Multiple reductions, foliations and the dynamics of cluster maps

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July 14, 2016

Abstract

Presymplectic and Poisson reduction of cluster maps are described in terms of the "canonical" foliations of presymplectic and Poisson manifolds. This approach to reduction leads to a geometric description, in terms of foliations, of the dynamics of the original (not reduced) map. The case where multiple reductions exist (presymplectic/Poisson or presymplectic/Poisson/Poisson) is further explored and examples illustrating several features of this approach are presented, including a nontrivial one in dimension seven which is comprehensively treated.

MSC 2010: 53D17, 53D05, 53C12 (Primary); 39A20, 13F60 (Secondary).

1 Introduction

The main aim of this work is to explain how to combine presymplectic reduction and Poisson reduction of a cluster map, and to translate this combined reduction into specific relations between the null and symplectic foliations of presymplectic and Poisson manifolds. This combination of reductions, their associated foliations and the study of the reduced systems naturally arising from them, are shown here to be powerful tools in the geometric description of the dynamics of the cluster map.

Cluster maps are birational maps defined via mutations, which are the main operations in the theory of cluster algebras ([7] and [17]). More precisely, cluster maps are maps associated to quivers (oriented graphs) satisfying a mutation-periodicity property. These maps were introduced by Fordy and Marsh in [11] and their study was the subject of some recent works (see for example [9], [10], [5] and [6]).

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The use of presymplectic and Poisson structures in the theory of cluster algebras was introduced by Gekhtman *et al.* (see the monograph [12] and references therein) and, among several other applications, they were used to study cluster maps from different points of view (see for example [14], [8], [10], [9] and [4]). The class of presymplectic and Poisson structures considered in that theory are meant to deal with mutations and are of a particular type. They are known, in cluster algebra theory, as *log-canonical* structures, since they are constant in logarithmic coordinates. Only this type of presymplectic and Poisson structures will be considered in the present work.

The most effective study of cluster maps makes use of presymplectic and Poisson structures, and is achieved through reduction to spaces of lower dimension. By *reduction* of a given map φ we mean the existence of a map ψ and of a submersion Π onto a lower dimensional space, such that $\Pi \circ \varphi = \psi \circ \Pi$. The map ψ will be called a *reduced map*.

Presymplectic reduction was proved by Fordy and Hone in [10] for cluster maps associated to mutation-periodic quivers of period 1 and by Cruz and Sousa-Dias [4] for mutation-periodic quivers with arbitrary period. It exists as long as the skew-symmetric matrix B defining the quiver is singular and reduces the cluster map to a map in a space with dimension equal to the rank of B.

Poisson reduction was also addressed in [10] for cluster maps associated to mutation-periodic quivers of period 1 and in [4] for quivers of higher period. Contrary to presymplectic reduction, Poisson reduction may fail to exist due to the lack of a nontrivial Poisson structure for which the cluster map is a Poisson map. Moreover, it may happen that there are different log-canonical Poisson structures leading to Poisson reduction of the same cluster map to spaces of different dimensions (and equal to the coranks of the skew-symmetric matrices defining these Poisson structures).

In this work we explore the coexistence of multiple reductions (presymplectic and Poisson) and its consequences to the dynamics of a cluster map. Each reduction process gives rise to two kind of objects: (a) a foliation of the domain $\mathbb{R}_+^N$ of the cluster map, which is a foliation defined by the fibres of a submersion Π; (b) a reduced system (ψ, Π), where ψ is the respective reduced map.

In order to be able to take full advantage of the existence of multiple reductions we require that the foliations satisfy some specific properties, namely that one of the foliations is a subfoliation of the other. The main results of the paper concern these properties and their implications to the global dynamics of the cluster map.

We show that the foliations arising from presymplectic and Poisson reduction are given by submersions with connected fibres and are, respectively, the null and the symplectic foliation of the domain of the cluster map. Necessary and sufficient conditions for the referred foliations to satisfy the aforementioned properties are proved.

The possibility of combining multiple reductions translates into the existence of a flag of foliations given by submersions, and leads to multiple reduced maps. As we will show, there are also specific relationships between the reduced maps arising from multiple reductions. These relationships are the key to draw conclusions about a reduced map from another reduced map, which proves to be particularly helpful when we are studying cluster maps (and reduced maps) with a considerable number of variables. Simultaneously, they also provide the geometrical arrangement of the dynamics of the cluster maps on the leaves of the foliations arising from the multiple reductions.

The structure of the paper is as follows. In Section 2, we recall the definition of mutation-periodic quiver and of the cluster map associated to it, as well as the construction of presymplectic and Poisson reductions of a cluster map.

In Section 3, we consider log-canonical presymplectic and Poisson structures and give a description, in terms of submersions, of the null foliation and of the symplectic foliation induced by these structures. We also introduce the notion of *simple subfoliation* and give necessary and sufficient conditions for a null foliation to be a simple subfoliation of a symplectic foliation (and vice-versa) and for a symplectic foliation to be a simple subfoliation of another symplectic foliation.

In Section 4, we consider cluster maps which admit multiple reductions and explain the implication that the dynamics of the reduced maps have on the dynamics of the original cluster map.

In the last section we illustrate our results with two examples: the Somos-5 cluster map and a cluster map in dimension 7 for which we have multiple reductions of the form presymplectic/Poisson/Poisson.

2 Background on cluster maps

A cluster map is a map associated to a mutation-periodic quiver, as introduced in [11]. A quiver is an oriented graph with N nodes and with (possibly multiple) arrows between the nodes. If the quiver has no loops nor 2-cycles, it is represented by an $N \times N$ skew-symmetric matrix $B = [b_{ij}]$ whose positive entries b_{ij} denote the number of arrows from node i to node j. To each node i of such a quiver one associates a variable x_i, called *cluster variable*. The N-tuple $\mathbf{x} = (x_1, \ldots, x_N)$ is known as the *initial cluster* and the pair $(B, \mathbf{x})$ is called the *initial seed*.

The *mutation* μ_k at the node k of any quiver defined by a skew-symmetric matrix $B = [b_{ij}]$, is an operation that transforms a seed $(B, \mathbf{x})$ into the seed $(B', \mathbf{x}')$ as follows:

- $\mu_k(B) = B' = \left[b'_{ij}\right]$ with

$$b'_{ij} = \begin{cases} -b_{ij}, & (k-i)(j-k) = 0 \\ b_{ij} + \frac{1}{2}\left(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|\right), & \text{otherwise.} \end{cases}$$

- $\mu_k(x_1, \ldots, x_N) = \mathbf{x}' = (x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_N)$, with

$$x'_k = \frac{\prod_{j:b_{kj}\geq 0} x_j^{b_{kj}} + \prod_{j:b_{kj}\leq 0} x_j^{-b_{kj}}}{x_k}.$$

A quiver Q_B, defined by a skew-symmetric matrix B, is said to be mutation-periodic if there exists a positive integer m such that

$$\mu_m \circ \cdots \circ \mu_1(B) = \sigma^{-m} B \sigma^m.$$

where σ is the permutation $\sigma : (1, 2, \ldots, N) \mapsto (2, 3, \ldots, N, 1)$. The mutation-period of B is then the smallest integer m for which the above identity holds. A mutation-periodic quiver of period m gives rise to a system of m recurrence relations of order N whose solutions correspond to the orbits (under iteration) of the following map:

$$\varphi(\mathbf{x}) = \sigma^m \circ \mu_m \circ \cdots \circ \mu_1(\mathbf{x}).$$

This map is called the *cluster map* associated to the quiver.

Example 1. If r and s are non-negative integers, the matrix

$$B = \begin{bmatrix} 0 & r & -1 & -1 & s \\ -r & 0 & r+s & r-1 & -1 \\ 1 & -r-s & 0 & r+s & -1 \\ 1 & 1-r & -r-s & 0 & r \\ -s & 1 & 1 & -r & 0 \end{bmatrix}$$

defines a 1-periodic quiver if $r = s$ and a 2-periodic quiver otherwise. The cluster map is

$$\varphi(x_1, \ldots, x_5) = \left(x_2, x_3, x_4, x_5, \frac{x_2^r x_5^r + x_3 x_4}{x_1} \right) \tag{1}$$

in the 1-periodic case (i.e. $r = s$) and

$$\varphi(x_1, x_2, x_3, x_4, x_5) = \left(x_3, x_4, x_5, \frac{x_2^r x_5^s + x_3 x_4}{x_1}, \frac{x_3^s (x_2^r x_5^s + x_3 x_4)^r + x_1^r x_4 x_5}{x_1^r x_2} \right) \tag{2}$$

in the 2-periodic case.

The cluster map (1) is associated to the recurrence relation of order 5

$$x_{n+5} x_n = x_{n+1}^r x_{n+4}^r + x_{n+2} x_{n+3},$$

which, when $r = 1$, is the well-known Somos-5 recurrence relation. On the other hand, the cluster map (2) is associated to the following system of two recurrence relations of order 5

$$\begin{cases} x_{2n+4} x_{2n-1} = x_{2n}^r x_{2n+3}^s + x_{2n+1} x_{2n+2} \\ x_{2n+5} x_{2n} = x_{2n+1}^s x_{2n+4}^r + x_{2n+2} x_{2n+3}. \end{cases}$$

We refer to [11] and [4] for details.

Cluster maps are birational maps, that is, rational with rational inverse. Our study of the dynamics of these maps will be restricted to $\mathbb{R}_+^N$, the set of points where all coordinates are strictly positive.

2.1 Presymplectic reduction

An important feature of any cluster map $\varphi : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is that it preserves the presymplectic form ω defined by the skew-symmetric matrix B which characterizes φ. In fact, as proved in [4, Theorem 3.1], the presymplectic form

$$\omega = \sum_{1\leq i<j\leq N} \frac{b_{ij}}{x_i x_j} dx_i \wedge dx_j$$

is preserved by the cluster map, that is $\varphi^*\omega = \omega$ with φ^* denoting the *pullback by* φ.

As a consequence of this invariance property, if the rank of the matrix B is $2k < N$ then there exist a submersion $\Pi_0 : \mathbb{R}_+^N \longrightarrow \mathbb{R}_+^{2k}$ and a map $\widehat{\varphi} : \mathbb{R}_+^{2k} \longrightarrow \mathbb{R}_+^{2k}$ such that

$$\Pi_0 \circ \varphi = \widehat{\varphi} \circ \Pi_0.$$

Moreover, $\widehat{\varphi}$ is symplectic with respect to the canonical symplectic form

$$\omega_0 = \sum_{m=1}^{k} \frac{dy_{2m-1}}{y_{2m-1}} \wedge \frac{dy_{2m}}{y_{2m}}$$

and $\Pi_0^*\omega_0 = \omega$.

We will call $\widehat{\varphi}$ the *symplectic reduced map* and the overall result as *presymplectic reduction* [4, Theorem 3.4].

As the submersion Π_0 and the reduced map $\widehat{\varphi}$ will be important in the following sections we have included their explicit construction in Appendix A.

Note that the relation $\Pi_0 \circ \varphi = \widehat{\varphi} \circ \Pi_0$ implies that the dynamics of the cluster map φ is related to the dynamics of the symplectic reduced map $\widehat{\varphi}$ by

$$\Pi_0 \circ \varphi^{(n)} = \widehat{\varphi}^{(n)} \circ \Pi_0,$$

where $f^{(n)}$ denotes the nth iterate of the map f.

2.2 Poisson reduction

The existence of a Poisson structure of non maximal rank which is invariant under the cluster map also allows for a reduction of the cluster map, that is, for semi-conjugation to a map on a lower dimensional space.

The cluster map φ is said to be a Poisson map with respect to a Poisson structure $\{\, ,\, \}$ if this structure is invariant under φ, that is, if

$$\{f \circ \varphi, g \circ \varphi\} = \{f, g\} \circ \varphi, \qquad \forall f, g \in C^\infty(\mathbb{R}_+^N).$$

Given a Poisson structure $\{\, ,\, \}$ of non maximal rank for which the cluster map $\varphi : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is a Poisson map, functions z_i satisfying

$$\{z_i, f\} = 0, \qquad \forall f \in C^\infty(\mathbb{R}_+^N),$$

known as *Casimir functions*, will provide "reduced" variables. In fact, let $\{z_1, \ldots, z_s\}$ be a maximal independent set of Casimir functions for P. As φ is a diffeomorphism, each function $z_i \circ \varphi$ is again a Casimir:

$$\{z_i \circ \varphi, f\} = \{z_i, f \circ \varphi^{-1}\} \circ \varphi = 0,$$

and the maximality of the set of Casimirs then implies that $z_i \circ \varphi$ is a function which depends only on $z_1, \ldots, z_s$. So

$$z_i \circ \varphi = \widetilde{\varphi}_i(z_1, \ldots, z_s), \quad i = 1, \ldots, s.$$

Let $\Pi : \mathbb{R}_+^N \to \mathbb{R}_+^s$ be the submersion defined by

$$\Pi(\mathbf{x}) = (z_1(\mathbf{x}), \ldots, z_s(\mathbf{x})).$$

The above argument implies that

$$\Pi \circ \varphi = \widetilde{\varphi} \circ \Pi,$$

with $\widetilde{\varphi} = (\widetilde{\varphi}_1, \ldots \widetilde{\varphi}_s)$. The map $\widetilde{\varphi} : \mathbb{R}_+^s \to \mathbb{R}_+^s$ will be called a *Poisson reduced map* and the overall process *Poisson reduction*.

In the context of cluster maps, there are Poisson structures of a particularly simple type which often allow for the above reduction. These are homogeneous quadratic Poisson structures of the form:

$$\{x_i, x_j\} = c_{ij} x_i x_j \qquad i, j \in \{1, ..., N\}, \tag{3}$$

where $C = [c_{ij}]$ is an $N \times N$ skew-symmetric singular matrix. Note that, concerning this structure, if $\mathbf{k} = (k_1, ..., k_N)$ belongs to $\ker C$ then the function $z = \mathbf{x}^{\mathbf{k}} = x_1^{k_1} \cdots x_N^{k_N}$ is a Casimir. It can easily be proved that a maximal independent set of Casimirs can be obtained in this way.

In order to be able to perform Poisson reduction, we require the Poisson structure to be invariant under the cluster map φ. This condition imposes several restrictions on the matrix C, namely the restrictions in the next lemma.

Lemma 1. *Let φ be a cluster map associated to a mutation-periodic quiver with period m and N nodes, and let C be the matrix defining the Poisson structure (3). If φ is a Poisson map with respect to (3) then*

$$c_{ij} = c_{i+m, j+m} \qquad \textit{for } 1 \leq i, j \leq N - m.$$

Proof. As $\varphi = \sigma^m \circ \mu_m \circ \cdots \circ \mu_1$, the first $N - m$ components of φ are the last $N - m$ components of $\mathbf{x}$, that is, for any $i \in \{1, \ldots, N - m\}$

$$x_i \circ \varphi = x_{i+m}.$$

Consequentely, for $i, j \in \{1, \ldots, N - m\}$,

$$\{x_i \circ \varphi, x_j \circ \varphi\} = \{x_{i+m}, x_{j+m}\} = c_{i+m, j+m} x_{i+m} x_{j+m},$$

whereas

$$\{x_i, x_j\} \circ \varphi = (c_{ij} x_i x_j) \circ \varphi = c_{ij} x_{i+m} x_{j+m}.$$

The result is now immediate from the above equations. □

There are examples where Poisson reduction by a structure of the form (3) cannot be achieved. For example, in Example 1, the only case where a Poisson structure of the form (3) exists for which the cluster map is a Poisson map is the case $r = s = 1$, that is, the Somos-5 map.

3 Foliations from presymplectic forms and from Poisson structures

Presymplectic manifolds and Poisson manifolds are natural generalizations of symplectic manifolds. On each of these manifolds there is a natural foliation, which is known as *null foliation* in the case of a presymplectic manifold and as *symplectic foliation* in the case of a Poisson manifold (see for example [15], [2] and references therein).

Without the assumption that the rank of the structure (presymplectic or Poisson) is constant, the corresponding foliation can be singular, meaning that distinct leaves of the foliation can have different dimensions. However, in the framework of cluster algebras, a special kind of presymplectic and Poisson structures are used, the so-called *log-canonical* structures. When restricting these structures to $\mathbb{R}_+^N$, not only their rank is constant, but also the associated foliations of $\mathbb{R}_+^N$ are given by (surjective) submersions, that is, they are *simple foliations*. Moreover the fibres of these submersions will be shown to be connected, which leads to the fact that each fibre of the submersion is a leaf of the foliation. This kind of simple foliations where all fibres are connected are usually called *strictly simple foliations* (see for example [16]) and we will adopt this terminology.

3.1 The null foliation from a presymplectic form

In the context of cluster algebras, there is a class of presymplectic forms of interest in the theory, the so-called *log-canonical* presymplectic forms.

A log-canonical presymplectic form is a 2-form given in cluster variables $(x_1, \ldots, x_N)$ by

$$\omega = \sum_{1 \leq i < j \leq N} \frac{\omega_{ij}}{x_i x_j} dx_i \wedge dx_j, \tag{4}$$

where $\Omega = [\omega_{ij}]$ is a skew-symmetric matrix. In the next proposition we describe the *null foliation* of $\mathbb{R}_+^N$ determined by this presymplectic form. Leaves of this foliation will be referred to as *null leaves* of the foliation.

Proposition 1. *Let $\Omega = [\omega_{ij}]$ be a singular skew-symmetric matrix of rank $2k$ and ω the presymplectic form on $\mathbb{R}_+^N$ defined by Ω as in (4). Then there exists a strictly simple foliation $\mathcal{F}^\omega$ of $\mathbb{R}_+^N$ of dimension $N - 2k$,*

$$\mathcal{F}^\omega = \bigsqcup_\alpha N_\alpha,$$

whose leaves N_α satisfy $\omega\big|_{N_\alpha} = 0$.

Moreover, if Ω is an integer matrix then the leaves N_α are algebraic varieties.

Proof. Let $\Phi : \mathbb{R}^N_+ \to \mathbb{R}^N$ be the diffeomorphism given by

$$\Phi(x_1, \dots, x_N) = (\ln x_1, \dots, \ln x_N). \tag{5}$$

The presymplectic form $\omega' = \left(\Phi^{-1}\right)^* \omega$ is the constant symplectic form on $\mathbb{R}^N$ given by

$$\omega' = \sum_{1 \le i < j \le N} \omega_{ij} dv_i \wedge dv_j.$$

By Élie Cartan's theorem (see [15] and Appendix A) there exist $2k$ independent linear functions on $\mathbb{R}^N$ of the form $f_i(\mathbf{v}) = \mathbf{u}_i \cdot \mathbf{v}$ with $\mathbf{u}_i \in \operatorname{Im}\Omega$, such that ω' is written as

$$\omega' = \sum_{m=1}^{k} df_{2m-1} \wedge df_{2m}.$$

Now consider the foliation of $\mathbb{R}^N$ given by the surjective submersion

$$\begin{aligned} \Pi_0' : \mathbb{R}^N &\longrightarrow \mathbb{R}^{2k} \\ \mathbf{v} &\longmapsto (f_1(\mathbf{v}), \dots, f_{2k}(\mathbf{v})) \end{aligned}$$

The fibre of Π_0' through $\mathbf{v_0}$ is given by $N' = \{\mathbf{v} \in \mathbb{R}^N :\ \Pi_0'(\mathbf{v}) = \Pi_0'(\mathbf{v_0})\}$, which coincides with the affine subspace

$$\mathbf{v_0} + (\operatorname{Im}\Omega)^{\perp}.$$

In particular: (a) each fibre of Π_0' is connected and therefore constitutes a leaf of the foliation; (b) this foliation depends only on $\operatorname{Im}\Omega$ and not on the particular choice of the linear functions $f_1, \dots, f_{2k}$.

Consider now the functions $y_i = \Phi^{-1} \circ f_i \circ \Phi$, which have the form

$$y_i(\mathbf{x}) = \mathbf{x}^{\mathbf{u}_i}.$$

Clearly, the expression for ω is given in terms of $y_1, \dots y_{2k}$ by:

$$\omega = \Phi^* \omega' = \sum_{m=1}^{k} \frac{dy_{2m-1}}{y_{2m-1}} \wedge \frac{dy_{2m}}{y_{2m}}. \tag{6}$$

Moreover, each fibre N_α of the surjective submersion

$$\begin{aligned} \Pi_0 : \mathbb{R}^N_+ &\longrightarrow \mathbb{R}^{2k}_+ \\ \mathbf{x} &\longmapsto (y_1(\mathbf{x}), \dots, y_{2k}(\mathbf{x})) \end{aligned} \tag{7}$$

is connected, since it is Φ-diffeomorphic to a connected fibre N' of Π_0'.

To show that ω vanishes on N_α note that $N_\alpha = \{\mathbf{x} \in \mathbb{R}^N_+ :\ \Pi_0(\mathbf{x}) = \alpha\}$ and consider $U, V \in T_{\mathbf{x}} N_\alpha$. As $dy_{i_{\mathbf{x}}}(U) = dy_{i_{\mathbf{x}}}(V) = 0$, then the identity (6) implies that $\omega_{\mathbf{x}}(U, V) = 0$.

Finally, if $\Omega = [\omega_{ij}]$ is an integer matrix then each f_i can be chosen to have rational coefficients (see [3] and Apendix A for the details). Then it can easily be checked that each fibre N_α is an algebraic variety. $\square$

Remark 1. The foliation described in the previous proposition coincides with the null foliation of the presymplectic manifold $(\mathbb{R}^N_+, \omega)$. This foliation is often described using distributions, but the description in the proposition is more convenient for our purposes.

3.2 The symplectic foliation from a Poisson structure

We will now consider a particular class of Poisson structures, known as *log-canonical* Poisson structures in the cluster algebra literature. Analogously to log-canonical presymplectic forms, these Poisson structures are particularly simple when restricted to $\mathbb{R}^N_+$, as they are diffeomorphic to constant Poisson structures on a vector space. Consequently, as we will prove, they have constant rank and their symplectic foliation can be defined by a submersion whose fibres are connected.

A log-canonical Poisson structure is a Poisson structure given in cluster variables $(x_1, \ldots, x_N)$ by a tensor of the form

$$P = \sum_{1 \le i < j \le N} c_{ij} x_i x_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{8}$$

where $C = [c_{ij}]$ is a skew-symmetric matrix. Equivalently, the Poisson bracket of the variables x_i and x_j is given by

$$\{x_i, x_j\} = c_{ij} x_i x_j, \quad i, j \in \{1, \ldots, N\}.$$

Proposition 2. *Let $C = [c_{ij}]$ be a singular skew-symmetric matrix with s-dimensional kernel and P the Poisson structure on $\mathbb{R}^N_+$ defined by C as in* (8)*. Then there is a strictly simple foliation $\mathcal{F}^P$ of $\mathbb{R}^N_+$ of dimension $N - s$,*

$$\mathcal{F}^P = \bigsqcup_{\beta} S_\beta,$$

whose leaves S_β are symplectic submanifolds of $(\mathbb{R}^N_+, P)$.

Moreover, if C is an integer matrix, then each leaf S_β is an algebraic subvariety of $\mathbb{R}^N_+$.

Proof. Consider again the diffeomorphism $\Phi : \mathbb{R}^N_+ \to \mathbb{R}^N$ given by (5). The pushforward of P by Φ is the Poisson structure given by

$$P' = \Phi_* P = \sum_{1 \le i < j \le N} c_{ij} \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_j},$$

which is a constant Poisson structure on the vector space $\mathbb{R}^N$. The symplectic leaves of such Poisson structures are well known: they are affine subspaces which coincide with the common level sets of a maximal set of independent linear Casimirs. Note that the linear function $f(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ is a Casimir of P' if and only if $\mathbf{u} \in \ker C$. Therefore the symplectic leaves of P' can be defined as the (connected) fibres of the surjective submersion

$$\begin{aligned} \Pi' : \mathbb{R}^N &\longrightarrow \mathbb{R}^s \\ \mathbf{v} &\longmapsto (f_1(\mathbf{v}), \ldots, f_s(\mathbf{v})) \end{aligned}$$

where $f_1, \ldots, f_s$ are s independent linear functions of the form

$$f_i(\mathbf{v}) = \mathbf{u}_i \cdot \mathbf{v}, \quad \mathbf{u}_i \in \ker C.$$

It can easily be checked that the fibre S' of Π' through $\mathbf{v_0}$ can be written as

$$\mathbf{v_0} + (\ker C)^{\perp}.$$

In particular this foliation depends only on $\ker C$ and not on the particular choice of the linear functions $f_1, \ldots, f_s$.

The functions $z_i = \Phi^{-1} \circ f_i \circ \Phi$, which have the form

$$z_i(\mathbf{x}) = \mathbf{x}^{\mathbf{u}_i},$$

form a maximal set of independent Casimirs of the Poisson structure P. The map

$$\begin{aligned} \Pi : \mathbb{R}^N_+ &\longrightarrow \mathbb{R}^s_+ \\ \mathbf{x} &\longmapsto (z_1(\mathbf{x}), \ldots, z_s(\mathbf{x})) \end{aligned} \tag{9}$$

is a surjective submersion and each of its fibres,

$$S_\beta = \left\{ \mathbf{x} \in \mathbb{R}^N_+ : \ \Pi(\mathbf{x}) = \beta \right\},$$

is connected since it is Φ-diffeomorphic to a fibre of Π'. As each S_β is connected and a common level set of s independent Casimirs of P, then S_β is a symplectic leaf of P.

Finally note that if $C = [c_{ij}]$ is an integer matrix, then each vector $\mathbf{u}$ in $\ker C$ can be chosen to have integer coefficients, which implies that the fibres S_β are algebraic varieties. □

Remark 2. The foliation described in the previous proposition coincides with the symplectic foliation of the Poisson manifold $(\mathbb{R}^N_+, P)$, whose leaves are called *symplectic leaves*. This foliation still exists if C is nonsingular (it consists of a unique leaf, the whole space $\mathbb{R}^N_+$), but this case is of no interest in our study of cluster maps.

3.3 Simple subfoliations of null and symplectic foliations

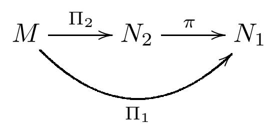
We now look for conditions under which the symplectic foliation $\mathcal{F}^P$ determined by a log-canonical Poisson structure P is a *subfoliation* of the null foliation $\mathcal{F}^\omega$ defined by a log-canonical presymplectic form ω (or vice-versa). We will also be interested in considering two distinct symplectic foliations, $\mathcal{F}^{P_1}$ and $\mathcal{F}^{P_2}$, and on conditions under which $\mathcal{F}^{P_2}$ is a subfoliation of $\mathcal{F}^{P_1}$.

Definition 1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two simple foliations of a manifold M given respectively by surjective submersions $\Pi_1 : M \to N_1$ and $\Pi_2 : M \to N_2$ with $\dim N_1 < \dim N_2$.

The foliation $\mathcal{F}_2$ is said to be a *simple subfoliation of* $\mathcal{F}_1$, and will be denoted by $\mathcal{F}_2 \prec \mathcal{F}_1$, if there exists a surjective submersion $\pi : N_2 \to N_1$ such that

$$\pi \circ \Pi_2 = \Pi_1$$

Schematically, a simple subfoliation $\mathcal{F}_2$ of $\mathcal{F}_1$ is described by the following diagram



Remark 3. It is clear from Definition 1 that if $\mathcal{F}_2 \prec \mathcal{F}_1$ then any leaf of $\mathcal{F}_2$ is contained in a leaf of $\mathcal{F}_1$, that is, $\mathcal{F}_2$ is indeed a subfoliation of $\mathcal{F}_1$.

Proposition 3. *Let Ω be an $N \times N$ skew-symmetric matrix and ω the presymplectic form on $\mathbb{R}_+^N$ determined by Ω as in* (4)*. Let C_1 and C_2 be two $N \times N$ skew-symmetric singular matrices and P_1 and P_2 the Poisson structures on $\mathbb{R}_+^N$ determined by C_1 and C_2 as in* (8)*. Then:*

1. *$\mathcal{F}^{P_i} \prec \mathcal{F}^\omega$ if and only if* $\operatorname{Im}\Omega \subset \ker C_i$.
2. *$\mathcal{F}^\omega \prec \mathcal{F}^{P_i}$ if and only if* $\ker C_i \subset \operatorname{Im}\Omega$.
3. *$\mathcal{F}^{P_2} \prec \mathcal{F}^{P_1}$ if and only if* $\ker C_1 \subset \ker C_2$.

Proof. We only prove the first statement since the proofs of the remaining statements are analogous.

The submersions Π_0 and Π_i whose fibres define (respectively) the foliations $\mathcal{F}^\omega$ and $\mathcal{F}^{P_i}$ were obtained in the proofs of propositions 1 and 2 by means of the diffeomorphism Φ given by (5).

Recall from those proofs that the leaf N of $\mathcal{F}^\omega$ through $\mathbf{x}_0 \in \mathbb{R}_+^N$ is Φ-diffeomorphic to the affine subspace of $\mathbb{R}^N$

$$N' = \Phi(\mathbf{x}_0) + (\operatorname{Im}\Omega)^\perp$$

whereas the leaf S^i of $\mathcal{F}^{P_i}$ through the same point $\mathbf{x}_0$ is Φ-diffeomorphic to

$$S^{i\prime} = \Phi(\mathbf{x}_0) + (\ker C_i)^\perp .$$

If $\mathcal{F}^{P_i} \prec \mathcal{F}^\omega$ then $S^i \subset N$ which is equivalent to $S^{i\prime} \subset N'$ and consequently $\operatorname{Im}\Omega \subset \ker C_i$.

Conversely, if $\operatorname{Im}\Omega \subset \ker C_i$ we can choose the submersions $\Pi_0 : \mathbb{R}_+^N \to \mathbb{R}_+^{2k}$ from (7) and $\Pi_i : \mathbb{R}_+^N \to \mathbb{R}_+^{s_i}$ from (9) in such a way that

$$\Pi_i(\mathbf{x}) = (\Pi_0(\mathbf{x}), z_{2k+1}(\mathbf{x}), \dots, z_{s_i}(\mathbf{x})) .$$

Then it it is clear that $\Pi_0 = \pi_i \circ \Pi_i$, where $\pi_i : \mathbb{R}_+^{s_i} \to \mathbb{R}_+^{s_i}$ is the projection onto the first $2k$ coordinates, showing that $\mathcal{F}^{P_i} \prec \mathcal{F}^\omega$. □

Remark 4. In the context of the theory of cluster algebras, Gekhtman *et al* [12] introduced the notion of Poisson structure *compatible with a cluster algebra*. In the particular case where the cluster algebra $\mathcal{A}(\Omega)$ is determined by a skew-symmetric integer matrix Ω, with Ω not invertible, Inoue and Nakanishi proved (see [14] and [12]) that the log-canonical Poisson structure P determined by a skew-symmetric matrix C is compatible with $\mathcal{A}(\Omega)$ if and only if $C\Omega = [0]$. A geometric interpretation of this result is then given by the first statement in the last proposition as follows: the Poisson structure P determined by C is compatible with the cluster algebra $\mathcal{A}(\Omega)$ if and only if the symplectic foliation determined by P is a subfoliation of the null foliation determined by ω.

Corollary 1. *Let Ω and C be $N \times N$ skew-symmetric matrices in the conditions of Proposition 3. If $\mathcal{F}^P \prec \mathcal{F}^\omega$ then $\omega_{|_S} \equiv 0$ for any symplectic leaf S of $\mathcal{F}^P$, that is, each symplectic leaf S of P is* isotropic *with respect to ω.*

Proof. This is an immediate consequence of the last proposition and the characterization of the null foliation $\mathcal{F}^\omega$ given in Proposition 1. □

We conclude this section by remarking that the condition $\operatorname{Im}\Omega \subset \ker C$ in the first statement of Proposition 3 is equivalent to $C\Omega = [0]$ and imposes several restrictions on the matrix C. For instance, just by dimension counting, it is clear that if Ω has rank equal to $N-1$ then the only possible matrix C satisfying this condition is the zero matrix.

On the other extreme, for some particular matrices Ω as in the next example, there exist nontrivial matrices C_1 and C_2, with different ranks, satisfying the above condition. This case will be explored in the last section, where two distinct Poisson structures are used to provide a better insight into the dynamics of a cluster map.

Example 2. Consider the skew-symmetric matrix Ω of rank 2:

$$\Omega = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

The following skew-symmetric matrices C_1 and C_2 both satisfy the condition $C_i\Omega = [0]$ (only the upper triangular part of the matrices is shown):

$$C_1 = \begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ & 0 & 1 & 1 & 2 & 3 & 3 \\ & & 0 & 1 & 1 & 2 & 3 \\ & & & 0 & 1 & 1 & 2 \\ & & & & 0 & 1 & 1 \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ & 0 & 1 & -1 & 0 & 1 & -1 \\ & & 0 & 1 & -1 & 0 & 1 \\ & & & 0 & 1 & -1 & 0 \\ & & & & 0 & 1 & -1 \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{bmatrix} \qquad (10)$$

Moreover, it can be checked that $\operatorname{rank} C_1 = 4$, $\operatorname{rank} C_2 = 2$ and $\ker C_1 \subset \ker C_2$. By Proposition 3 we have

$$\mathcal{F}^{P_2} \prec \mathcal{F}^{P_1} \prec \mathcal{F}^{\omega},$$

where the symplectic leaves of $\mathcal{F}^{P_2}$ are 2-dimensional, the symplectic leaves of $\mathcal{F}^{P_1}$ are 4-dimensional and the null leaves of $\mathcal{F}^{\omega}$ are 5-dimensional. Submersions defining these foliations are given respectively by:

$$\Pi_2(x_1,\ldots,x_7) = (y_1,y_2,y_3,y_4,y_5), \quad \Pi_1(x_1,\ldots,x_7) = (y_1,y_2,y_3)$$

and

$$\Pi_0(x_1,\ldots,x_7) = (y_1,y_2),$$

with

$$y_1 = \frac{x_1x_6}{x_3x_4},\ y_2 = \frac{x_2x_7}{x_4x_5},\ y_3 = \frac{x_1x_7}{x_4^2},\ y_4 = x_1x_2x_3,\ y_5 = x_2x_3x_4$$

(we used row 5 and row 1 of Ω to obtain the submersion Π_0 by the procedure described in Appendix A).

Note that $\mathcal{F}^{P_2} \prec \mathcal{F}^{P_1} \prec \mathcal{F}^{\omega}$ since

$$\pi \circ \Pi_2 = \Pi_1 \quad \text{and} \quad \pi_1 \circ \Pi_1 = \Pi_0$$

with:

$$\begin{array}{rcl} \pi : \mathbb{R}_+^5 & \longrightarrow & \mathbb{R}_+^3 \\ (y_1,\ldots,y_5) & \longmapsto & (y_1,y_2,y_3) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \pi_1 : \mathbb{R}_+^3 & \longrightarrow & \mathbb{R}_+^2 \\ (y_1,y_2,y_3) & \longmapsto & (y_1,y_2). \end{array}$$

4 Foliations and reduction of cluster maps

In this section we will focus on the consequences to the dynamics of a cluster map φ, of presymplectic reduction and of Poisson reduction (when it exists). We will show that associated to these reductions there are foliations which play an important role in the dynamics of φ.

We start with a general definition of “reduction” and some properties derived from it.

Definition 2. Let $\mathcal{F}$ be a strictly simple foliation of a manifold M given by a surjective submersion $\Pi : M \to N$ and $f : M \to M$ a given map. A map $g : N \to N$ is called a *reduced map* of f if

$$\Pi \circ f = g \circ \Pi. \tag{11}$$

The pair (g, Π) is then called a *reduced system of* f.

Before stating the next proposition we recall that a *first integral* of a map f is a real-valued function I such that $I \circ f = I$.

Proposition 4. *Let* $\mathcal{F}$ *be a strictly simple foliation of* M *given by a surjective submersion* $\Pi : M \to N$ *and suppose that* (g, Π) *is a reduced system of* $f : M \to M$. *Then*

1. *the map f sends leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$, more precisely*

$$f(L_\alpha) \subset L_{g(\alpha)};$$

2. *the leaf L_α of $\mathcal{F}$ is invariant under the map $f^{(p)}$ if and only if $\alpha \in N$ is a p-periodic point of g, that is, $g^{(p)}(\alpha) = \alpha$;*

3. *g is a globally p-periodic map, i.e., $g^{(p)} = Id$, if and only if Π is a vector-valued first integral of $f^{(p)}$.*

Proof. All the statements follow from (11) and from the fact that any leaf L_α of the foliation is given by

$$L_\alpha = \{\mathbf{x} \in M : \Pi(\mathbf{x}) = \alpha\}.$$

For the first statement assume that $\mathbf{x}$ belongs to L_α. Then (11) shows that $\Pi(f(\mathbf{x})) = g(\alpha)$, that is, $f(\mathbf{x})$ belongs to $L_{g(\alpha)}$.

Concerning the second and third statements, note that (11) implies

$$\Pi \circ f^{(n)} = g^{(n)} \circ \Pi, \quad n = 1, 2, \dots$$

and the results follow directly from this identity. □

We will make use of this proposition to describe the dynamics of a cluster map for which presymplectic or Poisson reduction exists.

Throughout the next subsections, $B = [b_{ij}]$ will denote an integer skew-symmetric $N \times N$ matrix representing a mutation-periodic quiver with period m and $\varphi = \sigma^m \circ \mu_m \circ \cdots \circ \mu_1$ will denote the associated cluster map defined on $\mathbb{R}_+^N$. The presymplectic form on $\mathbb{R}_+^N$ given by the matrix B will be denoted by ω, that is

$$\omega = \sum_{1 \le i < j \le N} \frac{b_{ij}}{x_i x_j} dx_i \wedge dx_j. \tag{12}$$

4.1 The null foliation and presymplectic reduction

Assume that the matrix B has rank $2k < N$. Then ω given by (12) is a presymplectic form on $\mathbb{R}_+^N$ of rank $2k$ and, as described in Proposition 1, it defines a strictly simple foliation $\mathcal{F}^\omega$ of $\mathbb{R}_+^N$ with null leaves N_α of dimension $N - 2k$.

It turns out that the surjective submersion $\Pi_0 : \mathbb{R}_+^N \to \mathbb{R}_+^{2k}$ defining this foliation also reduces the cluster map φ to a symplectic map $\widehat{\varphi}$ on $\mathbb{R}_+^{2k}$ as described in Subsection 2.1.

Proposition 5. *Let B be the $N \times N$ matrix associated to a mutation-periodic quiver of period m such that* $\operatorname{rank}(B) = 2k < N$.

Let $\mathcal{F}^\omega$ denote the $(N-2k)$-dimensional null foliation of $\mathbb{R}_+^N$ associated to ω and $\Pi_0 : \mathbb{R}_+^N \to \mathbb{R}_+^{2k}$ a submersion defining $\mathcal{F}^\omega$.

Then there exists a (symplectic) map $\widehat{\varphi} : \mathbb{R}_+^{2k} \to \mathbb{R}_+^{2k}$ such that $(\widehat{\varphi}, \Pi_0)$ is a reduced system of φ. Moreover,

1. *the cluster map φ sends each leaf N_α of $\mathcal{F}^\omega$ to the leaf $N_{\widehat{\varphi}(\alpha)}$ of the same foliation;*

2. *a null leaf N_α is invariant under $\varphi^{(p)}$ if and only if α is a p-periodic point of $\widehat{\varphi}$.*

Proof. The submersion (7) in Proposition 1 which defines the foliation $\mathcal{F}^\omega$ is precisely the submersion Π_0 that leads to the reduction of φ as described in Subsection 2.1. The relation

$$\Pi_0 \circ \varphi = \widehat{\varphi} \circ \Pi_0$$

follows from the reduction procedure described in the referred subsection. The fact that the reduced map $\widehat{\varphi}$ is symplectic was proved in [4, Theorem 3.4].

The remaining statements are a direct consequence of Proposition 4 applied to the reduced system $(\widehat{\varphi}, \Pi_0)$. □

4.2 The symplectic foliation and Poisson reduction

We will now be interested in log-canonical Poisson structures P which allow us to reduce the cluster map φ (as previously remarked, such Poisson structures may fail to exist).

We start by considering a singular skew-symmetric matrix C and the associated Poisson structure P as in (8). Let $s = \dim \ker C$ and recall from Proposition 2 that there exists a submersion $\Pi : \mathbb{R}_+^N \to \mathbb{R}_+^s$ defining the strictly simple foliation $\mathcal{F}^P$ whose leaves are symplectic manifolds of dimension $N - s$.

A sufficient condition to obtain a reduced system is that the Poisson structure P is invariant under the cluster map φ, that is, all cluster variables satisfy the identity

$$\{x_i \circ \varphi, x_j \circ \varphi\} = \{x_i, x_j\} \circ \varphi.$$

Proposition 6. *Let C be a singular skew-symmetric matrix defining a log-canonical Poisson structure P and $\Pi : \mathbb{R}_+^N \to \mathbb{R}_+^s$ a surjective submersion defining the symplectic foliation $\mathcal{F}^P$.*

If φ is a cluster map on $\mathbb{R}_+^N$ which is a Poisson map with respect to P, then there exists a map $\widetilde{\varphi} : \mathbb{R}_+^s \to \mathbb{R}_+^s$ such that $(\widetilde{\varphi}, \Pi)$ is a reduced system of φ.

Proof. The proof follows the same lines as the proof of the previous proposition. The submersion Π in (9) defining the foliation $\mathcal{F}^P$ satisfies the conditions described in Subsection 2.2 to obtain a Poisson reduction of φ. The relation

$$\Pi \circ \varphi = \widetilde{\varphi} \circ \Pi$$

follows from the reduction procedure there described. □

Proposition 7. *In the conditions of Proposition 6, if β is a a p-periodic point of the reduced map $\widetilde{\varphi}$, then:*

1. *the symplectic leaf S_β is invariant under $\varphi^{(p)}$;*

2. the restriction of $\varphi^{(p)}$ to S_β is a symplectic map.

Proof. By Proposition 6, $(\widetilde{\varphi}, \Pi)$ is a reduced system of φ and so the first statement is just a rephrasing of item *2* in Proposition 4.

The second statement is a much more interesting property of Poisson reduction. Its proof follows from classical theory on Poisson and symplectic manifolds (see for example [15] and [1]) as we now describe.

By definition, the symplectic structure on S_β is given by the nondegenerate Poisson structure induced by the Poisson structure P on $\mathbb{R}_+^N$. This means that the inclusion $i : S_\beta \hookrightarrow \mathbb{R}_+^N$ is a Poisson map.

The fact that φ is a Poisson map implies that $\varphi^{(p)}$ is also a Poisson map. The composition $\varphi^{(p)} \circ i$, which is precisely the restriction $\varphi^{(p)}\big|_{S_\beta}$, is therefore a Poisson map. Since the Poisson structure on S_β is nondegenerate, then any map preserving this Poisson structure must preserve the associated symplectic structure. In other words $\varphi^{(p)}\big|_{S_\beta}$ is symplectic. □

Remark 5. Assume that, in the conditions of Proposition 6, the Poisson reduced map $\widetilde{\varphi}$ is globally p-periodic. Then all the symplectic leaves of P are invariant under $\varphi^{(p)}$, which means that the $\varphi^{(p)}$-orbit of any point $\mathbf{x} \in \mathbb{R}_+^N$ stays in the symplectic leaf through that point.

Note that this behaviour is the discrete analogue to a well-known behaviour in the continuous setting, where the orbit of any point under the flow of a Hamiltonian vector field remains in the symplectic leaf through the point.

4.3 Simple subfoliations and multiple reductions

We now consider the case where we have two strictly simple foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, with $\mathcal{F}_2 \prec \mathcal{F}_1$, both leading to reduction of the same map $f : M \to M$.

In this setting a better understanding of the dynamics of f is achieved, since we can study the dynamics of the reduced map g_1 by using the map g_2 and then study the dynamics of f using the map g_1.

This procedure is made precise in the next proposition.

Proposition 8. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be strictly simple foliations of M given by the surjective submersions $\Pi_1 : M \to N_1$ and $\Pi_2 : M \to N_2$, respectively. Suppose that $\mathcal{F}_2 \prec \mathcal{F}_1$ and let $\pi : N_2 \to N_1$ be the submersion in the conditions of Definition 1. If (g_1, Π_1) and (g_2, Π_2) are reduced systems of $f : M \to M$, then (g_1, π) is a reduced system of g_2.*

Proof. Because (g_1, Π_1) is a reduced system of f, we have $\Pi_1 \circ f = g_1 \circ \Pi_1$. As $\mathcal{F}_2 \prec \mathcal{F}_1$, we also have $\Pi_1 = \pi \circ \Pi_2$ which leads to

$$\pi \circ \Pi_2 \circ f = g_1 \circ \pi \circ \Pi_2.$$

As (g_2, Π_2) is also a reduced system of f we obtain

$$\pi \circ g_2 \circ \Pi_2 = g_1 \circ \pi \circ \Pi_2$$

and the conclusion follows from surjectivity of Π_2. □

The procedure described in the last proposition can be used with great advantage to study the original map f when we have several foliations (and corresponding reduced systems) verifying

$$\mathcal{F}_p \prec \cdots \prec \mathcal{F}_2 \prec \mathcal{F}_1.$$

Such a set of simple foliations will be called a *flag of simple subfoliations.*

An example of this type, where we have two symplectic foliations and a null foliation satisfying

$$\mathcal{F}^{P_2} \prec \mathcal{F}^{P_1} \prec \mathcal{F}^{\omega},$$

will be comprehensively treated in the next section. We also include an application of Proposition 8 to the Somos-5 cluster map.

5 Examples

In this section we apply the results obtained in the previous sections to draw conclusions on the dynamics of two cluster maps.

Our first example is the cluster map given by (1) with $r = 1$, which is the map associated to the Somos-5 recurrence $x_{n+5}x_n = x_{n+1}x_{n+4} + x_{n+2}x_{n+3}$.

Although this recurrence has been extensively studied (see for example [10], [13] and references therein) a direct application of Proposition 8 allows us to identify a symplectic leaf which is invariant under the cluster map φ and a family of symplectic leaves which are invariant under its second iterate. The restriction of φ and of $\varphi^{(2)}$ to these invariant leaves lead to special solutions of the Somos-5 recurrence which are identified in Proposition 9.

The second example concerns the study of a cluster map φ in dimension 7 associated to a mutation-periodic quiver of period 1: the quiver associated to the matrix $B = \Omega$ in Example 2. Our interest in this cluster map resides in the fact that, besides presymplectic reduction, it also admits reduction by two Poisson structures (as described in the referred example). In this case we obtain a flag of simple subfoliations, $\mathcal{F}^{P_2} \prec \mathcal{F}^{P_1} \prec \mathcal{F}^{\omega}$, which will allow us to make a very complete description of the dynamics of φ.

Example 3. Somos-5

Consider the cluster map

$$\varphi(x_1, \ldots, x_5) = \left(x_2, x_3, x_4, x_5, \frac{x_2x_5 + x_3x_4}{x_1}\right), \tag{13}$$

which is the map associated to the Somos-5 recurrence

$$x_{n+5}x_n = x_{n+1}x_{n+4} + x_{n+2}x_{n+3}. \tag{14}$$

As described in Example 1 this cluster map is associated to the mutation-periodic quiver of period 1 given by the skew-symmetric matrix

$$B = \begin{bmatrix} 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

The procedure described in Subsection 2.1 (see also Appendix A) leads to presymplectic reduction of φ. We note that $\operatorname{rank} B = 2$ and choose the fifth and first rows of B to obtain

$$y_1 = \frac{x_1 x_4}{x_2 x_3},\ y_2 = \frac{x_2 x_5}{x_3 x_4}. \tag{15}$$

This leads to the submersion

$$\begin{aligned} \Pi_0 : \mathbb{R}_+^5 &\longrightarrow \mathbb{R}_+^2 \\ \mathbf{x} &\longmapsto (y_1, y_2) \end{aligned}$$

and to the (presymplectic) reduced system $(\widehat{\varphi}, \Pi_0)$ where

$$\widehat{\varphi}(y_1, y_2) = \left(y_2, \frac{1 + y_2}{y_1 y_2} \right).$$

Consider the log-canonical Poisson structure P on $\mathbb{R}_+^5$ given by the skew-symmetric matrix

$$C = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 & 3 \\ -2 & -1 & 0 & 1 & 2 \\ -3 & -2 & -1 & 0 & 1 \\ -4 & -3 & -2 & -1 & 0 \end{bmatrix}.$$

This matrix has a 3-dimensional kernel and moreover $\operatorname{Im} B \subset \ker C$. The symplectic foliation $\mathcal{F}^P$, which by Proposition 3 is a simple subfoliation of $\mathcal{F}^\omega$, is given by the submersion

$$\begin{aligned} \Pi : \mathbb{R}_+^5 &\longrightarrow \mathbb{R}_+^3 \\ \mathbf{x} &\longmapsto (y_1, y_2, y_3) \end{aligned}$$

where y_1 and y_2 are given by (15) and $y_3 = \dfrac{x_3 x_5}{x_4^2}$. The symplectic leaf $S_{(a,b,c)}$ is then given by

$$S_{(a,b,c)} = \{\mathbf{x} \in \mathbb{R}_+^5 : x_1 x_4 = a x_2 x_3,\ x_2 x_5 = b x_3 x_4, x_3 x_5 = c x_4^2\}.$$

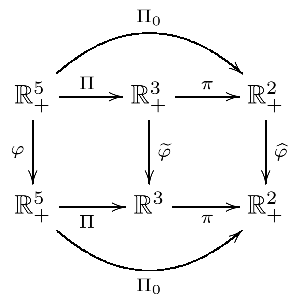
It can be checked that the Somos-5 map (13) is a Poisson map with respect to P (note that C satisfies the conditions in Lemma 1 with $m = 1$, that is, C is a Toeplitz matrix). By Proposition 6 we obtain the (Poisson) reduced system $(\widetilde{\varphi}, \Pi)$, with

$$\widetilde{\varphi}(y_1, y_2, y_3) = \left(y_2, \frac{1+y_2}{y_1 y_2}, \frac{1+y_2}{y_1 y_2 y_3} \right).$$

By Proposition 8, $(\widehat{\varphi}, \pi)$ is a reduced system of $\widetilde{\varphi}$ with

$$\pi(y_1, y_2, y_3) = (y_1, y_2).$$

Schematically:



It is easy to see that the map $\widehat{\varphi}$ has a unique fixed (i.e., 1-periodic) point, the point (r, r), where r is the real (positive) root of $x^3 = 1 + x$, and that $\widehat{\varphi}$ has no points of minimal period 2. A direct application of Proposition 4 to the reduced system $(\widehat{\varphi}, \pi)$ of $\widetilde{\varphi}$ shows that the leaf

$$L = \{\mathbf{y} \in \mathbb{R}^3_+ : y_1 = r, y_2 = r\}$$

is invariant under $\widetilde{\varphi}$. Let h denote the restricted map $\widetilde{\varphi}\big|_L$. Using the natural y_3 coordinate on L we obtain the expression for h as

$$h(y_3) = \frac{r}{y_3},$$

which is a globally 2-periodic map with a unique fixed point: $y_3 = \sqrt{r}$.

Summing up, the point $(r, r, \sqrt{r})$ is the unique fixed point of the map $\widetilde{\varphi}$ and the 2-periodic points of the same map are precisely the points on $\mathbb{R}^3_+$ of the form (r, r, λ).

By Proposition 7, applied to the Poisson reduced system $(\widetilde{\varphi}, \Pi)$ of φ, two conclusions follow:

- the symplectic leaf of $\mathcal{F}^P$

$$S_{(r,r,\sqrt{r})} = \{\mathbf{x} \in \mathbb{R}^5_+ : x_1 x_4 = r x_2 x_3,\ x_2 x_5 = r x_3 x_4, x_3 x_5 = \sqrt{r} x_4^2\}$$

  is invariant under φ and the restriction of φ to $S_{(r,r,\sqrt{r})}$ is symplectic;

- any symplectic leaf $\mathcal{F}^P$ of the form

$$S_{(r,r,\lambda)} = \{\mathbf{x} \in \mathbb{R}^5_+ : x_1 x_4 = r x_2 x_3,\ x_2 x_5 = r x_3 x_4, x_3 x_5 = \lambda x_4^2\}$$

  is invariant under $\varphi^{(2)}$ and the restriction of $\varphi^{(2)}$ to $S_{(r,r,\lambda)}$ is symplectic.

Choosing natural coordinates (x_3, x_4) on each symplectic leaf, we can compute the restricted maps

$$h_1 = \varphi\big|_{S_{(r,r,\sqrt{r})}} \quad \text{and} \quad h_2 = \varphi^{(2)}\big|_{S_{(r,r,\lambda)}},$$

obtaining, respectively

$$h_1(x_3, x_4) = \left(x_4, \sqrt{r}\frac{x_4^2}{x_3}\right) \quad \text{and} \quad h_2(x_3, x_4) = \lambda\left(\frac{x_4^2}{x_3}, \frac{r\,x_4^3}{x_3^2}\right).$$

Remark 6. The maps h_1 and h_2 above belong to the group of birational maps from $\mathbb{R}^2_+$ to itself of the form

$$f(x, y) = (kx^m y^n, lx^p y^q)\,, \tag{16}$$

with k, l real positive constants and m, n, p, q integers satisfying $mq - np = 1$. These maps preserve the log-canonical symplectic form

$$\widetilde{\omega} = \frac{1}{xy} dx \wedge dy$$

and are therefore symplectic maps.

The expression of the iterates of h_1 and h_2 can be obtained by a result proved in [5] which we quote in Appendix B. They are given by

$$h_1^{(n)} = r^{n(n-1)/4}\left(\frac{x_4^n}{x_3^{n-1}}, r^{n/2}\frac{x_4^{n+1}}{x_3^n}\right), \quad n \geq 1$$

and

$$h_2^{(n)} = \lambda^n r^{n(n-1)}\left(\frac{x_4^{2n}}{x_3^{2n-1}}, r^n\frac{x_4^{2n+1}}{x_3^{2n}}\right), \quad n \geq 1.$$

Therefore we are lead to the following proposition.

Proposition 9. *Consider the Somos-5 recurrence*

$$x_{n+5}x_n = x_{n+1}x_{n+4} + x_{n+2}x_{n+3}, \quad n \geq 1$$

and let r be the real root of $x^3 = 1 + x$. Then

1. *if the initial data $(x_1, \ldots, x_5)$ satisfies the identities*

$$x_1x_4 = rx_2x_3,\ x_2x_5 = rx_3x_4, x_3x_5 = \sqrt{r}x_4^2$$

then the solution of the Somos-5 recurrence is given by

$$x_{n+5} = r^{(n+2)(n+1)/4}\ \frac{x_4^{n+2}}{x_3^{n+1}}, \quad n \geq 1;$$

2. *if the initial data* $(x_1, \ldots, x_5)$ *satisfies the identities*

$$x_1x_4 = rx_2x_3,\ x_2x_5 = rx_3x_4, x_3x_5 = \lambda x_4^2$$

(λ is an arbitrary positive real number), then the solution of the Somos-5 recurrence is given by

$$x_{2n+4} = \lambda^n r^{n^2}\ \frac{x_4^{2n+1}}{x_3^{2n}} \quad x_{2n+5} = \lambda^{n+1} r^{n(n+1)}\ \frac{x_4^{2n+2}}{x_3^{2n+1}}, \quad n \geq 1.$$

Remark 7. The general solution of the initial value problem for the Somos-5 recurrence (14) was given in [13] in terms of sigma functions. We were not able to match the results in that reference with those in the above proposition. It seems to us that the solutions in [13] miss those in Proposition 9, which we think correspond to an isolated point of the elliptic curve used in that reference.

Example 4. Dynamics of a cluster map in dimension 7

We now consider the quiver of 7 nodes represented by the matrix $B = \Omega$ in Example 2. This quiver is 1-periodic and gives rise to the cluster map

$$\varphi(x_1, \ldots, x_7) = \left(x_2, x_3, \ldots, x_7, \frac{x_2x_7 + x_4x_5}{x_1}\right).$$

As seen in the referred example there exists a flag of simple subfoliations

$$\mathcal{F}^{P_2} \prec \mathcal{F}^{P_1} \prec \mathcal{F}^{\omega}$$

where ω is the presymplectic form defined by the matrix Ω, and P_1 and P_2 are Poisson structures of the form (8) defined respectively by the matrices C_1 and C_2 in (10). Submersions defining these foliations were also given in Example 2 and are given by:

$$\Pi_0(\mathbf{x}) = (y_1, y_2), \quad \Pi_1(\mathbf{x}) = (y_1, y_2, y_3), \quad \Pi_2(\mathbf{x}) = (y_1, y_2, y_3, y_4, y_5),$$

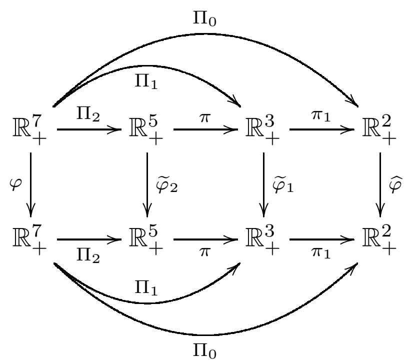
with

$$y_1 = \frac{x_1x_6}{x_3x_4},\ y_2 = \frac{x_2x_7}{x_4x_5},\ y_3 = \frac{x_1x_7}{x_4^2},\ y_4 = x_1x_2x_3,\ y_5 = x_2x_3x_4.$$

As B is the matrix of a mutation-periodic quiver, the use of Proposition 5 leads to a (presymplectic) reduced system $(\widehat{\varphi}, \Pi_0)$ of φ. It can also be checked, with some straightforward computations, that the map φ is a Poisson map with respect to both P_1, P_2 and therefore, by Proposition 6 we obtain two further reduced systems: $(\widetilde{\varphi}_1, \Pi_1)$ and $(\widetilde{\varphi}_2, \Pi_2)$. The respective reduced maps can easily be computed and have the form:

$$\begin{aligned} \widehat{\varphi}(y_1, y_2) &= \left(y_2, \frac{1 + y_2}{y_1}\right) \\ \widetilde{\varphi}_1(y_1, y_2, y_3) &= \left(\widehat{\varphi}(y_1, y_2); \frac{y_2(1 + y_2)}{y_3}\right) \\ \widetilde{\varphi}_2(y_1, \ldots, y_5) &= \left(\widetilde{\varphi}_1(y_1, y_2, y_3); y_5, \frac{y_3y_5^2}{y_2y_4}\right). \end{aligned} \tag{17}$$

This can be summarised in the following commutative diagram:



(18)

where the maps $\pi : \mathbb{R}^5_+ \to \mathbb{R}^3_+$ and $\pi_1 : \mathbb{R}^3_+ \to \mathbb{R}^2_+$ are the canonical projections

$$\pi(y_1, \ldots, y_5) = (y_1, y_2, y_3), \quad \pi_1(y_1, y_2, y_3) = (y_1, y_2) \tag{19}$$

also obtained in Example 2.

In order to describe the dynamics of φ we study sequentially several reduced systems starting with the lowest dimensional one.

As $\mathcal{F}^{P_1} \prec \mathcal{F}^{\omega}$, Proposition 8 implies that $(\widehat{\varphi}, \pi_1)$ is a reduced system of $\widetilde{\varphi}_1$. The map $\widehat{\varphi}$ is the well-known Lyness map which is globally 5-periodic and has exactly one fixed point (ϕ, ϕ) in $\mathbb{R}^2_+$, where ϕ is the golden number $\phi = \frac{1+\sqrt{5}}{2}$ (i.e., the positive root of $x^2 = 1 + x$). All the other points in $\mathbb{R}^2_+$ have minimal period 5.

The use of Proposition 4 then leads to the $\widetilde{\varphi}_1$-invariance properties of the leaves

$$L_{(a,b)} = \{\mathbf{y} \in \mathbb{R}^3_+ : y_1 = a, y_2 = b\}, \tag{20}$$

which are the leaves of the 1-dimensional foliation $\mathcal{F}$ of $\mathbb{R}^3_+$ given by π_1. These properties are as follows:

1. $L_{(\phi,\phi)}$ is invariant under $\widetilde{\varphi}_1$;
2. for any $(a, b) \neq (\phi, \phi)$, $L_{(a,b)}$ is invariant under $\widetilde{\varphi}_1^{(5)}$ and not invariant under $\widetilde{\varphi}_1^{(n)}$ for $1 \leq n < 5$, that is, the $\widetilde{\varphi}_1$-orbit of any point in $L_{(a,b)}$ circulates between five distinct leaves of $\mathcal{F}$:

$$L_{(a,b)} \to L_{\widehat{\varphi}(a,b)} \to L_{\widehat{\varphi}^{(2)}(a,b)} \to L_{\widehat{\varphi}^{(3)}(a,b)} \to L_{\widehat{\varphi}^{(4)}(a,b)}.$$

By studying the restrictions of $\widetilde{\varphi}_1$ to $L_{(\phi,\phi)}$ and of $\widetilde{\varphi}_1^{(5)}$ to $L_{(a,b)}$ we can obtain the full description of the dynamics of $\widetilde{\varphi}_1$, as stated in the next lemma.

Lemma 2. *Consider the map* $\widetilde{\varphi}_1 : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ *given by*

$$\widetilde{\varphi}_1(y_1, y_2, y_3) = \left(y_2, \frac{1 + y_2}{y_1}, \frac{y_2(1 + y_2)}{y_3}\right)$$

and the 1-dimensional foliation $\mathcal{F}$ of $\mathbb{R}^3_+$ whose leaves $L_{(a,b)}$ are given by (20)*. Let $\phi = \frac{1+\sqrt{5}}{2}$. Then*

1. *$L_{(\phi,\phi)}$ contains the unique fixed point of $\widetilde{\varphi}_1$, the point $P = (\phi, \phi, \sqrt{\phi^3})$, and all the other points in $L_{(\phi,\phi)}$ are periodic points of $\widetilde{\varphi}_1$ with minimal period 2;*

2. *if $(a,b) \neq (\phi,\phi)$ then $L_{(a,b)}$ contains precisely one periodic point of $\widetilde{\varphi}_1$ with minimal period 5, the point $(a, b, \sqrt{g(a,b)})$ with*

$$g(a,b) = \frac{ab(a+1)(b+1)}{(a+b+1)}. \tag{21}$$

   Any other point in $L_{(a,b)}$ is a periodic point of $\widetilde{\varphi}_1$ with minimal period 10.

In particular $\widetilde{\varphi}_1$ is globally 10-periodic.

Proof. The restriction h_1 of $\widetilde{\varphi}_1$ to $L_{(\phi,\phi)}$ is given in the natural coordinate y_3 by

$$h_1(y_3) = \frac{\phi^3}{y_3},$$

which is a globally 2-periodic map with exactly one fixed point: $\sqrt{\phi^3}$. We immediately conclude that $\left(\phi, \phi, \sqrt{\phi^3}\right)$ is the only fixed point of $\widetilde{\varphi}_1$ and all the other points of $L_{(\phi,\phi)}$ are periodic points of $\widetilde{\varphi}_1$ with minimal period 2.

For each $(a,b) \in \mathbb{R}^2_+ \setminus \{(\phi,\phi)\}$ the restriction of $\widetilde{\varphi}_1^{(5)}$ to $L_{(a,b)}$ is given by

$$h_5(y_3) = \frac{g(a,b)}{y_3},$$

where

$$g(a,b) = \frac{ab(a+1)(b+1)}{(a+b+1)}.$$

The map h_5 is globally 2-periodic and has exactly one fixed point: $\sqrt{g(a,b)}$. Consequently the restriction of $\widetilde{\varphi}_1$ to $L_{(a,b)}$ has precisely one point of minimal period 5, the point $(a, b, \sqrt{g(a,b)})$, and all the remaining points of $L_{(a,b)}$ are periodic points of $\widetilde{\varphi}_1$ with minimal period 10. $\square$

Since $(\widetilde{\varphi}_1, \pi)$, with π given in (19), is a reduced system associated to $\widetilde{\varphi}_2$ (see also the commutative diagram (18)), the knowledge of the dynamics of $\widetilde{\varphi}_1$ permits us to study the dynamics of $\widetilde{\varphi}_2$.

By combining the previous lemma with Proposition 4 we are lead to the $\widetilde{\varphi}_2$-invariance properties of the leaves

$$L'_{(a,b,c)} = \{\mathbf{y} \in \mathbb{R}^5_+ : y_1 = a, y_2 = b, y_3 = c\} \tag{22}$$

of the 2-dimensional foliation $\mathcal{F}'$ of $\mathbb{R}^5_+$ given by π. More precisely:

1. L'_P, with $P = (\phi, \phi, \sqrt{\phi^3})$, is invariant under $\widetilde{\varphi}_2$;

2. if $Q = (\phi, \phi, c)$ is distinct from P, then the $\widetilde{\varphi}_2$-orbit of any point in L'_Q circulates between the leaves L'_Q and $L'_{\widetilde{\varphi}_1(Q)}$ of $\mathcal{F}'$;

3. if $Q = (a, b, \sqrt{g(a,b)})$ with $(a, b) \neq (\phi, \phi)$, then the $\widetilde{\varphi}_2$-orbit of any point in L'_Q circulates between five distinct leaves of $\mathcal{F}'$:

$$L'_Q \to L'_{\widetilde{\varphi}_1(Q)} \to L'_{\widetilde{\varphi}_1^{(2)}(Q)} \to L'_{\widetilde{\varphi}_1^{(3)}(Q)} \to L'_{\widetilde{\varphi}_1^{(4)}(Q)};$$

4. in all other cases the $\widetilde{\varphi}_2$-orbit of any point in L'_Q circulates between ten distinct leaves of $\mathcal{F}'$:

$$L'_Q \to L'_{\widetilde{\varphi}_1(Q)} \to \ldots \to L'_{\widetilde{\varphi}_1^{(9)}(Q)}.$$

Again, studying the restrictions of $\widetilde{\varphi}_2$ to L'_P, and of $\widetilde{\varphi}_2^{(2)}, \widetilde{\varphi}_2^{(5)}$ or $\widetilde{\varphi}_2^{(10)}$ to other leaves L'_Q, we can draw conclusions about the reduced map $\widetilde{\varphi}_2$.

In this case the relevant conclusion is that the map $\widetilde{\varphi}_2$ has no periodic points of any period. In spite of the apparent simplicity of the result, some intermediate results and nontrivial computations are needed to deduce it. We refer the reader to [6] for ommited details in the proof.

Lemma 3. *The map $\widetilde{\varphi}_2 : \mathbb{R}^5_+ \to \mathbb{R}^5_+$ given by*

$$\widetilde{\varphi}_2(y_1, y_2, \ldots, y_5) = \left(y_2, \frac{1 + y_2}{y_1}, \frac{y_2(1 + y_2)}{y_3}, y_5, \frac{y_3 y_5^2}{y_2 y_4} \right)$$

does not have any periodic points.

Proof. From the leaf-invariance described above, it is enough to study the restrictions of $\widetilde{\varphi}_2$, $\widetilde{\varphi}_2^{(2)}$, $\widetilde{\varphi}_2^{(5)}$ and $\widetilde{\varphi}_2^{(10)}$ to the appropriate leaves of $\mathcal{F}'$. All these restricted maps will lead to maps of the form (16), which were comprehensively studied in [6].

The restriction h_1 of $\widetilde{\varphi}_2$ to L'_P is given in the natural coordinates (y_4, y_5) by

$$h_1(y_4, y_5) = \left(y_5, \sqrt{\phi} \frac{y_5^2}{y_4} \right).$$

By Lemma 2 in[6] this map has no periodic points and all its components go to infinity. Consequently $\widetilde{\varphi}_2$ has no periodic points in L'_P.

The restriction of $\widetilde{\varphi}_2^{(2)}$ to L'_Q, where $Q = (\phi, \phi, c) \neq P$, is given by

$$h_2(y_4, y_5) = c \left(\frac{1}{\phi} \frac{y_5^2}{y_4}, \frac{y_5^3}{y_4^2} \right),$$

and the restriction of $\widetilde{\varphi}_2^{(5)}$ to L'_Q, where $Q = (a, b, \sqrt{g(a,b)})$ and $(a, b) \neq (\phi, \phi)$, is given by

$$h_5(y_4, y_5) = \frac{(1 + b) g(a, b)}{b} \left(a \frac{y_5^5}{y_4^4}, \frac{(1 + a + b)\sqrt{g(a, b)}}{b} \frac{y_5^6}{y_4^5} \right). \qquad (23)$$

Finally the restriction of $\widetilde{\varphi}_2^{(10)}$ to any other L'_Q is given by

$$h_{10}(y_4, y_5) = k(a,b,c)\left(\frac{y_5^{10}}{y_4^9}, \frac{(1+a)(1+a+b)(1+b)}{ab}\frac{y_5^{11}}{y_4^{10}}\right),$$

with $k(a,b,c) = \frac{(1+a)^2(1+a+b)^3(1+b)^4c^5}{ab^5}$.

All three restricted maps h_2, h_5 and h_{10} belong to the group of maps of the form (16) and by Theorem 1 in [6] they are conjugate to the map

$$f(x,y) = (y, \xi\frac{y^2}{x})$$

with $\xi = \phi^2 > 1$ in the first case, $\xi = \left(\frac{(1+a)(1+b)(1+a+b)}{ab}\right)^{\frac{5}{2}} > 1$ in the second case and $\xi = \left(\frac{(1+a)(1+b)(1+a+b)}{ab}\right)^{10} > 1$ in the third case.

By Lemma 2 in the same reference the maps h_2, h_5 and h_{10} have no periodic points and all their components go to infinity. Consequently the map $\widetilde{\varphi}_2$ has no periodic point in any leaf of the form L'_Q and the conclusion follows. □

Finally we will describe some dynamical properties of the cluster map φ. This will be accomplished from the dynamics of the Lyness map $\widehat{\varphi}$ and from the conclusions drawn in lemmas 2 and 3 for the dynamics of the maps $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$, respectively.

Recall that we have a flag of simple subfoliations of $\mathbb{R}^7_+$,

$$\mathcal{F}^{P_2} \prec \mathcal{F}^{P_1} \prec \mathcal{F}^{\omega},$$

with the leaves of the 5-dimensional null foliation $\mathcal{F}^{\omega}$ being given by

$$N_{(a,b)} = \{\mathbf{x} \in \mathbb{R}^7_+ : x_1x_6 = ax_3x_4,\ x_2x_7 = bx_4x_5\} \tag{24}$$

and the leaves of the 4-dimensional symplectic foliation $\mathcal{F}^{P_1}$ being given by

$$S_{(a,b,c)} = \{\mathbf{x} \in N_{(a,b)} : x_1x_7 = cx_4^2\} \tag{25}$$

(the leaves of the 2-dimensional symplectic foliation $\mathcal{F}^{P_2}$ are not invariant under any iterate of φ, therefore will not be used in the description of the dynamics of φ).

The corresponding reduced systems of φ are $(\widehat{\varphi}, \Pi)$, $(\widetilde{\varphi}_1, \Pi_1)$ and $(\widetilde{\varphi}_2, \Pi_2)$ (cf. commutative diagram (18)), and assembling the results obtained in Lemma 2 and Lemma 3 we arrive at the following description.

Proposition 10. *Consider the cluster map $\varphi : \mathbb{R}^7_+ \to \mathbb{R}^7_+$ given by*

$$\varphi(x_1, \ldots, x_7) = \left(x_2, x_3, \ldots, x_7, \frac{x_2x_7 + x_4x_5}{x_1}\right),$$

and its reduced maps $\widehat{\varphi}$ and $\widetilde{\varphi}_1$ in (17).

Let ϕ denote the golden number $\frac{1+\sqrt{5}}{2}$ and consider the null leaves of $\mathcal{F}^{\omega}$ and the symplectic leaves of $\mathcal{F}^{P_1}$ given respectively by (24) *and* (25).

Then there are four distinct types of orbits of φ with respect to the 5-dimensional foliation $\mathcal{F}^{\omega}$ and its 4-dimensional subfoliation $\mathcal{F}^{P_1}$. More precisely, a φ-orbit is either entirely contained in the null leaf $N_{(\phi,\phi)}$, or it circulates between five distinct null leaves:

$$N_{(a,b)} \to N_{\widehat{\varphi}(a,b)} \to N_{\widehat{\varphi}^{(2)}(a,b)} \to N_{\widehat{\varphi}(3)(a,b)} \to N_{\widehat{\varphi}(4)(a,b)}.$$

In the null leaf $N_{(\phi,\phi)}$ there exist two exclusive cases, which are:

1. *the φ-orbit is entirely contained in the symplectic leaf $S_{(\phi,\phi,\sqrt{\phi^3})}$;*
2. *the φ-orbit circulates between two distinct symplectic leaves:*
$$S_{(\phi,\phi,c)} \to S_{\widetilde{\varphi}_1(\phi,\phi,c)},$$
with $c \neq \sqrt{\phi^3}$.

In any other null leaf $N_{(a,b)}$ there are another two exclusive cases, namely:

3. *the φ-orbit circulates between five distinct symplectic leaves:*
$$S_Q \to S_{\widetilde{\varphi}_1(Q)} \to S_{\widetilde{\varphi}_1^{(2)}(Q)} \to S_{\widetilde{\varphi}_1^{(3)}(Q)} \to S_{\widetilde{\varphi}_1^{(4)}(Q)},$$
with $Q = (a, b, \sqrt{g(a,b)})$ and $g(a,b)$ given by (21)*;*
4. *the φ-orbit circulates between ten distinct symplectic leaves*
$$S_Q \to S_{\widetilde{\varphi}_1(Q)} \to \cdots \to S_{\widetilde{\varphi}_1^{(9)}(Q)},$$
with $Q = (a, b, c)$ and $c \neq \sqrt{g(a,b)}$, in the following way: beginning in a leaf $S_Q \subset N_{(a,b)}$ the orbit comes back to $N_{(a,b)}$ to a different leaf $S_{\widetilde{\varphi}_1^{(5)}(Q)}$ after 5 iterations, and returns to the same leaf S_Q after 10 iterations.

Moreover, the map φ does not have any periodic points.

In Figure 1 we illustrate the contents of the previous proposition. To avoid overloading the picture we have omitted case *3.* of the proposition.

Acknowledgements.

The work of I. Cruz and H. Mena-Matos was partially funded by FCT under the project PEst-C/MAT/UI0144/2013.

The work of M. E. Sousa-Dias was partially funded by FCT/Portugal through the projects UID/MAT/04459/2013 and EXCL/MAT-GEO/0222/2012.

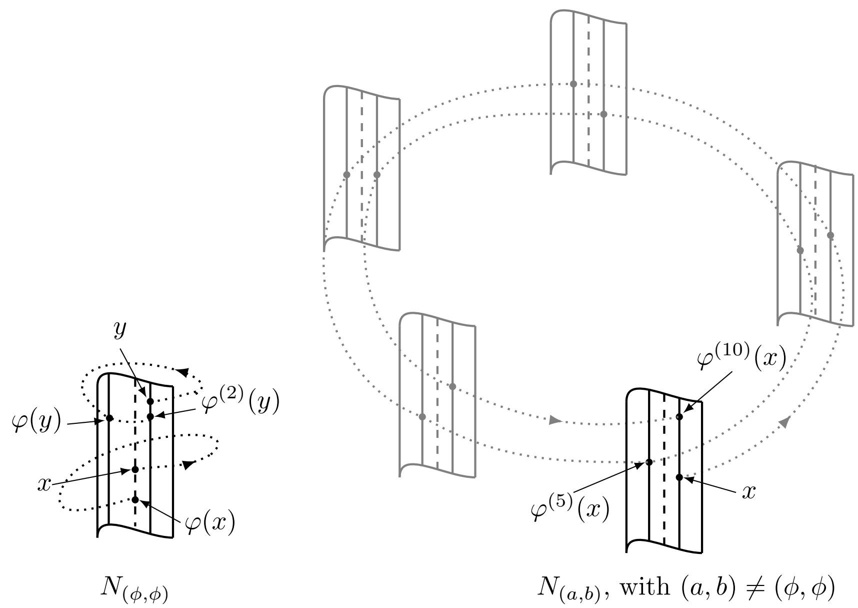


Figure 1: The foliation of the 5D null leaves $N_{(a,b)}$ by 4D symplectic leaves $S_{(a,b,c)}$ in Example 4, and the dynamics of φ on these leaves. On the left, the φ-invariant leaf $N_{(\phi,\phi)}$ and its $\varphi^{(2)}$-invariant leaves $S_{(\phi,\phi,c)}$. On the right, a $\varphi^{(5)}$-invariant leaf $N_{(a,b)}$ and its $\varphi^{(10)}$-invariant leaves $S_{(a,b,c)}$.

Appendix A Presymplectic reduction

Let B be an $N \times N$ skew-symmetric matrix of rank $2k$ and ω the presymplectic form

$$\omega = \sum_{1 \leq i < j \leq N} \frac{b_{ij}}{x_i x_j} dx_i \wedge dx_j.$$

Taking $v_i = \ln x_i$, the form ω is written as

$$\omega = \sum_{1 \leq i < j \leq N} b_{ij} dv_i \wedge dv_j.$$

As B has rank $2k$, Cartan's Theorem [15] guarantees that there exist $2k$ functions $g_1, \ldots, g_{2k}$ depending linearly on the variables v_i such that

$$\omega = \sum_{m=1}^{k} dg_{2m-1} \wedge dg_{2m}.$$

In fact, reordering if necessary the v_i-coordinates, we can assume that $b_{12} \neq 0$. Let

$$g_1 = \frac{1}{b_{12}} \sum_{k=1}^{N} b_{1k} v_k, \qquad g_2 = \sum_{k=1}^{N} b_{2k} v_k.$$

Then

$$dg_1 \wedge dg_2 = b_{12} dv_1 \wedge dv_2 + \sum_{i=3}^{N} b_{1i} dv_1 \wedge dv_i + \sum_{j=3}^{N} b_{2j} dv_2 \wedge dv_j + \alpha,$$

where α depends only on $v_3, ..., v_N$. Then $\widetilde{\omega} = \omega - dg_1 \wedge dg_2$ is a closed 2-form with rank equal to $2k-2$ on the $(N-2)$-dimensional space with coordinates $v_3, ..., v_N$. The same procedure can be now applied to $\widetilde{\omega}$ and after repeating it k times all the functions g_i will be obtained.

As each function g_i is a linear function of the variables v_j and $v_j = \ln x_j$, we have:

$$g_i(\ln x_1, ..., \ln x_N) = k_i^1 \ln x_1 + \cdots + k_i^N \ln x_N = \ln(f_i(\mathbf{x})),$$

where $f_i(\mathbf{x}) = \mathbf{x}^{\mathbf{k}_i} = x_1^{k_i^1} \cdots x_N^{k_i^N}$.

The submersion Π_0 and the reduced map $\widehat{\varphi}$ are then given respectively by:

$$\Pi_0(x_1, ..., x_N) = (f_1(\mathbf{x}), ..., f_{2k}(\mathbf{x})) \qquad \widehat{\varphi}(f_1, ..., f_N) = (f_1 \circ \varphi, ..., f_{2k} \circ \varphi).$$

Remark 8. The submersion Π_0 and the reduced map $\widehat{\varphi}$ are not unique. In fact, any two rows i and j for which $b_{ij} \neq 0$ can be used to produce g_1 and g_2. In this case we define

$$g_1 = \frac{1}{b_{ij}} \sum_{k=1}^{N} b_{ik} v_k, \qquad g_2 = \sum_{k=1}^{N} b_{jk} v_k,$$

and the procedure goes on as described above, now with α depending on all v coordinates except v_i and v_j.

Appendix B Technical lemma

The following lemma and its proof can be found in [5].

Lemma 4 (Lemma 2 of [5]). *Let $g : \mathcal{S} \subset \mathbb{R}^n \longrightarrow \mathcal{S}$ be a map of the form $g(\mathbf{x}) = G(\mathbf{x}) D\mathbf{x}$, with D a constant diagonal matrix and G a real-valued function defined on $\mathcal{S}$. If $G(g(\mathbf{x})) = c\, G(\mathbf{x})$ for some constant $c \in \mathbb{R}$ then*

$$g^{(n)}(\mathbf{x}) = c^{\frac{n(n-1)}{2}} G^n(\mathbf{x}) D^n \mathbf{x},$$

for all $n \in \mathbb{N}_0$.

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