Stratifications on the Moduli Space of Higgs Bundles

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Abstract. The moduli space of Higgs bundles has two stratifications. The Bialynicki-Birula stratification comes from the action of the non-zero complex numbers by multiplication on the Higgs field, and the Shatz stratification arises from the Harder–Narasimhan type of the vector bundle underlying a Higgs bundle. While these two stratification coincide in the case of rank two Higgs bundles, this is not the case in higher rank. In this paper we analyze the relation between the two stratifications for the moduli space of rank three Higgs bundles.

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1 Introduction

Higgs bundles and their moduli were first studied by Hitchin and Simpson and have been around for almost 30 years. They continue to be the subject of intensive investigations with links to diverse areas of mathematics such as non-abelian Hodge theory, integrable systems, mirror symmetry, the Langlands programme, among others.

In this paper we focus on the moduli space of Higgs bundles on a compact Riemann surface X. The topology of this moduli space has been studied extensively. Some early calculations of Betti numbers were carried out by Hitchin [17] for rank 2 and the first author [6] for rank 3. Further significant progress has been made by a number authors, see, e.g., [15, 16, 18, 1, 19, 13, 5, 12, 11, 8, 9]. Recently Schiffmann [22] has completely determined the additive cohomology in the case of Higgs bundles with rank and degree co-prime.

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On the other hand, the homotopy theory of the moduli space of Higgs bundles has not been the subject of a lot of interest. Hausel [10] in his thesis studied the case of rank 2 Higgs bundles, while in [4] some results were obtained for general rank. The latter paper used the Bialynicki-Birula stratification of the Higgs bundle moduli space coming from the $\mathbb{C}^*$-action given by multiplying the Higgs field by scalars. In rank 2 this stratification coincides with the Shatz stratification, which is given by the Harder–Narasimhan type of the vector bundle underlying a Higgs bundle. As already observed by Hitchin and exploited by Hausel and Thaddeus [10, 15] this makes the case of rank 2 Higgs bundles akin to a finite dimensional version of the infinite dimensional situation of Atiyah–Bott [2].

However, in general the Bialynicki-Birula and Shatz stratifications do not coincide, and it is therefore of interest to study their relationship. In this paper we carry out such a study in the case of rank 3 Higgs bundles, where it turns out that the situation is alreay fairly complicated. Indeed, our main result, Theorem 4.2, shows that each Shatz stratum is intersected by several different Bialynicki-Birula strata. Moreover, knowledge of the underlying vector bundle of a Higgs bundle is not sufficient to determine its Bialynicki-Birula stratum, one also needs knowledge on the Higgs field. However, for sufficiently unstable underlying vector bundles the situation is simpler and the Shatz strata coincide with Bialynicki-Birula strata: this is described in Theorem 4.6.

Our results should serve as a useful pointer to the general situation for higher rank Higgs bundles and one may also hope that it could shed light on the homotopy theory of higher rank Higgs bundle moduli spaces.

This paper is organized as follows. In Section 2 we give some preliminaries about Higgs bundles and their moduli spaces and we explain the Bialynicki-Birula and Shatz stratifications of the moduli space. Then in Section 3 we give some bounds on the Harder–Narasimhan types which occur in the moduli space of rank 3 Higgs bundles. Finally, in Section 4 we give our main results on the relation of the two stratifications.

2 Preliminaries

2.1 Higgs bundles and their moduli

Let X be a closed Riemann surface of genus g and let $K = K_X = TX^*$ be the canonical line bundle of X.

Definition 2.1. A *Higgs bundle* over X is a pair (E, Φ) where the *underlying vector bundle* $E \to X$ is a holomorphic vector bundle and the *Higgs field* $\Phi : E \to E \otimes K$ is a holomorphic endomorphism of E twisted by K.

The *slope* of a vector bundle E is the quotient between its degree and its rank: $\mu(E) = \deg(E)/\operatorname{rk}(E)$. Recall that a vector bundle E is *semistable* if $\mu(F) \leqslant \mu(E)$ for all non-zero subbundles $F \subset E$, *stable* if it is semistable and strict inequality holds for all non-zero proper F, and *polystable* if it is the direct sum of stable bundles, all of the same slope. The corresponding stability notions for Higgs bundles are defined in exactly the same way, except that only Φ-invariant subbundles $F \subset E$ (satisfying $\Phi(F) \subset F \otimes K$) are considered.

The moduli space $\mathcal{M}(r, d)$ of rank r and degree d Higgs bundles was constructed by Nitsure [20]. When r and d are coprime any semistable Higgs bundle is automatically stable and $\mathcal{M}(r, d)$ is smooth. For the remainder of the paper we shall asumme that this is the case. There are no stable Higgs bundles when $g \leqslant 1$, and so we shall also assume that $g \geqslant 2$.

We shall need to consider the moduli space from the complex analytic point of view. For this, fix a complex C^∞ vector bundle $\mathcal{E}$ of rank r and degree d on X. A holomorphic structure on $\mathcal{E}$ is given by a $\bar{\partial}$-operator $\bar{\partial}_E\colon A^0(\mathcal{E}) \to A^{0,1}(\mathcal{E})$ and we thus obtain a holomorphic vector bundle $E = (\mathcal{E}, \bar{\partial}_E)$. From this point of view, a Higgs bundle (E, Φ) arises from a pair $(\bar{\partial}_E, \Phi)$ consisting of a $\bar{\partial}$-operator and a Higgs field $\Phi \in A^{1,0}(\mathrm{End}(\mathcal{E}))$ such that $\bar{\partial}_E\Phi = 0$. The natural symmetry group of the situation is the *complex gauge group* $\mathcal{G}^{\mathbb{C}} = \{g\colon \mathcal{E} \to \mathcal{E} \mid g \text{ is a } C^\infty \text{ bundle isomorphism}\}$, which acts on pairs $(\bar{\partial}_E, \Phi)$ in the standard way:

$$g \cdot (\bar{\partial}_E, \Phi) = (g \circ \bar{\partial}_E \circ g^{-1}, g \circ \Phi \circ g^{-1}).$$

The moduli space can then be viewed as the quotient[3]

$$\mathcal{M}(r, d) = \{(\bar{\partial}_E, \Phi) \mid \bar{\partial}_E\Phi = 0 \text{ and } (E, \Phi) \text{ is stable}\}/\mathcal{G}^{\mathbb{C}}.$$

2.2 Harder–Narasimhan filtrations and the Shatz stratification

The Harder–Narasimhan filtration of a vector bundle was introduced in [7] and studied systematically by Shatz [21]. It plays an important role in the work of Atiyah and Bott [2]. We refer the reader to these references for details on what follows.

Let E be a holomorphic vector bundle on X. A *Harder-Narasimhan Filtration* of E, is a filtration of the form

$$\mathrm{HNF}(E):\; E = E_s \supset E_{s-1} \supset \cdots \supset E_1 \supset E_0 = 0 \tag{2.1}$$

which satisfies the following two properties:

(i) $\mu(E_{j+1}/E_j) < \mu(E_j/E_{j-1})$ for $1 \leqslant j \leqslant s-1$.

(ii) E_j/E_{j-1} is semistable for $1 \leqslant j \leqslant s$.

For brevity, when we have a filtration $E = E_s \supset E_{s-1} \supset \cdots \supset E_1 \supset E_0 = 0$ we shall sometimes write $\bar{E}_j = E_j/E_{j-1}$ for the subquotients. The associated graded vector bundle is

$$\mathrm{Gr}(E) = \bigoplus_{j=1}^{s} E_j/E_{j-1} = \bigoplus_{j=1}^{s} \bar{E}_j.$$

Any vector bundle E has a unique Harder–Narasimhan filtration. The *Harder–Narasimhan polygon* is the polygon in the first quadrant of the (r, d)-plane with vertices $(\mathrm{rk}(E_j), \deg(E_j))$ for $j = 0, \ldots, s$. The slope of the line joining $(\mathrm{rk}(E_{j-1}), \deg(E_{j-1}))$ and $(\mathrm{rk}(E_j), \deg(E_j))$ is $\mu(\bar{E}_j)$. Condition (i) above says that the Harder–Narasimhan polygon is convex. Clearly this is equivalent to saying that $\mu(E_j) < \mu(E_{j-1})$ for $j = 2, \ldots, s$.

The *Harder–Narasimhan type* of E is the vector in $\mathbb{R}^r$:

$$\mathrm{HNT}(E) = \mu = (\mu(\bar{E}_1), \ldots, \mu(\bar{E}_1), \ldots, \mu(\bar{E}_s), \ldots, \mu(\bar{E}_s))$$

where the slope of each $\bar{E}_j$ is repeated $\mathrm{rk}(\bar{E}_j)$ times.

[3]Strictly speaking one should use appropriate Sobolev completions as in Atiyah and Bott [2]; see, for example, Hausel and Thaddeus [15] for the case of Higgs bundles.

As a consequence of Shatz [21, Propositions 10 and 11], there is a finite stratification of $\mathcal{M}(r,d)$ by the Harder–Narasimhan type of the underlying vector bundle E of a Higgs bundle (E,Φ):

$$\mathcal{M}(r,d) = \bigcup_{\mu} U'_{\mu}$$

where $U'_{\mu} \subset \mathcal{M}(r,d)$ is the subspace of Higgs bundles (E,Φ) whose underlying vector bundle E has Harder–Narasimhan type μ. This stratification is known as the *Shatz stratification*. Note that there is an open dense stratum corresponding to Higgs bundles (E,Φ) for which the underlying vector bundle E is itself stable. Since $\Phi \in H^0(\mathrm{End}(E) \otimes K) \cong H^1(\mathrm{End}(E))^*$ (by Serre duality) such a Higgs bundle represents a point in the cotangent bundle of the moduli space of stable bundles $\mathcal{N}(r,d)$. In other words,

$$U'_{(r/d,\ldots,r/d)} = T^*\mathcal{N}(r,d) \subset \mathcal{M}(r,d).$$

2.3 The $\mathbb{C}^*$-action and the Bialynicki-Birula stratification

We review some standard facts about the $\mathbb{C}^*$-action on $\mathcal{M}(r,d)$. For more details see, e.g., Simpson [23].

The holomorphic action of the multiplicative group $\mathbb{C}^*$ on $\mathcal{M}(r,d)$ is defined by the multiplication:

$$z \cdot (E,\Phi) \mapsto (E, z \cdot \Phi).$$

The limit $(E_0,\varphi_0) = \lim_{z\to 0} z \cdot (E,\Phi)$ exists for all $(E,\Phi) \in \mathcal{M}(r,d)$. Moreover, this limit is fixed by the $\mathbb{C}^*$-action. A Higgs bundle (E,Φ) is a fixed point of the $\mathbb{C}^*$-action if and only if it is a *Hodge bundle*, i.e. there is a decomposition $E = \bigoplus_{j=1}^{p} E_j$ with respect to which the Higgs field has weight one: $\Phi\colon E_j \to E_{j+1} \otimes K$. The *type* of the Hodge bundle (E,Φ) is $(\mathrm{rk}(E_1),\ldots,\mathrm{rk}(E_p))$

Let $\{F_\lambda\}$ be the irreducible components of the fixed point locus of $\mathbb{C}^*$ on $\mathcal{M}(r,d)$. Let

$$U^+_\lambda := \{(E,\Phi) \in \mathcal{M} \ \mid \ \lim_{z\to 0} z \cdot (E,\Phi) \in F_\lambda\}.$$

Then we have the *Bialynicki-Birula stratification* (cf. [3]) of $\mathcal{M}(r,d)$:

$$\mathcal{M} = \bigcup_{\lambda} U^+_\lambda.$$

3 Bounds on Harder–Narasimhan types in rank 3

Let (E,Φ) be a rank 3 Higgs bundle. Let (μ_1,μ_2,μ_3) be the Harder–Narasimhan type of E, so that $\mu_1 \geqslant \mu_2 \geqslant \mu_3$ and $\mu_1+\mu_2+\mu_3 = 3\mu$, where $\mu = \mu(E)$. We can write the Harder–Narasimhan filtration of the vector bundle E as follows:

$$\mathrm{HNF}(E):\ 0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E,$$

where we have made the convention that $E_i = E_j$ if $\mu_i = \mu_j$. Thus, for example, if $\mu_1 = \mu_2 > \mu_3$ then the Harder–Narasimhan filtration is

$$\mathrm{HNF}(E):\ 0 = E_0 \subset E_1 = E_2 \subset E_3 = E$$

and $\mathrm{rk}(E_1) = \mathrm{rk}(E_2) = 2$. Similarly, if $\mu_1 > \mu_2 = \mu_3$ then $\mathrm{rk}(E_1) = 1$ and $\mathrm{rk}(E_2) = 3$.

We shall next introduce some notation which will be used throughout the remainder of the paper.

Let $\varphi_{21}\colon E_1 \to E/E_1 \otimes K$ be the map induced by Φ and let

$$I \subset E/E_1 \tag{3.1}$$

be the subbundle defined by saturating the subsheaf $\varphi_{21}(E_1) \otimes K^{-1} \subset E/E_1$. Similarly, let $\varphi_{32}\colon E_2 \to E/E_2 \otimes K$ be the map induced by Φ and let

$$N = \ker(\varphi_{32}) \subset E_2 \tag{3.2}$$

viewed as a subbundle.

Remark 3.1. Let (E, Φ) be a stable Higgs bundle such that E is an unstable vector bundle of Harder–Narasimhan type (μ_1, μ_2, μ_3). Then $E_1 \subset E_2$ is destabilizing and hence, by stability of (E, Φ), we have $\varphi_{21} \neq 0$. Similarly $E_2 \subset E$ is destabilizing and so $\varphi_{32} \neq 0$ (unless $\mu_2 = \mu_3 \iff E_2 = E$).

Proposition 3.2. *Let (E, Φ) be a semistable rank 3 Higgs bundle of Harder–Narasimhan type (μ_1, μ_2, μ_3). Then*

$$0 \leqslant \mu_1 - \mu_2 \leqslant 2g - 2, \tag{3.3}$$
$$0 \leqslant \mu_2 - \mu_3 \leqslant 2g - 2. \tag{3.4}$$

Proof. The fact that the differences $\mu_{i+1} - \mu_i$ are non-negative is just the convexity of the Harder–Narasimhan polygon.

If E is semistable the result is clear, so we may assume that this is not the case.

If $\mu_1 > \mu_2$ then $\mathrm{rk}(E_1) = 1$, and $I \subset E/E_1$ is a line bundle, since $\varphi_{21} \neq 0$ by Remark 3.1. It follows that we have a non-zero map of line bundles $E_1 \to I \otimes K$ and so

$$\mu(I) + 2g - 2 \geqslant \mu(E_1) = \mu_1.$$

Also, since $E_2/E_1 \subset E/E_1$ is maximal destabilizing, we have that

$$\mu(I) \leqslant \mu(E_2/E_1) = \mu_2$$

(note that this inequality also holds if $\mu_2 = \mu_3$). Combining these two inequalities proves (3.3).

If $\mu_2 > \mu_3$ then $\mathrm{rk}(E_2) = 2$, and $N \subset E_2$ is line bundle, since $\varphi_{32} \neq 0$ by Remark 3.1. It follows that we have a non-zero map of line bundles $E_2/N \to E/E_2 \otimes K$ and so

$$\begin{aligned} &\mu(E/E_2) + 2g - 2 \geqslant \mu(E_2/N) \\ \iff\ &\mu_3 + 2g - 2 \geqslant \deg(E_2) - \mu(N) = \mu_1 + \mu_2 - \mu(N). \end{aligned}$$

Also, since $E_1 \subset E_2$ is maximal destabilising, we have that

$$\mu(N) \leqslant \mu(E_1) = \mu_1$$

(note that this inequality also holds if $\mu_1 = \mu_2$). Combining these two inequalities proves (3.4). ∎

Note that the proof of the preceding Proposition gives the following bounds on the slopes of the bundles I and N.

Proposition 3.3. *Let (E, Φ) be a semistable rank 3 Higgs bundle of Harder–Narasimhan type (μ_1, μ_2, μ_3) and define $I \subset E/E_1$ and $N \subset E_2$ as above.*

(1) If $\mu_1 > \mu_2$ then $I \subset E/E_1$ is a line subbundle of a rank 2 bundle and $\mu_1 - (2g-2) \leqslant \mu(I) \leqslant \mu_2$.

(2) If $\mu_2 > \mu_3$ then $N \subset E_2$ is a line subbundle of a rank 2 bundle and $\mu_1 + \mu_2 - \mu_3 - (2g-2) \leqslant \mu(N) \leqslant \mu_1$.

□

4 Limits of the $\mathbb{C}^*$-action

The purpose of the present section is to analyse the limit as $z \to 0$ of $z \cdot (E, \Phi)$ as a function of the Harder–Narasimhan type of E.

4.1 Trivial Harder–Narasimhan filtrations

Let (E, Φ) be a stable Higgs bundle. When the underlying vector bundle E is itself stable, clearly $\lim_{z \to 0} z \cdot (E, \Phi) = (E, 0)$. Hence we have the following result, valid for any rank.

Proposition 4.1. *Let $(E, \Phi) \in \mathcal{M}(r, d)$. Then $\lim_{z \to 0} z \cdot (E, \Phi) = (E, 0)$ if and only if E is stable.*

Proof. It only remains to observe that if $(E, 0) = \lim_{z \to 0} z \cdot (E, \Phi)$ is a stable Higgs bundle then E is a stable vector bundle. □

4.2 Non-trivial Harder–Narasimhan filtrations

Again we limit ourselves to considering rank 3 stable Higgs bundles (E, Φ) and assume that $(r, d) = 1$, i.e., that d is not divisible by 3. We shall use the notation introduced in Section 3.

Theorem 4.2. *Let $(E, \Phi) \in \mathcal{M}(3, d)$ be such that E is an unstable vector bundle of slope μ and with Harder–Narasimhan type (μ_1, μ_2, μ_3). Then the limit $(E_0, \Phi_0) = \lim_{z \to 0}(E, z \cdot \Phi)$ is given as follows.*

(1) Assume that $\mu_2 < \mu$. Then $\mu_1 > \mu_2 \geqslant \mu_3$ and one of the following alternatives holds.

(1.1) The slope of I satisfies $\mu_1 - (2g-2) \leqslant \mu(I) < -\frac{1}{3}\mu_1 + \frac{2}{3}\mu_2 + \frac{2}{3}\mu_3$ and (E_0, Φ_0) is the following Hodge bundle of type $(1, 2)$:

$$(E_0, \Phi_0) = \Big(E_1 \oplus E/E_1, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}\Big).$$

The associated graded vector bundle is $\mathrm{Gr}(E_0) = \mathrm{Gr}(E)$.

(1.2) *The slope of* I *satisfies* $-\frac{1}{3}\mu_1 + \frac{2}{3}\mu_2 + \frac{2}{3}\mu_3 < \mu(I) \leqslant \mu_3$ *and* (E_0, Φ_0) *is the following Hodge bundle of type* $(1,1,1)$*:*

$$(E_0, \Phi_0) = \left(E_1 \oplus I \oplus (E/E_1)/I, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}\right),$$

where φ_{21} *and* φ_{32} *are induced from* Φ*. The associated graded vector bundle is* $\mathrm{Gr}(E_0) = E_1 \oplus (E/E_1)/I \oplus I$ *and its Harder–Narasimhan type is* $\mathrm{HNT}(E_0) = (\mu_1, \mu_2 + \mu_3 - \mu(I), \mu(I))$*.*

(1.3) *The slope of* I *satisfies* $\mu(I) = \mu_2$ *and the strict inequality* $\mu_3 < \mu_2$ *holds. Moreover, the line bundle* $I = E_2/E_1$ *and* (E_0, Φ_0) *is the following Hodge bundle of type* $(1,1,1)$*:*

$$(E_0, \Phi_0) = \left(E_1 \oplus E_2/E_1 \oplus E/E_2, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}\right),$$

where φ_{32} *is induced from* Φ*. The associated graded vector bundle is* $\mathrm{Gr}(E_0) = \mathrm{Gr}(E)$*.*

(2) *Suppose that* $\mu_2 > \mu$*. Then* $\mu_1 \geqslant \mu_2 > \mu_3$ *and one of the following alternatives holds.*

(2.1) *The slope of* N *satisfies* $\mu_1 + \mu_2 - \mu_3 - (2g-2) \leqslant \mu(N) < \mu$ *and* (E_0, Φ_0) *is the following Hodge bundle of type* $(2,1)$*:*

$$(E_0, \Phi_0) = \left(E_2 \oplus E/E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{32} & 0 \end{pmatrix}\right).$$

The associated graded vector bundle is $\mathrm{Gr}(E_0) = \mathrm{Gr}(E)$*.*

(2.2) *The slope of* N *satisfies* $\mu < \mu(N) \leqslant \mu_2$ *and* (E_0, Φ_0) *is the following Hodge bundle of type* $(1,1,1)$*:*

$$(E_0, \Phi_0) = \left(N \oplus E_2/N \oplus E/E_2, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}\right)$$

where φ_{21} *and* φ_{32} *are induced from* Φ*. The associated graded vector bundle is* $\mathrm{Gr}(E_0) = E_2/N \oplus N \oplus E/E_2$ *and its Harder–Narasimhan type is* $\mathrm{HNT}(E_0) = (\mu_1 + \mu_2 - \mu(N), \mu(N), \mu_3)$*.*

(2.3) *The slope of* N *satisfies* $\mu(N) = \mu_1$ *and the strict inequality* $\mu_1 > \mu_2$ *holds. Moreover the line bundle* $N = E_1$ *and* (E_0, Φ_0) *is the following Hodge bundle of type* $(1,1,1)$*:*

$$(E_0, \Phi_0) = \left(E_1 \oplus E_2/E_1 \oplus E/E_2, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}\right),$$

where φ_{21} *is induced from* Φ*. The associated graded vector bundle is* $\mathrm{Gr}(E_0) = \mathrm{Gr}(E)$*.*

Remark 4.3. Note that the condition $\mu_2 < \mu$ is equivalent to $\mu_3 > -\frac{1}{3}\mu_1 + \frac{2}{3}\mu_2 + \frac{2}{3}\mu_3$. In particular the range for $\mu(I)$ in Case (1.2) is non-empty.

Before proceeding with the proof of Theorem 4.2 we deduce a couple of interesting consequences. The theorem shows that, in general, knowledge of the Harder–Narasimhan type of E does not suffice to determine the underlying bundle E_0 of the limit $(E_0, \Phi_0) = \lim_{z\to 0}(E, z\cdot\Phi)$: indeed in cases (1.2) and (2.2) of the theorem one also needs knowledge of Φ. However, there are some Harder–Narasimhan types (μ_1, μ_2, μ_3) for which E_0 is determined by E. We note that, by Proposition 3.2, one has $0 \leqslant \mu_1 - \mu_3 \leqslant 4g - 4$.

Corollary 4.4. *Let $(E, \Phi) \in \mathcal{M}(3, d)$ be such that E is an unstable vector bundle of slope μ and Harder–Narasimhan type (μ_1, μ_2, μ_3). Assume that $\mu_1 - \mu_3 > 2g - 2$. Then the limit $(E_0, \Phi_0) = \lim_{z\to 0}(E, z\cdot\Phi)$ is given by (1.3) of Theorem 4.2 if $\mu_2 < \mu$, and by (2.3) of Theorem 4.2 if $\mu_2 > \mu$.*

Proof. We only have to observe that in all the other case of Theorem 4.2 we have $\mu_1 - \mu_3 \leqslant 2g - 2$.

In Cases (1.1) and (1.2) we have $\mu(I) \leqslant \mu_3$ (cf. Remark 4.3). Moreover, by (1) of Proposition 3.3, we have $\mu_1 - (2g - 2) \leqslant \mu(I)$. It follows that $\mu_1 - (2g - 2) \leqslant \mu_3$ as desired.

Similarly, in Cases (2.1) and (2.2) we have $\mu(N) \leqslant \mu_2$ and, by (2) of Proposition 3.3, $\mu_1 + \mu_2 - \mu_3 - (2g - 2) \leqslant \mu(N)$. Hence $\mu_1 + \mu_2 - \mu_3 - (2g - 2) \leqslant \mu_2$ which gives the conclusion. □

In a similar vein, we shall next see that certain types of Hodge bundle can only be the limit of a Higgs bundle whose underlying vector bundle has the same Harder–Narasimhan type as that of the Hodge bundle.

Before stating the result we recall (see, e.g., [6] or Hausel–Thaddeus [14]) that fixed points of type $(1, 1, 1)$ of the form

$$(E_0, \Phi_0) = (L_1 \oplus L_2 \oplus L_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix})$$

are usually parametrised by the numerical invariants

$$\begin{aligned} m_1 &= \deg(L_2) - \deg(L_1) + 2g - 2, \\ m_2 &= \deg(L_3) - \deg(L_2) + 2g - 2, \end{aligned}$$

subject to the conditions

$$\begin{aligned} m_i &\geqslant 0, \quad i = 1, 2, \\ 2m_1 + m_2 &< 6g - 6, \\ m_1 + 2m_2 &< 6g - 6, \\ m_1 + 2m_2 &\equiv 0 \pmod 3. \end{aligned}$$

For our purposes it is more natural to translate to the invariants (l_1, l_2, l_3) with $l_i = \mu(L_i) = \deg(L_i)$ (subject to the condition $l_1 + l_2 + l_3 = 3\mu$). We then have corresponding components $F_{(l_1,l_2,l_3)}$ of the fixed locus and the invariants (l_1,l_2,l_3) are subject to the constraints

$$\begin{aligned} &l_{i+1} - l_i + 2g - 2 \geqslant 0, \quad i = 1, 2, \\ &\frac{1}{3}l_1 + \frac{1}{3}l_2 - \frac{2}{3}l_3 > 0, \\ &\frac{2}{3}l_1 - \frac{1}{3}l_2 - \frac{1}{3}l_3 > 0. \end{aligned}$$

Corollary 4.5. *Let* $(E_0 = L_1 \oplus L_2 \oplus L_3, \Phi_0 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{smallmatrix}\right))$ *be a Hodge bundle of type* $(1,1,1)$ *with* $\mu(L_1) - \mu(L_3) > 2g-2$*. Then* $\mu(L_1) > \mu(L_2) > \mu(L_3)$ *and any* (E,Φ) *such that* $\lim_{z\to 0}(E, z\cdot\Phi) = (E_0,\Phi_0)$ *satisfies* $\mathrm{Gr}(E) = \mathrm{Gr}(E_0)$*.*

Proof. Since φ_{21} is non-zero we have $\mu(L_2)+2g-2 \geqslant \mu(L_1)$. This, together with $\mu(L_1) - \mu(L_3) > 2g-2$ implies that $\mu(L_2) > \mu(L_3)$. Since in Cases (1.2) and (2.2) of Theorem 4.2 one has $\mu(L_2) \leqslant \mu(L_3)$, it follows that the limit $(E_0, \Phi_=)$ must arise either from Case (1.3) or from Case (2.3). The conclusion follows since in these cases $\mathrm{Gr}(E) = \mathrm{Gr}(E_0)$. □

The two previous corollaries lead to an identification between Shatz and Bialynicki–Birula strata in some cases. Recall that $U^+_{(l_1,l_2,l_3)}$ denotes the Bialynicki-Birula stratum of Higgs bundles whose limits lie in $F_{(l_1,l_2,l_3)}$ and that $U'_{(l_1,l_2,l_3)}$ denotes the Shatz stratum of Higgs bundles whose Harder–Narasimhan type is (l_1,l_2,l_3).

Theorem 4.6. *Let* (l_1,l_2,l_3) *be such that* $l_1 - l_3 > 2g-2$*. Then the corresponding Shatz and Bialynicki-Birula strata in* $\mathcal{M}(3,d)$ *coincide:*

$$U'_{(l_1,l_2,l_3)} = U^+_{(l_1,l_2,l_3)}.$$

□

4.3 Proof of Theorem 4.2

For the proof, we adopt the complex analytic point of view as explained in Section 2.1. Let $\mathcal{E}$ be the C^∞ bundle underlying E and consider the pair $(\bar\partial_E, \Phi)$ representing (E,Φ) in the configuration space of all Higgs bundles. Our strategy of proof is to find a family of gauge transformations $g(z) \in \mathcal{G}^{\mathbb{C}}$, parametrized by $z \in \mathbb{C}^*$, such that the limit in the configuration space

$$(\bar\partial_{E_0}, \Phi_0) = \lim_{z\to 0}\Big(g(z)\cdot(\bar\partial_E, z\cdot\Phi)\Big)$$

gives a *stable* Higgs bundle (E_0,Φ_0). It will then follow that (E_0,Φ_0) represents the limit in the moduli space.

We now need to consider several cases.

4.3.1 Proof of Theorem 4.2 – Case (1)

Suppose that $\mu_2 < \mu$. Then, since $\mu_1 > \mu$, we must have $\mu_1 > \mu_2 \geqslant \mu_3$. It follows from (1) of Proposition 3.3 that $I \subset E/E_1$ is a line bundle and that $\mu_1 - (2g-2) \leqslant \mu(I) \leqslant \mu_2$.

We consider two separate cases.

Case A: $\mu_1 - (2g-2) \leqslant \mu(I) < -\frac{1}{3}\mu_1 + \frac{2}{3}\mu_2 + \frac{2}{3}\mu_3$.

We have a short exact sequence $0 \to E_1 \to E \to E/E_1 \to 0$. Let $\mathcal{E}$, $\mathcal{E}_1$ and $\mathcal{E}_2$ be the C^∞ vector bundles underlying E, E_1 and E/E_1, respectively. Then

$$\mathcal{E} \cong \mathcal{E}_1 \oplus \mathcal{E}_2 \tag{4.1}$$

and the holomorphic structure on $\mathcal{E}$ is given by the $\bar{\partial}$-operator:

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \beta \\ 0 & \bar{\partial}_2 \end{pmatrix},$$

where $\bar{\partial}_1$ and $\bar{\partial}_2$ are $\bar{\partial}$-operators defining the holomorphic structures on $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, and $\beta \in A^{0,1}(\mathrm{Hom}(\mathcal{E}_2, \mathcal{E}_1))$. With respect to the smooth decomposition (4.1), the Higgs field $\Phi \in A^{1,0}(\mathrm{End}(\mathcal{E}))$ takes the form:

$$\Phi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}.$$

Consider, for each $z \in \mathbb{C}^*$, the constant gauge transformation $g(z) \in \mathcal{G}^{\mathbb{C}}$ defined by

$$g(z) := \begin{pmatrix} 1 & 0 \\ 0 & z \cdot I \end{pmatrix},$$

with respect to the decomposition (4.1). Then:

$$g(z) \cdot (z \cdot \Phi) = g(z)^{-1}(z \cdot \Phi) g(z) = \begin{pmatrix} z \cdot \varphi_{11} & z^2 \cdot \varphi_{12} \\ \varphi_{21} & z \cdot \varphi_{22} \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix} \text{ when } z \to 0$$

and, moreover,

$$g(z) \cdot \bar{\partial}_E = g(z)^{-1} \circ \bar{\partial}_E \circ g(z) = \begin{pmatrix} \bar{\partial}_1 & z \cdot \beta \\ 0 & \bar{\partial}_2 \end{pmatrix} \to \begin{pmatrix} \bar{\partial}_1 & 0 \\ 0 & \bar{\partial}_2 \end{pmatrix} \text{ when } z \to 0.$$

Note that this simple formula for the gauge transformed $\bar{\partial}$-operator is valid because the gauge transformation is constant on X. Thus, in the configuration space of all Higgs bundles the limit $\lim_{z\to 0} z \cdot (E, \Phi)$ is gauge equivalent to

$$(E_0, \Phi_0) = \left(E_1 \oplus E/E_1, \begin{pmatrix} 0 & 0 \\ \varphi_{21} & 0 \end{pmatrix}\right).$$

This Higgs bundle will represent the limit in the moduli space $\mathcal{M}(3, d)$ provided that it is stable.

To show stability, we note that there are three kinds of Φ_0-invariant subbundles of E_0, namely $E_1 \oplus I$, E/E_1, and an arbitrary line bundle $L \subset E/E_1$. We deal with each case in turn:

1. *The subbundle* $E_1 \oplus I \subset E_1 \oplus E/E_1$. By hypothesis $\mu(I) < -\frac{1}{3}\mu_1 + \frac{2}{3}\mu_2 + \frac{2}{3}\mu_3$ which is equivalent to $\mu(E_1 \oplus I) < \mu(E) = \mu(E_0)$ as required.

2. *The subbundle* $E/E_1 \subset E_1 \oplus E/E_1$. It is immediate from the properties of the Harder–Narasimhan filtration that $\mu(E/E_1) < \mu(E) = \mu(E_0)$.

3. *A line subbundle* $L \subset E/E_1$. From the properties of the Harder–Narasimhan filtration we have that either $E_2/E_1 \subset E/E_1$ is maximal destabilizing (if $\mu_2 < \mu_3$) or E/E_1 is semistable (if $\mu_2 = \mu_3$). Either way we have that $\mu(L) \leqslant \mu_2$. Since $\mu_2 < \mu = \mu(E)$ by hypothesis, it follows that $\mu(L) < \mu(E) = \mu(E_0)$.

Finally note that, clearly, $\mathrm{Gr}(E_0) = E_1 \oplus E_2/E_1 \oplus E/E_2 = \mathrm{Gr}(E)$. Altogether we have seen that, under the given conditions on the slope of I, the limiting bundle (E_0, Φ_0) is as stated in Case (1.1) of the theorem.

Case B: $-\frac{1}{3}\mu_1 + \frac{2}{3}\mu_2 + \frac{2}{3}\mu_3 < \mu(I) \leqslant \mu_2$**.**

Define $Q = (E/E_1)/I$ so that we have a short exact sequence $0 \to I \to E/E_1 \to Q \to 0$. Let $\mathcal{E}_1$, $\mathcal{I}$ and $\mathcal{Q}$ be the C^∞ bundles underlying E_1, I and Q, respectively, so that we have a C^∞-decomposition

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{I} \oplus \mathcal{Q}. \tag{4.2}$$

Recalling that $\mathcal{I}$ comes from $\Phi(E_1) \otimes K^{-1}$, we may write the Higgs field Φ as:

$$\Phi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}$$

with respect to the decomposition (4.2). Moreover, the holomorphic structure on E is of the form

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \beta_{12} & \beta_{13} \\ 0 & \bar{\partial}_2 & \beta_{23} \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix}.$$

Now, for each $z \in \mathbb{C}^*$ take the following constant gauge transformation:

$$g(z) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix}$$

of $\mathcal{E}$ with respect to the decomposition (4.2). Then

$$\begin{aligned} g(z) \cdot (z \cdot \Phi) &= g(z)^{-1}(z \cdot \Phi) g(z) \\ &= \begin{pmatrix} z \cdot \varphi_{11} & z^2 \cdot \varphi_{12} & z^3 \cdot \varphi_{13} \\ \varphi_{21} & z \cdot \varphi_{22} & z^2 \cdot \varphi_{23} \\ 0 & \varphi_{32} & z \cdot \varphi_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix} \text{ when } z \to 0 \end{aligned}$$

and

$$\begin{aligned} g(z) \cdot \bar{\partial}_E &= g(z)^{-1} \circ \bar{\partial}_E \circ g(z) \\ &= \begin{pmatrix} \bar{\partial}_1 & z \cdot \beta_{12} & z^2 \cdot \beta_{13} \\ 0 & \bar{\partial}_2 & z \cdot \beta_{23} \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{\partial}_1 & 0 & 0 \\ 0 & \bar{\partial}_2 & 0 \\ 0 & 0 & \bar{\partial}_3 \end{pmatrix} \text{ when } z \to 0. \end{aligned}$$

Hence, in the configuration space, $\lim_{z \to 0} z \cdot (E, \Phi)$ is gauge equivalent to

$$(E_0, \Phi_0) = \left(E_1 \oplus I \oplus (E/E_1)/I, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix} \right).$$

It remains to prove that (E_0, Φ_0) is a stable Higgs bundle. The Φ_0-invariant subbundles of E_0 are the following:

1. *The subbundle* $(E/E_1)/I \subset E_0$. The condition $\mu((E/E_1)/I) < \mu(E)$ is equivalent to $-\frac{1}{3}\mu_1 + \frac{2}{3}\mu_2 + \frac{2}{3}\mu_3 < \mu(I)$ which holds by assumption.

2. *The subbundle* $I \oplus (E/E_1)/I \subset E_0$. In this case we note that $\mu(I \oplus (E/E_1)/I) < \mu(E) \iff \mu(E_1) > \mu(E)$, which holds by properties of the Harder–Narasimhan filtration.

Next we analyze the Harder–Narasimhan type. For brevity we continue to write $Q = (E/E_1)/I$. There are two situations to consider.

The first situation is when $\mu(I) \leqslant \mu(Q)$. Then the Harder–Narasimhan type of E_0 is $\mathrm{HNT}(E_0) = (\mu(E_1), \mu(Q), \mu(I))$. Hence, using Shatz's theorem [21, Theorem 3] that the Harder–Narasimhan polygon rises under specialization, we conclude that $\mu(I) \leqslant \mu(E/E_2)$. This leads to the description given in Case (1.2).

The second situation is when $\mu(I) > \mu(Q)$. Then the Harder–Narasimhan type of E_0 is $\mathrm{HNT}(E_0) = (\mu(E_1), \mu(I), \mu(Q))$. Hence, from Shatz's theorem we deduce that $\mu(I) \geqslant \mu(E_2/E_1)$. But $I \subset E/E_1$ so, from the properties of the Harder–Narasimhan filtration, we conclude that in fact $\mu(I) = \mu_2$. If $\mu_3 = \mu_2$ it follows that $\mu(I) = \mu(Q)$, contradicting $\mu(I) > \mu(Q)$. Hence $\mu_3 < \mu_2$ and $I \subset E/E_1$ is the unique maximal destabilizing subbundle, i.e., $I = E_2/E_1$ and so Case (1.3) occurs.

This completes the proof of Case (1).

4.3.2 Proof of Theorem 4.2 – Case (2)

Suppose that $\mu_2 > \mu$. Then, since $\mu_3 < \mu$, we must have $\mu_1 \geqslant \mu_2 > \mu_3$. It follows from (2) of Proposition 3.3 that $N \subset E_2$ is a line bundle and that $\mu_1 + \mu_2 - \mu_3 - (2g-2) \leqslant \mu(N) \leqslant \mu_1$.

We consider two separate cases.

Case C: $\mu_1 + \mu_2 - \mu_3 - (2g-2) \leqslant \mu(N) < \mu$**.**

We have a short exact sequence $0 \to E_2 \to E \to E/E_2 \to 0$. Let $\mathcal{E}$, $\mathcal{E}_2$ and $\mathcal{E}_3$ be the C^∞ vector bundles underlying E, E_2 and E/E_2, respectively. Then $\mathcal{E} \cong \mathcal{E}_2 \oplus \mathcal{E}_3$ and the holomorphic structure on $\mathcal{E}$ is given by a $\bar\partial$-operator of the form $\bar\partial_E = \left(\begin{smallmatrix} \bar\partial_2 & \beta \\ 0 & \bar\partial_3 \end{smallmatrix}\right)$, while the Higgs field $\Phi \in A^{1,0}(\mathrm{End}(\mathcal{E}))$ takes the form: $\Phi = \left(\begin{smallmatrix} \varphi_{22} & \varphi_{23} \\ \varphi_{32} & \varphi_{33} \end{smallmatrix}\right)$. The same calculation as in Case A shows that in the configuration space of all Higgs bundles, $\lim_{z\to 0} z \cdot (E, \Phi)$ is gauge equivalent to

$$(E_0, \Phi_0) = \left(E_2 \oplus\ E/E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{32} & 0 \end{pmatrix}\right).$$

This Higgs bundle will represent the limit in the moduli space $\mathcal{M}(3, d)$ if it is stable. There are three kinds of Φ_0-invariant subbundles to check:

1. *The subbundle* $N \subset E_2 \oplus E/E_2$. By hypothesis $\mu(N) < \mu = \mu(E) = \mu(E_0)$.

2. *The subbundle* $E/E_2 \subset E_2 \oplus E/E_2$. It is immediate from the properties of the Harder–Narasimhan filtration that $\mu(E/E_2) < \mu(E) = \mu(E_0)$.

3. *Subbundles* $L \oplus E/E_2 \subset E_2 \oplus E/E_2$ *for* $L \subset E_2$ *a line subbundle.* From the properties of the Harder–Narasimhan filtration we have that either $E_1 \subset E_2$ is

maximal destabilizing (if $\mu_1 > \mu_2$) or E_2 is semistable (if $\mu_1 = \mu_2$). Either way we have that $\mu(L) \leqslant \mu_1$. It follows that

$$\begin{aligned} 2\mu(L \oplus E/E_2) &= \mu(L) + 3\mu - \mu_1 - \mu_2 \\ &\leqslant 3\mu - \mu_2 \\ &< 2\mu, \end{aligned}$$

where we have used the hypothesis $\mu_2 > \mu$ in the last step. Hence $\mu(L \oplus E/E_2) < \mu = \mu(E) = \mu(E_0)$ as desired.

Finally note that, clearly, $\mathrm{Gr}(E_0) = E_1 \oplus E_2/E_1 \oplus E/E_2 = \mathrm{Gr}(E)$. Altogether we have seen that, under the given conditions on the slope of I, the limiting bundle (E_0, Φ_0) is as stated in Case (2.1) of the theorem.

Case D: $\mu < \mu(N) \leqslant \mu_1$**.**
Define $R = E_2/N$ so that we have a short exact sequence $0 \to N \to E_2 \to R \to 0$. Let $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{E}_3$ be the C^∞ bundles underlying N, R and E/E_2, respectively, so that we have a decomposition of C^∞-bundles

$$\mathcal{E} = \mathcal{N} \oplus \mathcal{Q} \oplus \mathcal{E}_3. \tag{4.3}$$

Recalling that $\mathcal{N}$ comes from $\ker(\varphi_{21})$, we may write the Higgs field Φ as:

$$\Phi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}$$

with respect to the decomposition (4.3). Moreover, the holomorphic structure on E is of the form

$$\bar\partial_E = \begin{pmatrix} \bar\partial_1 & \beta_{12} & \beta_{13} \\ 0 & \bar\partial_2 & \beta_{23} \\ 0 & 0 & \bar\partial_3 \end{pmatrix}.$$

Now take the constant gauge transformation $g(z) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{smallmatrix}\right)$ of $\mathcal{E}$ with respect to the decomposition (4.3). The same calculation as in Case B shows that in the configuration space $\lim_{z\to 0} z \cdot (E, \Phi)$ is gauge equivalent to

$$(E_0, \Phi_0) = \left(N \oplus E_2/N \oplus E/E_2, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & 0 \\ 0 & \varphi_{32} & 0 \end{pmatrix}\right).$$

It remains to prove that (E_0, Φ_0) is a stable Higgs bundle. The Φ_0-invariant subbundles of E_0 are the following:

1. *The subbundle* $E/E_2 \subset E_0$. From the properties of the Harder–Narasimhan filtration we have $\mu(E/E_2) < \mu(E) = \mu(E_0)$.

2. *The subbundle* $E_2/N \oplus E/E_2 \subset E_0$. The hypothesis $\mu(N) > \mu$ is equivalent to $\mu(E_2/N \oplus E/E_2) < \mu = \mu(E) = \mu(E_0)$.

Next we analyze the Harder–Narasimhan type. For brevity we continue to write $R = E_2/N$. There are two situations to consider.

The first situation is when $\mu(N) \leqslant \mu(R)$. Then the Harder–Narasimhan type of E_0 is $\mathrm{HNT}(E_0) = (\mu(R), \mu(N), \mu_3)$. Hence, once again using Shatz's theorem, we conclude that $\mu(N) \leqslant \mu_2$. This leads to the description given in Case (2.2).

The second situation is when $\mu(N) > \mu(R)$. Then the Harder–Narasimhan type of E_0 is $\mathrm{HNT}(E_0) = (\mu(N), \mu(R), \mu_3)$. Hence, from Shatz's theorem we deduce that $\mu(N) \geqslant \mu_1$. But $N \subset E_2$ so, from the properties of the Harder–Narasimhan filtration, we conclude that in fact $\mu(N) = \mu_1$. If $\mu_2 = \mu_1$ it follows that $\mu(N) = \mu(R)$, contradicting $\mu(N) > \mu(R)$. Hence $\mu_2 < \mu_1$ and so $N \subset E_2$ is the unique maximal destabilizing subbundle, i.e., $N = E_1$ and Case (2.3) occurs.

This completes the proof of Case (2) and thus the proof of Theorem 4.2.

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