NOTE ON THE BIJECTIVITY OF THE PAK-STANLEY LABELLING

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1. Introduction

This article has the sole purpose of presenting a simple, self-contained and direct proof of the fact that the Pak-Stanley labeling is a bijection. The construction behind the proof is subsumed in a forthcoming paper [1], but an actual self-contained proof is not explicitly included in that paper.

Let n be a natural number and consider the *Shi arrangement of order* n, the union $\mathcal{S}_n$ of the hyperplanes of $\mathbb{R}^n$ defined, for every $1 \leq i < j \leq n$, either by equation $x_i - x_j = 0$ or by equation $x_i - x_j = 1$. The *regions* of the arrangement are the connected components of the complement of $\mathcal{S}_n$ in $\mathbb{R}^n$. Jian Yi Shi [5] introduced in literature this arrangement of hyperplanes and showed that the number of regions is $(n+1)^{n-1}$.

On the other hand, $(n+1)^{n-1}$ is also the number of *parking functions of size* n, which were defined (and counted) by Alan Konheim and Benjamin Weiss [3]. These are the functions $f\colon [n] \to [n]$ such that,

$$\forall_{i \in [n]}\,,\ |f^{-1}([i])| \geq i$$

or, equivalently, such that, for some $\pi \in \mathfrak{S}_n$, $f(i) \leq \pi(i)$ for every $i \in [n]$ (as usual, $[n] := \{1, \dots, n\}$ and $\mathfrak{S}_n$ is the set of permutations of $[n]$).

The Pak-Stanley labeling [7] consists of a function λ from the set of regions of $\mathcal{S}_n$ to the set of parking functions of size n.

We define $[0] := \varnothing$ and, for $i, j \in \mathbb{N}$, $[i, j] := [j] \setminus [i-1]$, so that $[i, j] = \{i, i+1, \dots, j\}$ if $i \leq j$ and $[i, j] = \varnothing$ otherwise. Finally, $[i] = [1, i]$ for every integer $i \geq 0$ as stated before.

Let $A \subseteq [n]$, say $A =: \{a_1, \dots, a_m\}$ with $a_1 < \cdots < a_m$ and let W_A be the set of words of form $w = a_{\alpha_1} \cdots a_{\alpha_m}$ for some permutation $\alpha \in \mathfrak{S}_m$. If $1 \leq i < j \leq m$, we distinguish the subword $w\langle i : j\rangle := a_{\alpha_i} \cdots a_{\alpha_j}$ from the set $w([i, j]) := \{a_{\alpha_i}, \dots, a_{\alpha_j}\}$. Similarly, we define $w^{-1}\colon A \to [m]$ through $w^{-1}(w_i) = i$ for every $i \in [m]$.

Definition 1.1. Given a word $w = w_1 \cdots w_k \in \mathrm{W}_A$ and a set $\mathfrak{I} = \{[o_1, c_1], \dots, [o_k, c_k]\}$ with $1 \leq o_i < c_i \leq m$ for every $i \in [k]$ and $o_1 < o_2 < \cdots < o_k$, we say that the pair $P = (w, \mathfrak{I})$ is a *valid pair* if

- $w_{o_i} > w_{c_i}$ for every $i \in [k]$;
- $c_1 < c_2 < \cdots < c_k$.

An *A-parking function* is a function $f\colon A \to [m]$ for which

$$\forall_{j \in [m]}\,,\ |f^{-1}([j])| \geq j\,. \tag{1.1}$$

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We denote by PF_A the set of A-parking functions. Of course, for $f\colon A \to [m]$, $f \in \mathrm{PF}_A$ if and only if $f \circ \iota_A$ is a parking function, where $\iota_A\colon [m] \to A$ is such that $\iota_A(i) = a_i$. A particular case occurs when

$$\forall_{j\in[m]}\,,\ f(a_j) \leq j\,.$$

In this case, we say that f is *A-central.* We denote by CF_A the set of A-central parking functions. We call *contraction of* w to the new function $\widehat{w}\colon A \to [m]$ such that

$$\widehat{w}(a) := w^{-1}(a) - \Big|\big\{b \in A \mid b > a\,,\ w^{-1}(b) < w^{-1}(a)\big\}\Big|\,. \tag{1.2}$$

Note that indeed $\widehat{w} \in \mathrm{CF}_A$, since $\widehat{w}(a) = \Big|w([w^{-1}(a)]) \cap [a]\Big|$.

For example, $\widehat{843967} = \overset{3\,4\,6\,7\,8\,9}{113414}$. In fact, $\widehat{843967}(3) = 1$ since $w^{-1}(3) = 3$ and $w([3]) \cap [3] = \{8,4,3\} \cap [3] = \{3\}$, but, for instance, $\widehat{843967}(6) = 3$ since $w^{-1}(6) = 5$ and $w([5]) \cap [6] = \{3,4,6\}$.

When $A = [n]$, the A-central parking functions are simply *central* parking functions.

2. The Pak-Stanley labeling

Igor Pak and Richard Stanley [7] created a (bijective) labeling of the regions of the Shi arrangement with parking functions that may be defined as follows.

Consider, for a point $x = (x_1,\dots,x_n) \in \mathbb{R}^n \setminus \mathcal{S}_n$, the (unique) permutation $w \in \mathfrak{S}_n$ such that $x_{w_1} < \cdots < x_{w_n}$[(1)], and consider the set $\mathfrak{I} = \big\{[o_1,c_1],\dots,[o_m,c_m]\big\}$ of all *maximal* intervals $I_i = [o_i,c_i]$ with $o_i < c_i$ for $i = 1,\dots,k$, such that

- $w_{o_i} > w_{c_i}$;
- for every $\ell, m \in I_i$ with $\ell < m$ and $w_\ell > w_m$, $0 < x_{w_m} - x_{w_\ell} < 1$[(2)].

Then, clearly $(w,\mathfrak{I})$ is a valid pair that does not depend on the particular point x that we have chosen. More precisely, if a similar construction is based on a different point $y \in \mathbb{R}^n \setminus \mathcal{S}_n$ then at the end we obtain the same valid pair if and only if x and y are in the same region of $\mathcal{S}_n$. Finally, it is not difficult to see that every valid pair corresponds in this way to a (unique) region of $\mathcal{S}_n$.

Example 2.1 ([6, example p. 484, ad.]). Let $w = 843967125$ and $\mathfrak{I} = \{[1,6],[3,8],[6,9]\}$. The valid pair $(w,\mathfrak{I})$ corresponds to the region

$$\Big\{(x_1,\dots,x_9) \in \mathbb{R}^9 \ \mid x_8 < x_4 < x_3 < x_9 < x_6 < x_7 < x_1 < x_2 < x_5,\\ x_8+1 > x_7,\ x_3+1 > x_2,\ x_7+1 > x_5\,,\\ x_4+1 < x_1,\ x_6+1 < x_5\Big\}$$

where also $x_8 + 1 > x_6$ (since $x_7 > x_6$) and $x_8 + 1 < x_1$ (since $x_8 < x_4$), for example.

Let R_0 be the region corresponding to the valid pair $(w,\mathfrak{I})$ where $w = n(n-1)\cdots 2\,1$ and $\mathfrak{I} = \{[1,n]\}$, so that $(x_1,\dots,x_n) \in R_0$ if and only if $0 < x_i - x_j < 1$ for every $0 \leq i < j \leq n$.

In the Pak-Stanley labeling λ, the label of R_0 is, using the one-line notation, $\lambda(R_0) = 1\,1\cdots 1$. Furthermore,

[(1)]Note that the order is reversed relatively to Stanley's paper [7].

[(2)]The fact that $0 < x_{w_m} - x_{w_\ell}$ already follows from the fact that $w_\ell > w_m$.

- if the only hyperplane that separates two regions, R and R', has equation $x_i = x_j$ $(i < j)$ and R_0 and R lie in the same side of this plane, then $\lambda(R') = \lambda(R) + e_j$ (as usual, the i-th coordinate of e_j is either 1, if $i = j$, or 0, otherwise);
- if the only hyperplane that separates two regions, R and R', has equation $x_i = x_j + 1$ $(i < j)$ and R_0 and R lie in the same side of this plane, then $\lambda(R') = \lambda(R) + e_i$.

Thus, given a region R of $\mathcal{S}_n$ with associated valid pair $P = (w, \{[o_1, c_1], \dots, [o_m, c_m]\})$, if $f = \lambda(R)$ and $\boldsymbol{i = w_j}$, then, counting the planes of equation $x_{w_k} - x_i = 0$ or $x_i - x_{w_k} = 1$ that separate R and R_0, respectively, we obtain (cf. [7])

$$\begin{aligned} f_i = 1 &+ \Big|\big\{k < j \mid w_k < i\big\}\Big| \\ &+ \Big|\big\{k < j \mid w_k > i\,,\ \text{no } \ell \in [m] \text{ satisfies } j, k \in [o_\ell, c_\ell]\big\}\Big|\,. \end{aligned} \tag{2.3}$$

Hence, if $j \notin [o_1, c_1], \dots, [o_m, c_m]$,

$$f_i = j\,; \tag{2.4}$$

in this case, *let* $o_P(i) = o_P(w_j) := j$. Otherwise, if $k \leq m$ is the least integer for which $j \in [o_k, c_k]$,

$$f_i = o_k - 1 + \widehat{w\langle o_k : c_k\rangle}(i)\,. \tag{2.5}$$

and we *define* $o_P(i) := o_k$.

In Figure 1, we represent $\mathcal{S}_3$ with each region R labeled with $\lambda(R)$.

By requiring the validity of equations (2.4) and (2.5) under the same conditions, we extend λ to every valid pair $P = (w, \mathfrak{I})$, where $w \in \mathrm{W}_A$ for some $A \subseteq [n]$. Note that in this way we still obtain an A-parking function $f = \lambda(w, \mathfrak{I})$.

Moreover, if $1 \leq k < \ell \leq |A|$ then $o_P(w_k) \leq o_P(w_\ell)$. If, in addition, $w_k > w_\ell$, then

$$f(w_k) \leq f(w_\ell)\,. \tag{2.6}$$

In fact, $f(w_\ell) = \ell - \big|\big\{o_P(w_\ell) \leq j \leq \ell \mid w_j > w_\ell\big\}\big| \geq k - \big|\big\{o_P(w_k) \leq j \leq k \mid w_j > w_k\big\}\big| = f(w_k)$, since the size of the set $\big\{o_P(w_\ell) \leq j \leq \ell \mid w_j > w_\ell\big\} \setminus \big\{o_P(w_k) \leq j \leq k \mid w_j > w_k\big\}$, which is equal to $\big\{k < j \leq \ell \mid w_\ell < w_j \leq w_k\big\}$, is clearly less than or equal to $\ell - k$.

Example 2.1 (continued). Let again R be the region of $\mathcal{S}_9$ associated with the valid pair $\big(843967125, \{[1,6],[3,8],[6,9]\}\big)$. Writing with a variant of Cauchy's two-line notation, we have, corresponding to the intervals $[1,6]$, $[3,8]$ and $[6,9]$, respectively, $w\langle 1{:}6\rangle = 843967$ and $f_1 = \widehat{843967} = \overset{3\,4\,6\,7\,8\,9}{113414}$, $f_2 = \widehat{396712} = \overset{1\,2\,3\,6\,7\,9}{121232}$, $f_3 = \widehat{7125} = \overset{1\,2\,5\,7}{1131}$ and, finally, $f = \lambda(R) = 341183414$, which we also write $\widehat{843967125}$ [(3)] (cf. Figure 1).

Similarly, for $A = [9] \setminus \{8,4\}$, we may consider $f = \lambda\big(3967125, \{[1,6],[4,7]\}\big)$, the A-parking function $\widehat{3967125} = \overset{1\,2\,3\,5\,6\,7\,9}{1216232}$.

3. Injectivity of λ

The proof of the injectivity of λ is based on the following lemma, where a particular case is considered. Beforehand, we introduce a new concept.

(3) Note that, for example, the central parking function $1132 = \widehat{2413}$ corresponds to $\widehat{2413}$.

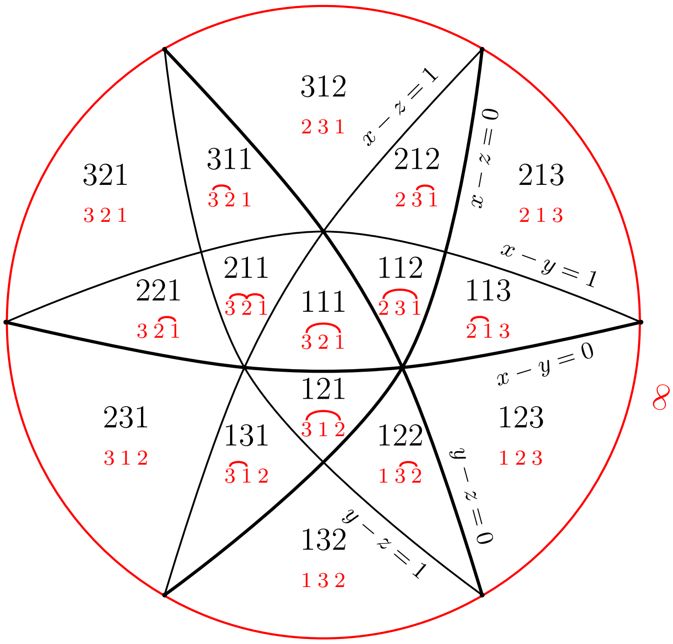

FIGURE 1. Pak-Stanley labeling for $n = 3$

Definition 3.1. Let $w \in \mathrm{W}_A$ for a subset A of $[n]$, consider the poset of *inversions of* w, $\mathrm{inv}(w) := \{(i,j) \mid i < j, w_i > w_j\}$, ordered so that $(i,j) \leq (k,\ell)$ if and only if $[i,j] \subseteq [k,\ell]$. Then, define $\mathrm{maxinv}(w)$ as the set of maximal elements of $\mathrm{inv}(w)$.

Lemma 3.2. *Let $A \subseteq [n]$, $v, w \in \mathrm{W}_A$, and suppose that $P = (v, \mathfrak{I})$ is a valid pair. If*

$$\lambda(v, \mathfrak{I}) = \widehat{w}\,,$$

then $v = w$ and $\mathfrak{I} = \mathrm{maxinv}(v)$.

Proof. We first prove that $v = w$. Let $A = \{a_1, \ldots, a_m\}$ with $a_1 < \cdots < a_m$, and suppose that, for $\pi, \rho \in \mathfrak{S}_m$, $v = a_{\pi_1} a_{\pi_2} \cdots a_{\pi_m}$ and $w = a_{\rho_1} a_{\rho_2} \cdots a_{\rho_m}$, and that, for some $1 \leq \ell \leq n$, $\pi_i = \rho_i$ whenever $1 \leq i < \ell$ but, contrary to our assumption, $\pi_\ell \neq \rho_\ell$. Finally, define $j, k > \ell$ such that $\rho_\ell = \pi_j$ and $\pi_\ell = \rho_k$ and $x := a_{\pi_\ell}$, $y := a_{\rho_\ell}$. Graphically, we have

$$\begin{array}{rcl} v & = & w_1 \ \cdots \ w_{\ell-1} \ x{=}v_\ell \ v_{\ell+1} \ \cdots \ y{=}v_j \ \cdots \ v_m \\ w & = & w_1 \ \cdots \ w_{\ell-1} \ y{=}w_\ell \ w_{\ell+1} \ \cdots \ x{=}w_k \ \cdots \ w_m \end{array}$$

Then, for $a = o_P(y) < j$,

$$\begin{aligned} \widehat{w}(y) &= \ell - \Big|\{1 \leq i < \ell \mid w_i > y\}\Big| \\ &= j - \Big|\{a \leq i < j \mid v_i > y\}\Big| \end{aligned}$$

and hence

$$j - \ell = \Big|\{\ell \leq i < j \mid v_i > y\}\Big| - \Big|\{1 \leq i < a \mid w_i > y\}\Big|.$$

This means that, for every i with $\ell \leq i < j$, $w_i > y$ (and, in particular, $x > y$) and that, for every i with $1 \leq i < a$, $w_i \leq y$. On the other hand, for $b = o_P(x) \leq \ell$,

$$\begin{aligned}\widehat{w}(x) &= k - \left|\left\{1 \leq i < k \mid w_i > x\right\}\right| \\ &= \ell - \left|\left\{b \leq i < \ell \mid w_i > x\right\}\right|\end{aligned}$$

and

$$k - \ell = \left|\left\{\ell \leq i < k \mid w_i > x\right\}\right| + \left|\left\{1 \leq i < b \mid w_i > x\right\}\right|$$

Note that $b \leq a$ since $\ell < j$ and P is a valid pair. Then, $\left\{1 \leq i < b \mid w_i > x\right\} = \varnothing$ and $w_i > x$ for every i with $\ell \leq i < j$. In particular, $y > x$, which is absurd. We now leave it to the reader to prove that $\mathfrak{I} = \operatorname{maxinv}(v)$. □

Corollary 3.3. *Let $A \subseteq [n]$. The function $\mathrm{C_A} \colon \mathrm{W}_A \to \mathrm{CF}_A \colon w \mapsto \widehat{w}$ is a bijection.*

Proof. Since $\left|\mathrm{W}_A\right| = \left|\mathrm{CF}_A\right| = |A|!$, the result follows from the last lemma, since $\mathrm{C_A}$ is injective. □

Definition 3.4.

- We denote the inverse of $\mathrm{C_A}$ by $\varphi_A \colon \mathrm{CF}_A \to \mathrm{W}_A$.
- Given an A-parking function $f \colon A \to [n]$, the center of f, $Z(f)$, is the (unique[(4)]) maximal subset Z of A such that the restriction of f to Z is Z-central. Let $\zeta := |Z|$ and note that $\zeta \neq 0$ since $f^{-1}(1) \subseteq Z$ and $\left|f^{-1}(1)\right| \geq 1$. Finally, let $f_Z \colon Z \to [n]$ be the restriction of f to its center.

Lemma 3.5. *Let $f = \lambda(w, \mathfrak{I})$ for a valid pair $P = (w, \mathfrak{I})$, where $w \in \mathrm{W}_A$ for $A \subseteq [n]$ with $m = |A|$.*

3.5.1. *Let, for some $p \geq 0$, $\mathfrak{I} = \left\{[o_1, c_1], \ldots, [o_p, c_p]\right\}$ with $o_1 < \cdots < o_p$. Then,*

$$f_Z = \widehat{w\langle 1\colon\zeta\rangle}$$

and, in particular, $w([\zeta]) = Z$. Moreover, $\operatorname{maxinv}(w\langle 1\colon\zeta\rangle) = \{[o_1, c_1], \ldots, [o_j, c_j]\}$ for some $0 \leq j \leq p$.

3.5.2. *For every $j \in [m]$, $w_j \in Z(f)$ if and only if*

$$f(w_j) = 1 + \left|\left\{k < j \mid w_k < w_j\right\}\right|.$$

Proof.
(3.5.1) We start by proving the second statement, namely that $w([\zeta]) = Z$. Note that $w_1 \in f^{-1}(\{1\}) \subseteq Z$ and suppose, contrary to our claim, that, for some $k < \zeta$ which we consider as small as possible, $w_k \notin Z$. Again, let $\ell > k$ be as small as possible with $w_\ell \in Z$ and define $v = w\langle 1\colon k\rangle$.

We now consider the "restriction" w^* of w to Z, that is, the subword of w obtained by deleting all the elements of $[n] \setminus Z$, and let

$$w' := \varphi_Z(f_Z) \in \mathrm{W}_Z\,.$$

(4) Note that if the restriction of f to X is X-central and the restriction of f to Y is Y-central for two subsets X and Y of A, then the restriction of f to $(X \cup Y)$ is also $(X \cup Y)$-central.

By Lemma 3.2, $w^* = w'$ and $k - f(w_\ell)$ is the number of integers greater than w_ℓ that precede it in w^*. This means that $w_k, \dots, w_{\ell-1} > w_\ell$ and that $o(w_\ell) \le k$. Hence, $k - f(w_k)$ is also the number of integers greater than w_k that precede it in w, and so $\widehat{v}$ is the restriction of f to $w([k])$, and $a \in Z$, a contradiction. Now, the result follows also from Lemma 3.2.
3.5.2 is a clear consequence of 3.5.1. □

We have proven that the "initial parts" of both w and $\mathfrak{I}$ are characterized by f. Let $m = |A|$, consider $c \in \mathbb{N}$ such that $1 < c \le \zeta$, and define $\tilde{w} := w\langle c\!:\!m\rangle$; define also $\tilde{\mathfrak{I}} := \varnothing$ if $j = p$, for j, p defined as in the statement of Lemma 3.5, and $\tilde{\mathfrak{I}} := \{\tilde{I}_1, \dots, \tilde{I}_{p-j}\}$, where

$$\tilde{I}_1 =: [1, c_{j+1} - c + 1], \dots, \tilde{I}_{p-j} := [o_p - c + 1, c_p - c + 1]\,,$$

if $p > j$. Suppose that, for some such c, f also determines $\tilde{f} := \lambda(\tilde{w}, \tilde{\mathfrak{I}})$. This proves our promised result (by induction on $|A|$) and shows how to proceed for actually finding $w \in \mathfrak{S}_n$ and $\mathfrak{I}$, given $f = \lambda(w, \mathfrak{I})$: we find the center Z of f, build $\varphi_Z(f_Z) \in \mathrm{W}_Z$ and $\tilde{f}$, find the center $\tilde{Z}$ of $\tilde{f}$, build $\varphi_{\tilde{Z}}(f_{\tilde{Z}}) \in \mathrm{W}_{\tilde{Z}}$ and $\tilde{\tilde{f}}$, etc.

Definition 3.6. Given a parking function $f \in \mathrm{PF}_A$, $f = \lambda(w, \mathfrak{I})$, $m := |A|$, $Z := Z(f)$, and $\zeta := |Z| < m$,

- let $b := \min f(A \setminus Z)$ and $a := \max\big(f^{-1}(\{b\}) \setminus Z\big)$;
- if $b > \zeta$, let $c := b$;
  if $b \le \zeta$, let c be the greatest integer $i \in [\zeta]$ for which

$$i + \big|w([i, \zeta]) \cap [a-1]\big| = b\,. \tag{3.7}$$

- let $X := w([c-1])$ ($X \subseteq Z$ by Lemma 3.5);
- let $$\tilde{f}\colon \quad A \setminus X \;\to\; [m - c + 1]$$ $$x \;\mapsto\; \begin{cases} f(x) - \big|X \cap [x-1]\big|\,, & \text{if } x \in Z\,; \\ f(x) - c + 1\,, & \text{otherwise}\,. \end{cases}$$

Lemma 3.7. *With the definitions above,*
3.7.1. $a = w_{\zeta+1}$ *and* $a \in Z(\tilde{f})$*;*
3.7.2. $Z \setminus X \subseteq Z(\tilde{f})$*;*
3.7.3. $c = o_{(\tilde{w}, \tilde{\mathfrak{I}})}(a)$ *and*
3.7.4. $\tilde{f} = \lambda\big(\tilde{w}, \tilde{\mathfrak{I}}\big)$.

Proof. If $b > \zeta$, then $X = Z$ and all the statements follow directly from the definitions. Hence, we consider that $b \le \zeta$. We start by seeing that c is well defined. Define $h\colon [\zeta] \to \mathbb{N}$ by

$$h(i) = i + \big|w([i, \zeta]) \cap [a-1]\big|\,.$$

Then, for every $i < \zeta$, since $w([i, \zeta]) = \{w_i\} \cup w([i+1, \zeta])$, $h(i+1)$ either equals h_i or $h_i + 1$, depending on whether w_i is either less than a or greater than a. Since $h(\zeta) \ge \zeta \ge b$, by definition, all we have to prove is that $h(1) < b$, or, equivalently, that $1 + \big|Z \cap [a-1]\big| < f_a$. But $f_a \le 1 + \big|Z \cap [a-1]\big|$ implies that the restriction of f to $Z' := Z \cup \{a\}$ is Z'-central, by Lemma 3.5.2, which, since $a \notin Z$, contradicts the maximality of Z. Note that the set of values of i for which (3.7) holds true is an interval, and that its maximum, c, is the only one that is greater than a. By definition of a and by Lemma 3.5.1, $a = w_{\zeta+1}$, for if $x = w_k$ and $a = w_\ell$ with $\ell > k$ and $x > a$, then $f(x) \le b$, by (2.6), and $x \in Z(f)$.

Now, let $g = \lambda(\tilde{w}, \tilde{\mathfrak{I}})$ for $\tilde{w}$ and $\tilde{\mathfrak{I}}$ as defined before. If $x \in A \setminus Z$, by definition of λ, viz. (2.3), $g(x) = f(x) - c + 1 = \tilde{f}(x)$. In particular, $g(a) = 1 + \left|\tilde{w}([\zeta - c + 1]) \cap [a-1]\right|$. Hence, by Lemma 3.5.2, $a \in Z(g)$. Now, Lemma 3.5.1 implies that $Z \setminus X$, the set of elements on the left side of a in $\tilde{w}$, is a subset of $Z(g)$, and that $c = o_{(\tilde{w}, \tilde{\mathfrak{I}})}(a)$. Now, the last result, viz. $g = \tilde{f}$, follows immediately, since for $x = w_j$ with $c \leq j \leq \zeta$, $f(x) = 1 + \left|w([j]) \cap [x-1]\right|$ and $g(x) = 1 + \left|\tilde{w}([j - c + 1]) \cap [x-1]\right|$. □

This concludes the proof of our main result.

Proposition 3.8. *The Pak-Stanley labeling is injective.* □

4. Inverse

It is easy to directly prove Corollary 3.3 and even to explicitly define φ_A, the inverse of $\mathrm{C_A}$. Nevertheless, we consider here a method that we find very convenient, and particularly well-suited to our purpose, the s-parking. Note that a similar method is given by the depth-first search version of Dhar's burning algorithm defined by Perkinson, Yang and Yu [4]. In fact, it may be proved that $Z(f)$ is the set of ζ visited vertices before the first back-tracking, and that $w\langle 1{:}\zeta\rangle$ is given by the order in which the vertices are visited.

Definition 4.1. Let again $A =: \{a_1, \ldots, a_m\}$ with $a_1 < \cdots < a_m$ and $f : A \to [m]$. For every $i \in [m]$, define the set $A_i := \{a_1, \ldots, a_i\}$, and define recursively the bijection $w^i \colon A_i \to [i]$ as follows.

- $w^1 : a_1 \mapsto 1$ (necessarily);
- for $1 < j \leq i \leq m$,
  - if $j < i$, $w^i(a_j) = \begin{cases} w^{i-1}(a_j), & \text{if } w^{i-1}(a_j) < f(a_i) \\ 1 + w^{i-1}(a_j), & \text{if } w^{i-1}(a_j) \geq f(a_i) \end{cases}$
  - $w^i(a_i) = f(a_i)$;

Finally, let $\psi : [m] \to A$ be the inverse of $w^m : A \to [m]$. We call $S(f) := \psi$ (viewed as the word $\psi(1) \cdots \psi(m)$) the *s-parking of f*.

This operation resembles placing books on a bookshelf, where in step i we want to put book a_i at position $f(a_i)$ — and so we must shift right every book already placed in a position greater than or equal to $f(a_i)$. For example, if $A = \{3, 4, 6, 7, 8, 9\} \subseteq [9]$ and $f = \overset{3\,4\,6\,7\,8\,9}{113414}$, then $S(f) = 843967$. On the other hand, if $B = \{1, 2, 3, 6, 7, 9\}$ and $g = \overset{1\,2\,3\,6\,7\,9}{121232}$, then $S(g) = 396712$. Finally, let $C = \{1, 2, 5, 7\}$ and $h = \overset{1\,2\,5\,7}{1231}$, so that $S(h) = 7125$. The three constructions are used in the next example. See Figure 2, where a parking function f is represented on the top rows by orderly stacking in column i the elements of $f^{-1}(i)$ (cf. [2]), and row j below the horizontal line is the inverse of w^j. Note that (1.1) implies that w^i is indeed a bijection for $i = 1, \ldots, m$.

Lemma 4.2. *Given A and f as in the previous definition, $f = \widehat{S(f)}$. Conversely, given A and $w \in \mathrm{W}_A$, $w = S(\widehat{w})$.*

Proof. Let $w = S(f)$ and $\psi = w^{-1}$ and note that, when we s-park f, each element a_i of A is put first at position $f(a_i)$, and it is shifted one position to the right by an element a_j if and only if $j > i$ and $\pi_j < \pi_i$; it ends at position ψ_i. Hence, $f = \widehat{S(f)} = \widehat{w}$. Then S is the inverse of $\mathrm{C_A}$, that is, $S = \varphi_A$. □

8	9	
4 9	3 6	7
3 6 7	1 2 7	1 2 5
3	1	1
4 3	1 2	1 2
4 3 6	3 1 2	1 2 5
4 3 6 7	3 6 1 2	7 1 2 5
8 4 3 6 7	3 6 7 1 2	
8 4 3 9 6 7	3 9 6 7 1 2	

FIGURE 2. S-parking

Example 2.1 (conclusion). Let us recover the valid pair $P = \lambda^{-1}(f)$ out of $f = 341183414$. In the first column, on the right, the elements of the center of f are written in italic and a is written in boldface. The last column may be obtained by s-parking, as represented in Figure 2.

f		a	b	c	f_Z		
$\overset{123456789}{341183414}$	$\mathit{8}\ \mathit{9}$ / $\mathit{4}\ \mathit{6}\ \mathit{7}$ / $\mathit{3}$ ␣ **1** 2 ␣␣␣ 5 ␣	1	3	3	$\overset{346789}{113414}$	$=$	$\widehat{843967}$
$\overset{1235679}{1216232}$	$\mathit{9}$ / $\mathit{3}\ \mathit{6}$ / $\mathit{1}\ \mathit{2}\ \mathit{7}$ ␣␣ **5** ␣	5	6	4	$\overset{123679}{121232}$	$=$	$\widehat{396712}$
$\overset{1257}{1231}$	$\mathit{7}$ / $\mathit{1}\ \mathit{2}\ \mathit{5}$	–	–	–	$\overset{1257}{1231}$	$=$	$\widehat{7125}$

In fact, as we know, $f = \widehat{843967125}$, that is $P = \big(843967125, \{[1,6],[3,8],[6,9]\}\big)$.

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