A multiparameter family of non-weight irreducible representations of the quantum plane and of the quantum Weyl algebra

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Abstract

We construct a family of irreducible representations of the quantum plane and of the quantum Weyl algebra over an arbitrary field, assuming the deformation parameter is not a root of unity. We determine when two representations in this family are isomorphic, and when they are weight representations, in the sense of [1].

1 Introduction

Assume throughout that $\mathbb{F}$ is a field of arbitrary characteristic, not necessarily algebraically closed, with group of units $\mathbb{F}^*$. Fix $q \in \mathbb{F}^*$ with $q \neq 1$. The *quantum plane* is the unital associative algebra

$$\mathbb{F}_q[x,y] = \mathbb{F}\{x,y\}/(yx - qxy) \tag{1.1}$$

with generators x and y subject to the relation $yx = qxy$.

Consider the operators τ_q and ∂_q defined on the polynomial algebra $\mathbb{F}[t]$ by

$$\tau_q(p)(t) = p(qt), \quad \text{and} \quad \partial_q(p)(t) = \frac{p(qt) - p(t)}{qt - t}, \quad \text{for } p \in \mathbb{F}[t]. \tag{1.2}$$

Then the assignment $x \mapsto \tau_q$, $y \mapsto \partial_q$ yields a (reducible) representation $\mathbb{F}_q[x,y] \to \mathrm{End}_{\mathbb{F}}(\mathbb{F}[t])$ of $\mathbb{F}_q[x,y]$, which is faithful if and only if q is not a root of unity. The operators τ_q and ∂_q are central in the theory of linear q-difference equations and ∂_q is also known as the *Jackson derivative*, as it appears in [4]. See e.g. [6], [5, Chap. IV] and references therein for further details.

The irreducible representations of the quantum plane $\mathbb{F}_q[x,y]$ have been classified in [1] using results from [2]. Following [1] we say that a representation of $\mathbb{F}_q[x,y]$ is a *weight representation* if it is semisimple as a representation of the polynomial subalgebra

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$\mathbb{F}[H]$ generated by the element $H = xy$. When q is a root of unity all irreducible representations of $\mathbb{F}_q[x, y]$ are finite-dimensional weight representations, and these are well understood. For example, if $\mathbb{F}$ is algebraically closed and q is a primitive n-th root of unity then the irreducible representations of $\mathbb{F}_q[x, y]$ are either 1 or n dimensional. When q is not a root of unity there are irreducible representations of $\mathbb{F}_q[x, y]$ that are not weight representations, and in particular are not finite dimensional. These turn out to be the $\mathbb{F}[H]$-*torsionfree* irreducible representations of $\mathbb{F}_q[x, y]$, as they remain irreducible (i.e. nonzero) upon localizing at the nonzero elements of $\mathbb{F}[H]$. In [1, Cor. 3.3] the torsionfree representations of $\mathbb{F}_q[x, y]$ are classified in terms of elements satisfying certain conditions, but no explicit construction of these representations is given.

We assume q is not a root of unity, and we give an explicit construction of a 3-parameter family $\mathsf{V}_f^{m,n}$ of infinite-dimensional representations of $\mathbb{F}_q[x, y]$ having the following properties (compare Propositions 2.4, 2.6 and 2.7):

- m and n are positive integers, and $f : \mathbb{Z} \to \mathbb{F}^*$ satisfies condition (2.1) below, which essentially encodes n independent parameters from $\mathbb{F}^*$;
- $\mathsf{V}_f^{m,n}$ is irreducible if and only if $\mathsf{gcd}(m, n) = 1$;
- if $(m, n) \neq (m', n')$ then $\mathsf{V}_f^{m,n}$ and $\mathsf{V}_{f'}^{m',n'}$ are not isomorphic;
- $\mathsf{V}_f^{m,n}$ is a weight representation if and only if $m = n$;
- if $\mathbb{F}$ is algebraically closed and V is an irreducible weight representation of $\mathbb{F}_q[x, y]$ that is infinite dimensional, then $V \simeq \mathsf{V}_f^{1,1}$ for some $f : \mathbb{Z} \to \mathbb{F}^*$.

Thus, in some sense weight and non-weight representations of $\mathbb{F}_q[x, y]$ are rejoined in the family $\mathsf{V}_f^{m,n}$.

The localization of $\mathbb{F}_q[x, y]$ at the multiplicative set generated by x contains a copy of the q-*Weyl algebra*, which is the algebra

$$\mathbb{A}_1(q) = \mathbb{F}\{X, Y\}/(YX - qXY - 1) \tag{1.3}$$

with generators X and Y subject to the relation $YX - qXY = 1$ (see (3.1) for details about this embedding). This is used in Subsection 3.1 to regard the representations $\mathsf{V}_f^{m,n}$ as infinite-dimensional irreducible representations of $\mathbb{A}_1(q)$. In contrast with the action of $\mathbb{F}_q[x, y]$ on $\mathsf{V}_f^{m,n}$ when $m = n$, it turns out that $\mathsf{V}_f^{m,n}$ is never a weight representation of $\mathbb{A}_1(q)$ in the sense of [1]. In Subsection 3.2 we pursue a dual approach by constructing representations W_g^n of $\mathbb{A}_1(q)$ and then restricting the action from the q-Weyl algebra to two distinct subalgebras of $\mathbb{A}_1(q)$ isomorphic to $\mathbb{F}_q[x, y]$.

2 A family $\mathsf{V}_f^{m,n}$ of infinite-dimensional irreducible representations of $\mathbb{F}_q[x, y]$ for q not a root of unity

Assume $q \in \mathbb{F}^*$ is not a root of unity. We introduce a family $\mathsf{V}_f^{m,n}$ of infinite-dimensional representations of $\mathbb{F}_q[x, y]$ which are not in general weight representations in the sense of [1], but which includes all irreducible infinite-dimensional weight representations of $\mathbb{F}_q[x, y]$ if we further assume $\mathbb{F}$ to be algebraically closed.

2.1 Structure of the representations $\mathsf{V}_f^{m,n}$

Fix positive integers $m, n \in \mathbb{Z}_{>0}$ and a function $f : \mathbb{Z} \to \mathbb{F}^*$ satisfying

$$f(i + n) = qf(i), \qquad \text{for all } i \in \mathbb{Z}. \tag{2.1}$$

Such functions are in one-to-one correspondence with elements of $(\mathbb{F}^*)^n$. Let $\mathsf{V}_f^{m,n}$ denote the representation of $\mathbb{F}_q[x, y]$ on the space $\mathbb{F}[t^{\pm 1}]$ of Laurent polynomials in t given by

$$x.t^i = t^{i+n}, \qquad y.t^i = f(i)t^{i-m}, \qquad \text{for all } i \in \mathbb{Z}. \tag{2.2}$$

Condition (2.1) ensures that the expressions (2.2) do define an action of $\mathbb{F}_q[x, y]$ on $\mathbb{F}[t^{\pm 1}]$ as, for all $i \in \mathbb{Z}$,

$$(yx - qxy).t^i = (f(i + n) - qf(i))t^{i+n-m} = 0.$$

Example 2.1. Fix $\mu \in \mathbb{F}^*$ and $m, n \in \mathbb{Z}_{>0}$. For $i \in \mathbb{Z}$ let $f(i) = \mu q^{\lfloor \frac{i}{n} \rfloor}$, where $\lfloor \frac{i}{n} \rfloor$ denotes the largest integer not exceeding $\frac{i}{n}$. Then $f : \mathbb{Z} \to \mathbb{F}^*$ satisfies condition (2.1) and thus there is a representation $\mathsf{V}_f^{m,n}$ of $\mathbb{F}_q[x, y]$ on $\mathbb{F}[t^{\pm 1}]$ with action

$$x.t^i = t^{i+n}, \qquad y.t^i = \mu q^{\lfloor \frac{i}{n} \rfloor} t^{i-m}, \qquad \text{for all } i \in \mathbb{Z}.$$

We begin the study of the representations $\mathsf{V}_f^{m,n}$ by first considering the case that the parameters m and n are coprime. The following consequence of (2.1) will be helpful.

Lemma 2.2. *Assume* $\mathsf{gcd}(m, n) = 1$ *and* $f : \mathbb{Z} \to \mathbb{F}^*$ *satisfies* (2.1). *For* $k \in \mathbb{Z}$ *define*

$$\mathsf{s}_f(k) = \prod_{i=0}^{n-1} f(k - im). \tag{2.3}$$

Then $\mathsf{s}_f(k) = \mathsf{s}_f(0)q^k$.

Proof. For $j \in \mathbb{Z}$ let $0 \leq \overline{\jmath} < n$ be the unique integer such that $\overline{\jmath} \equiv j \bmod n$. Then the formula $f(j) = f(\overline{\jmath})q^{\frac{j-\overline{\jmath}}{n}}$ can be verified by induction on $\left|\frac{j-\overline{\jmath}}{n}\right|$. Thus,

$$\mathsf{s}_f(k) = \prod_{i=0}^{n-1} f(k - im) = \prod_{i=0}^{n-1} f\left(\overline{k - \imath m}\right) \prod_{i=0}^{n-1} q^{\frac{k-im-\overline{k-\imath m}}{n}}.$$

Since m and n are coprime, the set $\left\{\overline{k - \imath m} \mid 0 \leq i < n\right\}$ consists of all the integers from 0 to $n - 1$, and is thus independent of k. Moreover,

$$\sum_{i=0}^{n-1} \frac{k - im - \overline{k - \imath m}}{n} = k + \sum_{i=0}^{n-1} \frac{-im - \overline{k - \imath m}}{n} = k + \sum_{i=0}^{n-1} \frac{-im - \overline{(-\imath m)}}{n}.$$

Hence,

$$\mathsf{s}_f(k) = q^k \prod_{i=0}^{n-1} f\left(\overline{-im}\right) \prod_{i=0}^{n-1} q^{\frac{-im-\overline{(-im)}}{n}} = q^k \mathsf{s}_f(0).$$

$\square$

Proposition 2.3. *Assume* $\mathsf{gcd}(m,n) = 1$ *and* $f : \mathbb{Z} \to \mathbb{F}^*$ *satisfies* (2.1)*. Then the representation* $\mathsf{V}_f^{m,n}$ *defined by* (2.2) *is an irreducible representation of* $\mathbb{F}_q[x,y]$*.*

Proof. We begin with a computation: for $k \in \mathbb{Z}$ we have, by Lemma 2.2,

$$x^m y^n.t^k = x^m \left(\prod_{i=0}^{n-1} f(k-im) \right) t^{k-nm} = \mathsf{s}_f(k) t^k = \mathsf{s}_f(0) q^k t^k. \tag{2.4}$$

Hence, $x^m y^n.p(t) = \mathsf{s}_f(0) p(qt)$ for all $p \in \mathbb{F}[t^{\pm 1}]$.

Let $\mathsf{W} \subseteq \mathsf{V}_f^{m,n}$ be a nonzero subrepresentation. If $p(t) \in \mathsf{W}$ then also $p(qt) \in \mathsf{W}$, by (2.4). As q is not a root of unity, the latter implies that $t^\ell \in \mathsf{W}$ for some $\ell \in \mathbb{Z}$. The coprimeness of m and n shows the existence of integers a and b so that $an - bm = 1$. By replacing a and b with $a + jm$ and $b + jn$ for a sufficiently large integer j, we can assume $a, b \in \mathbb{Z}_{>0}$. Then $x^a y^b.t^k = \lambda_k t^{k+1}$ for some $\lambda_k \in \mathbb{F}^*$, showing that $t^k \in \mathsf{W}$ for all $k \geq \ell$. A similar argument shows that $t^k \in \mathsf{W}$ for all $k \leq \ell$. Hence $\mathsf{W} = \mathsf{V}_f^{m,n}$, establishing the irreducibility of $\mathsf{V}_f^{m,n}$. $\square$

Next we describe $\mathsf{V}_f^{m,n}$ in terms of a maximal left ideal of $\mathbb{F}_q[x,y]$. Recall that for a representation V of $\mathbb{F}_q[x,y]$ and an element $v \in \mathsf{V}$, the annihilator of v in $\mathbb{F}_q[x,y]$ is $\mathsf{ann}_{\mathbb{F}_q[x,y]}(v) = \{r \in \mathbb{F}_q[x,y] \mid r.v = 0\}$, a left ideal of $\mathbb{F}_q[x,y]$.

Proposition 2.4. *Assume* $\mathsf{gcd}(m,n) = 1$ *and* $f : \mathbb{Z} \to \mathbb{F}^*$ *satisfies* (2.1)*.*

(a) *For* $1 \in \mathsf{V}_f^{m,n}$, $\mathsf{ann}_{\mathbb{F}_q[x,y]}(1) = \mathbb{F}_q[x,y]\left(x^m y^n - \mathsf{s}_f(0)\right)$ *and*

$$\mathsf{V}_f^{m,n} \simeq \mathbb{F}_q[x,y] / \mathbb{F}_q[x,y]\left(x^m y^n - \mathsf{s}_f(0)\right).$$

(b) *For positive integers* m', n'*, and* $f' : \mathbb{Z} \to \mathbb{F}^*$ *satisfying* (2.1) *(with* n *replaced by* n'*), we have* $\mathsf{V}_f^{m,n} \simeq \mathsf{V}_{f'}^{m',n'}$ *if and only if* $m = m'$, $n = n'$ *and* $\mathsf{s}_{f'}(0) = q^k \mathsf{s}_f(0)$ *for some* $k \in \mathbb{Z}$*.*

Proof. (a) Let $\theta = x^m y^n$. First we show that

$$\mathsf{ann}_{\mathbb{F}_q[x,y]}(1) = \mathbb{F}_q[x,y]\left(\mathbb{F}[\theta] \cap \mathsf{ann}_{\mathbb{F}_q[x,y]}(1)\right). \tag{2.5}$$

The inclusion $\supseteq$ is clear, so suppose $u \in \mathsf{ann}_{\mathbb{F}_q[x,y]}(1)$. Write $u = \sum_{i \geq 0} \mu_i x^{a_i} y^{b_i} = \sum_{k \in \mathbb{Z}} u_k$, where $u_k = \displaystyle\sum_{na_i - mb_i = k} \mu_i x^{a_i} y^{b_i}$. Since $u_k.1$ is in $\mathbb{F}t^k$, it follows that $u_k \in \mathsf{ann}_{\mathbb{F}_q[x,y]}(1)$ for all $k \in \mathbb{Z}$, and it suffices to prove $u_k \in \mathbb{F}_q[x,y]\left(\mathbb{F}[\theta] \cap \mathsf{ann}_{\mathbb{F}_q[x,y]}(1)\right)$.

If $na_i - mb_i = na_j - mb_j$ then, as $\mathsf{gcd}(m,n) = 1$, we deduce that $(a_i, b_i) = (a_j, b_j) + \xi(m,n)$ for some $\xi \in \mathbb{Z}$. Thus, by the normality of x and y, there are $a, b \geq 0$ with $na - mb = k$ such that $u_k = x^a y^b w_0$, where $w_0 = \sum_{j\geq 0} \nu_j x^{\xi_j m} y^{\xi_j n} \in \mathbb{F}[\theta]$. Notice that for any $\ell \in \mathbb{Z}$, $x^a y^b.t^\ell$ is a nonzero scalar multiple of $t^{\ell+k}$, so $x^a y^b w_0 = u_k \in \mathsf{ann}_{\mathbb{F}_q[x,y]}(1)$ implies that $w_0 \in \mathsf{ann}_{\mathbb{F}_q[x,y]}(1)$. Hence, $u_k \in \mathbb{F}_q[x,y]\left(\mathbb{F}[\theta] \cap \mathsf{ann}_{\mathbb{F}_q[x,y]}(1)\right)$ and (2.5) is established.

Now (2.4) implies that $\theta - \mathsf{s}_f(0) \in \mathbb{F}[\theta] \cap \mathsf{ann}_{\mathbb{F}_q[x,y]}(1)$. Since $\mathbb{F}[\theta]\,(\theta - \mathsf{s}_f(0))$ is a maximal ideal of $\mathbb{F}[\theta]$ it follows that $\mathbb{F}[\theta] \cap \mathsf{ann}_{\mathbb{F}_q[x,y]}(1) = \mathbb{F}[\theta]\,(\theta - \mathsf{s}_f(0))$ and $\mathsf{ann}_{\mathbb{F}_q[x,y]}(1) = \mathbb{F}_q[x,y]\,(\theta - \mathsf{s}_f(0))$. This proves (a) as $1 \in \mathsf{V}_f^{m,n}$ generates $\mathsf{V}_f^{m,n}$.

(b) We observe that the arguments above also show that for $t^k \in \mathsf{V}_f^{m,n}$, $\mathsf{ann}_{\mathbb{F}_q[x,y]}(t^k) = \mathbb{F}_q[x,y]\left(\theta - q^k \mathsf{s}_f(0)\right)$ and

$$\mathsf{V}_f^{m,n} \simeq \mathbb{F}_q[x,y]/\mathbb{F}_q[x,y]\left(x^m y^n - q^k \mathsf{s}_f(0)\right),$$

for any $k \in \mathbb{Z}$. This establishes the *if* part of (b). For the direct implication, suppose $\mathsf{V}_f^{m,n} \simeq \mathsf{V}_{f'}^{m',n'}$. We have, for $a, b \geq 0$ and $t^k \in \mathsf{V}_f^{m,n}$,

$$x^a y^b.t^k = \left(\prod_{i=0}^{b-1} f(k - im)\right) t^{k+na-mb}$$

and $\prod_{i=0}^{b-1} f(k - im) \neq 0$. This implies that $x^a y^b$ is diagonalizable on $\mathsf{V}_f^{m,n}$ if and only if $na = mb$. As $\mathsf{gcd}(m,n) = 1$ this amounts to having $(a,b) = \xi(m,n)$ for some $\xi \geq 0$.

Since $\mathsf{V}_f^{m,n} \simeq \mathsf{V}_{f'}^{m',n'}$, then $x^{m'} y^{n'}$ is diagonalizable on $\mathsf{V}_f^{m,n}$ and similarly $x^m y^n$ is diagonalizable on $\mathsf{V}_{f'}^{m',n'}$. By the relation above we conclude that $(m,n) = (m',n')$. Moreover, the eigenvalues of $x^m y^n$ on $\mathsf{V}_f^{m,n}$ are of the form $q^k \mathsf{s}_f(0)$, whereas $\mathsf{s}_{f'}(0)$ is an eigenvalue of $x^{m'} y^{n'} = x^m y^n$ on $\mathsf{V}_{f'}^{m',n'}$. Hence $\mathsf{s}_{f'}(0) = q^k \mathsf{s}_f(0)$ for some $k \in \mathbb{Z}$, which concludes the proof. $\square$

Remark 2.5. By Proposition 2.4 above, for $\mathsf{gcd}(m,n) = 1$ and $f : \mathbb{Z} \to \mathbb{F}^*$ satisfying (2.1), the isomorphism class of $\mathsf{V}_f^{m,n}$ depends only on m, n and $\mathsf{s}_f(0) \in \mathbb{F}^*$.

Fix $\lambda \in \mathbb{F}^*$. Since $\mathsf{gcd}(m,n) = 1$ there is a unique $f_\lambda : \mathbb{Z} \to \mathbb{F}^*$ such that (2.1) holds and $f_\lambda(km) = \lambda$ if $k = 0$ and $f_\lambda(km) = 1$ if $-(n-1) \leq k \leq -1$. Then $\mathsf{s}_{f_\lambda}(0) = \lambda$, $\mathsf{V}_{f_\lambda}^{m,n} \simeq \mathbb{F}_q[x,y]/\mathbb{F}_q[x,y]\,(x^m y^n - \lambda)$ and, for $\lambda' \in \mathbb{F}^*$, $\mathsf{V}_{f_\lambda}^{m,n} \simeq \mathsf{V}_{f_{\lambda'}}^{m,n}$ if and only if $\lambda/\lambda' \in \langle q \rangle$, where $\langle q \rangle$ is the subgroup of $\mathbb{F}^*$ generated by q.

If $\mathbb{F}$ contains an n-th root of λ, say μ, there is a more natural construction for the irreducible representation $\mathbb{F}_q[x,y]/\mathbb{F}_q[x,y]\,(x^m y^n - \lambda)$. Define $f^\mu(i) = \mu q^{\lfloor \frac{i}{n} \rfloor}$, as in Example 2.1. Then $\mathsf{s}_{f^\mu}(0) = q^k \mu^n = q^k \lambda$, for some $k \in \mathbb{Z}$. It follows from Proposition 2.4 that $\mathsf{V}_{f^\mu}^{m,n} \simeq \mathbb{F}_q[x,y]/\mathbb{F}_q[x,y]\,(x^m y^n - \lambda)$ and $\mathsf{V}_{f^\mu}^{m,n}$ depends only on m, n and λ, and not on the particular n-th root of λ that was chosen.

Finally we consider the general case of arbitrary $m, n \in \mathbb{Z}_{>0}$.

Proposition 2.6. *Let $m, n \in \mathbb{Z}_{>0}$ be arbitrary, with $d = \mathsf{gcd}(m,n)$, and assume $f : \mathbb{Z} \to \mathbb{F}^*$ satisfies* (2.1)*. Then there is a direct sum decomposition*

$$\mathsf{V}_f^{m,n} \simeq \bigoplus_{k=0}^{d-1} \mathsf{V}_{f_k}^{m/d,n/d} \tag{2.6}$$

into irreducible representations, where $f_k(i) = f(k+id)$, for $0 \le k < d$ and $i \in \mathbb{Z}$.

Moreover, suppose $m', n' \in \mathbb{Z}_{>0}$, and $f' : \mathbb{Z} \to \mathbb{F}^*$ satisfies (2.1) *(with n replaced by n'). If $\mathsf{V}_f^{m,n} \simeq \mathsf{V}_{f'}^{m',n'}$ then $m = m'$ and $n = n'$.*

Proof. For $0 \le k < d$, the subspace $t^k\mathbb{F}[t^{\pm d}]$ of $\mathsf{V}_f^{m,n}$ is readily seen to be invariant under the actions of x and y, and we have $\mathsf{V}_f^{m,n} = \bigoplus_{k=0}^{d-1} t^k\mathbb{F}[t^{\pm d}]$. Thus, next we argue that the subrepresentation $t^k\mathbb{F}[t^{\pm d}]$ is isomorphic to $\mathsf{V}_{f_k}^{m/d,n/d}$, where $f_k(i) = f(k+id)$ for all $i \in \mathbb{Z}$. First notice that $f_k(i+n/d) = f(k+id+n) = qf(k+id) = qf_k(i)$, so $\mathsf{V}_{f_k}^{m/d,n/d}$ is defined. Consider the map $\phi : \mathsf{V}_{f_k}^{m/d,n/d} \to t^k\mathbb{F}[t^{\pm d}]$ given by $\phi(p)(t) = t^k p(t^d)$, for all $p \in \mathbb{F}[t^{\pm 1}]$. In particular, $\phi(t^i) = t^{k+id}$ for $i \in \mathbb{Z}$. Still viewing $t^k\mathbb{F}[t^{\pm d}]$ as a subrepresentation of $\mathsf{V}_f^{m,n}$, we have:

$$\begin{aligned}
\phi(x.t^i) &= \phi(t^{i+n/d}) = t^{k+id+n} = x.t^{k+id} = x.\phi(t^i),\\
\phi(y.t^i) &= \phi(f_k(i)t^{i-m/d}) = f(k+id)t^{k+id-m} = y.t^{k+id} = y.\phi(t^i).
\end{aligned}$$

Since ϕ is clearly bijective, the calculations above show that ϕ is an isomorphism of representations, and $\mathsf{V}_f^{m,n} \simeq \bigoplus_{k=0}^{d-1} \mathsf{V}_{f_k}^{m/d,n/d}$. The fact that each summand $\mathsf{V}_{f_k}^{m/d,n/d}$ is irreducible follows from $\mathsf{gcd}(m/d, n/d) = 1$ and Proposition 2.3, which will be established independently.

Finally, assume $\mathsf{V}_f^{m,n} \simeq \mathsf{V}_{f'}^{m',n'}$ for positive integers m' and n', and $f' : \mathbb{Z} \to \mathbb{F}^*$ satisfying $f'(i+n') = qf'(i)$, for all $i \in \mathbb{Z}$. Then, up to isomorphism, $\mathsf{V}_f^{m,n}$ and $\mathsf{V}_{f'}^{m',n'}$ have the same composition factors, and in particular the same composition length. This proves that $d = \mathsf{gcd}(m,n) = \mathsf{gcd}(m',n')$ and that $\mathsf{V}_{f_0}^{m/d,n/d} \simeq \mathsf{V}_{f'_k}^{m'/d,n'/d}$ for some k. By Proposition 2.4, which will also be established independently, we have $m/d = m'/d$ and $n/d = n'/d$, so $m = m'$ and $n = n'$.

□

2.2 Weight representations of the form $\mathsf{V}_f^{m,n}$

Let us now determine when $\mathsf{V}_f^{m,n}$ is a weight representation in the sense of [1]. Recall that this occurs when $\mathsf{V}_f^{m,n}$ is semisimple as a representation over the polynomial subalgebra $\mathbb{F}[H]$, where $H = xy$. Assume first that $m = n = 1$ and fix $\lambda \in \mathbb{F}^*$. The map f_λ defined in Remark 2.5 is given by $f_\lambda(i) = \lambda q^i$ for all $i \in \mathbb{Z}$, and the corresponding representation $\mathsf{V}_{f_\lambda}^{1,1} \simeq \mathbb{F}_q[x,y]/\mathbb{F}_q[x,y]\,(H-\lambda)$ is irreducible. Since $H.t^i = xy.t^i = \lambda q^i t^i$ for all i, the decomposition $\mathsf{V}_{f_\lambda}^{1,1} = \bigoplus_{i\in\mathbb{Z}} \mathbb{F}\, t^i$ shows that $\mathsf{V}_{f_\lambda}^{1,1}$ is semisimple over $\mathbb{F}[H]$.

Moreover, for $\nu \in \mathbb{F}^*$, $\mathsf{V}_{f_\lambda}^{1,1} \simeq \mathsf{V}_{f_\nu}^{1,1}$ if and only if $\lambda/\nu \in \langle q \rangle$, the multiplicative subgroup of $\mathbb{F}^*$ generated by q, by Proposition 2.4. In case $\mathbb{F}$ is algebraically closed, these are all the infinite-dimensional irreducible weight representations of $\mathbb{F}_q[x,y]$, by [1, Cor. 3.2]. Combined with Proposition 2.4(b) the above yields the classification of irreducible weight representations in the family $\mathsf{V}_f^{m,n}$.

Proposition 2.7. *Assume* $\mathsf{gcd}(m,n) = 1$ *and* $f : \mathbb{Z} \to \mathbb{F}^*$ *satisfies* (2.1). *Then* $\mathsf{V}_f^{m,n}$ *is a weight representation if and only if* $m = n = 1$.

For completeness, we include a brief and direct proof of Proposition 2.7 not assuming that $\mathbb{F}$ is algebraically closed, a condition that was used implicitly at the end of the previous paragraph.

Proof. Assume first that $m = n = 1$. Then since f satisfies (2.1) we have $f = f_\lambda$ for $\lambda = f(0)$ and the discussion above shows that $\mathsf{V}_f^{m,n}$ is a weight representation of $\mathbb{F}_q[x,y]$. Conversely, suppose $\mathsf{V}_f^{m,n}$ is a weight representation of $\mathbb{F}_q[x,y]$. Then clearly $\dim_{\mathbb{F}} \mathbb{F}[H].v < +\infty$ for any $v \in \mathsf{V}_f^{m,n}$. Notice that, for all $i \in \mathbb{Z}$, $H.t^i = xy.t^i = f(i)t^{i+n-m}$. Thus, for $\ell \in \mathbb{Z}$, $H^\ell.t^i = \zeta t^{i+\ell(n-m)}$ for some $\zeta \in \mathbb{F}^*$. But then the condition $\dim_{\mathbb{F}} \mathbb{F}[H].1 < +\infty$ immediately implies $m = n$, and hence $m = n = 1$, as $\mathsf{gcd}(m,n) = 1$. ☐

Remark 2.8. Given arbitrary positive integers m and n, and f satisfying (2.1), the representation $\mathsf{V}_f^{m,n}$ is a weight representation if and only if $m = n$. The direct implication follows from the proof of Proposition 2.7. For the converse implication, recall that $\mathsf{V}_f^{m,m}$ is the direct sum of m representations of the form $\mathsf{V}_{f_k}^{1,1}$, for $0 \leq k < m$, by Proposition 2.6, so the claim follows as each of these is a weight representation.

3 Connections with the representation theory of the q-Weyl algebra $\mathbb{A}_1(q)$

We continue to assume $q \in \mathbb{F}^*$ is not a root of unity. Let $\mathbb{A}_1(q)$ be the q-Weyl algebra given by generators X and Y and defining relation $YX - qXY = 1$, as in (1.3). It is straightforward to show that $\{x^k \mid k \geq 0\}$ is a right and left Ore set consisting of regular elements of $\mathbb{F}_q[x,y]$, and we denote the corresponding localization by $\mathbb{F}_q[x^{\pm 1}, y]$. The calculation

$$\big(x^{-1}(y-1)\big)x - qx\big(x^{-1}(y-1)\big) = x^{-1}yx - q(y-1) - 1 = qy - q(y-1) - 1 = q - 1$$

shows that there is an algebra map

$$\mathbb{A}_1(q) \to \mathbb{F}_q[x^{\pm 1}, y], \quad \text{with} \quad X \mapsto x, \quad Y \mapsto \tfrac{1}{q-1}x^{-1}(y-1). \tag{3.1}$$

To see that the map in (3.1) is injective we can argue as follows. The multiplicative subset $\{X^k \mid k \geq 0\}$ of $\mathbb{A}_1(q)$ is a right and left Ore set of regular elements and we denote the corresponding localization by $\widehat{\mathbb{A}}_1(q)$. Then the map in (3.1) extends to a map $\widehat{\mathbb{A}}_1(q) \to \mathbb{F}_q[x^{\pm 1}, y]$, which has an inverse $\mathbb{F}_q[x^{\pm 1}, y] \to \widehat{\mathbb{A}}_1(q)$ with $x^{\pm 1} \mapsto X^{\pm 1}$ and $y \mapsto (q-1)XY + 1$. It follows that (3.1) induces an isomorphism $\widehat{\mathbb{A}}_1(q) \simeq \mathbb{F}_q[x^{\pm 1}, y]$, and in particular (3.1) is injective. In view of the above we will identify X with x, Y with $\frac{1}{q-1}x^{-1}(y-1)$ and $\mathbb{A}_1(q)$ with the corresponding subalgebra of $\mathbb{F}_q[x^{\pm 1}, y]$. Since $y = (q-1)XY + 1 = YX - XY$, we have the embeddings

$$\mathbb{F}_q[x, y] \subseteq \mathbb{A}_1(q) \subseteq \mathbb{F}_q[x^{\pm 1}, y] = \widehat{\mathbb{A}}_1(q). \tag{3.2}$$

3.1 Extension of the representations $\mathsf{V}_f^{m,n}$ to $\mathbb{A}_1(q)$

Our aim in this subsection is to extend the action of $\mathbb{F}_q[x, y]$ on $\mathsf{V}_f^{m,n}$ to an action of the q-Weyl algebra $\mathbb{A}_1(q)$. Assume thus that m, n are positive integers and $f : \mathbb{Z} \to \mathbb{F}^*$ satisfies (2.1). If $\rho_f^{m,n} : \mathbb{F}_q[x, y] \to \mathrm{End}_{\mathbb{F}}(\mathsf{V}_f^{m,n})$ is the representation of $\mathbb{F}_q[x, y]$ on $\mathsf{V}_f^{m,n}$, we first observe that $\rho_f^{m,n}(x)$ is an invertible linear map on $\mathsf{V}_f^{m,n}$, a fact which is clear from (2.2). Therefore $\rho_f^{m,n}$ extends to the localization $\mathbb{F}_q[x^{\pm 1}, y]$, and $\mathsf{V}_f^{m,n}$ can be seen as a representation of $\mathbb{F}_q[x^{\pm 1}, y]$ with $x^{-1}.t^i = t^{i-n}$ for all $i \in \mathbb{Z}$. Now we get an action of $\mathbb{A}_1(q)$ on $\mathsf{V}_f^{m,n} = \mathbb{F}[t^{\pm 1}]$ by restricting $\rho_f^{m,n}$:

$$\begin{aligned} &X.t^i = x.t^i = t^{i+n}, \\ &Y.t^i = \frac{1}{q-1}x^{-1}(y-1).t^i = \frac{1}{q-1}(f(i)t^{i-m-n} - t^{i-n}), \qquad \text{for all } i \in \mathbb{Z}. \end{aligned} \tag{3.3}$$

In our next result we view $\mathsf{V}_f^{m,n}$ as a representation of $\mathbb{A}_1(q)$, as above.

Proposition 3.1. *Assume* $\mathsf{gcd}(m, n) = 1$ *and* $f : \mathbb{Z} \to \mathbb{F}^*$ *satisfies* (2.1). *Then:*

(a) $\mathsf{V}_f^{m,n}$ *defined by* (3.3) *is an irreducible representation of* $\mathbb{A}_1(q)$.

(b) *For positive integers* m', n', *and* $f' : \mathbb{Z} \to \mathbb{F}^*$ *satisfying* (2.1) *(with* n *replaced by* n'*), we have* $\mathsf{V}_f^{m,n} \simeq \mathsf{V}_{f'}^{m',n'}$ *as representations of* $\mathbb{A}_1(q)$ *if and only if* $m = m'$, $n = n'$ *and* $\mathsf{s}_{f'}(0) = q^k \mathsf{s}_f(0)$ *for some* $k \in \mathbb{Z}$.

(c) $\mathsf{V}_f^{m,n}$ *is not semisimple as a representation over the polynomial subalgebra of* $\mathbb{A}_1(q)$ *generated by* XY*; hence,* $\mathsf{V}_f^{m,n}$ *is not a weight representation of* $\mathbb{A}_1(q)$ *in the sense of [1].*

Proof. Part (a) and the direct implication in (b) follow from the embedding (3.2), and from Propositions 2.3 and 2.4.

Suppose now $f' : \mathbb{Z} \to \mathbb{F}^*$ satisfies (2.1), and there is $k \in \mathbb{Z}$ so that $\mathsf{s}_{f'}(0) = q^k \mathsf{s}_f(0)$. Then by Proposition 2.4 there is an isomorphism $\phi : \mathsf{V}_f^{m,n} \to \mathsf{V}_{f'}^{m,n}$ as representations of $\mathbb{F}_q[x, y]$. For $v \in \mathsf{V}_f^{m,n}$ we have $\phi(v) = \phi(xx^{-1}.v) = x.\phi(x^{-1}.v)$, thus $\phi(x^{-1}.v) = x^{-1}.\phi(v)$. Whence ϕ is an isomorphism of representations of $\mathbb{F}_q[x^{\pm 1}, y]$. The other implication in (b) now follows from (3.2).

Observe that $XY = \frac{1}{q-1}(y-1)$, so the polynomial subalgebra of $\mathbb{A}_1(q)$ generated by XY is just $\mathbb{F}[y]$. Given $0 \neq v \in \mathsf{V}_f^{m,n}$, the formula $y.t^i = f(i)t^{i-m}$ for $i \in \mathbb{Z}$ implies $\dim_{\mathbb{F}} \mathbb{F}[y].v = +\infty$. Hence, $\mathsf{V}_f^{m,n}$ is not semisimple over $\mathbb{F}[y] = \mathbb{F}[XY]$, and therefore it is not a weight representation of $\mathbb{A}_1(q)$ in the sense of [1]. $\square$

Remark 3.2. In [3] the authors introduce Whittaker representations for generalized Weyl algebras. For the cases covered in this note, a representation V is a *Whittaker representation* for $\mathbb{F}_q[x,y]$ (respectively, for $\mathbb{A}_1(q)$) if V is generated by an element $v \in \mathsf{V}$ which is an eigenvector for the action of $x \in \mathbb{F}_q[x,y]$ (respectively, for the action of $X \in \mathbb{A}_1(q)$). Since $m, n \geq 1$, it is immediate that the operators $x, y \in \mathbb{F}_q[x,y]$ (respectively, $X, Y \in \mathbb{A}_1(q)$) have no eigenvectors in $\mathsf{V}_f^{m,n}$, so $\mathsf{V}_f^{m,n}$ is not a Whittaker representation for the quantum plane (respectively, for the q-Weyl algebra).

3.2 The representations W_g^n of $\mathbb{A}_1(q)$ and their restriction to $\mathbb{F}_q[x,y]$

We will now use a similar idea to construct representations of the q-Weyl algebra on the Laurent polynomial algebra $\mathbb{F}[t^{\pm 1}]$. Fix positive integers $m, n \in \mathbb{Z}_{>0}$ and a function $g : \mathbb{Z} \to \mathbb{F}$. Then the formulas

$$X.t^i = t^{i+n}, \qquad Y.t^i = g(i)t^{i-m}, \qquad \text{for all } i \in \mathbb{Z} \tag{3.4}$$

yield a representation of $\mathbb{A}_1(q)$ on $\mathbb{F}[t^{\pm 1}]$ if and only if $m = n$ and g satisfies

$$g(i+n) = qg(i) + 1, \qquad \text{for all } i \in \mathbb{Z}. \tag{3.5}$$

We denote the corresponding representation of $\mathbb{A}_1(q)$ by W_g^n. Notice that for all $i \in \mathbb{Z}$

$$XY.t^i = g(i)t^i, \qquad (YX - XY).t^i = (g(i+n) - g(i))\, t^i = ((q-1)g(i) + 1)\, t^i, \tag{3.6}$$

so W_g^n is a weight representation of $\mathbb{A}_1(q)$ in the sense of [1].

Remark 3.3. It follows from the computations at the beginning of Section 3 that the element $YX - XY$ is normal in $\mathbb{A}_1(q)$ and it is sometimes referred to as a Casimir element, in spite of not being central. The equality $YX - XY = (q-1)XY + 1$ shows that $YX - XY$ and $(q-1)XY + 1$ generate the same subalgebra of $\mathbb{A}_1(q)$ and thus a weight representation of $\mathbb{A}_1(q)$ could be defined in an equivalent manner as a representation which is semisimple over the subalgebra generated by the Casimir element $YX - XY$.

Our first observation is the analogue of Proposition 2.6.

Lemma 3.4. *Let $n \in \mathbb{Z}_{>0}$ and assume $g : \mathbb{Z} \to \mathbb{F}$ satisfies* (3.5)*. There is a direct sum decomposition*

$$\mathsf{W}_g^n \simeq \bigoplus_{k=0}^{n-1} \mathsf{W}_{g_k}^1, \tag{3.7}$$

where $g_k(i) = g(k + in)$, for $0 \leq k < n$ and $i \in \mathbb{Z}$.

Proof. For $0 \leq k < n$, the subspace $t^k\mathbb{F}[t^{\pm n}]$ is invariant under the actions of X and Y and $\mathsf{W}_g^n = \bigoplus_{k=0}^{n-1} t^k\mathbb{F}[t^{\pm n}]$. Moreover, the map $\phi : \mathsf{W}_{g_k}^1 \to t^k\mathbb{F}[t^{\pm n}]$ given by $\phi(p)(t) = t^k p(t^n)$, for all $p \in \mathbb{F}[t^{\pm 1}]$ is easily checked to be an isomorphism. $\square$

In view of the above, it is enough to study the structure of the representations W_g^1, where $g : \mathbb{Z} \to \mathbb{F}$ satisfies $g(i+1) = qg(i) + 1$ for all $i \in \mathbb{Z}$. Equivalently, $g(i) = g(0)q^i + [i]_q$, where $[i]_q = \frac{q^i - 1}{q-1}$ for all $i \in \mathbb{Z}$.

Proposition 3.5. *Let $g, g' : \mathbb{Z} \to \mathbb{F}$ satisfy* (3.5) *with $n = 1$. Then:*

(a) $\mathsf{W}_g^1 \simeq \mathsf{W}_{g'}^1$ *if and only if $g(0) = g'(i)$ for some $i \in \mathbb{Z}$;*

(b) W_g^1 *is irreducible if and only if $g(0) \notin \{[i]_q \mid i \in \mathbb{Z}\} \cup \left\{-\frac{1}{q-1}\right\}$.*

Proof. For (a), suppose $\mathsf{W}_g^1 \simeq \mathsf{W}_{g'}^1$. By (3.6) the eigenvalues of XY on W_g^1 are $g(i)$, for $i \in \mathbb{Z}$ and similarly the eigenvalues of XY on $\mathsf{W}_{g'}^1$ are $g'(i)$, for $i \in \mathbb{Z}$. Thus, g and g' must have the same image and in particular $g(0) = g'(i)$ for some $i \in \mathbb{Z}$. Conversely, if the latter holds then the map $\phi : \mathsf{W}_g^1 \to \mathsf{W}_{g'}^1$ given by $\phi(p)(t) = t^i p(t)$ for all $p \in \mathbb{F}[t^{\pm 1}]$ is an isomorphism.

For (b), first observe that for $i \in \mathbb{Z}$ we have $g(0) = [i]_q \iff g(-i) = 0$. Thus, if $g(0) = [i]_q$ for some $i \in \mathbb{Z}$, then $t^{-i}\mathbb{F}[t]$ is invariant under the actions of X and Y, so W_g^1 is not irreducible in this case. Next observe that $g(0) = -\frac{1}{q-1} \iff g$ is not injective $\iff g$ is constant. It follows that if $g(0) = -\frac{1}{q-1}$, then $(t-1)\mathbb{F}[t^{\pm 1}]$ is a proper subrepresentation and hence W_g^1 is not irreducible. This proves the direct implication in (b). For the converse, by the observations above, we can assume that $g(i) \neq 0$ for all $i \in \mathbb{Z}$ and that g is injective. Let S be a nonzero subrepresentation of W_g^1. By repeatedly applying the operator X to a chosen nonzero element of S, we will obtain a nonzero element of $\mathsf{S} \cap \mathbb{F}[t]$. Let p be one such element, chosen so that it has minimum degree, say $p = \sum_{k=0}^{d} a_k t^k$, with $a_d \neq 0$. Since $g(i) \neq 0$ for all $i \in \mathbb{Z}$, the minimality of p implies that $a_0 \neq 0$. Then

$$\mathsf{S} \cap \mathbb{F}[t] \ni (XY - g(d)).p = \sum_{k=0}^{d-1} (g(k) - g(d)) a_k t^k.$$

By the minimality of p we must have $(XY - g(d)).p = 0$. Hence, $g(0) = g(d)$ and the injectivity of g gives $d = 0$. It follows that $t^0 \in \mathsf{S}$ and thus $\mathsf{S} = \mathsf{W}_g^1$. $\square$

Now that we understand the representations W_g^n, we will consider their restriction to $\mathbb{F}_q[x, y]$ via each of the two embeddings

$$\sigma : \mathbb{F}_q[x, y] \to \mathbb{A}_1(q), \qquad x \mapsto X, \quad y \mapsto YX - XY = (q-1)XY + 1; \tag{3.8}$$

$$\tau : \mathbb{F}_q[x, y] \to \mathbb{A}_1(q), \qquad x \mapsto YX - XY = (q-1)XY + 1, \quad y \mapsto Y. \tag{3.9}$$

We consider first the restriction relative to σ. In this case, the action of $\mathbb{F}_q[x, y]$ on W_g^n is given by

$$x.t^i = t^{i+n}, \qquad y.t^i = ((q-1)g(i) + 1)\, t^i, \qquad \text{for all } i \in \mathbb{Z}. \tag{3.10}$$

Lemma 3.6. *Consider the restriction map σ given in* (3.8) *to view the representations W_g^n as representations of $\mathbb{F}_q[x, y]$.*

(a) *Let* $n \in \mathbb{Z}_{>0}$ *and assume* $g : \mathbb{Z} \to \mathbb{F}$ *satisfies* (3.5). *Then* $\mathsf{W}_g^n \simeq \bigoplus_{k=0}^{n-1} \mathsf{W}_{g_k}^1$ *as representations of* $\mathbb{F}_q[x, y]$, *where* $g_k(i) = g(k + in)$, *for* $0 \le k < n$ *and* $i \in \mathbb{Z}$.

(b) *Let* $g, g' : \mathbb{Z} \to \mathbb{F}$ *satisfy* (3.5) *with* $n = 1$. *Then* $\mathsf{W}_g^1 \simeq \mathsf{W}_{g'}^1$ *as representations of* $\mathbb{F}_q[x, y]$ *if and only if* $g(0) = g'(i)$ *for some* $i \in \mathbb{Z}$.

(c) *Assume* $g : \mathbb{Z} \to \mathbb{F}$ *satisfies* (3.5) *with* $n = 1$. *Then* W_g^1 *has trivial socle as a representation of* $\mathbb{F}_q[x, y]$, *i.e., it has no irreducible* $\mathbb{F}_q[x, y]$*-subrepresentations.*

Proof. Part (a) follows directly from Lemma 3.4 and part (b) follows from the proof of Proposition 3.5(a), as the argument for the direct implication in Proposition 3.5(a) used only the restriction of the action to the subalgebra generated by XY, which coincides with the subalgebra generated by $YX - XY$.

For part (c), suppose by way of contradiction that S is an irreducible $\mathbb{F}_q[x, y]$-subrepresentation of W_g^1. Let $0 \ne s \in \mathsf{S}$. Then $x.s \ne 0$ and thus $\mathbb{F}_q[x, y]x.s = \mathsf{S}$, which is a contradiction as $s \notin \mathbb{F}_q[x, y]x.s$. $\square$

Remark 3.7. In the conditions of Lemma 3.6, it can be checked that W_g^1 has maximal $\mathbb{F}_q[x, y]$-subrepresentations if and only if g is constant.

Now we consider the restriction of W_g^n to $\mathbb{F}_q[x, y]$ relative to the map τ defined in (3.9). In this case, the action of $\mathbb{F}_q[x, y]$ on W_g^n is given by

$$x.t^i = ((q-1)g(i) + 1)\, t^i, \qquad y.t^i = g(i)t^{i-n}, \qquad \text{for all } i \in \mathbb{Z}. \tag{3.11}$$

Lemma 3.8. *Consider the restriction map* τ *given in* (3.9) *to view the representations* W_g^n *as representations of* $\mathbb{F}_q[x, y]$.

(a) *Let* $n \in \mathbb{Z}_{>0}$ *and assume* $g : \mathbb{Z} \to \mathbb{F}$ *satisfies* (3.5). *Then* $\mathsf{W}_g^n \simeq \bigoplus_{k=0}^{n-1} \mathsf{W}_{g_k}^1$ *as representations of* $\mathbb{F}_q[x, y]$, *where* $g_k(i) = g(k + in)$, *for* $0 \le k < n$ *and* $i \in \mathbb{Z}$.

(b) *Let* $g, g' : \mathbb{Z} \to \mathbb{F}$ *satisfy* (3.5) *with* $n = 1$. *Then* $\mathsf{W}_g^1 \simeq \mathsf{W}_{g'}^1$ *as representations of* $\mathbb{F}_q[x, y]$ *if and only if* $g(0) = g'(i)$ *for some* $i \in \mathbb{Z}$.

(c) *Assume* $g : \mathbb{Z} \to \mathbb{F}$ *satisfies* (3.5) *with* $n = 1$. *If* $g(0) \notin \{[i]_q \mid i \in \mathbb{Z}\}$ *then* W_g^1 *has trivial socle as a representation of* $\mathbb{F}_q[x, y]$, *i.e., it has no irreducible* $\mathbb{F}_q[x, y]$*-subrepresentations. If* $g(0) = [i]_q$ *for some* $i \in \mathbb{Z}$ *then* $\mathbb{F}t^{-i}$ *is the unique irreducible* $\mathbb{F}_q[x, y]$*-subrepresentation of* W_g^1.

Proof. The proof is the same as the proof of Lemma 3.6, except for part (c). For this part, suppose that S is an irreducible $\mathbb{F}_q[x, y]$-subrepresentation of W_g^1. If there is $0 \ne s \in \mathsf{S}$ such that $y.s \ne 0$, then we obtain a contradiction as in the proof of Lemma 3.6(c), showing that no such irreducible $\mathbb{F}_q[x, y]$-subrepresentation of W_g^1 exists. If $g(0) \notin \{[i]_q \mid i \in \mathbb{Z}\}$ then $g(i) \ne 0$ for all $i \in \mathbb{Z}$, so $y.s \ne 0$ for all $s \ne 0$ and the first claim follows. Now suppose $g(0) = [i]_q$ for some $i \in \mathbb{Z}$. Then $g(k) = 0 \iff k = -i$. In particular, $x.t^{-i} = t^{-i}$ and $y.t^{-i} = 0$, so that $\mathbb{F}t^{-i}$ is an irreducible $\mathbb{F}_q[x, y]$-subrepresentation of W_g^1. If S is any irreducible $\mathbb{F}_q[x, y]$-subrepresentation of W_g^1, then the argument above implies that $y.s = 0$ for all $s \in \mathsf{S}$, and this in turn implies that $\mathsf{S} \subseteq \mathbb{F}t^{-i}$, which establishes the second claim in (c). $\square$

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References

[1] V. Bavula, *Classification of the simple modules of the quantum Weyl algebra and the quantum plane*, Quantum groups and quantum spaces (Warsaw, 1995), Banach Center Publ., vol. 40, Polish Acad. Sci., Warsaw, 1997, pp. 193–201.

[2] V. Bavula and F. van Oystaeyen, *The simple modules of certain generalized crossed products*, J. Algebra **194** (1997), no. 2, 521–566.

[3] G. Benkart and M. Ondrus, *Whittaker modules for generalized Weyl algebras*, Represent. Theory **13** (2009), 141–164.

[4] F.H. Jackson, *q-difference equations*, Amer. J. Math. **32** (1910), no. 4, 305–314.

[5] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995.

[6] Yu.I. Manin, *Quantum groups and noncommutative geometry*, Université de Montréal, Centre de Recherches Mathématiques, Montreal, QC, 1988.

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