SRB MEASURES FOR PARTIALLY HYPERBOLIC SYSTEMS WHOSE CENTRAL DIRECTION IS WEAKLY EXPANDING

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ABSTRACT. We consider partially hyperbolic C^{1+} diffeomorphisms of compact Riemannian manifolds of arbitrary dimension which admit a partially hyperbolic tangent bundle decomposition $E^s \oplus E^{cu}$. Assuming the existence of a set of positive Lebesgue measure on which f satisfies a weak nonuniform expansivity assumption in the centre unstable direction, we prove that there exists at most a finite number of transitive attractors each of which supports an SRB measure. As part of our argument, we prove that each attractor admits a Gibbs-Markov-Young geometric structure with integrable return times. We also characterize in this setting SRB measures which are liftable to Gibbs-Markov-Young structures.

1. Introduction

A key outstanding question in the theory of Dynamical Systems, which motivates a large amount of research, is the *Palis conjecture* [31], which states that "typically" dynamical systems on finite dimensional manifolds have a finite number of "ergodic attractors". More specifically, a Borel probability measure μ on M is a *physical measure* if there exists a positive Lebesgue measure set of points $x \in M$ such that for any continuous $\varphi : M \to \mathbb{R}$ one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) \longrightarrow \int \varphi \, d\mu.$$

The set of points $\mathcal{B}(\mu)$ for which the above convergence holds is called the *basin* of μ. The Palis conjecture then says that typical dynamical systems admit at least one and at most a finite number of physical measures and that their basins have full Lebesgue measure in M.

The Palis conjecture has been proved in the setting of smooth one-dimensional maps for various notions of "typical" [27, 6, 26], but it is still completely open in higher dimensions where the general problem of proving ergodicity or even finiteness of the number of ergodic components of a measure is extremely difficult. In this paper we give a contribution to this area of research by proving that certain natural classes of partially hyperbolic systems admit SRB measures which can have at most a finite number of ergodic components. As part of our argument we also show that these SRB measures are associated to particular geometric structures. We give the precise definitions below as well as a more detailed discussion and comparison of existing results.

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1.1. **SRB measures.** Throughout this paper we let M be a finite dimensional compact Riemannian manifold and $f : M \to M$ a diffeomorphism of class C^{1+}, meaning that f is C^1 with Hölder continuous derivative. We denote by Leb a normalized volume form on the Borel sets of M that we call *Lebesgue measure.* Given a submanifold $\gamma \subset M$ we use Leb_γ to denote the measure on γ induced by the restriction of the Riemannian structure to γ.

In our setting we will work with a particular class of physical measures: we say that an ergodic f-invariant probability measure μ is an *SRB measure* if it has no zero Lyapunov exponents μ almost everywhere and the conditional measures on unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these manifolds. It follows by standard results on the absolute continuity of the stable foliation that SRB measures are a particular form of physical measures [11, 32, 34, 43].

1.2. **Hyperbolicity.** The only general class of dynamical systems in higher dimensions for which the problem of the number of physical measures has been solved is that of *uniformly hyperbolic* diffeomorphisms, i.e. systems which admit a continuous invariant tangent bundle decomposition $TM = E^s \oplus E^u$ such that the differential map is uniformly contracting on E^s and uniformly expanding on E^u. Indeed, it has long been known, since the classical work of Anosov, Smale, Ruelle and Bowen [11, 12, 39, 40] in the 1970's, that such systems can admit at most a finite number of physical measures.

The problem remains wide open for the natural generalisations of uniform hyperbolicity, namely *nonuniform hyperbolicity* where the decomposition $TM = E^s \oplus E^u$ is only measurable and the contraction and expansion estimates are only asymptotic and not uniform [7, 32], and *partial hyperbolicity* where the tangent bundle decomposition takes the form $TM = E^s \oplus E^c \oplus E^u$ which is still assumed to be continuous and to admit uniform contraction and expansion estimates in E^s and E^u respectively, but also includes a "central" direction on which very little is assumed [13, 33], we give a more precise definition below. A vast literature exists concerning the properties of systems satisfying such weak hyperbolicity conditions and several papers address specifically the existence of SRB measures under various kinds of additional assumptions, most of which include the assumption that the system be *both* partially hyperbolic and nonuniformly hyperbolic (in the central direction), see [1, 3, 5, 15, 10, 16, 17, 18, 19, 20, 21, 23, 37, 34, 44, 46].

The main purpose of this paper is to relax the conditions of nonuniform hyperbolicity in the central direction assumed in [1], thus obtaining a significantly more general result which we stress is not contained in any other existing result. This generalization requires a completely different approach which is more powerful in this setting and probably more suited to potential further extensions.

1.3. **Partial hyperbolicity.** We say that a forward invariant compact set $K \subseteq M$ admits a *dominated decomposition* if there is a continuous Df-invariant splitting $T_K M = E^{cs} \oplus E^{cu}$ and there exists a Riemannian metric on M and a constant $\lambda < 1$ such that for all $x \in K$

$$\|Df|_{E^{cs}_x}\| \cdot \|Df^{-1}|_{E^{cu}_{f(x)}}\| < \lambda. \tag{1}$$

We say that f is *partially hyperbolic* if moreover we have for all $x \in K$

$$\|Df|_{E^{cs}_x}\| < \lambda.$$

To emphasize the uniform contraction we shall write $E^{cs} = E^s$, so that we have

$$T_K M = E^s \oplus E^{cu}.$$

We remark that, more generally, a diffeomorphism with dominated decomposition is said to be partially hyperbolic if at least one of the sub-bundles E^{cs} or E^{cu} admits uniform contraction or uniform expansion, respectively. In this paper we are assuming that it is the stable sub-bundle which admits uniform estimates.

1.4. **Nonuniform expansion.** Without additional assumptions on the centre-unstable bundle E^{cu} it is very difficult to obtain any results at all about the dynamics. In [1], the existence of physical (SRB) measures was proved under the assumption that there exists a set H of positive Lebesgue measure on which f is *non-uniformly expanding* along E^{cu}: there exists an $\epsilon > 0$ and some choice of Riemannian metric on M such that for all $x \in H$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|_{E^{cu}_{f^j(x)}}\| < -\epsilon. \tag{2}$$

In this paper we address a subtle but non-trivial generalization of this result which has remained open for over a decade. We say that the map f is *weakly non-uniformly expanding* along E^{cu} on a set H if there exists an $\epsilon > 0$ and some choice of Riemannian metric on M such that for all $x \in H$

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1}|_{E^{cu}_{f^j(x)}}\| < -\epsilon. \tag{3}$$

We emphasize that the lim inf condition (3) implies that the growth only needs to be verified on a subsequence of iterates, in contrast to the limsup condition (2), where the condition needs to be verified for all sufficiently large times. The techniques and methods that we use below to deal with this weaker assumptions are completely different from those used in [1].

Theorem A. *Let $f : M \to M$ be a C^{1+} diffeomorphism, $K \subseteq M$ a forward invariant compact set on which f is partially hyperbolic with splitting $T_K M = E^s \oplus E^{cu}$, and $H \subseteq K$ a set with positive Lebesgue measure on which f is weakly non-uniformly expanding along E^{cu}. Then*

1. *there exist closed invariant transitive sets $\Omega_1, ..., \Omega_\ell$ such that for Lebesgue almost every $x \in H$ we have $\omega(x) = \Omega_j$ for some $1 \le j \le \ell$;*
2. *there exist SRB measures $\mu_1, ..., \mu_\ell$ supported on the sets $\Omega_1, ..., \Omega_\ell$, whose basins have nonempty interior, such that for Lebesgue almost every $x \in H$ we have $x \in \mathcal{B}(\mu_j)$ for some $1 \le j \le \ell$.*

We remark that our argument works also if we let $\epsilon = 0$ in (3), in which case we get a countable number of transitive sets and corresponding SRB measures supported on them. As the ergodic basins $\mathcal{B}(\mu_j)$ have non-empty interior, in the special case in which f is transitive, partially hyperbolic and weakly nonuniformly expanding along E^{cu} on the whole manifold M, we get the following consequence.

Corollary B. *Let $f : M \to M$ be a C^{1+} transitive partially hyperbolic diffeomorphism with splitting $TM = E^s \oplus E^{cu}$ which is weakly nonuniformly expanding along E^{cu} on a subset of full Lebesgue measure. Then $\omega(x) = M$ for Lebesgue almost every x and f has a unique SRB measure whose basin has full Lebesgue measure in M.*

It remains an interesting question whether these results hold true under the weaker assumption that the map f has *positive Lyapunov exponents along* E^{cu}: there exists a set H of positive Lebesgue measure and there exists some $\epsilon > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| > \epsilon, \tag{4}$$

for every $x \in H$ and every non-zero vector $v \in E^{cu}_x$. If $\dim(E^{cu}) = 1$ then (4) is equivalent to (3) and therefore analogous results as in Theorem A and Corollary B hold under this weaker assumption. Observe that condition (4), unlike condition (3), does not depend on the choice of metric. Moreover, condition (4) is strictly weaker than (3) for a given norm; see for instance the example in [2, Section 4].

Conjecture. Assume that f is partially hyperbolic and satisfies (4) along E^{cu} on a set H. Then there exists a Riemannian metric on M such that f satisfies (3) on H.

If this conjecture is true, we will immediately obtain the conclusions of Theorem A and of Corollary B under the a-priori weaker condition of positive Lyapunov exponents.

1.5. **GMY structures.** The strategy which we will use to prove the results above, and which is completely different from the approach in [1], is the construction of certain geometric structures whose existence is of independent interest. These geometric structures were introduced in [42] and have been applied to study the existence and properties of physical measures in certain classes of dynamical systems. We give here the precise definitions.

An embedded disk $\gamma \subset M$ is called an *unstable manifold* if $d(f^{-n}(x), f^{-n}(y)) \to 0$ exponentially fast as $n \to \infty$ for all $x, y \in \gamma$; similarly $\gamma \subset M$ is called a *stable manifold* if $d(f^n(x), f^n(y)) \to 0$ exponentially fast as $n \to \infty$. We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable manifolds* if there is a compact set K^s, a unit disk D^u of some $\mathbb{R}^n$, and a map $\Phi^u \colon K^s \times D^u \to M$ such that

i) $\gamma^u = \Phi^u(\{x\} \times D^u)$ is an unstable manifold;
ii) Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
iii) $x \mapsto \Phi^u|(\{x\} \times D^u)$ defines a continuous map from K^s into $\mathrm{Emb}^1(D^u, M)$.

Here $\mathrm{Emb}^1(D^u, M)$ denotes the space of C^1 embeddings from D^u into M. Continuous families of C^1 stable manifolds are defined similarly.

We say that a set $\Lambda \subset M$ has a *hyperbolic product structure* if there exist a continuous family of local unstable manifolds $\Gamma^u = \{\gamma^u\}$ and a continuous family of local stable manifolds $\Gamma^s = \{\gamma^s\}$ such that

i) $\Lambda = (\cup \gamma^u) \cap (\cup \gamma^s)$;
ii) $\dim \gamma^u + \dim \gamma^s = \dim M$;
iii) each γ^s meets each γ^u in exactly one point;
iv) stable and unstable manifolds are transversal with angles bounded away from 0.

If $\Lambda \subset M$ has a product structure, we say that $\Lambda_0 \subset \Lambda$ is an *s-subset* if Λ_0 also has a product structure and its defining families Γ^s_0 and Γ^u_0 can be chosen with $\Gamma^s_0 \subset \Gamma^s$ and $\Gamma^u_0 = \Gamma^u$; *u-subsets* are defined analogously. For convenience we shall use the following notation: given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, for $* = s, u$. Also, for each $n \geq 1$ let $(f^n)^u$ denote the restriction of the map f^n to γ^u-disks and let $\det D(f^n)^u$ be the Jacobian of $D(f^n)^u$.

We say that f admits a *Gibbs-Markov-Young (GMY) structure* if there exist a set Λ with hyperbolic product structure and constants $C > 0$ and $0 < \beta < 1$, depending on f and Λ, satisfying the following additional properties:

(P_0) *Detectable*: $\mathrm{Leb}_\gamma(\Lambda) > 0$ for each $\gamma \in \Gamma^u$.
(P_1) *Markov*: there are pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that
(a) $\mathrm{Leb}_\gamma\left((\Lambda \setminus \cup \Lambda_i) \cap \gamma\right) = 0$ on each $\gamma \in \Gamma^u$;

(b) for each $i \in \mathbb{N}$ there is $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is u-subset, and for all $x \in \Lambda_i$

$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$

(P_2) *Contraction on stable leaves*: for all $\gamma^s \in \Gamma^s$, $x, y \in \gamma^s$ and $n \geq 1$

$$\operatorname{dist}(f^n(y), f^n(x)) \leq C\beta^n.$$

(P_3) *Backward contraction on unstable leaves*: for all $\gamma^u \in \Gamma^u$, $x, y \in \Lambda_i \cap \gamma^u$ and $0 \leq n < R_i$

$$\operatorname{dist}(f^n(y), f^n(x)) \leq C\beta^{R_i - n} \operatorname{dist}(f^{R_i}(x), f^{R_i}(y)).$$

(P_4) *Bounded distortion*: for all $\gamma^u \in \Gamma^u$ and $x, y \in \Lambda_i \cap \gamma^u$

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \leq C \operatorname{dist}(f^{R_i}(x), f^{R_i}(y)).$$

(P_5) *Regularity of the foliations*:

(a) for all $\gamma^s \in \Gamma^s$, $x, y \in \gamma^s$ and $n \geq 1$

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C\beta^n;$$

(b) given $\gamma, \gamma' \in \Gamma^u$, we define $\Theta\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ by taking $\Theta(x)$ equal to $\gamma^s(x) \cap \gamma'$. Then Θ is absolutely continuous and

$$\frac{d(\Theta_* \operatorname{Leb}_\gamma)}{d \operatorname{Leb}_{\gamma'}}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\Theta^{-1}(x)))}.$$

We define a return time function $R : \Lambda \to \mathbb{N}$ by $R|_{\Lambda_i} = R_i$ and we say that the GMY structure has *integrable return times* if for some (and hence all) $\gamma \in \Gamma^u$, we have

$$\int_{\gamma \cap \Lambda} R d \operatorname{Leb}_\gamma < \infty. \tag{5}$$

Most of the paper will be dedicated to the proof of the following result on the existence of GMY structures, from which we will deduce the other results, and whose interest goes beyond the applications presented here.

Theorem C. *Let $f : M \to M$ be a C^{1+} diffeomorphism, $K \subseteq M$ a forward invariant compact set on which f is partially hyperbolic with splitting $T_K M = E^s \oplus E^{cu}$, and $H \subseteq K$ a set with positive Lebesgue measure for which f is weakly non-uniformly expanding along E^{cu}. If there exists a closed invariant transitive set Ω such that $\omega(x) = \Omega$ for every $x \in H$, then there exists a GMY structure $\Lambda \subseteq \Omega$ with integrable return times.*

1.6. **Liftable measures.** Associated to a GMY structure we have an *induced map* $F : \Lambda \to \Lambda$ defined by

$$F|_{\Lambda_i} = f^{R_i}|_{\Lambda_i}.$$

It is well known that F has a unique SRB measure ν. Assuming the integrability condition (5) we can define the measure

$$\hat{\mu} = \sum_{j=0}^{\infty} f_*^j(\nu | \{R > j\}), \tag{6}$$

which is a finite measure whose normalization μ is an SRB measure for f, see [42, Section 2].

Probability measures μ obtained from GMY structures through (6) often give a substantial amount of information on the statistical properties of the dynamics with respect to μ such as decay of correlations, large deviations, limit theorems, etc.; see for instance [42, 22, 29, 30, 38]. This motivates a general question as to whether any SRB measure μ is of the form (6) for a suitable GMY structure. In this case we say that μ is *liftable* (to a GMY structure). As a consequence of the techniques used here we will obtain that for the partially hyperbolic systems considered in this paper, every SRB measure is liftable.

Theorem D. *Let $f : M \to M$ be a C^{1+} diffeomorphism and $K \subset M$ a compact forward invariant set on which f is partially hyperbolic with splitting $T_K M = E^s \oplus E^{cu}$. Then μ is an SRB measure with positive Lyapunov exponents in the E^{cu} direction supported on K if and only if μ is liftable to a GMY structure on K.*

We emphasize that the assumption on the dynamics in center-unstable direction in the statement of Theorem D is that the map has *positive Lyapunov exponents* as in (4) rather than the stronger *weak non-uniform expansion* as in (3) which we need to assume instead for Theorem A, see also discussion following Corollary B. In both cases we need to construct a Gibbs-Markov-Young structure with integrable return times which implies the existence of an SRB measure. The key difference however is that in Theorem D we already have an SRB measure by assumption and we just need to show that the two measures coincide. This allows us to prove, using a non-trivial argument, that some power of f is weakly non-uniformly expanding and thus apply the same construction as in the proof of Theorem A.

The paper is organized as follows. In Section 2 we give an abstract criterion for verifying that at most a finite number of topological attractors exist for a given set. In Section 3 we show that this criterion is satisfied by the set H in Theorem A, thus proving item (1). We then prove Theorem C in Sections 4, 5, 6 and 7, which in particular applies to the sets Ω obtained in item (1) of Theorem A. Combining the conclusion of Theorem C with the comments in the beginning of Section 1.6 we get item (2) of Theorem A and the "if" direction of Theorem D. We prove the other implication of Theorem D in Section 8.

2. Ergodic components

Let X be a compact metric space and μ a Borel probability measure on X. Let $f : X \to X$ be a measurable map, not necessarily preserving the measure μ. Given $x \in X$, the *stable set* of x is

$$W^s(x) = \left\{ y \in X : \operatorname{dist}(f^j(x), f^j(y)) \to 0, \text{ as } j \to \infty \right\}.$$

Notice that the relation $x \sim y$ if and only if $y \in W^s(x)$ defines an equivalence relation on X. In particular, we will use below the transitive property of this relation. If $U \subset X$, let

$$W^s(U) = \bigcup_{x \in U} W^s(x).$$

We recall that a set $U \subseteq X$ is *invariant* if $f^{-1}(U) = U$. We now formulate a notion which is key to our argument. We say that $Y \subseteq X$ is *μ-unshrinkable* if it is an invariant set with $\mu(Y) > 0$ and there exists a $\delta > 0$ such that for every invariant set $U \subseteq Y$ we have

$$\mu(U) > 0 \quad \Rightarrow \quad \mu(W^s(U)) > \delta.$$

Proposition 2.1. *Suppose $Y \subseteq X$ is μ-unshrinkable. Then there exists a finite number of closed invariant subsets $\Omega_1, ..., \Omega_\ell$ of X such that for μ almost every $x \in Y$ we have $\omega(x) = \Omega_j$, for some $1 \le j \le \ell$.*

We will split the proof of Proposition 2.1 itself into two lemmas. To do this we need to introduce some additional concepts. We say that a set S is *s-saturated* if $W^s(S) = S$. We say that S is a *u-ergodic component* if it is invariant, s-saturated, and any subset $S' \subset S$ which is also invariant and s-saturated, satisfies $\mu(S) = \mu(S')$ or $\mu(S') = 0$.

Lemma 2.2. *Suppose $Y \subseteq X$ is μ-unshrinkable. Then Y is contained (μ mod 0) in the union of a finite number of u-ergodic components.*

Proof. Let $Y_1 = Y$ and let

$$\mathcal{F}(Y_1) := \{W^s(U) \,:\, U \subseteq Y_1,\ f^{-1}(U) = U,\ \text{and } \mu(W^s(U)) > 0\,\}$$

Note that $\mathcal{F}(Y_1)$ is non-empty because $W^s(Y_1) \in \mathcal{F}(Y_1)$. Moreover, we claim that

$$W, W' \in \mathcal{F}(Y_1) \quad \text{and} \quad \mu(W \setminus W') > 0 \quad \Rightarrow \quad W \setminus W' \in \mathcal{F}(Y_1) \tag{7}$$

To see this, let $U, U' \subseteq Y_1$ be invariant sets such that $W = W^s(U)$ and $W' = W^s(U')$. We claim that

$$W \setminus W' = W^s(U \setminus W^s(U')), \tag{8}$$

Notice that $U \setminus W^s(U') \subseteq Y_1$, and also $U \setminus W^s(U')$ is invariant because both $U, W^s(U')$ are invariant. Therefore (8) implies (7). To prove (8), we prove first of all that $W \setminus W' \subseteq W^s(U \setminus W^s(U'))$. Suppose $x \in W \setminus W'$, i.e. $x \in W^s(U)$ and $x \notin W^s(U')$. This means that there exists $u \in U$ such that $x \in W^s(u)$ and also that $x \notin W^s(u')$ for any $u' \in U'$, which implies that $x \notin W^s(z)$ for any $z \in W^s(U')$ by the transitivity of the equivalence relation $\sim$ mentioned above. This proves the first inclusion. To prove $W \setminus W' \supseteq W^s(U \setminus W^s(U'))$, let $x \in W^s(U \setminus W^s(U'))$. Then clearly $x \in W$ and $x \in W^s(y)$ for some $y \in U \setminus W^s(U')$. It just remains to show that $x \notin W'$. Arguing by contradiction, suppose that $x \in W' = W^s(U')$, then $x \in W^s(u')$ for some $u' \in U'$ and so, as $x \sim y$, we have $y \in W^s(u')$ which contradicts the fact that $y \in U \setminus W^s(U')$. This completes the proof of (8) and hence of (7).

Now consider the partial order on $\mathcal{F}(Y_1)$ defined by *strict inclusion*, meaning that $W \succ W'$ if $W \supset W'$ and $\mu(W \setminus W') > 0$. We claim that for this partial order relation, every totally ordered subset of $\mathcal{F}(Y_1)$ is finite, and in particular it has a lower bound. Indeed, arguing by contradiction, suppose that there is an infinite sequence $W_1 \succ W_2 \succ \cdots$ in $\mathcal{F}(Y_1)$, i.e. $W_1 \supset W_2 \supset \cdots$ with $\mu(W_k \setminus W_{k+1}) > 0$, for all $k \geq 1$. Then

$$\sum_{k \geq 1} \mu(W_k \setminus W_{k+1}) = \mu(W_1) < \infty,$$

and therefore $\mu(W_k \setminus W_{k+1}) \to 0$ as $k \to \infty$. Since $W_k \setminus W_{k+1} \in \mathcal{F}(Y_1)$ by (7), this contradicts our assumptions that $Y_1 = Y$ is μ-unshrinkable. This shows that every totally ordered subset of $\mathcal{F}(Y_1)$ has a lower bound. Thus by Zorn's Lemma there exists at least one minimal element $W^s(U_1) \in \mathcal{F}(Y_1)$, which therefore must necessarily be a u-ergodic component.

We now let $Y_2 := Y_1 \setminus W^s(U_1)$, which is again invariant. If $\mu(Y_2) = 0$ then $Y = Y_1$ is essentially contained in $W^s(U_1)$, which is a u-ergodic component, and thus we are done. On the other hand, if $\mu(Y_2) > 0$ we can repeat the entire argument above to obtain a set $U_2 \subseteq Y_2$ and a u-ergodic component $W^s(U_2)$. Inductively, we then construct a collection of disjoint u-ergodic components $W^s(U_1), ..., W^s(U_r)$ and continue as long as $\mu(Y \setminus W^s(U_1) \cup ... \cup W^s(U_r)) > 0$. But, as $\mu(W^s(U_j)) \geq \delta$ for all $1 \leq j \leq r$ by the assumption that Y is μ-unshrinkable, this process will stop and we will get the conclusion. □

Lemma 2.3. *Suppose $S \subseteq X$ is a u-ergodic component. Then there exists a closed invariant set $\Omega \subseteq X$ such that $\omega(x) = \Omega$ for μ-almost every $x \in S$.*

Proof. Given any open set $B \subset X$, let

$$B_\omega := \{x \in S : \omega(x) \cap B \neq \emptyset\}.$$

Then B_ω is invariant and s-saturated and therefore, by the assumption that S is u-ergodic, $\mu(B_\omega) = 0$ or $\mu(B_\omega) = \mu(S)$. Now, let $Z_1 = X$ and $\mathcal{C}_1$ be any finite covering of X by open balls of radius 1. By the previous considerations, for every $B \in \mathcal{C}_1$ we have $\mu(B_\omega) = 0$ or $\mu(B_\omega) = \mu(S)$ and therefore, since we only have a finite number of elements in $\mathcal{C}_1$, there exists at least one $B_\omega \in \mathcal{C}_1$ such that $\mu(B_\omega) = \mu(S)$. Let

$$\mathcal{C}_1' = \{B \in \mathcal{C}_1 : \mu(B_\omega) = 0\} \quad \text{and} \quad Z_2 = Z_1 \setminus \bigcup_{B \in \mathcal{C}_1'} B.$$

Then Z_2 is a non-empty compact set and $\omega(x) \subseteq Z_2$ for μ-almost every $x \in S$. We can therefore repeat the procedure with a finite cover $\mathcal{C}_2$ of Z_2 by open balls of radius $1/2$, and, by induction, construct sequences $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_1', \mathcal{C}_2', \ldots$ and $Z_1, Z_2, \ldots$ such that $Z_1 \supset Z_2 \supset \ldots$ is a sequence of non-empty compact sets and $\omega(x) \subset Z_j$ for almost every $x \in S$. In particular we have

$$\omega(x) \subseteq \Omega := \bigcap_{n \geq 1} Z_n$$

It just remains to show that $\Omega \subseteq \omega(x)$ for μ almost every $x \in S$. Indeed, given $y \in \Omega$ we have that $y \in Z_n$ for every $n \geq 1$, and therefore there is some $B^{(n)} \in \mathcal{C}_n \setminus \mathcal{C}_n'$ such that $y \in B^{(n)}$. Since $\operatorname{diam}(B^{(n)}) \to 0$ as $n \to \infty$, this implies that $\bigcap_n B^{(n)} = \{y\}$. Moreover, as $B^{(n)} \in \mathcal{C}_n \setminus \mathcal{C}_n'$ we have that $\mu(B_\omega^{(n)}) = \mu(S)$ and therefore $\omega(x) \cap B^{(n)} \neq \emptyset$ for μ almost all $x \in S$. This implies that $y \in \omega(x)$ for μ almost all $x \in S$ and, as $\omega(x)$ is closed and invariant, the statement follows. □

3. Transitive attractors

Here we prove the topological part of Theorem A. We assume throughout this section the assumptions of that theorem. Let $f : M \to M$ be a C^{1+} diffeomorphism, $K \subset M$ a forward invariant compact set on which f is partially hyperbolic, and $H \subseteq K$ a set with $\mathrm{Leb}(H) > 0$ on which f is weakly non-uniformly expanding along E^{cu}. The main result of this section is the following proposition.

Proposition 3.1. *There exist closed invariant sets $\Omega_1, ..., \Omega_\ell \subseteq K$ such that for Lebesgue almost every $x \in H$ we have $\omega(x) = \Omega_j$ for some $1 \leq j \leq \ell$. Moreover, each Ω_j is transitive and contains a cu-disk Δ_j of radius $\delta_1/4$ on which f is weakly non-uniformly expanding along E^{cu} for Leb_{Δ_j} almost every point in Δ_j.*

We first prove some preliminary lemmas. We remark that K is not assumed to contain any open sets. We therefore fix continuous extensions of the two sub-bundles E^s and E^{cu} to some compact neighborhood V of K, that we still denote E^s and E^{cu}. We do not require these extensions to be Df invariant. Given $0 < a < 1$, we define the *centre-unstable cone field* $C_a^{cu} = (C_a^{cu}(x))_{x \in V}$ *of width* a by

$$C_a^{cu}(x) = \left\{v_1 + v_2 \in E_x^s \oplus E_x^{cu} : \|v_1\| \leq a\|v_2\|\right\}. \tag{9}$$

We define the *stable cone field* $C_a^s = (C_a^s(x))_{x \in V}$ *of width* a in a similar way, just reversing the roles of the sub-bundles in (9). We fix $a > 0$ and V small enough so that the domination condition (1) remains valid in the two cone fields:

$$\|Df(x)v^s\| \cdot \|Df^{-1}(f(x))v^{cu}\| < \lambda \|v^s\| \cdot \|v^{cu}\|$$

for every $v^s \in C_a^s(x)$, $v^{cu} \in C_a^{cu}(f(x))$ and any point $x \in V \cap f^{-1}(V)$. Note that the centre-unstable cone field is forward invariant: $Df(x)C_a^{cu}(x) \subset C_a^{cu}(f(x))$, whenever $x, f(x) \in V$. Actually, the domination property together with the invariance of $E^{cu}|_K$ imply that $Df(x)C_a^{cu}(x) \subset C_{\lambda a}^{cu}(f(x)) \subset C_a^{cu}(f(x))$, for every $x \in K$, and this extends to any $x \in V \cap f^{-1}(V)$ just by continuity.

Given $0 < \sigma < 1$, we say that n is a *σ-hyperbolic time* for $x \in K$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^k, \qquad \text{for all } 1 \leq k \leq n.$$

The next result gives the existence of (infinitely many) σ-hyperbolic times for points satisfying the weak nonuniform expansion condition (3). For a proof see [4, Corollary 5.3].

Lemma 3.2. *There are $\sigma > 0$ and $\theta > 0$ such that if (3) holds for $x \in K$, then*

$$\limsup_{n\to\infty} \frac{1}{n} \#\{1 \leq j \leq n : j \text{ is a } \sigma\text{-hyperbolic time for } x\} \geq \theta.$$

Under the stronger assumption (2), it was proved in [1] that there is positive frequency at infinity of hyperbolic times, which means taking "lim inf" instead of "lim sup" in Lemma 3.2. The positive frequency of σ-hyperbolic times at infinity plays a crucial role in the argument used in [1, Corollary 3.2] to prove the existence of SRB measures, and this is the reason why we cannot use those arguments here.

Hyperbolic times are defined pointwise but, as we shall see below, some important properties can be derived for a neighbourhood of the reference point at a hyperbolic time. From now on we fix σ and θ as in Lemma 3.2. Now observe that, by continuity of the derivative, we can choose $a > 0$ and $\delta_1 > 0$ sufficiently small so that the δ_1-neighborhood of K is contained in V and

$$\|Df^{-1}(f(y))v\| \leq \sigma^{-1/4} \|Df^{-1}|E^{cu}_{f(x)}\| \, \|v\| \tag{10}$$

for all $x \in K, y \in V$ with $\operatorname{dist}(x, y) \leq \delta_1$ and $v \in C_a^{cu}(y)$. From now on we fix these values of a, δ_1 so that (10) holds.

We say that an embedded C^1 submanifold $D \subset V$ is a *cu-disk* if the tangent subspace to D at each point $x \in D$ is contained in the corresponding cone $C_a^{cu}(x)$. Then $f(D)$ is also a *cu*-disk, if it is contained in V, by the domination property. Given any disk $D \subset M$, we use $\operatorname{dist}_D(x, y)$ to denote the distance between $x, y \in D$, measured along D.

Lemma 3.3. *Let D be a cu-disk. There exists $C_1 > 1$ such that if n is a σ-hyperbolic time for $x \in K \cap D$, then there exists a neighbourhood $V_n^+(x)$ of x in D such that f^n maps $V_n^+(x)$ diffeomorphically onto a cu-disk $B^u_{2\delta_1}(f^n(x))$ of radius $2\delta_1$ around $f^n(x)$. Moreover, for every $1 \leq k \leq n$ and $y, z \in V_n(x)^+$ we have*

(1) $\operatorname{dist}_{f^{n-k}(V_n^+(x))}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{3k/4} \operatorname{dist}_{f^n(V_n^+(x))}(f^n(y), f^n(z))$;

(2) $\log \dfrac{|\det Df^n \mid T_y D|}{|\det Df^n \mid T_z D|} \leq C_1 \operatorname{dist}_{f^n(D)}(f^n(y), f^n(z))$;

(3) *for any Borel sets $X, Y \subset V_n^+(x)$,*

$$\frac{\operatorname{Leb}_{f^n(V_n^+(x))}(f^n(X))}{\operatorname{Leb}_{f^n(V_n^+(x))}(f^n(Y))} \leq C_1 \frac{\operatorname{Leb}_{V_n^+(x)}(X)}{\operatorname{Leb}_{V_n^+(x)}(Y)}.$$

The first two items are proved in [1, Lemma 2.7 & Proposition 2.8], and the third item is a standard consequence of the second one. Notice that the factor $\sigma^{3/4}$ in the first item of this lemma differs from the factor $\sigma^{1/2}$ in [1, Lemma 2.7] simply because we have chosen $\delta_1 > 0$ sufficiently small so that (10) holds, contrarily to estimate (6) in [4] where $\delta_1 > 0$ is chosen so that a similar conclusion holds with $\sigma^{1/2}$ in the place of $\sigma^{1/4}$.

Remark 3.4. Notice that if we replace the assumption that n is a σ-hyperbolic time in Lemma 3.3 with the assumption that n is a σ^{α}-hyperbolic time for some $\alpha > 1/4$, then the conclusions of the lemma continue to hold with $\sigma^{\alpha-1/4}$ instead of $\sigma^{3/4}$ in item (1), where the term $1/4$ comes from (10).

Now we define $V_n(x) \subseteq V_n^+(x)$, where $f^n(V_n(x)) = B^u_{\delta_1}(f^n(x))$ is the *cu*-disk of radius δ_1 around $f^n(x)$ contained in $B^u_{2\delta_1}(f^n(x))$ as in Lemma 3.3. The sets $V_n(x)$ are called *hyperbolic pre-disks* and their images $f^n(V_n(x))$ *hyperbolic disks*. The following result is proved in [4, Proposition 5.5].

Lemma 3.5. *Let D be a cu-disk and $U \subseteq H$ with $\mathrm{Leb}_D(U) > 0$. Then there exists a sequence of sets ... $\subseteq W_2 \subseteq W_1 \subseteq D$ and a sequence of integers $n_1 < n_2 < \cdots$ such that:*

(1) *W_k is contained in some hyperbolic pre-disk with hyperbolic time n_k;*
(2) *$D_k := f^{n_k}(W_k)$ is a cu-disk of radius $\delta_1/4$;*
(3) $\displaystyle\lim_{k\to\infty} \frac{\mathrm{Leb}_{D_k} f^{n_k}(U \cap D)}{\mathrm{Leb}_{D_k}(D_k)} = 1.$

Now we are in conditions to prove Proposition 3.1. We define

$$\widetilde{H} := \bigcup_{n\in\mathbb{Z}} f^n(H).$$

Then $\widetilde{H}$ is clearly invariant and $\mathrm{Leb}(\widetilde{H}) > 0$.

Lemma 3.6. *$\widetilde{H}$ is* Leb*-unshrinkable.*

Proof. To prove that $\widetilde{H}$ is Leb-unshrinkable it is sufficient to show that there exists $\delta > 0$ such that for every f-invariant set $U \subseteq \widetilde{H}$ with $\mathrm{Leb}(U) > 0$ we have $\mathrm{Leb}(W^s(U)) > \delta$. We remark that in the proof of this assertion, to be given in the following paragraphs, we will only use the assumption that U is forward invariant. This allows us to assume without loss of generality that $U \subseteq K$. Indeed, if U is invariant with positive Lebesgue measure, then it must intersect K in a set of positive Lebesgue measure and, as K is forward invariant, also $U \cap K$ is forward invariant. Clearly, if $\mathrm{Leb}(W^s(U \cap K)) > \delta$ then we also have $\mathrm{Leb}(W^s(U)) > \delta$. In particular, as $U \subseteq K$ it admits a partially hyperbolic structure and, as also $U \subseteq \widetilde{H}$, it is weakly non-uniformly expanding along E^{cu}.

Now we show that there exists a *cu*-disk $D \subseteq V$ such that $\mathrm{Leb}_D(U) > 0$. Recall that V is the neighbourhood of K introduced in the beginning of this section. To see this, consider a Lebesgue density point p of U. Notice that T_pM has a partially hyperbolic splitting $E^s_p \oplus E^{cu}_p$ and we can consider a neighbourhood of the origin foliated by disks parallel to the E^{cu} subspace whose images under the exponential map $\exp_p$ are *cu*-disks in the manifold. Since $\exp_p$ is a local diffeomorphism, the preimage of U under the exponential map has positive volume in T_pM and full density in the origin. By Fubini at least one of the disks above must intersect this set in positive relative volume, and the same must hold for its image under the exponential map.

Now let $D \subseteq V$ be a cu-disk satisfying $\mathrm{Leb}_D(U) > 0$, as in the previous paragraph. Consider the sequences $\cdots \subseteq W_2 \subseteq W_1 \subseteq D$ and $n_1 < n_2 < \cdots$ given by Lemma 3.5. By the third item of Lemma 3.5 it follows that the relative measure of $f^{n_k}(U \cap D)$ in D_k converges to 1. Since U is forward invariant we conclude that the relative measure of U in D_k converges to 1 and therefore $\mathrm{Leb}_{D_k}(U) \to \delta_1/4$ as $k \to \infty$. Since $U \subseteq K$, all points of U have local stable manifolds of uniform size and the foliation defined by these local stable manifolds is absolutely continuous, it follows that $\widetilde{H}$ is Leb-unshrinkable. □

The previous result, together with Proposition 2.1, imply that there exist closed invariant sets $\Omega_1, ..., \Omega_\ell$ such that for Lebesgue almost every $x \in H$ we have $\omega(x) = \Omega_j$ for some $1 \le j \le \ell$. This gives the first assertion of Proposition 3.1. We leave the proof of the remaining part of Proposition 3.1 to the next two lemmas.

Lemma 3.7. *Each $\Omega = \Omega_j$ contains a cu-disk Δ of radius $\delta_1/4$ on which f is weakly non-uniformly expanding along E^{cu} for Leb_Δ almost every point in Δ.*

Proof. Let

$$A^{(n)} = \{x \in H : \mathrm{dist}(f^k(x), \Omega) \le 1/n \text{ for every } k \ge 0\}.$$

Since the set of points $x \in H$ with $\omega(x) = \Omega$ has positive Lebesgue measure, we clearly have $\mathrm{Leb}(A^{(n)}) > 0$ for every $n \ge 1$. Then, by the same arguments used in the proof of Lemma 3.6, with $A^{(n)}$ playing the role of U, there exists a cu-disk $D^{(n)} \subseteq V$ such that $\mathrm{Leb}_{D^{(n)}}(A^{(n)}) > 0$, and corresponding sequences $\cdots \subseteq W_2^{(n)} \subseteq W_1^{(n)} \subseteq D^{(n)}$, $n_1 < n_2 < \cdots$ (also depending on n, but we omit the superscript here for obvious reasons...) and cu-disks $D_k^{(n)} = f^{n_k}(W_k^{(n)})$ such that

$$\mathrm{Leb}_{D_k^{(n)}}(A^{(n)}) \to \delta_1/4, \quad \text{as } k \to \infty. \tag{11}$$

Let $p_k^{(n)}$ denote the center of each disk $D_k^{(n)}$. Up to taking a subsequence, we may assume that the sequence $\{p_k^{(n)}\}$ converges to a point $p^{(n)} \in K$, and up to taking a further subsequence, and using Ascoli-Arzelà and the fact that the disks $D_k^{(n)}$ have tangent directions contained in the cu-cones, we may assume that the sequence $\{D_k^{(n)}\}$ converges uniformly, as $k \to \infty$, to some cu-disk $\Delta^{(n)}$ of radius $\delta_1/4$. Notice that each $\Delta^{(n)}$ is necessarily contained in a neighbourhood of Ω of radius $1/n$.

We claim the f is weakly non-uniformly expanding along E^{cu} for $\mathrm{Leb}_{\Delta^{(n)}}$ almost every point in $\Delta^{(n)}$. To see this, recall first of all that the property of weak non-uniform expansion is an asymptotic property and therefore if it is satisfied by a point x, then it is satisfied by every point $y \in W^s(x)$. Moreover, every point of $\Delta^{(n)}$ has a local stable manifold of uniform size, and the foliation by those local stable manifolds is absolutely continuous. Since the sequence $\{D_k^{(n)}\}$ converges uniformly to $\Delta^{(n)}$, for large k, the disks $D_k^{(n)}$ will intersect the stable foliation through points of $\Delta^{(n)}$, and therefore, by (11) and the fact that $A^{(n)} \subseteq H$, it follows that f is weakly non-uniformly expanding along E^{cu} for $\mathrm{Leb}_{\Delta^{(n)}}$ almost every point in $\Delta^{(n)}$.

Now, arguing as above, we can consider a subsequence of $\Delta^{(n)}$'s converging uniformly to some cu-disk Δ of radius $\delta_1/4$ and f is weakly non-uniformly expanding along E^{cu} for Leb_Δ almost every point in Δ. As each $\Delta^{(n)}$ is contained in a neighbourhood of Ω of radius $1/n$ and Ω is closed, it follows that $\Delta \subseteq \Omega$. □

Lemma 3.8. *$f|_\Omega$ is transitive.*

Proof. Recall that by construction there exists some point (in fact a positive Lebesgue measure set of points) in H whose ω-limit set coincides with Ω. The orbit of any such point must eventually hit the stable manifold of some point in $\Delta \subseteq \Omega$. As points in the same stable manifold have the same ω-limit sets, we conclude that there exists a point of Ω whose orbit is dense in Ω. □

4. Construction on a reference leaf

In this section we describe an algorithm for the construction of a partition of some subdisk of Δ which is the basis of the construction of the GMY structure. We first fix some arbitrary $1 \leq j \leq \ell$ and for the rest of the paper we let $\Omega = \Omega_j$ and $\Delta = \Delta_j$ as in Proposition 3.1. We also fix a constant $\delta_s > 0$ so that local stable manifolds $W^s_{\delta_s}(x)$ are defined for all points $x \in K$. For any subdisk $\Delta' \subset \Delta$ we define

$$\mathcal{C}(\Delta') = \bigcup_{x \in \Delta'} W^s_{\delta_s}(x).$$

Let π denote the projection from $\mathcal{C}(\Delta')$ onto Δ' along local stable leaves. We say that a centre-unstable disk $\gamma^u \subset M$ *u-crosses* $\mathcal{C}(\Delta')$ if $\pi(\gamma) = \Delta'$ for some connected component γ of $\gamma^u \cap \mathcal{C}(\Delta')$.

Remark 4.1. We will often be considering *cu*-disks which u-cross $\mathcal{C}(\Delta')$. By continuity of the stable foliation, choosing δ_s sufficiently small, the diameter and Lebesgue measure of such disks intersected with $\mathcal{C}(\Delta')$ are very close to those of Δ', respectively. To simplify the notation and the calculations below we will ignore this difference as it has no significant effect on the estimates.

Lemma 4.2. *Given $N \in \mathbb{N}$, there exists $\delta_2 = \delta_2(N, \delta_1) > 0$ such that if $\gamma^u \subset \Omega$ is a cu-disk of radius $\delta_1/2$ centred at z, then $f^m(\gamma^u)$ contains a cu-disk of radius δ_2 centred at $f^m(z)$, for each $1 \leq m \leq N$.*

Proof. We first prove the result for $j = 1$. Let z be the center of γ^u. Let $f(y)$ be a point in $\partial f(\gamma^u)$ minimizing the distance from $f(z)$ to $\partial f(\gamma^u)$, and let η_1 be a curve of minimal length in $f(\gamma^u)$ connecting $f(z)$ to $f(y)$. Letting $\eta_0 = f^{-1}(\eta_1)$ and $\dot\eta_1(x)$ be the tangent vector to the curve η_1 at the point x, we have

$$\|Df^{-1}(w)\dot\eta_1(x)\| \leq C\,\|\dot\eta_1(x)\|,$$

where

$$C = \max_{x \in M}\left\{\|Df^{-1}(x)\|\right\} \geq 1.$$

Hence,

$$\text{length}(\eta_0) \leq C\,\text{length}(\eta_1).$$

Noting that η_0 is a curve connecting z to $y \in \partial\gamma^u$, this implies that $\text{length}(\eta_0) \geq \delta_1/2$, and so

$$\text{length}(\eta_1) \geq C^{-1}\,\text{length}(\eta_0) \geq C^{-1}\delta_1/2.$$

Thus $f(\gamma^u)$ contains the *cu*-disk γ^u_1 of radius $C^{-1}\delta_1/2$ around $f(z)$.

Making now γ^u_1 play the role of γ^u and $f^2(z)$ play the role of $f(z)$, with the argument above we prove that $f(\gamma^u_1)$ contains a *cu*-disk of radius $C^{-2}\delta_1/2^2$ centered at $f^2(z)$. Inductively, we prove that $f^m(\gamma^u)$ contains a *cu*-disk of radius $C^{-m}\delta_1/2^m \geq C^{-N}\delta_1/2^N$ around $f^m(z)$, for each $1 \leq m \leq N$. We take $\delta_2 = C^{-N}\delta_1/2^N$. □

Lemma 4.3. *There are $p \in \Delta$ and $N_0 \geq 1$ such that for all $\delta_0 > 0$ sufficiently small and each hyperbolic pre-disk $V_n(x) \subseteq \Delta$ there is $0 \leq m \leq N_0$ such that $f^{n+m}(V_n(x))$ intersects $W^s_{\delta_s/2}(p)$ and u-crosses $\mathcal{C}(B^u_{\delta_0}(p))$, where $B^u_{\delta_0}(p)$ is the ball in Δ of radius δ_0 centred at p.*

Proof. First of all we observe that, as the sub-bundles in the dominated spliting have angles uniformly bounded away from zero, given any $\rho > 0$ there is $\alpha = \alpha(\rho) > 0$, with $\alpha \to 0$ as $\rho \to 0$, for which the following holds: if $x, y \in \Omega$ satisfy $\operatorname{dist}(x, y) < \rho$ and $\operatorname{dist}_{\gamma^u}(y, \partial\gamma^u) > \delta_1$ for some cu-disk $\gamma^u \subset \Omega$, then $W^s_{\delta_s}(x)$ intersects γ^u in a point z with

$$\operatorname{dist}_{W^s_{\delta_s}(x)}(z, x) < \alpha \quad \text{and} \quad \operatorname{dist}_{\gamma^u}(z, y) < \delta_1/2.$$

Take $\rho > 0$ small enough so that $4\alpha < \delta_s$. Since $f|_\Omega$ is transitive, we may choose $q \in \Omega$ and $N_0 \in \mathbb{N}$ such that both:

(1) $W^s_{\delta_s/4}(q)$ intersects Δ in a point p with $\operatorname{dist}_\Delta(p, \partial\Delta) > 0$; and
(2) $\{f^{-N_0}(q), \ldots, f^{-1}(q), q\}$ is ρ-dense in Ω.

Given a hyperbolic pre-disk $V_n(x) \subseteq \Delta$ we have by definition that $f^n(V_n(x))$ is a cu-disk of radius δ_1 centred at $y = f^n(x)$ inside Ω. Consider $0 \leq m \leq N_0$ such that $\operatorname{dist}(f^{-m}(q), y) < \rho$. Then, by the choice of ρ and α, we have that $W^s_{\delta_s}(f^{-j}(q))$ intersects $f^n(V_n(x))$ in a point z with $\operatorname{dist}_{W^s_{\delta_s}(f^{-j}(q))}(z, f^{-j}(q)) < \alpha < \delta_s/4$ and $\operatorname{dist}_{f^n(V_n(x))}(z, y) < \delta_1/2$. In particular, $f^n(V_n(x))$ contains a cu-disk γ^u of radius $\delta_1/2$ centred at z. It follows from Lemma 4.2 that $f^m(\gamma^u)$ contains a cu-disk of radius $\delta_2 = \delta_2(N_0, \delta_1) > 0$ centered at $f^m(z) \in W^s(p)$. Moreover, as distances are not expanded under iterations of points in the same stable manifold, we have

$$\operatorname{dist}_{W^s(p)}(f^m(z), p) \leq \operatorname{dist}_{W^s(p)}(f^m(z), q) + \operatorname{dist}_{W^s(p)}(q, p) \leq \frac{\delta_s}{4} + \frac{\delta_s}{4},$$

which means that $f^{n+m}(V_n(x))$ intersects $W^s_{\delta_s/2}(p)$. Also, choosing $\delta_0 > 0$ sufficiently small (depending only on δ_2) we have u-crosses $\mathcal{C}(B^u_{\delta_0}(p))$. □

We now fix $p \in \Delta, N_0 \geq 1$ and $\delta_0 > 0$ sufficiently small so that the conclusions of Lemma 4.3 hold. Considering the constant

$$K_0 = \max_{x \in M} \left\{ \|Df^{-1}(x)\|, \|Df(x)\| \right\}, \tag{12}$$

we choose in particular $\delta_0 > 0$ small so that

$$2\delta_0 K_0^{N_0} \sigma^{-N_0} < \delta_1 K_0^{-N_0}. \tag{13}$$

Now we define

$$\Delta_0 = B^u_{\delta_0}(p) \quad \text{and} \quad \mathcal{C}_0 = \mathcal{C}(\Delta_0). \tag{14}$$

We also choose $\delta_0 > 0$ small so that any cu-disk intersecting $W^s_{3\delta_s/4}$ cannot reach the top or bottom parts of $\mathcal{C}_0$, i.e. the boundary points of the local stable manifolds $W^s_{\delta_s}(x)$ through points $x \in \Delta_0$. For every $n \geq 1$ we define

$$H_n = \{x \in \Delta \cap H \colon \ n \text{ is a hyperbolic time for } x\ \}.$$

It follows from Lemma 3.3 that for each $x \in H_n \cap \Delta_0$ there exists a hyperbolic pre-disk $V_n(x) \subset \Delta$. Then, by Lemma 4.3 there are $0 \leq m \leq N_0$ and a centre-unstable disk $\omega^x_n \subseteq \Delta$ such that

$$\pi(f^{n+m}(\omega^x_n)) = \Delta_0. \tag{15}$$

We remark that condition (15) may in principle hold for several values of m. For definiteness, we shall always assume that m takes the smallest possible value. Notice that ω_n^x is associated to x by construction, but does not necessarily contain x.

In the sequel we describe an inductive partitioning algorithm which gives rise to a (Leb mod 0) partition $\mathcal{P}$ of the cu-disk Δ_0.

First step of induction. Notice that since $\|Df\|$ is uniformly bounded, for any $n \geq 1$, all hyperbolic pre-disks $V_n(x)$ contain a ball of some radius $\tau_n > 0$ depending only on n. In particular, by compactness, the set $H_n \cap \Delta_0$ is covered by a finite number of hyperbolic pre-disks $V_n(x)$. We fix some large $n_0 \in \mathbb{N}$ and ignore any dynamics occurring up to time n_0. Then there exist ℓ_{n_0} and points $z_1, \ldots, z_{\ell_{n_0}} \in H_{n_0}$ such that

$$H_{n_0} \cap \Delta_0 \subset V_{n_0}(z_1) \cup \cdots \cup V_{n_0}(z_{\ell_{n_0}}).$$

We now choose a maximal subset of points $x_1, \ldots, x_{j_{n_0}} \in \{z_1, \ldots, z_{\ell_{n_0}}\}$ such that the corresponding sets $\omega_{n_0}^{x_i}$ of type (15) are pairwise disjoint and contained in Δ_0, and let

$$\mathcal{P}_{n_0} = \{\omega_{n_0}^{x_1}, \ldots, \omega_{n_0}^{x_{j_{n_0}}}\}.$$

These are the elements of the partition $\mathcal{P}$ constructed in the n_0-step of the algorithm. Let

$$\Delta_{n_0} = \Delta \setminus \bigcup_{\omega \in \mathcal{P}_{n_0}} \omega.$$

For each $0 \leq i \leq j_{n_0}$, we define the inducing time

$$R|_{\omega_{n_0}^{x_i}} = n_0 + m_i$$

where $0 \leq m_i \leq N$ is the integer associated to $\omega_{n_0}^{x_i}$ as in (15). Let now Z_{n_0} be the set of points in $\{z_1, \ldots, z_{\ell_{n_0}}\}$ which were not chosen in the construction of $\mathcal{P}_{n_0}$, i.e.

$$Z_{n_0} = \{z_1, \ldots, z_{\ell_{n_0}}\} \setminus \{x_1, \ldots, x_{j_{n_0}}\}.$$

We remark that for every $z \in Z_{n_0}$, the set $\omega_{n_0}^z$ associated to z must either intersect some $\omega_{n_0}^{x_i} \in \mathcal{P}_{n_0}$ or intersect the complement of Δ_0 in Δ, since otherwise it would have been included in the set $\mathcal{P}_{n_0}$. We now introduce some notation to keep track of which one of the above reasons is responsible for the fact that z belongs to Z_{n_0}. We let $\Delta_0^c = \Delta \setminus \Delta_0$ and for each $\omega \in \mathcal{P}_{n_0} \cup \{\Delta_0^c\}$ we define

$$Z_{n_0}^{\omega} = \left\{x \in Z_{n_0} : \omega_{n_0}^x \cap \omega \neq \emptyset\right\}$$

and the associated n_0-*satellite* set

$$S_{n_0}^{\omega} = \bigcup_{x \in Z_{n_0}^{\omega}} V_{n_0}(x).$$

Finally let

$$V_{n_0} = \bigcup_{i=1}^{j_{n_0}} V(x_i)$$

and

$$S_{n_0} = \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \{\Delta_0^c\}} S_{n_0}^{\omega} \cup V_{n_0}.$$

Notice that S_{n_0}

$$H_{n_0} \cap \Delta_0 \subset S_{n_0} \cup \bigcup_{\omega \in \mathcal{P}_{n_0}} \omega.$$

General step of induction. We now proceed inductively and assume that the construction has been carried out up to time $n-1$ for some $n > n_0$. More precisely, for each $n_0 \le k \le n-1$ we have a collection of pairwise disjoint sets $\mathcal{P}_k = \{\omega_k^{x_1}, ..., \omega_k^{x_{j_k}}\}$ which "return" at time $k+m$ with $0 \le m \le N$, and such that for any $k \ne k'$, any two sets $\omega \in \mathcal{P}_k$ and $\omega' \in \mathcal{P}_{k'}$ we have $\omega \cap \omega' = \emptyset$. We also have a set Δ_k which is the set of points which do not yet have an associated return time. To construct all relevant objects at time n, we note first all, as before, that there are $z_1, \dots, z_{\ell_n} \in H_n \cap \Delta_{n-1}$ such that

$$H_n \cap \Delta_{n-1} \subset V_n(z_1) \cup \dots \cup V_n(z_{\ell_n}),$$

and we choose a maximal subset of points $x_1, \dots, x_{j_n} \in \{z_1, \dots, z_{\ell_n}\}$ such that the corresponding sets of type (15) are pairwise disjoint and contained in Δ_{n-1}. Then we let

$$\mathcal{P}_n = \{\omega_n^{x_1}, \dots, \omega_n^{x_{j_n}}\}$$

These are the elements of the partition $\mathcal{P}$ constructed in the n-step of the algorithm. We also define the set of points of Δ_0 which do not belong to partition elements constructed up to this point:

$$\Delta_n = \Delta_0 \setminus \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \dots \cup \mathcal{P}_n} \omega.$$

For each $0 \le i \le j_n$ we set

$$R|_{\omega_n^{x_i}} = n + m_i,$$

where $0 \le m_i \le N$ is the integer associated to $\omega_{n_0}^{x_i}$ as in (15). Let

$$Z_n = \{z_1, \dots, z_{\ell_n}\} \setminus \{x_1, \dots, x_{j_n}\}$$

and for any $\omega \in \mathcal{P}_{n_0} \cup \dots \cup \mathcal{P}_n \cup \{\Delta_0^c\}$ define

$$Z_n^\omega = \{z \in Z_n : \omega_n^z \cap \omega \ne \emptyset\}$$

and its n-*satellite*

$$S_n^\omega = \bigcup_{z \in Z_n^\omega} V_n(z).$$

Finally let

$$V_n = \bigcup_{i=1}^{j_n} V(x_i)$$

and

$$S_n = \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \dots \cup \mathcal{P}_n \cup \{\Delta_0^c\}} S_n^\omega \cup V_n.$$

Note that for each $n \ge n_0$ one has

$$H_n \cap \Delta_{n-1} \subset S_n \cup \bigcup_{\omega \in \mathcal{P}_{n_0} \cup \dots \cup \mathcal{P}_n} \omega. \tag{16}$$

More specifically we have that $H_n \cap \Delta_{n-1} \subset S_n$, i.e. all points in Δ_{n-1} which have a hyperbolic time at time n are "covered" by S_n while the points which have a hyperbolic time at time n but which are already contained in previously constructed partition elements, are trivially 'covered" by the union of these partition elements. The inclusion (16) will be crucial in Section 7 to prove the integrability of the return times.

This inductive construction allows us to define the family

$$\mathcal{P} = \bigcup_{n \geq n_0} \mathcal{P}_n$$

of pairwise disjoint subsets of Δ_0. At this point there is no guarantee that $\mathcal{P}$ forms a Leb mod 0 partition of Δ_0. This will follow as a corollary of Proposition 5.4 below.

5. Partition on the reference leaf

In this section we prove that the elements of $\mathcal{P}$ defined in the previous section form a Leb_Δ mod 0 partition of the disk Δ_0 introduced in (14). Some of the partial technical estimates will also be used later to prove the integrability of the return times with respect to Leb_Δ.

Lemma 5.1. *There exists $C_2 > 0$ such that for any $n \geq k \geq n_0$ and any $\omega \in \mathcal{P}_k$ we have*

$$\mathrm{Leb}_\Delta(S_n^\omega) \leq C_2 \, \mathrm{Leb}_\Delta \left(\bigcup_{z \in Z_n^\omega} \omega_n^z \right).$$

Proof. We note first of all that, from the construction above, two distinct points z_1, z_2 with the same hyperbolic time n can give rise to the same associated disks $\omega_n^{z_1} = \omega_n^{z_2}$. We prove here that the measure of the union of the hyperbolic pre-disks $V_n(z)$ associated to points $z \in Z_n^\omega$ which give rise to the same disk ω_n^z is comparable to the measure of ω_n^z. More precisely, we will show that for every $n \geq 1$ and $z_1, \dots, z_N \in H_n$ with $\omega_n^{z_i} = \omega_n^{z_1}$ for $1 \leq i \leq N$ we have

$$\mathrm{Leb}_\Delta \left(\bigcup_{i=1}^{N} V_n(z_i) \right) \leq C_2 \, \mathrm{Leb}_\Delta(\omega_n^{z_1}). \tag{17}$$

Notice that (17) implies the statement in the lemma. Indeed, consider a subdivision of the set S_n^ω of all hyperbolic pre-disks associated to the points in Z_n^ω into a finite number of classes such that all hyperbolic pre-disks in each class have the same associated set ω_n^z. Then apply(17) to each one. This gives the statement in the lemma.

Thus we just need to prove (17). For simplicity of notation, for $1 \leq i \leq N$, we write $U_i = V_n(z_i)$ and $B_i = f^n(V_i)$. We define

$$X_1 = U_1 \quad \text{and} \quad X_i = U_i \setminus \bigcup_{j=1}^{i-1} U_j, \quad \text{for } 2 \leq i \leq N.$$

Similarly

$$Y_1 = B_1 \quad \text{and} \quad Y_i = B_i \setminus \bigcup_{j=1}^{i-1} B_j, \quad \text{for } 2 \leq i \leq N.$$

Observe that the X_i's are pairwise disjoint sets whose union coincides with the union of the U_i's, and similarly for the Y_i's and B_i's. Recalling that $\omega_n^{z_i} = \omega_n^{z_1}$ for $1 \leq i \leq N$, by the third item of Lemma 3.3 we have

$$\frac{\mathrm{Leb}_\Delta(X_i)}{\mathrm{Leb}_\Delta(\omega_n^{z_1})} \leq C_1 \frac{\mathrm{Leb}_{f^n(\Delta)}(Y_i)}{\mathrm{Leb}_{f^n(\Delta)} \, f^n(\omega_n^{z_1})}.$$

Hence

$$\begin{aligned}\frac{\mathrm{Leb}_\Delta(U_1\cup\ldots\cup U_N)}{\mathrm{Leb}_\Delta(\omega_n^{z_1})} &= \frac{\sum_{i=1}^N \mathrm{Leb}_\Delta(X_i)}{\mathrm{Leb}_\Delta(\omega_n^{z_1})}\\ &\le C_1\frac{\sum_{i=1}^N \mathrm{Leb}_{f^n(\Delta)}(Y_i)}{\mathrm{Leb}_{f^n(\Delta)}(f^n(\omega_n^{z_1}))}\\ &= C_1\frac{\mathrm{Leb}_{f^n(\Delta)}(B_1\cup\ldots\cup B_N)}{\mathrm{Leb}_{f^n(\Delta)}(f^n(\omega_n^{z_1}))}.\end{aligned}$$

We just need to show that the right hand side is bounded above, and for this it is sufficient to show that the denominator $\mathrm{Leb}_{f^n(\Delta)}(f^n(\omega_n^{z_1}))$ on the right hand side is bounded below. This is clearly true, because by definition of $\omega_n^{z_1}$ we have $m\le N_0$ such that $f^{n+m}(\omega_n^{z_1})$ is a *cu*-disk of radius δ_0. □

Remark 5.2. The argument used to prove (17) gives in particular that for each $1\le i\le j_n$ we have $\mathrm{Leb}_\Delta(V(x_i))\le C_2\,\mathrm{Leb}_\Delta(\omega_n^{x_i})$.

The next lemma shows that, for each n and m fixed, the Lebesgue measure on the disk Δ of the union of sets ω_n^z which intersects an element of partition is proportional to the Lebesgue measure of that element. The proportion constant can actually be made uniformly summable in n.

Lemma 5.3. *There exists $C_3>0$ such that for all $n\ge k\ge n_0$ and $\omega\in\mathcal{P}_k$ we have*

$$\mathrm{Leb}_\Delta\left(\bigcup_{z\in Z_n^\omega}\omega_n^z\right)\le C_3\sigma^{n-k}\,\mathrm{Leb}_\Delta(\omega).$$

Proof. By construction, given $\omega\in\mathcal{P}_k$, there is some hyperbolic pre-disk $V_k(y)$ such that

$$\omega\subset V_k(y)\subset V_k^+(y)$$

and whose images under f^k are respectively *cu*-disks $B^u_{\delta_1}\subset B^u_{2\delta_1}$ centred at $f^k(y)$. Moreover, there exists some integer $0\le\ell\le N_0$ such that $f^{k+\ell}(V_k(y))$ u-crosses $\mathcal{C}_0$ and $f^{k+\ell}(\omega)$ is that part of $f^{k+\ell}(V_k(y))$ which projects onto Δ_0. Moreover, $f^k(V_k^+(y))$ is a δ_1-neighbourhood of $f^k(V_k(y))$ and so $f^{k+\ell}(V_k^+(y))$ contains a $\delta_1K_0^{-N_0}$-neighbourhood of $f^{k+\ell}(V_k(y))$, where is defined in (12). In particular,

$$f^{k+\ell}(V_k^+(y))\text{ contains a }\delta_1K_0^{-N_0}\text{-neighbourhood of }\partial f^{k+\ell}(\omega). \tag{18}$$

For any $n\ge k$ we let

$$A^0_{n,k}=\left\{z\in f^{k+\ell}(V_k^+(y)) : \mathrm{dist}_{f^{k+\ell}(V_k^+(y))}(z,\partial f^{k+\ell}(\omega))\le 2\delta_0K_0^{N_0}\sigma^{n-(k+N_0)}\right\}$$

and

$$A^1_{n,k}=\left\{z\in f^{k+\ell}(\omega) : \mathrm{dist}_{f^{k+\ell}(V_k^+(y))}(z,\partial f^{k+\ell}(\omega))\le 2\delta_1K_0^{N_0}\sigma^{n-(k+N_0)}\right\}.$$

Observe that $A^0_{n,k}$ and $A^1_{n,k}$ are both annuli surrounding the boundary of ω in $f^{k+\ell}(V_k^+(y))$, with the particularity that $A^1_{n,k}$ surrounds only inside $f^{k+\ell}(\omega)$. A straightforward calculation gives that there is a constant $C>0$, independent of k and n, such that

$$\mathrm{Leb}_{f^{k+\ell}(V_k^+(y))}(A^i_{n,k})\le C\sigma^{n-k},\quad i=1,2. \tag{19}$$

Now we see that for $z \in Z_n^\omega$ we have $f^{k+\ell}(\omega_n^z)$ contained in $A^0_{n,k}$ or $A^1_{n,k}$, depending on the following two possible cases:

(1) $\omega_n^z \subseteq \omega$.
By the first item of Lemma 3.3 (see also Remark 4.1), for each ω_n^z with $z \in Z_n^\omega$ we have

$$\operatorname{diam}_{f^{k+\ell}(\omega_n^z)}(f^{k+\ell}(\omega_n^z)) \le \operatorname{diam}_{f^{k+\ell}(\omega_n^z)}(f^{k+\ell}(V_n(z))) \le 2\delta_1 K_0^{N_0}\sigma^{n-(k+N_0)}, \tag{20}$$

Noting that as $z \notin \omega$ and $\omega_n^z \subseteq \omega$, then $V_n(z)$ necessarily intersects the boundary of ω, and so $f^{k+\ell}(V_n(z))$ intersects $\partial f^{k+\ell}(\omega)$. It follows from (20) that

$$f^{k+\ell}(\omega_n^z) \subseteq A^1_{n,k}. \tag{21}$$

(2) $\omega_n^z \not\subseteq \omega$.
In this case, ω_n^z necessarily intersects the boundary of ω because $z \in Z_n^\omega$. Once more by the first item of Lemma 3.3 (see also Remark 4.1), we have

$$\operatorname{diam}_{f^{k+\ell}(\omega_n^z)}(f^{k+\ell}(\omega_n^z)) \le 2\delta_0 K_0^{N_0}\sigma^{n-(k+N_0)}, \tag{22}$$

where we have used the fact that ω_n^z is contained in some hyperbolic pre-disk $V_n(z)$ and the term $K_0^{N_0}$ comes from the fact that ω_n^z may require up to a maximum of N_0 iterates to go from $f^n(V_n(z))$ to the cylinder $\mathcal{C}_0$, u-crossing it. Since ω_n^z intersects the boundary of ω, then $f^{k+\ell}(\omega_n^z)$ intersects the boundary of $f^{k+\ell}(\omega)$. Recalling that $\sigma < 1$ and (13), it follows from (18) and (22) that

$$f^{k+\ell}(\omega_n^z) \subseteq A^0_{n,k}. \tag{23}$$

Therefore

$$\begin{aligned}\frac{\operatorname{Leb}_{V_k^+(y)}\left(\bigcup_{z\in Z_n^\omega}\omega_n^z\right)}{\operatorname{Leb}_{V_k^+(y)}(\omega)} &\le \widetilde{C}\,\frac{\operatorname{Leb}_{f^{k+\ell}(V_k^+(y))}\left(f^{k+\ell}\left(\bigcup_{z\in Z_n^\omega}\omega_n^z\right)\right)}{\operatorname{Leb}_{f^{k+\ell}(V_k^+(y))}(f^{k+\ell}(\omega))}\\ &\le \widetilde{C}\,\frac{\operatorname{Leb}_{f^{k+\ell}(V_k^+(y))}\left(A^0_{n,k}\right)+\operatorname{Leb}_{f^{k+\ell}(V_k^+(y))}\left(A^1_{n,k}\right)}{\operatorname{Leb}_{f^{k+\ell}(V_k^+(y))}(f^{k+\ell}(\omega))},\end{aligned}$$

where $\widetilde{C} > 0$ is a uniform constant that incorporates the distortion at the hyperbolic time k given by Lemma 3.3 and the distortion of f^ℓ with $\ell \le N_0$. Recalling that $f^{k+\ell}(\omega))$ u-crosses $\mathcal{C}_0$, the result then follows by (19), (21) and (23). □

Proposition 5.4. $\displaystyle\sum_{n=n_0}^{\infty}\operatorname{Leb}_\Delta(S_n) < \infty.$

Proof. Observe that

$$\sum_{n=n_0}^{\infty}\operatorname{Leb}_\Delta(S_n) \le \sum_{n=n_0}^{\infty}\operatorname{Leb}_\Delta\left(S_n^{\Delta_0^c}\right) + \sum_{k=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_k}\sum_{n=k}^{\infty}\operatorname{Leb}_\Delta(S_n^\omega) + \sum_{n=n_0}^{\infty}\operatorname{Leb}_\Delta(V_n). \tag{24}$$

We start by estimating the sum with respect to the satellites of Δ_0^c. Notice that from Lemma 3.3 it follows that all hyperbolic pre-disks $V_n(x)$ have diameter $\le 2\delta_1\sigma^n$. Therefore

$$S_n^{\Delta_0^c} \subset \{x \in \Delta_0 : \operatorname{dist}(x, \partial\Delta_0) < 2\delta_1\sigma^n\},$$

and so we can find $\zeta > 0$ such that

$$\mathrm{Leb}_\Delta\left(S_n^{\Delta_0^c}\right) \leq \zeta\sigma^n.$$

This obviously implies that the part of the sum related to Δ_0^c in (24) is finite.

Consider now $n \geq k \geq n_0$. By Lemmas 5.1 and 5.3, for any $\omega \in \mathcal{P}_k$ we have

$$\mathrm{Leb}_\Delta(S_n^\omega) \leq C_2 C_3 \sigma^{n-k}\,\mathrm{Leb}_\Delta(\omega).$$

It follows that

$$\begin{aligned}
\sum_{k=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_k}\sum_{n=k}^{\infty}\mathrm{Leb}_\Delta(S_n^\omega) &\leq C_2C_3\sum_{k=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_k}\sum_{j=0}^{\infty}\sigma^j\,\mathrm{Leb}_\Delta(\omega)\\
&= C_2C_3\frac{1}{1-\sigma}\sum_{k=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_k}\mathrm{Leb}_\Delta(\omega)\\
&\leq C_2C_3\frac{1}{1-\sigma}\,\mathrm{Leb}_\Delta(\Delta).
\end{aligned}$$

Finally, by Remark 5.2 we have

$$\sum_{n=n_0}^{\infty}\mathrm{Leb}_\Delta(V_n) \leq C_2\sum_{n=n_0}^{\infty}\sum_{\omega\in\mathcal{P}_n}\mathrm{Leb}_\Delta(\omega) \leq C_2\,\mathrm{Leb}_\Delta(\Delta)$$

and this gives the conclusion. □

We are now ready to show that our inductive construction gives rise to a Leb_Δ mod 0 partition of Δ_0. Recall that $\Delta_0 \supset \Delta_{n_0} \supset \Delta_{n_0+1} \supset ...$, where Δ_n is the set of points which does not belong to any element of the collection $\mathcal{P}$ constructed up to time n. It is enough to show that

$$\mathrm{Leb}_\Delta\left(\bigcap_n \Delta_n\right) = 0. \tag{25}$$

To prove this, notice that by Proposition 5.4, the sum of the Leb_Δ measures of the sets S_n is finite. It follows from Borel-Cantelli Lemma that Leb_Δ almost every $x \in \Delta_0$ belongs only to finitely many S_n's, and therefore one can find n such that $x \notin S_j$ for $j \geq n$. Since Leb_Δ almost every $x \in \Delta_0$ has infinitely many hyperbolic times, it follows from (16) that $x \in \omega$ for some $\omega \in \mathcal{P}_{n_0} \cup \cdots \cup \mathcal{P}_n$ and therefore (25) holds.

6. The GMY structure

We are now ready to define the GMY structure on Ω as in the beginning of Section 4. Consider the center-unstable disk $\Delta_0 \subset \Delta$ as in (14) and the Leb_Δ mod 0 partition $\mathcal{P}$ of Δ_0 defined in Section 4. We define

$$\Gamma^s = \left\{W^s_{\delta_s}(x) : x \in \Delta_0\right\}.$$

Moreover, we define Γ^u as the set of all local unstable manifolds contained in $\mathcal{C}_0$ which u-cross $\mathcal{C}_0$. Clearly, Γ^u is nonempty because $\Delta_0 \in \Gamma^u$. We need to see that the union of the leaves in Γ^u is compact. This follows ideas that we have already used to prove Proposition 3.1. By the domination property and Ascoli-Arzelà Theorem, any limit leaf γ_∞ of leaves in Γ^u is still a cu-disk u-crossing $\mathcal{C}_0$. Thus, by definition of Γ^u, we have $\gamma_\infty \in \Gamma^u$. We thus define our set Λ with hyperbolic product structure as the intersection of these families of stable and unstable leaves. The cylinders $\{\mathcal{C}(\omega)\}_{\omega\in\mathcal{P}}$ then clearly form a countable collection of

s-subsets of Λ that play the role of the sets $\Lambda_1, \Lambda_2, \dots$ in (P_1) with the corresponding return times $R(\omega)$. It just remains to check that conditions (P_1)-(P_5) hold.

6.1. **Markov and contraction on stable leaves.** Condition (P_1) is essentially an immediate consequence of the construction. We just need to check that $f^{R(\omega)}(\mathcal{C}(\omega))$ is a u-subset, for any $\omega \in \mathcal{P}$. Indeed, choosing the integer n_0 in the first step of the inductive algorithm sufficiently large, and using the fact that the local stable manifolds are uniformly contracted by forward iterations under f, we can easily see that the "height" of $f^{R(\omega)}(C(\omega))$ is at most $\delta_s/4$. Hence, by the choice of δ_0 we have $f^{R(\omega)}(\mathcal{C}(\omega))$ made by cu-unstable disks contained in $\mathcal{C}_0$. Moreover, as $f^{R(\omega)}(\omega)$ u-crosses $\mathcal{C}_0$ the same occurs with the local unstable leaves that form $\mathcal{C}(\omega)$, and so (P_1) holds. (P_2) is clearly verified under our assumptions.

6.2. **Backward contraction and bounded distortion.** The backward contraction on unstable leaves and bounded distortion, respectively properties (P_3) and (P_4), follow from Lemma 3.3. Indeed, by construction, for each $\omega \in \mathcal{P}$ there is a hyperbolic pre-ball $V_{n(\omega)}(x)$ containing ω associated to some point $x \in D$ with σ-hyperbolic time $n(\omega)$ satisfying $R(\omega) - N_0 \le n(\omega) \le R(\omega)$. It is sufficient to prove the (P_3) and (P_4) at the time $n = n(\omega)$ instead of $R(\omega)$ since the two differ by a finite and uniformly bounded number of iterations whose contribution to the estimates is also uniformly controlled.

An immediate consequence of (10) is that if $y \in K$ satisfies $\operatorname{dist}(f^j(x), f^j(y)) \le \delta_1$ for $0 \le j \le n-1$, then n is a $\sigma^{3/4}$-hyperbolic time for y, i.e.

$$\prod_{j=n-k+1}^{n} \left\| Df^{-1} | E^{cu}_{f^j(y)} \right\| \le \sigma^{3k/4}, \qquad \text{for all } 1 \le k \le n.$$

Therefore, taking $\delta_s, \delta_0 < \delta_1/2$, for any $\gamma \in \Gamma^u$ we have that n is a $\sigma^{3/4}$-hyperbolic time for every point in $\mathcal{C}_\omega \cap \gamma$. The backward contraction on unstable leaves and bounded distortion are then consequence of Lemma 3.3, recall Remark 3.4.

6.3. **Regularity of the foliations.** Property (P_5) is standard for uniformly hyperbolic attractors. In the rest of this section we shall adapt classical ideas to our setting.

We begin with the statement of a useful lemma on vector bundles whose proof can be found in [24, Theorem 6.1]. Let us recall that a metric d on E is *admissible* if there is a complementary bundle E' over X, and an isomorphism $h\colon E \oplus E' \to X \times B$ to a product bundle, where B is a Banach space, such that d is induced from the product metric on $X \times B$.

Lemma 6.1. *Let $p\colon E \to X$ be a vector bundle over a metric space X endowed with an admissible metric. Let $D \subset E$ be the unit ball bundle, and $F\colon D \to D$ a map covering a Lipschitz homeomorphism $f\colon X \to X$. Assume that there is $0 \le \kappa < 1$ such that for each $x \in X$ the restriction $F_x\colon D_x \to D_x$ satisfies $\operatorname{Lip}(F_x) \le \kappa$. Then*

(1) *there is a unique section $\sigma_0\colon X \to D$ whose image is invariant under F;*
(2) *if $\kappa \operatorname{Lip}(f)^\alpha < 1$ for some $0 < \alpha \le 1$, then σ_0 is Hölder continuous with exponent α.*

Proposition 6.2. *Let $f : M \to M$ be a C^1 diffeomorphism and $\Omega \subset M$ a compact invariant set with a dominated splitting $T_\Omega M = E^{cs} \oplus E^{cu}$. Then the fiber bundles E^{cs} and E^{cu} are Hölder continuous on Ω.*

Proof. We consider only the centre-unstable bundle as the other one is similar. For each $x \in \Omega$ let L_x be the space of bounded linear maps from E^{cu}_x to E^{cs}_x and let L^1_x denote the unit ball around $0 \in L_x$. We define $\Gamma_x : L^1_x \to L^1_{f(x)}$ as the graph transform induced by $Df(x)$:

$$\Gamma_x(\mu_x) = (Df|E^{cs}_x) \cdot \mu_x \cdot (Df^{-1}|E^{cu}_{f(x)}).$$

Consider L the vector bundle over Ω whose fiber over each $x \in \Omega$ is L_x, and let L^1 be its unit ball bundle. Then $\Gamma : L^1 \to L^1$ is a bundle map covering $f|\Omega$ with

$$\mathrm{Lip}(\Gamma_x) \leq \|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda < 1.$$

Let c be a Lipschitz constant for $f|_\Omega$, and choose $0 < \alpha \leq 1$ small so that $\lambda c^\alpha < 1$. By Lemma 6.1 there exists a unique section $\sigma_0 \colon M \to L^1$ whose image is invariant under Γ and it satisfies a Hölder condition of exponent α. This unique section is necessarily the null section. □

The next result gives precisely (P_5)(a).

Corollary 6.3. *There are $C > 0$ and $0 < \beta < 1$ such that for all $y \in \gamma^s(x)$ and $n \geq 0$*

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C\beta^n.$$

Proof. As we are assuming that Df is Hölder continuous, it follows from Proposition 6.2 that $\log|\det Df^u|$ is Hölder continuous. The conclusion is then an immediate consequence of the uniform contraction on stable leaves. □

To prove (P_5)(b) we introduce some useful notions. We say that $\phi : N \to P$, where N and P are submanifolds of M, is *absolutely continuous* if it is an injective map for which there exists $J : N \to \mathbb{R}$ such that

$$\mathrm{Leb}_P(\phi(A)) = \int_A J d\,\mathrm{Leb}_N\,.$$

J is called the *Jacobian* of ϕ. Property (P_5)(b) can be restated in the following terms:

Proposition 6.4. *Given $\gamma, \gamma' \in \Gamma^u$, define $\phi\colon \gamma' \to \gamma$ by $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous and the Jacobian of ϕ is given by*

$$J(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

One can easily deduce from Corollary 6.3 that this infinite product converges uniformly. The remaining of this section is devoted to the proof of Proposition 6.4. We start with a general result about the convergence of Jacobians whose proof is given in [28, Theorem 3.3].

Lemma 6.5. *Let N and P be manifolds, P with finite volume, and for each $n \geq 1$, let $\phi_n : N \to P$ be an absolutely continuous map with Jacobian J_n. Assume that*

(1) *ϕ_n converges uniformly to an injective continuous map $\phi : N \to P$;*
(2) *J_n converges uniformly to an integrable function $J : N \to \mathbb{R}$.*

Then ϕ is absolutely continuous with Jacobian J.

For the sake of completeness, we observe that there is a slight difference in our definition of absolute continuity. Contrarily to [28], and for reasons that will become clear below, we do not impose the continuity of the maps ϕ_n. However, the proof of [28, Theorem 3.3] uses only the continuity of the limit function ϕ, and so it still works in our case.

Consider now $\gamma, \gamma' \in \Gamma^u$ and $\phi\colon \gamma' \to \gamma$ as in Proposition 6.4. The proof of the next lemma is given in [28, Lemma 3.4] for uniformly hyperbolic diffeomorphisms. Nevertheless, one can easily see that it is obtained as a consequence of [28, Lemma 3.8] whose proof uses only the existence of a dominated splitting.

Lemma 6.6. *For each $n \geq 1$, there is an absolutely continuous $\pi_n : f^n(\gamma) \to f^n(\gamma')$ with Jacobian G_n satisfying*

(1) $\lim_{n\to\infty} \sup_{x\in\gamma} \left\{\operatorname{dist}_{f^n(\gamma')}(\pi_n(f^n(x)), f^n(\phi(x))\right\} = 0$;
(2) $\lim_{n\to\infty} \sup_{x\in f^n(\gamma)} \{|1 - G_n(x)|\} = 0$.

We consider the sequence of consecutive return times for points in Λ,

$$r_1 = R \quad \text{and} \quad r_{n+1} = r_n + R \circ f^{r_n}, \quad \text{for } n \geq 1.$$

Notice that these return time functions are defined Leb_γ almost everywhere on each $\gamma \in \Gamma^u$ and are piecewise constant.

Remark 6.7. Using the sequence of return times one can easily construct a sequence of Leb_γ mod 0 partitions $(\mathcal{Q}_n)_n$ by s-subsets of Λ with r_n constant on each element of $\mathcal{Q}_n$, for which (P_1)-(P_5) hold when we take r_n playing the role of R and the elements of $\mathcal{Q}_n$ playing the role of the s-subsets. Moreover, the constants $C > 0$ and $0 < \beta < 1$ can be chosen not depending on n.

We define, for each $n \geq 1$, the map $\phi_n : \gamma \to \gamma'$ as

$$\phi_n = f^{-r_n} \pi_{r_n} f^{r_n}. \tag{26}$$

It is straightforward to check that ϕ_n is absolutely continuous with Jacobian

$$J_n(x) = \frac{|\det(Df^{r_n})^u(x)|}{|\det(Df^{r_n})^u(\phi_n(x))|} \cdot G_{r_n}(f^{r_n}(x)). \tag{27}$$

Observe that these functions are defined Leb_γ almost everywhere. So, we may find a Borel set $A \subset \gamma$ with full Leb_γ measure on which they are all defined. We extend ϕ_n to γ simply by considering $\phi_n(x) = \phi(x)$ and $J_n(x) = J(x)$ for all $n \geq 1$ and $x \in \gamma \setminus A$. Since A has zero Leb_γ measure one still has that J_n is the Jacobian of ϕ_n.

Proposition 6.4 is now a consequence of Lemma 6.5 together with the next one.

Lemma 6.8. *$(\phi_n)_n$ converges uniformly to ϕ and $(J_n)_n$ converges uniformly to J.*

Proof. It is sufficient to prove the convergence of each sequence restricted to A described above. In particular, the expressions of ϕ_n and J_n are given by (26) and (27) respectively.

Let us prove first the case of $(\phi_n)_n$. Using the backward contraction on unstable leaves given by (P_3) and recalling Remark 6.7, we may write for each $x \in \gamma$

$$\begin{aligned} \operatorname{dist}_{\gamma'}(\phi_n(x), \phi(x)) &= \operatorname{dist}_{\gamma'}(f^{-r_n}\pi_{r_n} f^{r_n}(x), f^{-r_n} f^{r_n}\phi(x)) \\ &\leq C\beta^{r_n} \operatorname{dist}_{f^{r_n}(\gamma')}(\pi_{r_n} f^{r_n}(x), f^{r_n}\phi(x)). \end{aligned}$$

Since $r_n \to \infty$ as $n \to \infty$ and $\operatorname{dist}_{f^{r_n}(\gamma')}(\pi_{r_n} f^{r_n}(x), f^{r_n}\phi(x))$ is bounded, by Lemma 6.6, we have the uniform convergence of ϕ_n to ϕ.

Let us prove now that the case of the Jacobians $(J_n)_n$. By (27), we have

$$J_n(x) = \frac{|\det(Df^{r_n})^u(x)|}{|\det(Df^{r_n})^u(\phi(x))|} \cdot \frac{|\det(Df^{r_n})^u(\phi(x))|}{|\det(Df^{r_n})^u(\phi_n(x))|} \cdot G_{r_n}(f^{r_n}(x)).$$

Using the chain rule and Corollary 6.3, it easily follows that the first term in the product above converges uniformly to $J(x)$. Moreover, by Lemma 6.6, the third term converges uniformly to 1. It remains to see that the middle term also converges uniformly to 1. Recalling Remark 6.7, by bounded distortion we have

$$\begin{aligned}\frac{|\det(Df^{r_n})^u(\phi(x))|}{|\det(Df^{r_n})^u(\phi_n(x))|} &\leq \exp\big(C\,\mathrm{dist}_{f^{r_n}(\gamma')}(f^{r_n}(\phi(x)), f^{r_n}(\phi_n(x)))^\eta\big)\\ &= \exp\big(C\,\mathrm{dist}_{f^{r_n}(\gamma')}(f^{r_n}(\phi(x)), \pi_{r_n}(f^{r_n}(x)))^\eta\big).\end{aligned}$$

Similarly we obtain

$$\frac{|\det(Df^{r_n})^u(\phi(x))|}{|\det(Df^{r_n})^u(\phi_n(x))|} \geq \exp\big(-C\,\mathrm{dist}_{f^{r_n}(\gamma')}(f^{r_n}(\phi(x)), \pi_{r_n}(f^{r_n}(x)))^\eta\big).$$

The conclusion then follows from Lemma 6.6. □

7. Integrability of the return time

In the previous sections we have constructed a GMY structure on Ω. To complete the proof of Theorem C it just remains to show that this GMY structure has integrable return times as in (5). Recall first that the existence of a GMY structure implies the existence of an induced map $F:\Lambda\to\Lambda$ with an invariant probability measure ν, see remarks following Theorem C. This measure can be disintegrated into a family of conditional measures on the unstable leaves $\{\gamma^u\}$ with conditional measures which are equivalent to Lebesgue measure with densities bounded by uniform constants above and below, see [42, Lemma 2]. We fix one such unstable leaf $\gamma\in\Gamma^u$ and let $\bar\nu$ denote the conditional measure associated to ν and equivalent to Lebesgue. The integrability of the return times with respect to Lebesgue as in (5) therefore follows immediately from the next result.

Proposition 7.1. *The inducing time function R is $\bar\nu$-integrable.*

Proof. We first introduce some notation. For $x\in\Delta$ we consider the orbit $x, f(x), ..., f^{n-1}(x)$ of the point x under iteration by f for some large value of n. In particular x may undergo several full returns to Δ before time n. Then we define the following quantities:

$$\begin{aligned}H^{(n)}(x) &:= \text{number of hyperbolic times for } x \text{ before time } n\\ S^{(n)}(x) &:= \text{number of times } x \text{ belongs to a satellite before time } n\\ R^{(n)}(x) &:= \text{number of returns of } x \text{ before time } n\end{aligned}$$

Each time that x has a hyperbolic time, it either then has a return within some finite and uniformly bounded number of iterations, or by definition it belongs to a satellite. Therefore there exists some constant $\kappa>0$ independent of x and n such that

$$R^{(n)}(x)+S^{(n)}(x)\geq\kappa H^{(n)}(x)$$

Notice that x may belong to a satellite or have a return without it having a hyperbolic time itself, since it may belong to a hyperbolic pre-disk of some other point y which has a hyperbolic time. Dividing the above equation through by n we get

$$\frac{R^{(n)}(x)}{n}+\frac{S^{(n)}(x)}{n}\geq\frac{\kappa H^{(n)}(x)}{n}$$

Recalling that hyperbolic times have uniformly positive asymptotic frequency, there exists a constant $\theta > 0$ such that $H^{(n)}(x)/n \geq \theta$ for all n sufficiently large, and therefore, rearranging the left hand side above gives

$$\frac{R^{(n)}(x)}{n}\left(1+\frac{S^{(n)}(x)}{R^{(n)}(x)}\right) \geq \kappa\theta > 0$$

Moreover $S^{(n)}(x)/R^{(n)}(x)$ converges by Birkhoff's ergodic theorem to precisely the average number of times $\int S d\nu$ that typical points belong to satellites before they return, and from Proposition 5.4 it follows that $\int S d\nu < \infty$. Therefore, we have

$$\frac{R^{(n)}(x)}{n} \geq \kappa' > 0 \tag{28}$$

for all sufficiently large n where κ' can be chosen arbitrarily close to $\kappa\theta/(1+\int S d\bar{\nu})$ which is independent of x and n. To conclude the proof notice that $n/R^{(n)}(x)$ is the average return time over the first n iterations and thus converges by Birkhoff's ergodic theorem to $\int \bar{R} d\bar{\nu}$. This holds even if we do not assume a priori that $\bar{R}$ is integrable since it is a positive function and thus $\int \bar{R} d\bar{\nu}$ is always well defined and lack of integrability necessarily implies $\int \bar{R} d\bar{\nu} = +\infty$. Thus. arguing by contradiction and assuming that $\int \bar{R} d\bar{\nu} = +\infty$ gives $n/R^{(n)}(x) \to \int \bar{R} d\bar{\nu} = +\infty$ and therefore $R^{(n)}(x)/n \to 0$. This contradicts (28) and therefore implies that we must have $\int \bar{R} d\bar{\nu} < +\infty$ as required. □

8. Liftabilty

In this section we complete the proof of Theorem D. The 'if" part of this result is well known and we refer to it in the comments preceeding Theorem D. We therefore just need to show that every SRB measure with positive Lyapunov exponents in the E^{cu} direction is liftable. To achieve this, first of all let Ω denote the support of the given SRB measure μ. Then Ω is invariant under f and thus under any positive iterate of f. We will show in the following proposition that there exists some $N \geq 1$ such that f^N on Ω is nonuniformly expanding, and thus weakly nonuniformly expanding, along E^{cu}. We can then apply the conclusions of Theorem C to obtain a GMY structure for f^N with integrable return time function R. This easily give a corresponding GMY structure for f with return time function NR which is therefore still integrable and therefore, as explained above, gives rise to an SRB measure. By uniqueness of SRB measures it follows that this measure coincides with μ, thus proving that μ is liftable.

Proposition 8.1. *There exists $N \geq 1$ such that f^N is non-uniformly expanding along E^{cu} on a set with positive Lebesgue measure.*

Proof. We prove first of all that there exists $N \geq 1$ such that

$$\int \log \|(Df^N|_{E^{cu}_x})^{-1}\| d\mu < 0. \tag{29}$$

Indeed, by assumption all Lyapunov exponents of f along E^{cu} are positive and therefore all Lyapunov exponents of the map f^{-1} along E^{cu} are negative. Thus, considering the cocycle $(x,v) \mapsto (f^{-1}(x), Df^{-1}(x)v)$, Oseledets' Theorem implies that there exists λ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \left\| Df^{-1}|_{E^{cu}_{f^{-n+1}(x)}} \cdots Df^{-1}|_{E^{cu}_x} \right\| = \lambda < 0 \tag{30}$$

where λ is the largest Lyapunov exponent of f^{-1}, see [8, Addendum 4]. By the chain rule and the inverse function theorem, we have

$$Df^{-1}|_{E^{cu}_{f^{-n+1}(x)}}\cdots Df^{-1}|_{E^{cu}_x}=\left(Df^n|_{E^{cu}_{f^{-n}(x)}}\right)^{-1}. \tag{31}$$

Since the sequence

$$\phi_n=\log\left\|\left(Df^n|_{E^{cu}_{f^{-n}(x)}}\right)^{-1}\right\|$$

satisfies $\phi_{n+m}\le\phi_n+\phi_m\circ f^{-n}$, using the invariance of μ with respect to f^{-1} and Kingmann's Subadditive Ergodic Theorem we have, for μ almost every x,

$$\lim_{n\to\infty}\frac{1}{n}\log\left\|\left(Df^n|_{E^{cu}_{f^{-n}(x)}}\right)^{-1}\right\|=\inf_{n\ge1}\frac{1}{n}\int\log\left\|\left(Df^n|_{E^{cu}_{f^{-n}(x)}}\right)^{-1}\right\|d\mu,$$

which, together with (30) and (31), gives (29).

Notice that μ may not be ergodic for f^N, but it can have at most N ergodic components. Indeed, notice first of all that any subset C which is f^N-invariant and has positive measure, satisfies $\mu(C)\ge 1/N$: assume by contradiction that $\mu(C)<1/N$ and consider the set $\cup_{j=0}^{N-1}f^{-j}(C)$. We have that

$$0<\mu\left(\bigcup_{j=0}^{N-1}f^{-j}(C)\right)\le\sum_{j=0}^{N-1}\mu(f^{-j}(C))<1.$$

This gives a contradiction, because the set is f- invariant and μ is ergodic. Now, if (f^N,μ) is not ergodic, then we decompose M into a union of two f^N-invariant disjoint sets with positive measure. If the restriction of μ to one of these sets is not ergodic, then we iterate this process. Note that this must stop after a finite number of steps with at most N disjoint subsets, since f^N-invariant sets with positive measure have its measure bounded from below by $1/N$.

Thus, we have that (f^N,μ) has at most N ergodic components. By (29), at least one of these ergodic components, whose support we denote by Σ, satisfies $\int_\Sigma\log\|(Df^N|E^{cu}_x)^{-1}\|d\mu<0$. Hence, by Birkhoff's Ergodic Theorem, for μ almost every $x\in\Sigma$ one has

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|(Df^N|E^{cu}_{f^{Nj}(x)})^{-1}\|=\int_\Sigma\log\|(Df^N|E^{cu}_x)^{-1}\|d\mu<0.$$

This proves that f^N is non-uniformly expanding along E^{cu} for μ almost every point in the set Σ. From the assumption that μ is an SRB measure we have that conditional measures of μ on local unstable manifolds are absolutely continuous with respect to Lebesgue. In particular there is some local unstable manifold γ^u on which we have non-uniform expansion for a set of points of positive Leb_{γ^u} measure. Considering the union of local stable manifolds through these points and the absolute continuity of the stable foliation we get the result. □

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