Dynamics and periodicity in a family of cluster maps

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Abstract

The dynamics of a 1-parameter family of cluster maps φ_r associated to mutation-periodic quivers in dimension 4, is studied in detail. The use of presymplectic reduction leads to a globally periodic symplectic map, and this enables us to reduce the problem to the study of maps belonging to a group of symplectic birational maps of the plane which is isomorphic to $SL(2,\mathbb{Z}) \ltimes \mathbb{R}^2$. We conclude that there are three different types of dynamical behaviour for φ_r characterized by the integer parameter values $r = 1$, $r = 2$ and $r > 2$. For each type, the periodic points, the structure and the asymptotic behaviour of the orbits are completely described. A finer description of the dynamics is provided by using first integrals.

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1 Introduction

We study the dynamics in $\mathbb{R}_+^4$ of a family of maps which arises in the context of the theory of cluster algebras associated to 4-node quivers with mutation-period equal to 2. This family depends on a positive integer parameter r and is defined by

$$\varphi_r(x_1, x_2, x_3, x_4) = \left(x_3, x_4, \frac{x_2^r + x_3^r}{x_1}, \frac{x_1^r x_4^r + (x_2^r + x_3^r)^r}{x_1^r x_2} \right). \tag{1}$$

The study performed in this work provides the full description of the dynamics of the family of maps (1) and enables us to conclude that there are

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three different types of dynamical behaviour characterised by $r = 1$, $r = 2$ and $r > 2$, respectively. The dynamics of φ_r in the cases $r = 1$ and $r > 2$ is described in Theorem 3 and Theorem 4 respectively, and for $r = 2$ it can be found in [4, Theorem 3]. In particular, in what concerns the existence of periodic points of the map φ_r: (a) φ_1 is globally 12-periodic, with a unique fixed point and 2-dimensional algebraic subvarieties of points with minimal period 4 and 6; (b) φ_2 has no periodic points; (c) if $r > 2$, φ_r has a unique fixed point and a 2-dimensional algebraic subvariety of periodic points of minimal period 4. Moreover, using first integrals, we also give a very detailed description of the φ_r-orbits.

The maps φ_r, whose iterates define the following system of difference equations:

$$\begin{cases} x_{2n+3}x_{2n-1} & = x_{2n}^r + x_{2n+1}^r \\ \\ x_{2n+4}\, x_{2n} & = x_{2n+3}^r + x_{2n+2}^r \end{cases} \qquad n = 1, 2, \dots,$$

are associated to the 4-node quivers Q_r in Figure 1, which have the property of being mutation-periodic quivers with period equal to 2. The notion of *mutation-periodic quiver* was introduced by Fordy and Marsh in [11] in the context of Fomin and Zelevinsky's theory of cluster algebras [7] and to such a quiver one associates a map whose iterates define a discrete dynamical system. A map associated to a mutation-periodic quiver is called a *cluster map*.

An important feature of any mutation-periodic quiver Q with N nodes represented by a skew-symmetric matrix B is the possibility of reducing the associated cluster map $\varphi : \mathbb{R}_+^N \to \mathbb{R}_+^N$ to a symplectic map $\widehat{\varphi} : \mathbb{R}_+^{2k} \to \mathbb{R}_+^{2k}$, where $2k$ is the rank of B. In fact, as proved by the first and last authors in [3], there exists a semiconjugacy $\Pi : \mathbb{R}_+^N \to \mathbb{R}_+^{2k}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{R}_+^N & \xrightarrow{\Pi} & \mathbb{R}_+^{2k} \\ \varphi \downarrow & & \downarrow \widehat{\varphi} \\ \mathbb{R}_+^N & \xrightarrow[\Pi]{} & \mathbb{R}_+^{2k} \end{array}$$

Also, Darboux type coordinates can be chosen so that the reduced map $\widehat{\varphi}$ preserves the symplectic form

$$\omega = \sum_{1 \le i \le k} \frac{dy_{2i-1} \wedge dy_{2i}}{y_{2i-1}y_{2i}}.$$

As the rank of the matrices B_r associated to the family of maps φ_r is equal to 2 (for any r), the reduced maps $\widehat{\varphi}_r$ are maps in $\mathbb{R}_+^2$. As will be proved in Proposition 1 the symplectic reduced maps $\widehat{\varphi}_r$ are all conjugate to a (parameter independent) globally 4-periodic map ψ. This property of global periodicity turns out to be the key feature to the successful description of the dynamics of the maps φ_r, since it allows us to study these maps by restricting them and their fourth iterates to 2-dimensional varieties of $\mathbb{R}_+^4$. Moreover, these restricted maps belong to a group of symplectic birational maps whose dynamical behaviour we are able to describe completely. The dynamics of the original family φ_r is then obtained from the dynamics of these restricted maps.

Since 2000, many connections and applications of the theory of cluster algebras to diverse areas of Mathematics and Physics have been unveiled. Namely, applications to integrable systems, Poisson geometry, algebraic geometry, quiver representations, Teichmüller theory and tropical geometry (see for instance [28], [13], [14], [22], and [19]). In particular, examples of mutation-periodic quivers appear in supersymmetric quiver gauge theories associated to complex cones over several surfaces such as the Hirzebruch 0 and the del Pezzo 0-3 surfaces (see [11, §11]). For instance, the quiver Q_2 (in Figure 1) is the one associated to the Hirzebruch 0 surface [6]. Moreover, the rule defining the mutation at a node of a quiver, as defined in 2000 by Fomin and Zelevinsky [7], coincides with the rule in supersymmetric quiver gauge theories for Seiberg-dualising a quiver at a given node (see [21, §3]).

Let us refer that in the context of cluster algebras, besides the notion of mutation-periodicity there are other periodicity notions (see [23] and [20, §7]). For instance the so-called categorical periodicity which has been used to reformulate and prove the Zamolodchikov's periodicity conjecture for Y-systems [27]. Although this conjecture arose from studies of the thermodynamic Bethe ansatz in mathematical physics, it was proved in the cluster algebras context by Fomin-Zelevinsky [8] for Dynkin diagrams and by Keller [16] for pairs of Dynkin diagrams.

There are not many works in the literature addressing the dynamics of cluster maps. In [9] and [10] we can find the study of integrability and algebraic entropy of cluster maps arising from 1-periodic quivers. Based on results in [3], which show how to constructively reduce cluster maps associated to mutation-periodic quivers of arbitrary period, we were able to study the dynamics of cluster maps arising from the quivers associated to the Hirzebruch 0 and del Pezzo 3 surfaces (see [4]). That work and the study presented here are, to the best of our knowledge, the only ones approaching the dynamics of cluster maps associated to higher periodic quivers.

The structure of the paper is as follows. In Section 2 we compute the reduced symplectic maps $\widehat{\varphi}_r$ and show that they are all conjugate to a globally 4-periodic map ψ. As a consequence, the dynamics of the maps φ_r reduces to the study of the restriction of φ_r and of its fourth iterate, $\varphi_r^{(4)}$, to certain 2-dimensional varieties (Proposition 2). All these restricted maps are then shown to belong to a specific group Γ of birational maps.

Section 3 is devoted to the study of the maps belonging to the group Γ which, we show, is isomorphic to the semidirect product $SL(2,\mathbb{Z}) \ltimes \mathbb{R}^2$. Firstly we find normal forms for maps of Γ up to conjugacy in $G \simeq GL(2,\mathbb{Z}) \ltimes \mathbb{R}^2$ (Theorem 1) and then describe the dynamics of those normal forms which are relevant to our study.

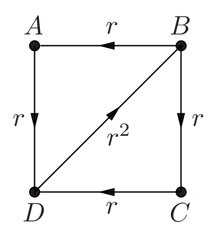
In Section 4, the dynamics of the family of maps φ_r is described. This is done by obtaining first the dynamics of the restricted maps from the results of the previous section.

2 Reduction and restriction

Each map φ_r of the family under study, that is

$$\varphi_r(x_1, x_2, x_3, x_4) = \left(x_3, x_4, \frac{x_2^r + x_3^r}{x_1}, \frac{x_1^r x_4^r + (x_2^r + x_3^r)^r}{x_1^r x_2} \right),$$

is a cluster map associated to a mutation-periodic quiver of period 2 and 4 nodes, in the sense introduced by Fordy and Marsh in [11]. This quiver, denoted by Q_r, and its associated skew-symmetric matrix $B_r = [b_{ij}]$ are both displayed in Figure 1: the labels on the arrows of the oriented graph Q_r denote the number of arrows between the corresponding nodes, and each entry b_{ij} of B_r is the number of arrows from node i to node j minus the number of arrows from j to i where the nodes are numbered as $(A, B, C, D) = (1, 2, 4, 3)$. For the notion of mutation-periodicity and the detailed construction of the respective cluster map we refer to [11], [9], [3] and [20].



$$B_r = \begin{bmatrix} 0 & -r & r & 0 \\ r & 0 & -r^2 & r \\ -r & r^2 & 0 & -r \\ 0 & -r & r & 0 \end{bmatrix}$$

Figure 1: The quiver Q_r and the associated skew-symmetric matrix B_r.

In this section we first show that each map φ_r is semiconjugate to a (parameter independent) map ψ which is globally 4-periodic. Then we explain how this reduction allows us to study the dynamics of φ_r by studying the restrictions of φ_r and of its fourth iterate, $\varphi_r^{(4)}$, to certain 2-dimensional subvarieties of $\mathbb{R}_+^4$ (cf. Proposition 2). Explicit expressions of these restrictions are given in Proposition 3, and it is shown that all of them belong to a group Γ of symplectic birational maps of the plane.

2.1 Reduction to a globally periodic map

The reduction of φ_r to a globally periodic map is achieved in two steps. First we use the reduction procedure in [3] to semiconjugate $\varphi_r : \mathbb{R}_+^4 \to \mathbb{R}_+^4$ to a symplectic map $\widehat{\varphi}_r : \mathbb{R}_+^2 \to \mathbb{R}_+^2$. Second, we show that each symplectic reduced map $\widehat{\varphi}_r$ is (topologically) conjugate to a parameter-independent map ψ which is globally 4-periodic. These steps can be schematically summarized as follows:

$$\begin{tikzcd} \mathbb{R}_+^4 \arrow[r, "\Pi_r"] \arrow[d, "\varphi_r"'] & \mathbb{R}_+^2 \arrow[r, "h_r"] \arrow[d, "\widehat{\varphi}_r"] & \mathbb{R}_+^2 \arrow[d, "\psi"] \\ \mathbb{R}_+^4 \arrow[r, "\Pi_r"'] & \mathbb{R}_+^2 \arrow[r, "h_r"'] & \mathbb{R}_+^2 \end{tikzcd}$$

For the first step we use the fact that the mutation-periodicity of the quiver Q_r represented by the skew-symmetric matrix $B_r = [b_{ij}]$ is equivalent (see [3, Theorem 3.1]) to the statement that φ_r preserves the *standard presymplectic form*

$$\omega_r = \sum_{1 \leq i < j \leq 4} \frac{b_{ij}}{x_i x_j} dx_i \wedge dx_j. \tag{2}$$

As the rank of B_r is equal to 2, the reduced symplectic map $\widehat{\varphi}_r$ is obtained by choosing coordinates such that the presymplectic form ω_r reduces to the *canonical symplectic* form

$$\omega = \frac{1}{xy} dx \wedge dy.$$

Following this procedure for the maps φ_r in (1), we obtain the following reduced maps in $\mathbb{R}_+^2$

$$\widehat{\varphi}_r(x,y) = \left(\frac{y\left(x^r + (1+y^r)^r\right)}{x^r}, \frac{1+y^r}{x} \right), \tag{3}$$

and the semiconjugacies $\Pi_r : \mathbb{R}_+^4 \to \mathbb{R}_+^2$

$$\Pi_r(x_1,x_2,x_3,x_4) = \left(\frac{x_1 x_4}{x_2^r}, \frac{x_3}{x_2} \right). \tag{4}$$

More details on these computations can be found in [3, §4].

Proposition 1. *Each map $\varphi_r : \mathbb{R}_+^4 \to \mathbb{R}_+^4$ in (1) is semiconjugate to the map $\psi : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ given by*

$$\psi(x,y) = \left(y, \frac{1}{x} \right) \tag{5}$$

by the semiconjugacy $\pi_r : \mathbb{R}_+^4 \to \mathbb{R}_+^2$ defined as

$$\pi_r(x_1,x_2,x_3,x_4) = \left(\frac{x_3}{x_2}, \frac{x_2^r + x_3^r}{x_1 x_4} \right). \tag{6}$$

That is, $\pi_r \circ \varphi_r = \psi \circ \pi_r$.

Proof. Each map φ_r is semiconjugate to the symplectic reduced map $\widehat{\varphi}_r$ in (3) by the semiconjugacy Π_r given by (4). That is, the map Π_r is surjective and satisfies $\Pi_r \circ \varphi_r = \widehat{\varphi}_r \circ \Pi_r$. Considering the conjugacy (i.e. the homeomorphism) $h_r : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ defined by

$$h_r(x, y) = \left(y, \frac{1 + y^r}{x} \right),$$

it is easy to check that $h_r \circ \widehat{\varphi}_r = \psi \circ h_r$, with ψ given by (5). Taking $\pi_r = h_r \circ \Pi_r$ we conclude that $\pi_r \circ \varphi_r = \psi \circ \pi_r$ as claimed. ☐

We note that the map ψ in (5) is globally 4-periodic, that is $\psi^{(4)} = Id$ where $\psi^{(m)} = \psi \circ \cdots \circ \psi$ (m compositions).

2.2 Restriction to invariant sets

We now describe the consequences of Proposition 1 to the orbits of the map φ_r.

Recall that, given a map $f : U \subset \mathbb{R}^m \to U$, a *periodic point* $\mathbf{x}$ *of* f is a point satisfying $f^{(m)}(\mathbf{x}) = \mathbf{x}$ and $f^{(k)}(\mathbf{x}) \neq \mathbf{x}$ for all $k < m$. The number m is called the *minimal period of* $\mathbf{x}$. The *(forward)* f*-orbit* of $\mathbf{x} \in U$ is the set

$$\mathcal{O}_f(\mathbf{x}) = \{ f^{(n)}(\mathbf{x}) : n \in \mathbb{N}_0 \},$$

where $f^{(0)}$ denotes the identity map.

The globally 4-periodic map ψ in (5) has only one point with minimal period less than 4 which is the fixed point $(1, 1)$. Moreover, the semiconjugacy between φ_r and ψ allows us to confine the orbits of φ_r to certain 2-dimensional subsets of $\mathbb{R}_+^4$. This is a consequence of Theorem 2 in [4] which is restated in the next proposition for the particular case of the maps φ_r.

Proposition 2. *Let* $\varphi_r : \mathbb{R}_+^4 \to \mathbb{R}_+^4$ *and* $\psi : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ *be the maps in Proposition 1. For* $(p, q) \in \mathbb{R}_+^2$ *let*

$$C^r_{(p,q)} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}_+^4 : \, x_3 = p x_2, \; q x_1 x_4 = (1 + p^r) x_2^r \right\}. \tag{7}$$

Then

1. $C^r_{(p,q)}$ *is invariant under* $\varphi_r^{(4)}$;
2. $C^r_{(1,1)}$ *is invariant under* φ_r.

Moreover, if $(p, q) \neq (1, 1)$ *the sets* $C^r_{\psi^{(i)}(p,q)}$*, with* $i = 0, 1, 2, 3$*, are pairwise disjoint and the* φ_r*-orbit of* $\mathbf{x} \in C^r_{(p,q)}$ *circulates cyclically between them as follows*

$$C^r_{(p,q)} \longrightarrow C^r_{(q,1/p)} \longrightarrow C^r_{(1/p,1/q)} \longrightarrow C^r_{(1/q,p)} \longrightarrow C^r_{(p,q)}$$

Proof. Note that

$$C^r_{(p,q)} = \{ \mathbf{x} \in \mathbb{R}_+^4 : \pi_r(\mathbf{x}) = (p, q) \}$$

where π_r is the map in (6). As $\pi_r \circ \varphi_r = \psi \circ \pi_r$ (cf. Proposition 1) and ψ is globally 4-periodic we have

$$\pi_r \circ \varphi_r^{(4)}(\mathbf{x}) = \psi^{(4)} \circ \pi_r(\mathbf{x}) = (p, q),$$

which means that $C^r_{(p,q)}$ is invariant under $\varphi_r^{(4)}$.

The invariance of $C^r_{(1,1)}$ under φ_r follows from the identity $\pi_r \circ \varphi_r(\mathbf{x}) = \psi \circ \pi_r(\mathbf{x})$ and from the fact that $(1,1)$ is a fixed point of ψ.

Finally the sets $C^r_{\psi^{(i)}(p,q)}$, with $i \in \{0,1,2,3\}$, are clearly disjoint and for $\mathbf{x} \in C^r_{(p,q)}$ one has

$$\pi_r \circ \varphi_r(\mathbf{x}) = \psi(p, q),$$

which means that $\varphi_r(\mathbf{x})$ belongs to $C^r_{\psi(p,q)}$. $\square$

Proposition 2 implies that each orbit of φ_r is either entirely contained in the 2-dimensional algebraic variety $C^r_{(1,1)}$ or circulates between four pairwise disjoint algebraic varieties of dimension 2. So the dynamics of φ_r can be studied through the restriction of φ_r to $C^r_{(1,1)}$ and through the restrictions of $\varphi_r^{(4)}$ to sets $C^r_{(p,q)}$ with $(p,q) \neq (1,1)$. In the next proposition we obtain explicit expressions of these restricted maps.

Proposition 3. *Let φ_r denote the map* (1) *and $C^r_{(p,q)}$ denote the algebraic variety of dimension 2 defined by* (7)*. Then,*

1. *$C^r_{(1,1)}$ is invariant under φ_r and the restriction $\bar{\varphi}_r = \varphi_r|_{C^r_{(1,1)}}$ is given in the coordinates (x_1, x_2) by*

$$\bar{\varphi}_r(x_1, x_2) = \left(x_2, 2\frac{x_2^r}{x_1}\right). \tag{8}$$

2. *$C^r_{(p,q)}$ is invariant under $\varphi_r^{(4)}$ and the restriction $\widetilde{\varphi}_r = \varphi_r^{(4)}|_{C^r_{(p,q)}}$ is given in the coordinates (x_1, x_4) by*

$$\widetilde{\varphi}_r(x_1, x_4) = \left(\lambda\frac{x_4^{r^2-2}}{x_1}, \lambda^{r^2-1}\frac{x_4^{(r^2-3)(r^2-1)}}{x_1^{r^2-2}}\right), \tag{9}$$

with

$$\lambda = \frac{(1+p^r)^2(1+q^r)^r}{q^2p^r}. \tag{10}$$

Proof. The restricted map $\bar{\varphi}_r$ is obtained by a straightforward computation. To obtain $\widetilde{\varphi}_r$, note that the computation of $\varphi_r^{(4)}(x_1, x_2, x_3, x_4) = (u_1, u_2, u_3, u_4)$

gives

$$u_1 = l(\mathbf{x})\frac{x_4^{r^2-2}}{x_1}, \qquad u_2 = l^r(\mathbf{x})\frac{x_2 x_4^{r^3-3r}}{x_1^r},$$

$$u_3 = l^r(\mathbf{x})\frac{x_3 x_4^{r^3-3r}}{x_1^r}, \qquad u_4 = l^{r^2-1}(\mathbf{x})\frac{x_4^{(r^2-3)(r^2-1)}}{x_1^{r^2-2}},$$

where

$$l(\mathbf{x}) = \frac{(x_1^r x_4^r + (x_2^r + x_3^r)^r)^r}{x_1^{r^2-2} x_2^r x_3^r x_4^{r^2-2}}. \tag{11}$$

It is easy to see that the function l is constant on each $C^r_{(p,q)}$ and given by

$$\lambda = l(\mathbf{x})\big|_{C^r_{(p,q)}} = \frac{(1+p^r)^2(1+q^r)^r}{q^2 p^r}.$$

This leads directly to the expression of $\widetilde{\varphi}_r$ in the coordinates (x_1, x_4). □

Remark 1. In the above proposition, the different choice of coordinates for $C^r_{(1,1)}$ and for $C^r_{(p,q)}$ has no particular meaning other than leading in each case to simpler expressions of the restricted maps.

It is easy to verify that the restricted maps (8) and (9) belong to the group of maps from $\mathbb{R}^2_+$ to itself of the form

$$f(x,y) = \left(\alpha x^a y^b, \beta x^c y^d\right),$$

with α and β real positive constants and a, b, c, d integers satisfying $ad - bc = 1$. This group will be denoted by Γ. The maps of Γ are birational (rational with rational inverse) and preserve the symplectic form

$$\omega = \frac{1}{xy} dx \wedge dy, \tag{12}$$

in the sense that the pullback of ω by f preserves ω, that is $f^*\omega = \omega$.

3 The group Γ

Using algebraic geometry techniques, it was proved in [2] that the group of birational transformations of $\mathbb{C}^2$ preserving the symplectic form (12) is generated by $SL(2,\mathbb{Z})$, the complex torus $(\mathbb{C}^*)^2$ and the globally 5-periodic (Lyness) map $(x,y) \mapsto (y, \frac{1+y}{x})$.

In this section we study the group of birational maps Γ and show that it is isomorphic to the semidirect product $SL(2,\mathbb{Z}) \ltimes \mathbb{R}^2$. We obtain simplified forms (*normal forms*) for the maps of Γ under conjugation and describe the dynamics of those normal forms which are relevant to our study.

Let Γ be the group of maps $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ defined by

$$f(x,y) = \left(\alpha x^a y^b, \beta x^c y^d\right), \quad \alpha, \beta \in \mathbb{R}_+, \ \ ad - bc = 1, \ a, b, c, d \in \mathbb{Z}. \tag{13}$$

The map $i : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2$ given by

$$i(x,y) = (\log x, \log y) \tag{14}$$

conjugates $f \in \Gamma$ to the affine map in $\mathbb{R}^2$

$$g(u,v) = (au + bv + \log\alpha, cu + dv + \log\beta).$$

Note that g is the composition of the translation by the vector $\mathbf{v} = (\log\alpha, \log\beta)$ and the area preserving linear map represented by the $SL(2,\mathbb{Z})$ matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Identifying g with $(M, \mathbf{v})$, the map i induces an isomorphism between Γ and the semidirect product

$$SL(2,\mathbb{Z}) \ltimes \mathbb{R}^2 = \{(M, \mathbf{v}) : M \in SL(2,\mathbb{Z}), \mathbf{v} \in \mathbb{R}^2\}$$

with group multiplication defined by $(M, \mathbf{v}) \cdot (N, \mathbf{w}) = (MN, \mathbf{v} + M\mathbf{w})$.

3.1 Normal forms

In the next theorem we obtain simplified forms for elements of Γ up to conjugation. We show that apart from the following maps

$$f^{\pm}_{(\alpha,\beta)}(x,y) = (\alpha x^{\pm 1}, \beta y^{\pm 1})$$

there are only two types of normal forms characterising the elements of Γ.

Note that the maps $f^{\pm}_{(\alpha,\beta)}$ (corresponding in (13) to $b = c = 0$) do not appear in Proposition 3 as restricted maps and furthermore their dynamical behaviour is trivial. In fact, the map $f^{+}_{(\alpha,\beta)}$ is conjugate to a translation and $f^{-}_{(\alpha,\beta)}$ is globally 2-periodic.

Theorem 1. *Let*

$$f(x,y) = \left(\alpha x^a y^b, \beta x^c y^d\right), \quad \alpha, \beta > 0, \ \ ad - bc = 1,$$

be an element of Γ with $b^2 + c^2 \neq 0$. Then f is conjugate to one of the following maps

1. $f_k(x,y) = (y, \frac{y^k}{x})$ *with* $k = a + d$ *if* $a + d \neq 2$*;*
2. $f_{2,\xi}(x,y) = (y, \xi\frac{y^2}{x})$ *if* $a + d = 2$*, where*

$$\xi = \begin{cases} \dfrac{\alpha^c}{\beta^{a-1}}, & \text{if } c \neq 0 \\ \beta^b, & \text{if } c = 0. \end{cases} \tag{15}$$

Proof. If $c \neq 0$, considering the homeomorphism π of $\mathbb{R}^2_+$ given by

$$\pi(x,y) = (y^a x^{-c}, \beta^a \alpha^{-c} y),$$

it is easy to check that $\pi \circ f = g \circ \pi$, where g is the map

$$g(x,y) = (y, K\frac{y^{a+d}}{x}) \quad \text{with} \quad K = \beta(\beta^a \alpha^{-c})^{1-(a+d)}.$$

If $a+d=2$ the map g is the map $f_{2,\xi}$ with $\xi = \frac{\alpha^c}{\beta^{a-1}}$. If $a+d \neq 2$, taking the following map Π

$$\Pi(x,y) = K^{\frac{1}{a+d-2}}(x,y),$$

we have $\Pi \circ g = f_{a+d} \circ \Pi$, that is $\Pi \circ \pi \circ f = f_{a+d} \circ \Pi \circ \pi$.

If $c=0$, the hypothesis $b^2+c^2 \neq 0$ implies that $b \neq 0$. Considering the involution $\sigma(x,y) = (y,x)$, which interchanges c and b, the problem reduces to the previous cases. In fact, $\sigma \circ f \circ \sigma = \left(\beta x^d, \alpha x^b y^a\right)$ is conjugate to f_{a+d} if $a+d \neq 2$ and to $f_{2,\xi}$ with $\xi = \frac{\beta^b}{\alpha^{d-1}} = \beta^b$ if $a+d=2$. □

Remark 2. It is worth noting that the conjugacies in the proof of the above theorem belong to a group G which is isomorphic to $GL(2,\mathbb{Z}) \ltimes \mathbb{R}^2$. The result in the theorem may be rephrased as follows. Up to conjugation in G, the elements $(M, \mathbf{v}) \in SL(2,\mathbb{Z}) \ltimes \mathbb{R}^2$, with $M \neq \pm I$, are parametrized by the trace of M if $\mathrm{tr}\, M \neq 2$, and by a real parameter ξ which depends on M and $\mathbf{v}$ through the expression (15) if $\mathrm{tr}\, M = 2$.

As we will see the explicit conjugacies given in the proof of Theorem 1 (and in Corollary 2 below) play a key role in the study of the dynamics performed in the following sections. We believe that if one was only interested in the normal forms *per se*, these might be obtained by quoting results from group theory scattered in the literature.

As a consequence of Theorem 1 the restricted maps (8) and (9) are conjugate to the normal forms given in the following corollary.

Corollary 2. *Let r and λ be a positive integer and a positive real number, respectively. Consider the maps*

$$\bar{\varphi}_r(x,y) = (y, 2\frac{y^r}{x}), \qquad \widetilde{\varphi}_r(x,y) = \left(\lambda\frac{y^{r^2-2}}{x}, \lambda^{r^2-1}\frac{y^{(r^2-3)(r^2-1)}}{x^{r^2-2}}\right).$$

1. *If $r=2$, then*

   i) *$\bar{\varphi}_2$ is already in normal form: $\bar{\varphi}_2 = f_{2,2}$;*

   ii) *$\widetilde{\pi}_2 \circ \widetilde{\varphi}_2 = f_{2,\lambda^4} \circ \widetilde{\pi}_2$ with*

$$\widetilde{\pi}_2(x,y) = \left(\frac{x^2}{y}, \frac{y}{\lambda}\right), \qquad f_{2,\lambda^4}(x,y) = \left(y, \lambda^4\frac{y^2}{x}\right). \qquad (16)$$

2. If $r \neq 2$ then,

i) $\bar{\pi}_r \circ \bar{\varphi}_r = f_r \circ \bar{\pi}_r$ *with*

$$\bar{\pi}_r(x, y) = 2^{\frac{1}{r-2}}(x, y), \qquad f_r(x, y) = \left(y, \frac{y^r}{x}\right); \tag{17}$$

ii) $\widetilde{\pi}_r \circ \widetilde{\varphi}_r = f_{(r^2-2)^2-2} \circ \widetilde{\pi}_r$ *with*

$$\widetilde{\pi}_r(x, y) = \lambda^{\frac{1}{r^2-4}} \left(\lambda \frac{x^{r^2-2}}{y}, y\right),$$

$$f_{(r^2-2)^2-2}(x, y) = \left(y, \frac{y^{(r^2-2)^2-2}}{x}\right). \tag{18}$$

Proof. Note that both maps $\bar{\varphi}_r$ and $\widetilde{\varphi}_r$ verify the hypotheses of Theorem 1 with $c \neq 0$, for any r. Furthermore, $a + d = r$ for $\bar{\varphi}_r$ and $a + d = (r^2 - 2)^2 - 2$ for $\widetilde{\varphi}_r$. Also, for both maps $a + d = 2$ if and only if $r = 2$. The result then follows from the proof of Theorem 1. $\square$

3.2 Dynamics of f_k and $f_{2,\xi}$

To understand the dynamics of the restricted maps (8) and (9) it is enough to analyse the dynamics of the maps f_k and $f_{2,\xi}$ given in Theorem 1. The dynamics of these maps can be better described by using first integrals and the graphical representation of their level sets.

A *first integral* of a given map $f : U \subset \mathbb{R}^m \to U$ is a non constant function $I : U \to \mathbb{R}$ which is constant on f-orbits, that is

$$I \circ f(\mathbf{x}) = I(\mathbf{x}), \quad \text{for all } \mathbf{x} \in U.$$

Any f-orbit is therefore confined to a level set of a first integral of f.

Dynamics of $f_k(x, y) = (y, \frac{y^k}{x}),\ k \neq 2$

We now study the dynamics of the maps $f_k(x, y) = (y, \frac{y^k}{x})$, with $k \neq 2$. Although this study can be performed for any integer k, we restrict (in Lemma 1 below) to the integers $k \geq -1$ since smaller values do not appear in the restricted maps of Proposition 3.

The dynamics in $\mathbb{R}^2_+$ of the map $f_k(x, y) = (y, \frac{y^k}{x})$ is easily obtained, via the isomorphism (14), from the properties of the linear map

$$g_k(u, v) = (v, -u + kv), \tag{19}$$

which is represented by a $SL(2, \mathbb{Z})$ matrix. Such linear maps leave invariant a quadratic form Q_k (see for instance [18]). Composing this quadratic form with the isomorphism $i(x,y) = (\log x, \log y)$ gives the following first integral of the map f_k:

$$I_k(x,y) = \log^2(x) - k\log(x)\log(y) + \log^2(y). \tag{20}$$

Consequently, each orbit of f_k lies on a level set of I_k. These level sets are displayed in Figure 2 for $k<2$ and in Figure 3 for $k>2$. Further information on the orbits of f_k can be obtained directly from the well known dynamical properties of the linear map g_k in (19). The next lemma summarizes the dynamics of the map f_k.

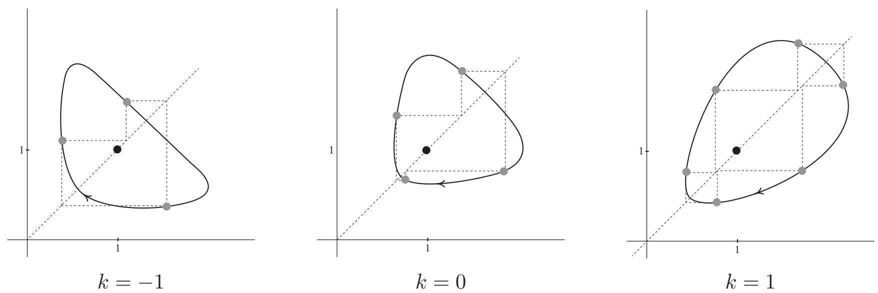


Figure 2: Level sets of I_k for $k \in \{-1,0,1\}$ and graphical construction of the f_k-orbit in the level set. The arrows indicate how the forward orbit moves along the level set.

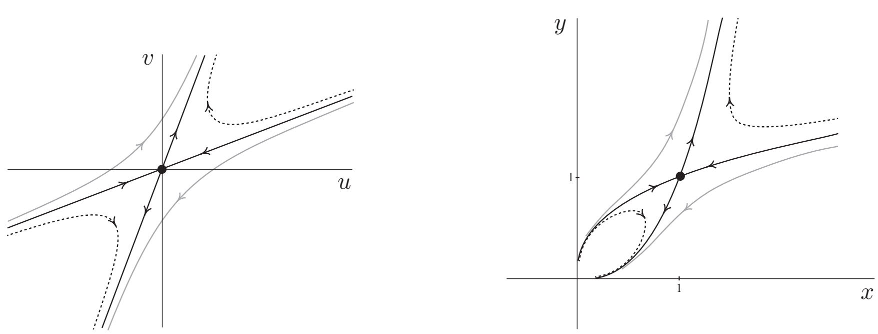


Figure 3: On the left the level sets of the quadratic form Q_k for $k>2$ and on the right the corresponding level sets of I_k. The arrows indicate how the forward orbit moves along the level set.

Lemma 1. *Let $f_k : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be given by $f_k(x,y) = (y, \frac{y^k}{x})$ with $k \in \mathbb{Z}$. Then,*

1. *f_{-1} is globally 3-periodic, $(1,1)$ is its unique fixed point and all the other points have minimal period 3.*

2. *f_0 is globally 4-periodic, $(1,1)$ is its unique fixed point and all the other points have minimal period 4.*

3. *f_1 is globally 6-periodic, $(1,1)$ is its unique fixed point and all the other points have minimal period 6.*

4. *If $k>2$ then $(1,1)$ is the unique fixed point of f_k and there are no more periodic points. Moreover, each f_k-orbit remains in the same connected component of a level curve of the first integral (20), and:*

   (a) *the f_k-orbit of $P \in \mathbb{R}^2_+$ converges to $(1,1)$ if P belongs to the subset of $\mathbb{R}^2_+$ defined by*

   $$\mathcal{F}_k = \{I_k(x,y)=0\} \cap \{(x<1,\, x<y) \, or \, (x>1,\, x>y)\} \tag{21}$$

   (b) *the f_k-orbit of $P \in \mathbb{R}^2_+$ converges to $(0,0)$ if P belongs to the subset of $\mathbb{R}^2_+$ defined by*

   $$\begin{aligned} \mathcal{Z}_k = &\{I_k(x,y)<0,\, x<1\} \cup \{I_k(x,y)=0,\, x<1,\, x>y\} \\ &\cup \{I_k(x,y)>0,\, x>y\}\,. \end{aligned} \tag{22}$$

   (c) *in any other case both components of $f_k^{(n)}(P)$ go to $+\infty$.*

Dynamics of $f_{2,\xi}(x,y) = (y, \xi \frac{y^2}{x})$

The dynamics of the map $f_{2,\xi}(x,y) = (y, \xi \frac{y^2}{x})$ with $\xi \in \mathbb{R}_+$ is obtained from the general expression of the iterates $f_{2,\xi}^{(n)}(x,y)$. This expression can be computed by applying Lemma 1 in [4]. An alternative way is to consider the conjugate affine map

$$g_{2,\xi}(u,v) = (v, -u+2v+\log\xi),$$

which can be identified with the $SL(3,\mathbb{R})$ matrix

$$X = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 2 & \log\xi \\ \hline 0 & 0 & 1 \end{array}\right].$$

By computing the nth power of X we arrive at the expression of $g_{2,\xi}^{(n)}$ from which we derive:

$$f_{2,\xi}^{(n)}(x,y) = \xi^{\frac{n(n-1)}{2}} \frac{y^n}{x^n} \left(x, \xi^n y\right), \quad n \geq 0.$$

The consequences for the dynamics of $f_{2,\xi}$ follow immediately from this expression and are summarised in the next lemma.

Lemma 2. *Let $f_{2,\xi} : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+$ be given by $f_{2,\xi}(x,y) = \left(y, \xi \frac{y^2}{x}\right)$ with $\xi \in \mathbb{R}_+$.*

1. *If $\xi < 1$, then the map $f_{2,\xi}$ has no periodic points and the $f_{2,\xi}$-orbit of any point converges to $(0,0)$.*

2. *If $\xi = 1$, then all the points of the form (x, x) are fixed points of $f_{2,\xi}$. Moreover $f_{2,\xi}^{(n)}(x, y)$ converges to $(0, 0)$ if $y < x$ and both components of $f_{2,\xi}^{(n)}(x, y)$ go to $+\infty$ if $y > x$.*

3. *If $\xi > 1$, then $f_{2,\xi}$ has no periodic points and both components of $f_{2,\xi}^{(n)}(P)$ go to $+\infty$ for any $P \in \mathbb{R}_{+}^{2}$.*

Remark 3. The maps f_k and $f_{2,\xi}$ in Theorem 1 are reversible symplectic birational maps. In fact, the map $R(x, y) = (y, x)$ is an involutory reversing symmetry of both f_k and $f_{2,\xi}$, that is

$$R \circ h \circ R^{-1} = h^{-1}, \quad \text{for } h = f_k \text{ or } h = f_{2,\xi}.$$

with $R^2 = Id$. The consequences of this reversing symmetry to the orbits of f_k and $f_{2,\xi}$ are clear from figures 2 and 3.

Moreover as the restricted maps (8) and (9) are conjugate to one of these normal forms by a homeomorphism π, they also admit the involutory reversing symmetry $\pi^{-1} \circ R \circ \pi$.

The existence of an involutory symmetry for a given map implies that the map can be written as a composition of two involutions [12]. This property plays a key role in the studies [5] and [15] of the QRT maps [24], which are birational maps of the plane admitting a biquadratic first integral.

Reversing symmetries of maps have been widely studied and have strong consequences to the dynamics of the map. We refer to [12] and references therein for a survey on the subject. For works on reversing symmetries related to the context of symplectic (birational) maps, we refer for instance to [1], [17], [26] and [25].

4 Dynamics of the family φ_r

The dynamics of the family φ_r given by (1) is based on the dynamics of the restricted maps $\bar{\varphi}_r$ and $\widetilde{\varphi}_r$ in Proposition 3. In turn, the dynamics of these restricted maps is easily obtained from the results in the previous section.

4.1 Dynamics of the restricted maps

In this subsection we combine the results in Corollary 2 with those in lemmas 1 and 2, to describe the dynamics of the restricted maps $\bar{\varphi}_r$ and $\widetilde{\varphi}_r$ (below, in propositions 4 and 5, respectively).

For future reference, we recall a basic property of conjugate maps. If two maps f and g are conjugate by a homeomorphism π, that is $\pi \circ g = f \circ \pi$, then the dynamics of g can be obtained from the dynamics of f through the identity

$$g^{(n)} = \pi^{-1} \circ f^{(n)} \circ \pi, \quad n = 1, 2, \ldots \tag{23}$$

Proposition 4. *Let r be a positive integer, $\bar{\varphi}_r : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ the map*

$$\bar{\varphi}_r(x, y) = \left(y, 2\frac{y^r}{x} \right),$$

$\bar{\pi}_r$ the homeomorphism given by $\bar{\pi}_r(x, y) = 2^{\frac{1}{r-2}}\,(x, y)$ with $r \neq 2$, and $\mathcal{F}_r, \mathcal{Z}_r \subset \mathbb{R}^2_+$ the sets defined by (21) and (22). Then,

i) $\bar{\varphi}_1$ is globally 6-periodic, the point $(2, 2)$ is a fixed point of $\bar{\varphi}_1$ and any other point $P \in \mathbb{R}^2_+$ is periodic with minimal period 6.

ii) $\bar{\varphi}_2$ has no periodic points and for any $P \in \mathbb{R}^2_+$ each component of $\bar{\varphi}_2^{(n)}(P)$ goes to $+\infty$.

iii) If $r > 2$ the map $\bar{\varphi}_r$ has a unique fixed point $(2^{\frac{1}{2-r}}, 2^{\frac{1}{2-r}})$ and no other periodic points.

If $P \in \bar{\pi}_r^{-1}(\mathcal{F}_r)$ then the $\bar{\varphi}_r$-orbit of P converges to the fixed point.

If $P \in \bar{\pi}_r^{-1}(\mathcal{Z}_r)$ then the $\bar{\varphi}_r$-orbit of P converges to $(0, 0)$.

For any other point $P \in \mathbb{R}^2_+$ each component of $\bar{\varphi}_r^{(n)}(P)$ goes to $+\infty$.

Proof. Statement *ii*) just follows from Lemma 2 with $\xi = 2$.

For the other statements, note that by Corollary 2 the map $\bar{\varphi}_r$ is conjugate to the map $f_r(x, y) = (y, y^r/x)$ by the conjugacy $\bar{\pi}_r$. The conclusions then follow from Lemma 1 and from the identity (23) by taking into account that $\bar{\pi}_r^{-1}(x, y) = 2^{\frac{1}{2-r}}\,(x, y)$. □

Proposition 5. *Let r be a positive integer and $\widetilde{\varphi}_r : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ the map*

$$\widetilde{\varphi}_r(x, y) = \left(\lambda \frac{y^{r^2-2}}{x}, \lambda^{r^2-1} \frac{y^{(r^2-3)(r^2-1)}}{x^{r^2-2}} \right),$$

where

$$\lambda = \frac{(1 + p^r)^2 (1 + q^r)^r}{q^2 p^r}, \quad (p, q) \in \mathbb{R}^2_+ \backslash \{(1, 1)\}.$$

Consider the homeomorphism $\widetilde{\pi}_r(x, y) = \lambda^{\frac{1}{r^2-4}} \left(\lambda \frac{x^{r^2-2}}{y}, y \right)$ with $r \neq 2$, and $\mathcal{F}_r, \mathcal{Z}_r \subset \mathbb{R}^2_+$ the sets defined by (21) and (22). Then,

i) $\widetilde{\varphi}_1$ is globally 3-periodic, the point $(\lambda^{1/3}, \lambda^{1/3})$ is its unique fixed point and any other point has minimal period 3.

ii) $\widetilde{\varphi}_2$ has no periodic points and both components of $\widetilde{\varphi}_2^{(n)}(P)$ go to $+\infty$ for any $P \in \mathbb{R}^2_+$.

iii) For $r > 2$, the map $\widetilde{\varphi}_r$ has a unique fixed point $\left(\lambda^{\frac{1}{4-r^2}}, \lambda^{\frac{1}{4-r^2}} \right)$ and no other periodic points.

If $P \in \widetilde{\pi}_r^{-1}(\mathcal{F}_{(r^2-2)^2-2})$ the $\widetilde{\varphi}_r$-orbit of P converges to the fixed point.

If $P \in \widetilde{\pi}_r^{-1}(\mathcal{Z}_{(r^2-2)^2-2})$ then the $\widetilde{\varphi}_r$-orbit of P converges to $(0,0)$.

For any other point P, both components of $\widetilde{\varphi}_r^{(n)}(P)$ go to $+\infty$.

Proof. By Corollary 2-*1.ii)*, the map $\widetilde{\varphi}_2$ is conjugate to the map f_{2,λ^4} by the conjugacy $\widetilde{\pi}_2$ in (16). Note that in this case $\lambda = \left(p+\frac{1}{p}\right)^2\left(q+\frac{1}{q}\right)^2 > 1$. Using (23) and noting that $\widetilde{\pi}_2^{-1}(x,y) = (\sqrt{\lambda xy}, \lambda y)$, statement $ii)$ follows from Lemma 2-*3* with $\xi = \lambda^4$.

If $r \neq 2$ then by Corollary 2-*2.ii)* the map $\widetilde{\varphi}_r$ is conjugate to the map $f_{(r^2-2)^2-2}$ in (18) by the conjugacy $\widetilde{\pi}_r$.

For statement $i)$, $f_{(r^2-2)^2-2} = f_{-1}$ and so the conclusion follows from Lemma 1-*1*.

Finally, if $r > 2$ then $(r^2-2)^2 - 2 > 2$ and so the last statement follows from Lemma 1-*4* using again the identity (23) and the fact that $\widetilde{\pi}_r^{-1}(x,y) = \lambda^{\frac{1}{4-r^2}}\left((xy)^{\frac{1}{r^2-2}}, y\right)$. □

4.2 Dynamics of φ_r

We now address the problem of describing the dynamics of the maps of the family (1). Recall that the maps of this family are given by

$$\varphi_r(x_1,x_2,x_3,x_4) = \left(x_3, x_4, \frac{x_2^r + x_3^r}{x_1}, \frac{x_1^r x_4^r + (x_2^r + x_3^r)^r}{x_1^r x_2}\right),$$

where the parameter r is a positive integer.

By Proposition 2, each orbit of φ_r is either entirely contained in the 2-dimensional algebraic variety $C^r_{(1,1)}$ or circulates between four pairwise disjoint 2-dimensional algebraic varieties:

$$C^r_{(p,q)}, \quad C^r_{(q,p^{-1})}, \quad C^r_{(p^{-1},q^{-1})}, \quad C^r_{(q^{-1},p)}$$

which are all invariant under $\varphi_r^{(4)}$. The dynamics of the restrictions of φ_r to $C^r_{(1,1)}$ and of $\varphi_r^{(4)}$ to C^r_P with $P \neq (1,1)$ were obtained in Proposition 4 and Proposition 5 respectively. From the results in those propositions we are able to describe the dynamics of the maps φ_r and conclude that there are three different types of dynamical behaviour depending on whether $r=1$, $r=2$ or $r>2$.

The dynamics of φ_r in the cases $r=1$ and $r>2$ is described respectively in Theorem 3 and Theorem 4 below. The case $r=2$ will not be considered in what follows, since it corresponds to one of the maps whose dynamics was described in detail in [4, Theorem 3]. For the sake of completeness we sum up the results from that reference: φ_2 has no periodic points and for any $\mathbf{x} \in \mathbb{R}^4_+$ each component of $\varphi_2^{(n)}(\mathbf{x})$ goes to $+\infty$. The explicit expression of $\varphi_2^{(n)}$ was also obtained in the referred work.

Theorem 3. *The map* $\varphi_1 : \mathbb{R}^4_+ \to \mathbb{R}^4_+$ *given by*

$$\varphi_1(x_1, x_2, x_3, x_4) = \left(x_3, x_4, \frac{x_2 + x_3}{x_1}, \frac{x_1x_4 + x_2 + x_3}{x_1x_2} \right)$$

is globally 12-periodic. Moreover,

1. φ_1 *has exactly one fixed point, the point* $F = (2, 2, 2, 2)$*.*
2. *Every point in the (punctured) algebraic variety* $V \setminus \{F\}$ *where*

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+ : x_4 = x_1,\ x_1x_2x_3 = x_1^2 + x_2 + x_3\},$$

   is periodic with minimal period 4.

3. *Every point in the (punctured) algebraic variety* $C^1_{(1,1)} \setminus \{F\}$ *where*

$$C^1_{(1,1)} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+ :\ x_3 = x_2,\ x_1x_4 = 2x_2\},$$

   is periodic with minimal period 6.

Any other point has minimal period 12.

Proof. By Proposition 2, the φ_1-orbit of any point is either entirely contained in the algebraic variety $C^1_{(1,1)}$ or moves between four pairwise disjoint algebraic varieties

$$C^1_{(p,q)}, \quad C^1_{(q,p^{-1})}, \quad C^1_{(p^{-1},q^{-1})}, \quad C^1_{(q^{-1},p)}$$

with $(p, q) \in \mathbb{R}^2_+ \setminus \{(1, 1)\}$.

The restriction $\bar{\varphi}_1$ of φ_1 to $C^1_{(1,1)}$ is globally 6-periodic by Proposition 4, and the restriction $\widetilde{\varphi}_1$ of $\varphi_1^{(4)}$ to any $C^1_{(p,q)}$ is globally 3-periodic by Proposition 5. Hence φ_1 is globally 12-periodic. Moreover, all the points in $C^1_{(1,1)}$ have minimal period 6 except the point $F = (2, 2, 2, 2)$ which is fixed.

Any point not belonging to $C^1_{(1,1)}$ is either a fixed point of $\varphi_1^{(4)}$ or a periodic point of $\varphi_1^{(4)}$ with minimal period 3.

To compute the fixed points of $\varphi_1^{(4)}$, which correspond to periodic points of φ_1 with minimal period 4, we refer again to Proposition 5-*i)*. Each of these points $\mathbf{x}$ belongs to a set $C^1_{(p,q)}$ with $(p, q) \neq (1, 1)$ and its coordinates (x_1, x_4) satisfy $x_1 = x_4 = \lambda^{1/3}$, where λ is given by (10) (with $r = 1$). On the other hand the constant λ is the value of the restriction to $C^1_{(p,q)}$ of the function $l(\mathbf{x})$ given in (11). To obtain the set V it is enough to eliminate λ from these relations, that is from

$$x_1 = x_4 = \lambda^{1/3}, \quad \lambda = \frac{x_1x_4 + x_2 + x_3}{x_1^{-1}x_2x_3x_4^{-1}}.$$

Finally the remaining points are periodic points of $\varphi_1^{(4)}$ with minimal period 3, and therefore they are periodic points of φ_1 with minimal period 12. □

Theorem 4. *For each integer $r > 2$, let $\varphi_r : \mathbb{R}^4_+ \to \mathbb{R}^4_+$ be the map*

$$\varphi_r(x_1, x_2, x_3, x_4) = \left(x_3, x_4, \frac{x_2^r + x_3^r}{x_1}, \frac{x_1^r x_4^r + (x_2^r + x_3^r)^r}{x_1^r x_2} \right)$$

and $\pi_r : \mathbb{R}^4_+ \to \mathbb{R}^2_+$ the map $\pi_r(\mathbf{x}) = \left(\frac{x_3}{x_2}, \frac{x_2^r + x_3^r}{x_1 x_4} \right)$. Then,

1. *φ_r has a unique fixed point, the point $F = (2^{\frac{1}{2-r}}, 2^{\frac{1}{2-r}}, 2^{\frac{1}{2-r}}, 2^{\frac{1}{2-r}})$.*
2. *Any point in the (punctured) algebraic variety $V \setminus \{F\}$ where*

$$V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+ : \, x_4 = x_1, \; x_1^r x_2 x_3 = x_1^{2r} + (x_2^r + x_3^r)^r \right\},$$

   is a periodic point of φ_r with minimal period 4.

3. *Any point $\mathbf{x} \notin V$ is non-periodic and*

   (a) *if $\mathbf{x}$ belongs to the φ_r-invariant set*

$$C^r_{(1,1)} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+ : x_3 = x_2, \; x_1 x_4 = 2x_2^r\},$$

   the φ_r-orbit of $\mathbf{x}$ is entirely contained in the curve:

$$L_{\mathbf{x}} = \{\mathbf{z} \in C^r_{(1,1)} : J(\mathbf{z}) = J(\mathbf{x})\},$$

   where $J(\mathbf{z}) = \log^2(z_1) - r\log(z_1)\log(z_2) + \log^2(z_2) - \log(2)\log(z_1 z_2)$. Moreover, the orbit of $\mathbf{x}$ either converges to the fixed point F or converges to $(0,0,0,0)$ or else every component of $\varphi_r^{(n)}(\mathbf{x})$ goes to $+\infty$.

   (b) *if $\mathbf{x}$ does not belong to $C^r_{(1,1)}$ then its φ_r-orbit circulates between the four pairwise disjoint curves $\mathcal{L}_{\mathbf{x}}$, $\mathcal{L}_{\varphi_r(\mathbf{x})}$, $\mathcal{L}_{\varphi_r^{(2)}(\mathbf{x})}$ and $\mathcal{L}_{\varphi_r^{(3)}(\mathbf{x})}$ where*

$$\mathcal{L}_{\mathbf{y}} = \{\mathbf{z} \in \mathbb{R}^4_+ : \pi_r(\mathbf{z}) = \pi_r(\mathbf{y}), \; J(\mathbf{z}) = J(\mathbf{y})\},$$

   with

$$J(\mathbf{z}) = \log^2(z_1 z_4) - r^2 \log(z_1)\log(z_4) - \log(l(\mathbf{z}))\log(z_1 z_4),$$

   and l the function defined in (11).
   Moreover, the orbit of $\mathbf{x}$ either converges to the orbit of a 4-periodic point, or converges to $(0,0,0,0)$ or else every component of $\varphi_r^{(n)}(\mathbf{x})$ goes to $+\infty$.

Proof. The proof follows the same lines as the proof of Theorem 3 substituting for each $(p,q) \in \mathbb{R}^2_+$ the set $C^1_{(p,q)}$ by

$$C^r_{(p,q)} = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+ : \, x_3 = p x_2, \; q x_1 x_4 = (1 + p^r) x_2^r \right\}.$$

As $C^r_{(1,1)}$ is invariant under φ_r and any $C^r_{(p,q)}$ is invariant under $\varphi_r^{(4)}$ (cf. Proposition 2), the orbit of any point $\mathbf{x} \in \mathbb{R}^4_+$ can be studied by looking at the restriction $\bar{\varphi}_r$ of φ_r to $C^r_{(1,1)}$ and at the restrictions $\widetilde{\varphi}_r$ of $\varphi_r^{(4)}$ to each $C^r_{(p,q)}$. The restriction $\bar{\varphi}_r$ is given in the coordinates (x_1, x_2) by (8) and $\widetilde{\varphi}_r$ is given in the coordinates (x_1, x_4) by (9).

By Proposition 4-*iii)*, $(x_1, x_2) = (2^{\frac{1}{2-r}}, 2^{\frac{1}{2-r}})$ is the unique fixed point of the map $\bar{\varphi}_r$ corresponding to the fixed point $F = (2^{\frac{1}{2-r}}, 2^{\frac{1}{2-r}}, 2^{\frac{1}{2-r}}, 2^{\frac{1}{2-r}})$ of φ_r.

By Proposition 5-*iii)*, for each $(p,q) \neq (1,1)$ the restriction $\widetilde{\varphi}_r$ of $\varphi_r^{(4)}$ to $C^r_{(p,q)}$ also has a unique fixed point: $(x_1, x_4) = (\lambda^{\frac{1}{4-r^2}}, \lambda^{\frac{1}{4-r^2}})$. Each of these fixed points is a periodic point of φ_r with minimal period 4. The full set of these 4-periodic points is a (punctured) 2-dimensional variety $V \backslash \{F\}$. Like in the proof of the previous theorem, the explicit form of V is easily obtained from the fact that the fixed point of $\widetilde{\varphi}_r$ satisfies $x_1 = x_4 = \lambda^{\frac{1}{4-r^2}}$ and from the fact that λ is the value of the restriction to $C^r_{(p,q)}$ of the function $l(\mathbf{x})$ given in (11). This completes the proof of statements *1.* and *2.*

As $\bar{\varphi}_r$ and $\widetilde{\varphi}_r$ do not have more periodic points, any other point $\mathbf{x}$ is a non-periodic point of φ_r.

If $\mathbf{x} \in C^r_{(1,1)}$ the φ_r-orbit of $\mathbf{x}$ remains in $C^r_{(1,1)}$. As $\bar{\varphi}_r$ is conjugate to f_r by $\bar{\pi}_r(x_1, x_2) = 2^{\frac{1}{r-2}}(x_1, x_2)$ (cf. Corollary 2), and f_r admits the first integral I_r given by (20), then $I_r \circ \bar{\pi}_r$ is a first integral of $\bar{\varphi}_r$. The first integral J in statement *3.a)* is, up to constants, $I_r \circ \bar{\pi}_r$. Hence, the orbit of any point $\mathbf{x}$ belonging to $C^r_{(1,1)}$ lies in the curve

$$L_{\mathbf{x}} = \{\mathbf{z} \in C^r_{(1,1)} : J(\mathbf{z}) = J(\mathbf{x})\}.$$

The remaining assertions in statement *3.a)* follow from Proposition 4-*iii)* and from the particular form of the map φ_r. Indeed, as the third and fourth coordinates of $\varphi_r^{(n)}$ are respectively the first and second coordinates of $\varphi_r^{(n-1)}$, it follows that, if the first and second coordinates of $\varphi_r^{(n)}$ tend to zero (or to $+\infty$), the same holds for the third and fourth coordinates.

If $\mathbf{x} \notin C^r_{(1,1)}$, then by Proposition 2 its φ_r-orbit moves cyclically between the following four pairwise disjoint algebraic varieties of dimension 2

$$C^r_{\pi_r(\mathbf{x})}, \quad C^r_{\pi_r(\varphi_r(\mathbf{x}))}, \quad C^r_{\pi_r(\varphi_r^{(2)}(\mathbf{x}))}, \quad C^r_{\pi_r(\varphi_r^{(3)}(\mathbf{x}))}.$$

Let $\widetilde{\varphi}^i_r$ denote the restriction of $\varphi_r^{(4)}$ to $C^r_{\pi_r(\varphi_r^{(i)}(\mathbf{x}))}$, for $i = 0, \ldots, 3$. By Corollary 2-*2.ii)*, each map $\widetilde{\varphi}^i_r$ is conjugate to the map $f_{(r^2-2)^2-2}$ in (18) by the conjugacy $\widetilde{\pi}^i_r$ given by

$$\widetilde{\pi}^i_r(x, y) = \lambda_i^{\frac{1}{r^2-4}} \left(\lambda_i \frac{x^{r^2-2}}{y}, y \right),$$

where λ_i is defined by the expression in (10) with (p,q) replaced by $(p_i, q_i) = \pi_r(\varphi^{(i)}(\mathbf{x}))$.

As the map $f_{(r^2-2)^2-2}$ admits the first integral $I_{(r^2-2)^2-2}$ in (20), then $I_{(r^2-2)^2-2} \circ \widetilde{\pi}_r^i$ is a first integral for $\widetilde{\varphi}_r^i$. Discarding multiplicative and additive constants in the computation of $I_{(r^2-2)^2-2} \circ \widetilde{\pi}_r^i$ and noting again that each parameter λ_i is the restriction to $C^r_{\pi_r(\varphi_r^{(i)}(\mathbf{x}))}$ of the function l in (11) we obtain the expression J in statement *3.b)*.

Finally note that for $i = 0, \ldots, 3$ the curve $\mathcal{L}_{\varphi_r^{(i)}(\mathbf{x})}$ lies on the 2-dimensional algebraic variety $C^r_{\pi_r(\varphi_r^{(i)}(\mathbf{x}))}$, and so the results in Proposition 5-*iii)* can be restated for the restriction of $\varphi_r^{(4)}$ to $C^r_{\pi_r(\varphi_r^{(i)}(\mathbf{x}))}$ as follows: either $\varphi_r^{(4n+i)}(\mathbf{x})$ converges to a 4-periodic point in V or it converges to $(0,0,0,0)$ or its four components go to $+\infty$. Using the expression of the map φ_r, we can then conclude that $\varphi_r(\varphi_r^{(4n+i)}(\mathbf{x}))$ has precisely the same behaviour, that is it converges to a 4-periodic point in V if that was the case with $\varphi_r^{(4n+i)}(\mathbf{x})$, or converges to $(0,0,0,0)$ if that was the the case with $\varphi_r^{(4n+i)}(\mathbf{x})$, or all its components go to $+\infty$ if that was the case with $\varphi_r^{(4n+i)}(\mathbf{x})$. □

5 Conclusions and discussion

The dynamics of cluster maps associated to mutation-periodic quivers is a recent subject of research. In [9] and [10] we can find the study of integrability and algebraic entropy of cluster maps arising from 1-periodic quivers. In [4] we have addressed the study of the dynamics of cluster maps arising from the 2-periodic quivers associated to the Hirzebruch 0 and del Pezzo 3 surfaces.

In the present work we completely described the dynamics of a parameter-dependent family of cluster maps in dimension 4 which arise from mutation-periodic quivers of period 2. This successful description relies on: (a) the possibility of reducing the original maps to symplectic maps of the plane (see [3]); (b) the global periodicity of the symplectic reduced maps, which allows us to study the dynamics of the original family by restricting appropriate maps to 2-dimensional symplectic varieties; (c) the fact that these restricted maps belong to a group Γ of birational (symplectic) maps of the plane isomorphic to $SL(2,\mathbb{Z}) \ltimes \mathbb{R}^2$. Parametrizing Γ by the trace of a $SL(2,\mathbb{Z})$ matrix and a real parameter (cf. Theorem 1) we obtained two classes of maps (normal forms) whose dynamics we described in detail. As a consequence we recovered the dynamics of the restricted maps (cf. propositions 4 and 5) and then that of the original maps (cf. theorems 3 and 4).

The techniques we have used apply to any cluster map defined on an N-dimensional domain for which the reduced symplectic map is a globally periodic map in dimension $2k$. However, the computations will become more involved as N or the global period of the symplectic map increases. Also, when $N - 2k$ is greater than 2, the respective restricted maps will belong to a group of symplectic maps with a more complicated structure than that of Γ.

A cluster map is a birational map which, in general, has a complicated expression due to the nature of the definition of a mutation. The existence of a (semi)conjugacy between a cluster map and a globally periodic map is not an obvious property of the cluster map, and there is no general procedure to prove its existence. In the case of the family treated here the presymplectic reduction enabled us to find an explicit semiconjugacy to a globally periodic map. The study of the integrability either of the symplectic reduced map or of the cluster map may shed light on the existence of such a semiconjugacy.

The search for first integrals of a cluster map is usually linked to the existence of Poisson structures preserved by the cluster map. For instance, such Poisson structures exist for a 3-parameter family of cluster maps associated to 2-periodic quivers of 4 nodes which include the family φ_r as a particular case (see [3, §2, §5] for more details). The study of these Poisson structures and first integrals arising from them, may provide answers to the difficult question of knowing which mutation-periodic quivers give rise to cluster maps that are (semi)conjugate to globally periodic maps and consequently to the description of their dynamics.

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