $i \leq i'$ and $h'+i' \leq h+i$. Hence $n - n' \in S$, because $h+i-h'-i' \geq 0$ and $\lfloor i'a/b \rfloor - \lfloor ia/b \rfloor \geq 0$.
2. If $i+h \geq b$, then $\{\mathfrak{m} \oplus i, \dots, \mathfrak{m} \oplus (i+h)\} = \{\mathfrak{m}, \dots, \mathfrak{m} \oplus (i+h-b)\} \cup \{\mathfrak{m} \oplus i, \dots, \mathfrak{m} \oplus (b-1)\}$, and we have to distinguish three sub-cases.
 - i. $0 \leq i' \leq i'+h' \leq i+h-b$. In this setting, $n - n' = (h+i-b-h'-i')a + (\lfloor i'a/b \rfloor - \lfloor ia/b \rfloor + a)b$, which is in S , since $h+i-b-h'-i' \geq 0$ and $\lfloor i'a/b \rfloor - \lfloor ia/b \rfloor + a \geq 0$.
 - ii. $i \leq i' \leq i'+h' < b (\leq i+h)$. Now, $n - n' = (h+i-h'-i')a + (\lfloor i'a/b \rfloor - \lfloor ia/b \rfloor)b$, which is in S .
 - iii. $i \leq i'$ and $0 \leq i'+h'-b \leq i+h-b < i-1$. In this case, $n - n' = (h+i-h'-i')a + (\lfloor i'a/b \rfloor - \lfloor ia/b \rfloor)b$, which belongs to S . \square

Next result tells us that triangles are in some sense maximal amenable sets with respect to their base.

Corollary 26. *Let $L = \{\mathfrak{m} \oplus i, \dots, \mathfrak{m} \oplus (i+h)\} \neq \{\mathfrak{m}, \dots, \mathfrak{m} + b - 1\}$, $0 \leq i, h < b$, be an amenable interval. Let M be an amenable set such that $M \cap [0, \mathfrak{m} + b) \subseteq L$. Then $M \subseteq D(\mathfrak{m} \oplus i + ha)$.*

Proof. Let $n = ia \bmod b + ha$ and take $\mathfrak{m} + n' \in M$. As M is amenable, $D(\mathfrak{m} + n') \subseteq M$, and $D(\mathfrak{m} + n') \cap [\mathfrak{m}, \mathfrak{m} + b) \subseteq L$. In light of Lemmas 23 and 25, we have $n' \leq_S n$. Hence $\mathfrak{m} + n' \in D(\mathfrak{m} + n) \cap [\mathfrak{m}, \infty) \subseteq D(\mathfrak{m} \oplus i + ha)$. \square

3.5. Moving triangles and optimal configurations. The results obtained so far allow us to assert that an amenable set is a union of triangles (not necessarily disjoint). We see in this section how to organize these triangles so that we get a configuration with the least possible number of divisors. First we prove that the size of the triangles increases as we increase their upper vertex.

Lemma 27. *Let $n, n' \in \mathbb{N}$, $n \leq n'$. Then $\#(D(\mathfrak{m} + n) \cap [\mathfrak{m}, \infty)) \leq \#(D(\mathfrak{m} + n') \cap [\mathfrak{m}, \infty))$.*

Proof. If $\mathfrak{m} + t \in D(\mathfrak{m} + n) \cap [\mathfrak{m}, \infty)$, then $t = n - s$ with $s \in S$. Since $t + (n' - n) = n' - s$, we get $\mathfrak{m} + t + (n' - n) \in D(\mathfrak{m} + n') \cap [\mathfrak{m}, \infty)$. \square

As we saw above, triangles are uniquely determined by their bases, which are amenable intervals. Moreover, the number of divisors of the elements in the triangles smaller than \mathfrak{m} depends only on the elements in their bases. We introduce a way to arrange amenable intervals that allows us to handle easily the elements in the ground of an amenable set.

Given L and L' amenable intervals, we write $L \prec L'$ if $L \cup L'$ is not an amenable interval and either

- $\mathfrak{m} \in L$ or
- $\mathfrak{m} \notin L \cup L'$, and for all $\mathfrak{m} \oplus x \in L$ and every $\mathfrak{m} \oplus y \in L'$, $x < y$ (the condition $L \cup L'$ is not an amenable interval then forces $x + 1 < y$, and also that $L \cap L' = \emptyset$).

Remark 28. Let M be an amenable set. The set $L = M \cap [\mathfrak{m}, \mathfrak{m} + b)$ can be expressed as union of disjoint amenable intervals, $L = L_1 \cup \dots \cup L_t$, such that $L_1 \prec L_2 \prec \dots \prec L_t$.

According to Proposition 9, when looking for an optimal configuration, we may choose M amenable with $\mathfrak{m} \in M$. This is why in the following results we may impose $\mathfrak{m} \in L_1$ without loosing generality. In light of this, we will also assume that $\mathfrak{m} \oplus (b - 1) \notin L_t$ for $t > 1$, since we will take $L_1 \prec L_t$.

Proposition 29. *Let L_1, \dots, L_t be a sequence of amenable intervals with $\mathfrak{m} \in L_1 \prec \dots \prec L_t$. For $1 < i < t$,*

$$D(L_i) \setminus D(L_1 \cup \dots \cup L_{i-1} \cup L_{i+1} \cup \dots \cup L_t) = D(L_i) \setminus D(\{\mathfrak{m}\} \cup L_{i-1} \cup L_{i+1}).$$

Proof. The inclusion $D(L_i) \setminus D(L_1 \cup \dots \cup L_{i-1} \cup L_{i+1} \cup \dots \cup L_t) \subseteq D(L_i) \setminus D(\{\mathfrak{m}\} \cup L_{i-1} \cup L_{i+1})$ is trivial. For the other inclusion, let $n_i \in \mathbb{N}$ be such that $D(\mathfrak{m} + n_i) \cap [\mathfrak{m}, \mathfrak{m} + b) = L_i$ (Lemma 23). As $i > 1$, $\mathfrak{m} \notin D(L_i)$, and thus $n_i \notin S$ (Remark 24).

Let $s \in D(L_i) \setminus D(\{\mathfrak{m}\} \cup L_{i-1} \cup L_{i+1})$. Assume that $s \in D(L_j)$ with $j \notin \{i-1, i, i+1\}$. Then there exist $u, v \in \{0, \dots, b-1\}$, such that $\mathfrak{m} \oplus u \in L_j$, $\mathfrak{m} \oplus v \in L_i$ and $s \in (D(\mathfrak{m} \oplus u) \cap D(\mathfrak{m} \oplus v)) \setminus D(\mathfrak{m})$. If $u < v$, then from the hypothesis it easily follows that for all $w \in \{0, \dots, b-1\}$ with $\mathfrak{m} \oplus w \in L_{i-1}$, we have $u < w < v$. Then, by taking any of such w and by Corollary 20, we deduce that $s \in D(\mathfrak{m} \oplus w) \setminus D(\mathfrak{m}) \subseteq D(L_{i-1})$, a contradiction. If $u > v$, we proceed analogously but with L_{i+1} . \square

This result allows us to focus in what happens when we have three disjoint triangles and we want to move the one in the middle. Our aim is to change $D(\mathfrak{m}, \mathfrak{m} + n_1, \mathfrak{m} + n_2, \mathfrak{m} + n_3)$ with $D(\mathfrak{m}, \mathfrak{m} + n_1 + (h_2 + 1)a, \mathfrak{m} + n_3)$. We are going to see that in this way, the number of divisors below \mathfrak{m} decreases, while we get more over \mathfrak{m} (see the picture in Example 36).

First we see how many new divisors $\mathfrak{m} + n_2$ adds to those of $\mathfrak{m} + n_1$ and $\mathfrak{m} + n_3$. To see this we will use (4).

Lemma 30. *Let $n_1, n_2, n_3 \in \mathbb{N}$, and set $L_j = D(\mathfrak{m} + n_j) \cap [\mathfrak{m}, \mathfrak{m} + b)$, $j \in \{1, 2, 3\}$. Assume that $\mathfrak{m} \in L_1 \prec L_2 \prec L_3$. Write $n_j = u_j a + v_j b$ with $u_j \in \{0, \dots, b-1\}$ and $v_j \in \mathbb{Z}$, $j \in \{1, 2, 3\}$. Then $u_1 < u_2$, $v_3 < v_2 < 0$ and*

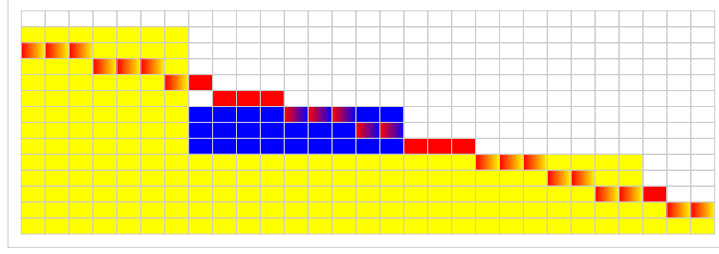
$$D(\mathfrak{m} + n_1, \mathfrak{m} + n_2, \mathfrak{m} + n_3) \setminus D(\mathfrak{m} + n_1, \mathfrak{m} + n_3) = \{\mathfrak{m} + xa + yb \mid u_1 < x \leq u_2, v_3 < y \leq v_2\}.$$

Proof. Observe that $D(\mathfrak{m} + n_1, \mathfrak{m} + n_2, \mathfrak{m} + n_3) \setminus D(\mathfrak{m} + n_1, \mathfrak{m} + n_3) = D(\mathfrak{m}, \mathfrak{m} + n_2) \setminus D(\mathfrak{m} + n_1, \mathfrak{m} + n_3) = (D(\mathfrak{m}, \mathfrak{m} + n_2) \setminus D(\mathfrak{m}, \mathfrak{m} + n_1)) \cap (D(\mathfrak{m}, \mathfrak{m} + n_2) \setminus D(\mathfrak{m}, \mathfrak{m} + n_3))$.

Let $i_j, h_j \in \{0, \dots, b-1\}$ be such that $n_j = i_j a \bmod b + h_j a$ as in Remark 24. Then we have that $L_j = \{\mathfrak{m} \oplus i_j, \dots, \mathfrak{m} \oplus (i_j + h_j)\}$. The condition $L_1 \prec L_2 \prec L_3$, implies that $n_2, n_3 \notin S$. And as $\mathfrak{m} \in L_1$, by Remark 24 again, if $i_1 \neq 0$, $i_1 + h_1 \geq b$. Hence $u_2 = i_2 + h_2$, $u_3 = i_3 + h_3$, $v_2 = -\lfloor i_2 a / b \rfloor$ and $v_3 = -\lfloor i_3 a / b \rfloor$. If $i_1 \neq 0$, then $u_1 = i_1 + h_1 - b$, $v_1 = a - \lfloor i_1 a / b \rfloor$, and if $i_1 = 0$, $u_1 = h_1$ and $v_1 = 0$. Hence $u_1 < u_2 < u_3$ and $v_1 - a < v_3 < v_2 < 0 \leq v_1$.

The proof now follows by using (4). \square

Example 31. The following picture illustrates the sets involved in the preceding lemma. It was made taking $n_1 = 6a + b, n_2 = 15a - 4b$ and $n_3 = 25a - 7b$.



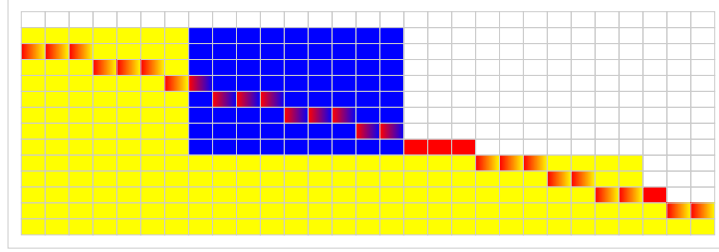
Now we see how many new divisors $\mathfrak{m} + n_1 + ka$ and $\mathfrak{m} + ka$ add to those of $\mathfrak{m} + n_1$ and $\mathfrak{m} + n_3$.

Lemma 32. *Let $n_1, n_3 \in \mathbb{N}$, and set $L_j = D(\mathfrak{m} + n_j) \cap [\mathfrak{m}, \mathfrak{m} + b)$, $j \in \{1, 3\}$. Assume that $\mathfrak{m} \in L_1 \prec L_3$, and let $k \in \mathbb{N}$ be such that $D(\mathfrak{m} + n_1 + ka) \cap [\mathfrak{m}, \mathfrak{m} + b) \prec L_3$. Write $n_j = u_j a + v_j b$ with $u_j \in \{0, \dots, b-1\}$ and $v_j \in \mathbb{Z}$, $j \in \{1, 3\}$. Then*

$$D(\mathfrak{m} + n_1 + ka, \mathfrak{m} + n_3) \setminus D(\mathfrak{m} + n_1, \mathfrak{m} + n_3) = \{\mathfrak{m} + xa + yb \mid u_1 < x \leq u_1 + k, v_3 < y \leq v_1\}.$$

Proof. As in Lemma 30 the proof follows from (4). □

Example 33. The following picture illustrates the sets involved in the preceding lemma. It was made by taking $n_1 = 6a + b, k = 9$ and $n_3 = 25a - 7b$.

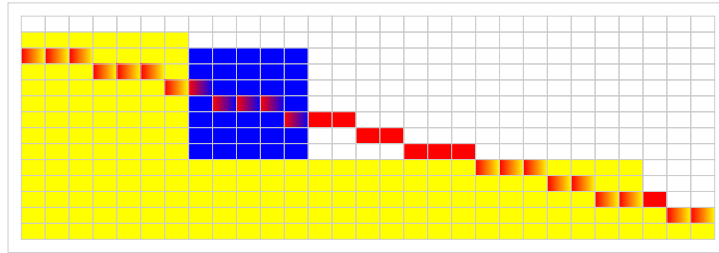


Lemma 34. *Let $n_1, n_3 \in \mathbb{N}$, and set $L_j = D(\mathfrak{m} + n_j) \cap [\mathfrak{m}, \mathfrak{m} + b)$, $j \in \{1, 3\}$. Write $n_j = u_j a + v_j b$ with $u_j \in \{0, \dots, b-1\}$ and $v_j \in \mathbb{Z}$, $j \in \{1, 3\}$. Assume that $\mathfrak{m} \in L_1 \prec L_3$, and let $k \in \mathbb{N}$ be such that $D(\mathfrak{m} + ka) \cap [\mathfrak{m}, \mathfrak{m} + b) \prec L_3$ and $k \geq u_1$. Then*

$$D(\mathfrak{m} + ka, \mathfrak{m} + n_3) \setminus D(\mathfrak{m} + n_1, \mathfrak{m} + n_3) = \{\mathfrak{m} + xa + yb \mid u_1 < x \leq k, v_3 < y \leq 0\}.$$

Proof. Again, the proof follows from (4). □

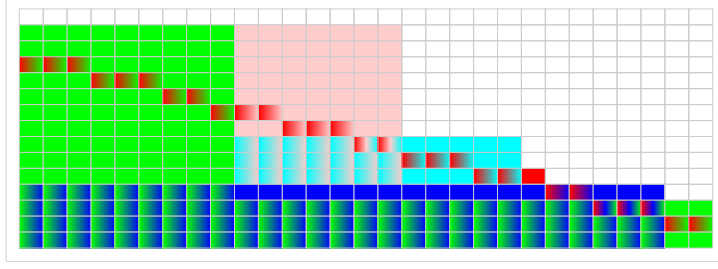
Example 35. The following figure represents the sets involved in Lemma 34. We used $n_1 = 6a + b, k = 11$ and $n_3 = 25a - 7b$.



The next task is to see that the number of divisors below \mathfrak{m} decreases when we change n_2 with $n_1 + (h_2 + 1)a$.

Example 36. Let us compare these sets in an example. Set $n_1 = 146, n_2 = 75$ and $n_3 = 54$, which corresponds to $u_1 = n_1 a^{-1} \bmod b = 8, v_1 = 2, u_2 = 20, v_2 = -5, u_3 = 26, v_3 = -8, i_1 = (n_1 \bmod$

$a)a^{-1} \bmod b = 24, h_1 = 13, i_2 = 14, h_2 = 6, i_3 = 22, h_3 = 4.$



Lemma 37. *Let $n_1, n_2, n_3 \in \mathbb{N}$, and set $L_j = D(m + n_j) \cap [m, m + b)$, $j \in \{1, 2, 3\}$. Assume that $m \in L_1 \prec L_2 \prec L_3$. Let $i_j, h_j \in \{0, \dots, b-1\}$ be such that $n_j = i_j a \bmod b + h_j a$ (as in Remark 24). Then*

$$\#(D(m + n_1, m + n_2, m + n_3) \cap [m, \infty)) \leq \#(D(m + n_1 + (h_2 + 1)a, m + n_3) \cap [m, \infty)).$$

Proof. By using Lemma 25 and that $L_1 \prec L_2 \prec L_3$, it is easy to prove that $\#(D(m + n_1, m + n_2, m + n_3) \cap [m, \infty)) = \sum_{j=1}^3 \#(D(m + n_j) \cap [m, \infty))$ and $\#(D(m + n_1 + (h_2 + 1)a, m + n_3) \cap [m, \infty)) = \#(D(m + n_1 + (h_2 + 1)a) \cap [m, \infty)) + \#(D(m + n_3) \cap [m, \infty))$.

As $m \in L_1 \prec L_2$, we get $m \notin L_2$, and thus by Remark 24, $n_2 \notin S$. Therefore, Remark 24, asserts that $0 < i_2 a \bmod b < a$. Thus, $h_2 a < n_2 = i_2 a \bmod b + h_2 a < (h_2 + 1)a$. Hence, from Lemma 27, $\#(D(m + n_2) \cap [m, \infty)) \leq \#(D(m + (h_2 + 1)a) \cap [m, \infty))$.

For every element $m+x$ in $D(m+(h_2+1)a) \cap [m, \infty)$, we have $m+n_1+x \in D(m+n_1+(h_2+1)a) \setminus D(m+n_1)$. Hence $\#(D(m + n_1 + (h_2 + 1)a) \cap [m, \infty)) - \#(D(m + n_1) \cap [m, \infty)) \geq \#(D(m + (h_2 + 1)a) \cap [m, \infty)) \geq \#(D(m + n_2) \cap [m, \infty))$. This proves $\#(D(m + n_1 + (h_2 + 1)a) \cap [m, \infty)) \geq \#(D(m + n_1) \cap [m, \infty)) + \#(D(m + n_2) \cap [m, \infty))$. \square

And now we show that we gain divisors over m .

Lemma 38. *Let $n_1, n_2, n_3 \in \mathbb{N}$, and set $L_j = D(m + n_j) \cap [m, m + b)$, $j \in \{1, 2, 3\}$. Assume that $m \in L_1 \prec L_2 \prec L_3$. Let $i_j, h_j \in \{0, \dots, b-1\}$ be such $n_j = i_j a \bmod b + h_j a$ (as in Remark 24). Then*

$$\begin{aligned} & \#((D(m + n_1, m + n_2, m + n_3) \setminus D(m + n_1, m + n_3)) \cap [0, m)) \\ & \geq \#((D(m + n_1 + (h_2 + 1)a, m + n_3) \setminus D(m + n_1, m + n_3)) \cap [0, m)). \end{aligned}$$

Proof. For $j \in \{1, 2, 3\}$, let $u_j \in \{0, \dots, b-1\}$ and $v_j \in \mathbb{Z}$ such that $n_j = u_j a + v_j b$. Then, as above, either $u_1 = i_1 + h_1 - b$ and $v_1 = a - \lfloor i_1 a / b \rfloor$ ($i_1 \neq 0$), or $u_1 = h_1$ and $v_1 = 0$ ($i_1 = 0$). Also $u_2 = i_2 + h_2$, $u_3 = i_3 + h_3$, $v_2 = -\lfloor i_2 a / b \rfloor$ and $v_3 = -\lfloor i_3 a / b \rfloor$. Remark 24 describes both L_1 and L_2 , and as a consequence of $L_1 \prec L_2$, we get $u_1 < i_2$. Thus $u_1 + h_2 + 1 \leq i_2 + h_2 = u_2$. Let

$$\begin{aligned} A &= D(m + n_1, m + n_2, m + n_3) \setminus D(m + n_1, m + n_3), \\ B &= D(m + (u_1 + h_2 + 1)a, m + n_3) \setminus D(m + n_1, m + n_3), \\ C &= D(m + u_2 a, m + n_3) \setminus D(m + n_1, m + n_3). \end{aligned}$$

From Lemmas 30 and 34, we deduce that

$$\begin{aligned} A &= \{m + xa + yb \mid u_1 < x \leq u_2, v_3 < y \leq v_2\}, \\ B &= \{m + xa + yb \mid u_1 < x \leq u_1 + h_2 + 1, v_3 < y \leq 0\}, \\ C &= \{m + xa + yb \mid u_1 < x \leq u_2, v_3 < y \leq 0\}. \end{aligned}$$

Notice that $A \subseteq C$, and $B \subseteq C$. Also

$$\begin{aligned} C \setminus A &= \{m + xa + yb \mid u_1 < x \leq u_2, v_2 < y \leq 0\}, \\ C \setminus B &= \{m + xa + yb \mid u_1 + h_2 + 1 < x \leq u_2, v_3 < y \leq 0\}. \end{aligned}$$

Define

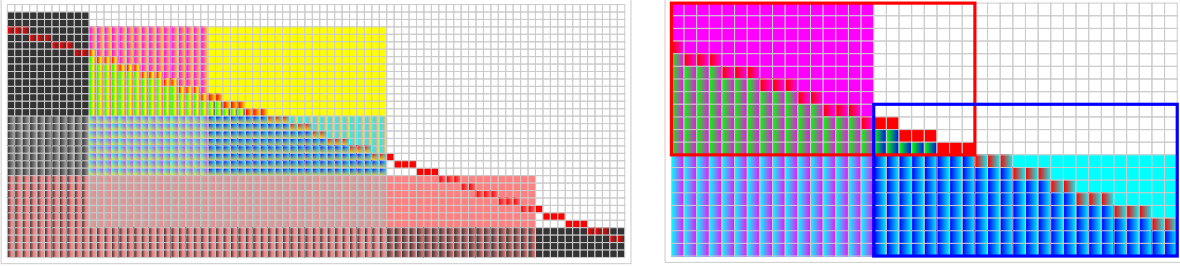
$$\begin{aligned} R_A &= \{m + xa + yb \mid u_1 < x \leq u_2 - h_2 - 1, v_2 < y \leq 0\}, \\ R_B &= \{m + xa + yb \mid u_1 + h_2 + 1 < x \leq u_2, v_3 < y \leq v_3 - v_2\}. \end{aligned}$$

Then we can write

$$R_A = v_A + D, \quad R_B = v_B + D,$$

where $D = \{xa + yb \mid u_1 - u_2 + h_2 + 1 < x \leq 0, 0 \leq y < -v_2\}$, $v_A = m + (u_2 - h_2 - 1)a + (v_2 + 1)b$ and $v_B = m + u_2a + (v_3 + 1)b$.

The following figure illustrates the regions involved in this proof. The one on the right corresponds to the region C where A , B , R_A , R_B , $R_A \cap [0, m)$ and $R_B \cap [0, m)$ are highlighted.



We claim that $(C \setminus A) \cap [0, m) \subseteq R_A \cap [0, m)$. Let us prove that $(C \setminus A) \setminus R_A \subseteq [m, \infty)$. To this end, observe that $(C \setminus A) \setminus R_A = \{m + xa + yb \mid u_2 - h_2 - 1 < x \leq u_2, v_2 < y \leq 0\} \subseteq v_A + \mathbb{N}$. In addition, $v_A = m + u_2a - h_2a - a + v_2b + b = m - h_2a + n_2 - a + b = m + i_2a \pmod{b - a + b}$. As $L_1 \prec L_2$, we have $m \notin L_2$, whence by Remark 24, $n_2 \notin S$, and by Remark 24, $0 < i_2a \pmod{b} < a$. Thus, $m + b - a < v_A < m + b$.

Since $R_B \subseteq C \setminus B$, we get trivially that $R_B \cap [0, m) \subseteq (C \setminus B) \cap [0, m)$.

Finally we prove that $R_A \cap [0, m) \leftrightarrow R_B \cap [0, m)$. First, $v_B = m + (i_2 + h_2)a - [i_3a/b]b + b = m + (i_2 + h_2 - i_3)a + i_3a \pmod{b + b}$. Again, by Remark 24, $i_3a \pmod{b} < a$, and thus $v_B < m + (i_2 + h_2 + 1 - i_3)a + b$. Moreover, $L_2 \prec L_3$, whence $i_2 + h_2 + 1 < i_3$, and consequently $v_B < m - a + b < v_A$. For every $n \in R_A \cap [0, m)$, $n = v_A + x$, $x \in D$, and $v_A + x < m$. Hence $v_B + x < v_A + x < m$. This implies that the map $R_A \rightarrow R_B$, $v_A + x \mapsto v_B + x$ is injective and maps elements in $R_A \cap [0, m)$ to elements in $R_B \cap [0, m)$.

Therefore, $\#((C \setminus A) \cap [0, m)) \leq \#(R_A \cap [0, m)) \leq \#(R_B \cap [0, m)) \leq \#((C \setminus B) \cap [0, m))$. Hence $\#(B \cap [0, m)) \leq \#(A \cap [0, m))$. In view of Lemma 32, $(D(m + n_1 + (h_2 + 1)a, m + n_3) \setminus D(m + n_1, m + n_3)) \cap [0, m) \subseteq B$. Thus $\#((D(m + n_1 + (h_2 + 1)a, m + n_3) \setminus D(m + n_1, m + n_3)) \cap [0, m)) \leq \#(B \cap [0, m)) \leq \#(A \cap [0, m)) = \#((D(m + n_1, m + n_2, m + n_3) \setminus D(m + n_1, m + n_3)) \cap [0, m))$. \square

Remark 39. Lemmas 30 to 38 also hold if we only take $n_1, n_2 \in \mathbb{N}$ with $m \in L_1 \prec L_2$. The role played by v_3 is in this setting played by $v_1 - a = -[i_1a/b]$ if $i_1 \neq 0$ and by $-a$ otherwise.

We are going to prove that in addition to the condition $m \in M$ (Proposition 9), in order to find an optimal configuration, we can also assume that the set $M \cap [m, m + b)$ is an amenable interval.

Lemma 40. *Let $M \subseteq [m, \infty)$ be an amenable set with $m \in M$, such that $\#D(M)$ is the minimum of $\#D(M')$ with $M' \subseteq [m, \infty)$ amenable, $m \in M'$ and $\#M = \#M'$. Then we can assume that $M \cap [m, m + b)$ is an amenable interval.*

Proof. We know that $L = M \cap [m, m + b)$ is of the form $L = L_1 \cup \dots \cup L_t$ with L_1, \dots, L_t amenable intervals such that $m \in L_1 \prec \dots \prec L_t$. Take M with t minimum. Assume that $t > 1$.

Let $r = \#M$. Let $n_1, \dots, n_t \in \{0, \dots, ab - 1\}$ be such that $D(m + n_i) \cap [m, m + b) = L_i$, and let $i_j, h_j \in \{0, \dots, b - 1\}$ be such that $n_j = i_ja \pmod{b} + h_ja$, $j \in \{1, \dots, t\}$ (Remark 24). Let $D = D(m + n_1, \dots, m + n_t) \cap [m, \infty)$. Then $D \cap [m, m + b) = L$, and by Lemma 25, $M \subseteq D$ and D is an amenable set.

Consider now $D' = D(m + n_1 + (h_2 + 1)a, m + n_3, \dots, m + n_t) \cap [m, \infty)$. Then $\#D' \geq r$ and $\#(D(D') \cap [0, m)) \leq \#(D(D) \cap [0, m)) = \#(D(M) \cap [0, m))$ in view of Lemmas 37 and 38, Propositions 29 and 10, and Remark 39. Observe that if we set $L' = D' \cap [m, m + b)$, then $\#L = \#L'$ by Lemma 23.

Finally we construct M' by changing D' with $D' \setminus \{\max(D')\}$ as many times as needed until M' has r elements. We can do this because $\#D' \geq r$. Then M' is amenable and $M' \cap [m, m + b) = L'$ (this last assertion holds because $\#L = \#L'$, and thus in the process of removing $\max(D')$ we never take elements in L'). By Proposition 10, $\#D(M') = \#(D(D') \cap [0, m)) + r \leq \#(D(D) \cap [0, m)) + r = \#(D(M) \cap [0, m)) + r = \#D(M)$. The minimality of $\#D(M)$ forces $\#D(M') = \#D(M)$. However, L' , in its decomposition as amenable intervals, has one interval less than L , contradicting the minimality of t . \square

Our next goal is to prove that the amenable set $D(\mathfrak{m} + \rho_r) \cap [m, \infty)$ is an optimal configuration (actually with r elements in light of Remark 2). First we need a result comparing the divisors below \mathfrak{m} while we move upwards in S .

Lemma 41. *For every $t, r \in \mathbb{N}$, with $t \geq r$, $\#(D(\mathfrak{m} + \rho_t) \cap [0, \mathfrak{m})) \geq \#(D(\mathfrak{m} + \rho_r) \cap [0, \mathfrak{m}))$.*

Proof. Observe that $\rho_t - \rho_{t-1} \geq 1$. Hence by induction $\rho_t - \rho_r \geq t - r$.

From Proposition 3, we deduce that $\#D(\mathfrak{m} + \rho_t) = \#D(\mathfrak{m}) + \rho_t$ and $\#D(\mathfrak{m} + \rho_r) = \#D(\mathfrak{m}) + \rho_r$. Hence, $\#D(\mathfrak{m} + \rho_r) + \rho_t - \rho_r = \#D(\mathfrak{m} + \rho_t)$.

By Proposition 10 and Remark 2, $\#D(\mathfrak{m} + \rho_t) = \#(D(\mathfrak{m} + \rho_t) \cap [0, \mathfrak{m})) + t$ and $\#D(\mathfrak{m} + \rho_r) = \#(D(\mathfrak{m} + \rho_r) \cap [0, \mathfrak{m})) + r$. Hence, $\#(D(\mathfrak{m} + \rho_t) \cap [0, \mathfrak{m})) = \#(D(\mathfrak{m} + \rho_r) \cap [0, \mathfrak{m})) + \rho_t - \rho_r - (t - r)$. As $\rho_t - \rho_r \geq t - r$ we conclude that $\#(D(\mathfrak{m} + \rho_t) \cap [0, \mathfrak{m})) \geq \#(D(\mathfrak{m} + \rho_r) \cap [0, \mathfrak{m}))$. \square

Lemma 42. *Let $M \subseteq [m, \infty)$ be an amenable set with $\mathfrak{m} \in M$, and such that $\#D(M)$ is the minimum of $\#D(M')$ with $M' \subseteq [m, \infty)$ amenable, $\mathfrak{m} \in M'$ and $\#M = \#M'$. Then $\#D(M) = \mathfrak{m} + 1 - 2g + \rho_r$, where $r = \#M$.*

Proof. In light of Lemma 40, we may assume that $L = M \cap [m, m + b)$ is an amenable interval. It may happen that L coincides with the ground or that it is strictly contained in it. We consider these two cases separately.

1. $L = \{m, \dots, m + b - 1\}$. Let $L' = D(\mathfrak{m} + \rho_r) \cap [m, m + b)$. In view of Proposition 10, $\#D(M) = \#(D(M) \cap [0, \mathfrak{m})) + r$. Also, $\#(D(M) \cap [0, \mathfrak{m})) = \#(D(L) \cap [0, \mathfrak{m})) \geq \#(D(L') \cap [0, \mathfrak{m})) = \#(D(\mathfrak{m} + \rho_r) \cap [0, \mathfrak{m}))$. Hence $\#D(M) \geq \#(D(\mathfrak{m} + \rho_r) \cap [0, \mathfrak{m})) + r = \#(D(\mathfrak{m} + \rho_r) \cap [0, \mathfrak{m})) + \#(D(\mathfrak{m} + \rho_r) \cap [m, \infty))$ (Remark 2). We conclude that $\#D(M) \geq \#D(\mathfrak{m} + \rho_r)$, and, by minimality of $\#D(M)$, the equality holds. Proposition 3 then asserts that $\#D(M) = \mathfrak{m} + 1 - 2g + \rho_r$.

2. $L \neq \{m, \dots, m + b - 1\}$. Let $n \in \mathbb{N}$ be such that $L = D(\mathfrak{m} + n) \cap [m, m + b)$. Such an element exists by Lemma 23. Since $\mathfrak{m} \in L \subseteq D(\mathfrak{m} + n)$, we have that $n = \mathfrak{m} + n - \mathfrak{m} \in S$. Let $D = D(\mathfrak{m} + n) \cap [m, \infty)$.

Let $t = \#D$. By Remark 2, we have that $n = \rho_t$. In view of Corollary 26, $r \leq t$. From Lemma 41 follows that $\#(D(\mathfrak{m} + \rho_t) \cap [0, \mathfrak{m})) \geq \#(D(\mathfrak{m} + \rho_r) \cap [0, \mathfrak{m}))$.

Proposition 10 ensures that $\#D(M) = \#(D(L) \cap [0, \mathfrak{m})) + r = \#(D(\mathfrak{m} + \rho_t) \cap [0, \mathfrak{m})) + r \geq \#(D(\mathfrak{m} + \rho_t) \cap [0, \mathfrak{m})) + r = \#D(\mathfrak{m} + \rho_t)$. By the minimality of $\#D(M)$, we get $\#D(M) = \#D(\mathfrak{m} + \rho_r)$.

Finally, it suffices to use the equality $\#D(\mathfrak{m} + \rho_r) = \mathfrak{m} + \rho_r + 1 - 2g$ (Proposition 3). \square

Now that we know that $D(\mathfrak{m} + \rho_r) \cap [m, \infty)$ is an optimal configuration with r elements, computing $E(S, r)$ is an easy task.

Theorem 43. *Let $S = \{0 = \rho_1 < \rho_2 < \dots < \rho_n < \dots\}$ be an embedding dimension two numerical semigroup. Then $E(S, r) = \rho_r$.*

Proof. Follows from the definition of $E(S, r)$, Proposition 9 and Lemma 42. \square

Since embedding dimension two numerical semigroups are symmetric, by using the fact that $\delta_{FR}^r(m) \geq m + 1 - 2g + E(S, r)$ for $m \geq c$, and equality holds if $m = 2g - 1 + \rho_k$ for some $k \geq 2$, one easily obtains the following consequence.

Corollary 44. *Let $S = \{0 = \rho_1 < \rho_2 < \dots < \rho_n < \dots\}$ be an embedding dimension two numerical semigroup. Then*

1. $\delta_{FR}^r(m) = \rho_r + \rho_k$ if $m = 2g - 1 + \rho_k$ with $k \geq 2$,
2. $\delta_{FR}^r(m) \geq \rho_r + \ell_i$ if $m = 2g - 1 + \ell_i$, where $\ell_i \in G(S)$ is a gap of S , for $i \in \{1, \dots, g\}$.

Remark 45. The above result, together with [17, Theorem 5.5] suggests the question of whether the following formula holds for $m \geq c$ in a numerical semigroup generated by two elements:

$$\delta_{FR}^r(m) = \min\{\rho_r + \rho_k \mid \rho_k \geq m + 1 - 2g\}.$$

However, this question has in general a negative answer. In fact, consider the semigroup $S = \langle 2, 5 \rangle$ (hyperelliptic) with genus $g = 2$ and conductor $c = 4$. If we take $r = 3$ and $m = 4 = (2g - 1) + 1$, the Feng-Rao number is $E_3 = \rho_3 = 4$, so that the Feng-Rao distance is

$$\delta_{FR}^3(4) \geq 5,$$

and the result of applying the above formula is 6. Nevertheless, the Feng-Rao distance is actually $\delta_{FR}^3(4) = 5$, since

$$D(4, 5, 7) = \{0, 2, 4, 5, 7\}.$$

4. EXAMPLES AND CONCLUSIONS

The results of the previous section, in particular Corollary 44, allows easily to prove the following Theorem 46, improving the Theorem 2.8 in [17]. We first recall the definition of the generalized (Hamming) weights. In fact, we define the support of a linear code C as

$$\text{supp}(C) := \{i \mid c_i \neq 0 \text{ for some } \mathbf{c} \in C\}.$$

Thus, the r th generalized weight of C is defined by

$$d_r(C) := \min\{\#\text{supp}(C') \mid C' \text{ is a linear subcode of } C \text{ with } \dim(C') = r\}.$$

Of course, the above definition only makes sense if $r \leq k$, where k is the dimension of C . The set of numbers

$$\text{GHW}(C) := \{d_1, \dots, d_k\}$$

is called the *weight hierarchy* of the code C (see [19]).

Theorem 46. *Let $S = \{0 = \rho_1 < \rho_2 < \dots < \rho_n < \dots\}$ be an embedding dimension two numerical semigroup. Then*

$$d_r(C_m) \geq \delta_{FR}(m+1) + \rho_r$$

for $r = 1, \dots, k_m$, where C_m is a code in an array of codes as in [17] (for example, C_m being a one-point AG code associated to a divisor of the form $G = mP$), and k_m is the dimension of C_m .

Proof. Since $E(S, r) = \rho_r$ and $\delta_{FR}^r(m) \geq m + 1 - 2g + E(S, r)$ for $m \geq c$, we just apply that $d_r(C_m) \geq \delta_{FR}^r(m+1)$. \square

Note that k_m depends not only on m , but also on the length of C_m in the array of codes. For example, if $C_m \equiv C_\Omega(D, mP)$ is again a one-point AG code, it depends on the number of points n that are used for evaluation, that is

$$k_m = n - \#\{\rho \in S \mid \rho \leq m\} = n - m + g - 1,$$

provided $2g - 2 < m < n$ and $m \in S$.

Remark 47. Theorem 46 improves [17, Theorem 2.8], which states

$$d_r(C_m) \geq \delta_{FR}(m+1) + (r-1).$$

This inequality is actually a consequence of the inequality $\delta_{FR}^r(m) \geq m + 1 - 2g + E(S, r)$, by taking into account that $E(S, r) \geq r - 1$. In fact $E(S, r) \geq r$ if the genus of S is $g > 0$ (see [10]). The improvement follows from the fact that $E(S, r) = \rho_r$ is larger than $r - 1$ if $r \geq 2$ and $g \geq 1$.

On the other hand, the generalized Griesmer bound for the generalized Hamming weights states that

$$d_r(C) \geq \sum_{i=1}^{r-1} \left\lceil \frac{d(C)}{q^i} \right\rceil,$$

where $d(C) \equiv d_1(C)$ is the minimum distance of the code C , which is defined over the finite field \mathbb{F}_q (see [15]). In particular, for $r = 2$ one has

$$d_2(C) \geq d(C) + \left\lceil \frac{d(C)}{q} \right\rceil.$$

Since we are just using the semigroup for estimating the generalized Hamming weights, we can substitute $d(C_m)$, C_m being in an array of codes as in [17], by the order bound $\delta_{FR}(m+1)$ obtaining the bound

$$d_r(C_m) \geq \sum_{i=1}^{r-1} \left\lceil \frac{\delta_{FR}(m+1)}{q^i} \right\rceil.$$

We may call this bound the *Griesmer order bound*. For $r = 2$ this bound becomes

$$d_2(C) \geq \delta_{FR}(m+1) + \left\lceil \frac{\delta_{FR}(m+1)}{q} \right\rceil.$$

The maximum values of these bounds are achieved in the binary case $q = 2$.

Remark 48. We have previously remarked that our bound in Theorem 46 is better than the one in [17]. The difference of both bounds is constant in $m \geq c$, when r is fixed.

In order to compare the bound in Theorem 46 with the Griesmer order bound, we first note that such a comparison depends on several parameters, namely the cardinality q of the finite field, the order r and the element m in the semigroup. Here we present some conclusions from our experimental results.

- We first note that in the following tables there will be a delay of one unit, because the bound for the code C_m corresponds to $m+1$ in the Feng-Rao distances. More precisely, in the first row of the tables m corresponds to the code C_m , whereas the Feng-Rao distances used in the second and third rows correspond to $m+1$.

On the other hand, note that for a semigroup generated by two elements, the minimum formula

$$(6) \quad \delta_{FR}(m+1) = \min\{\rho_k \mid \rho_k \geq m+2-2g\}$$

holds for $m \geq c$ (see [17]). Thus, the classical Feng-Rao distance comes in bursts of repeated values, according to intervals of gaps (deserts) of the form $m+2-2g$ preceding the ρ_k achieving the minimum in Formula (6). As a consequence, the corresponding Griesmer order bound also comes in bursts, and jumps just after the corresponding ρ_k .

- Our bound is increasing one by one with m , while the Griesmer order bound jumps at values of m corresponding to gaps of the form $m+2-2g$ starting a desert. Moreover, when there is no such a gap, the Griesmer order bound increases by one or more. Therefore, this bound tends to improve our bound as m becomes large, or if m corresponds to a gap at the beginning of a long desert. Nevertheless, our bound seems to be better for small values of m of the form $2g-2+\rho_k$, and also at the end of the desert preceding to such a ρ_k . For example, for $S = \langle 7, 11 \rangle$ and $r = 2$ we obtain with GAP the following results for $q = 2$,

m	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	...
GFR	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	...
GOB	11	11	11	11	11	11	17	17	17	17	21	21	21	27	27	27	27	...

where the row GFR corresponds to our bound with the generalized Feng-Rao distance, and the row GOB corresponds to the Griesmer order bound.

- On the other hand, if we increase r the difference of both bounds for $m = c$ becomes larger, so that our bound GFR remains better for more values of m . For instance, if we take $r = 10$ in the previous example, GFR is better for $m \leq 50$ and GOB is better for $m \geq 51$:

m	30	31	32	33	34	35	36	37	38	39	40	...	50	51	52	53
GFR	31	32	33	34	35	36	37	38	39	40	41	...	51	52	53	54
GOB	20	20	20	20	20	20	28	28	28	28	33	...	49	56	56	56

- Finally, as the size q of the finite field increases, the jumps of the Griesmer order bound become smaller, so that our bound is better for much more values of m . For example, if we switch in the last example to $q = 16$, our bound is much better in the whole interval $c \leq m \leq 2c - 1$.
- In general, the experimental results above suggest that a good strategy to estimate the generalized Hamming weights by means of the underlying numerical semigroup S is to combine both, the generalized Feng-Rao distances and the Griesmer order bound, depending on the parameters q , r and m . Roughly speaking, our bound GFR is better for q and r large, whereas the bound GOB is better otherwise, provided m is large or it corresponds to a gap of the form $m+1-2g$ at the beginning of a long desert.

We finally test these bounds in the case of Hermitian codes.

Example 49. Consider the Hermitian codes over \mathbb{F}_{16} (see [16] for further details). The involved semigroup is $S = \langle 4, 5 \rangle$ and the length of the codes is $n = 64$. Since the conductor is $c = 12$ and the genus is $g = 6$,

the dimension of the codes is $69 - m$, for $12 \leq m \leq 63$. Our computations with GAP show that our bound GFR is always better than (or equal to) the Griesmer order bound. For example, if $r = 2$ we obtain

m	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	...	58	59	...	63
GFR	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	...	52	53	...	57
GOB	5	5	5	6	9	9	9	10	11	13	13	14	15	16	17	19	20	...	51	53	...	57

and our bound GFR improves as r gets higher. In fact, for $r \geq 4$ the GFR bound is strictly better than the Griesmer one.

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CMUP, DEPARTAMENTO DE MATEMÁTICA, FACULDADE DE CIÊNCIAS, UNIVERSIDADE DO PORTO, RUA DO CAMPO ALEGRE 687, 4169-007 PORTO, PORTUGAL

E-mail address: `mdelgado@fc.up.pt`

DEPARTAMENTO DE MATEMÁTICA APLICADA, ESCUELA UNIVERSITARIA DE INFORMÁTICA, CAMPUS DE SEGOVIA - UNIVERSIDAD DE VALLADOLID, PLAZA DE SANTA EULALIA 9 Y 11 - 40005 SEGOVIA, SPAIN

E-mail address: `jifarran@eii.uva.es`

DEPARTAMENTO DE ÁLGEBRA, UNIVERSIDAD DE GRANADA, 18071 GRANADA, ESPAÑA

E-mail address: `pedro@ugr.es`

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE ALMERÍA, 04120 ALMERÍA, ESPAÑA

E-mail address: `dllena@ual.es`