TORSION LINE BUNDLES AND BRANES ON THE HITCHIN SYSTEM

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Abstract. We study the fixed loci for the action of tensorisation by a line bundle of order n on the moduli space of Higgs bundles for the Langlands self-dual group $\mathrm{GL}(n,\mathbb{C})$. We equip these loci with a hyperholomorphic bundle so that they can be viewed as BBB-branes, and we introduce corresponding BAA-branes which can be described via Hecke modifications. We then show how these branes are naturally dual via explicit Fourier-Mukai transform. It is noteworthy that these branes lie over the singular locus of the Hitchin fibration.

1. Introduction

Motivated by a dimensional reduction of the self-dual equations on a 4-manifold, N. Hitchin introduced in [Hi1] Higgs bundles over a smooth projective complex curve X of genus $g \geqslant 2$. These are pairs (E, φ), where E is a holomorphic vector bundle over X and φ is a holomorphic one-form with values in $\mathrm{End}(E)$. The moduli space of Higgs bundles $\mathbf{M}_X(n,d)$ of rank n and degree d is a holomorphic symplectic manifold carrying a hyperkähler metric. Moreover, it admits the structure of an algebraic completely integrable system given by the Hitchin map $h\colon \mathbf{M}_X(n,d) \to B_{X,n}$. Here the Hitchin base $B_{X,n}$ is an affine space whose dimension is half that of $\mathbf{M}_X(n,d)$, and the components of h are the coefficients of the characteristic polynomial of φ. The fibre of h over a generic point of the Hitchin base is a torsor for an abelian variety, namely the Jacobian of an associated spectral curve.

The concept of a G-Higgs bundle can be defined for any complex (and even real) reductive Lie group G and the definition above given above is then that of a $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle. N. Hitchin [Hi2] showed that his original construction gives an algebraically completely integrable system for any classical complex Lie group G. The Hitchin system has since been extensively studied by many authors, in particular it was generalized to arbitrary complex reductive groups G (see, for example, R. Donagi and E. Markman [DM]).

A new development arose with the discovery by T. Hausel and M. Thaddeus [HT] of a close relation between Higgs bundles, mirror symmetry and the Langlands correspondence. They proved that the moduli spaces of Higgs bundles for, respectively, the group $\mathrm{SL}(n,\mathbb{C})$ and its Langlands dual group $\mathrm{PGL}(n,\mathbb{C})$ form a pair of SYZ-mirror partners [SYZ], in the sense that the respective Hitchin maps have naturally isomorphic bases and their fibres over corresponding points are, generically, half-dimensional torsors for a pair of dual abelian varieties. This was subsequently generalised by N. Hitchin [Hi3] for the self-dual group G_2 and then by R. Donagi and T. Pantev [DP] for any pair $(G, {}^LG)$ of Langlands dual groups. The duality is reflected by a Fourier-Mukai transform between the moduli spaces interchanging fibres of the Hitchin map over corresponding points in the base. These dualities were obtained over the locus of the Hitchin base where the corresponding spectral curves are smooth.

As mentioned above, the moduli space $\mathbf{M}_X(n,d)$ is hyperkähler. This means that it carries three natural complex structures I_1, I_2 and I_3 verifying the quaternionic relations and a metric which is Kähler with respect to all three holomorphic structures. In the present case, I_1 is the natural complex structure on the moduli space of Higgs bundles $\mathbf{M}_X(n,d)$, while the complex structures I_2 and $I_3 = I_1 I_2$ arise via the non-abelian Hodge Theorem, which identifies $\mathbf{M}_X(n,d)$ with the moduli space of flat $\mathrm{GL}(n,\mathbb{C})$-connections (see [Hi1, Si1]).

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A. Kapustin and E. Witten considered in [KW] certain special subvarieties of $\mathbf{M}_X(n,d)$, equipped with special sheaves. The pair composed by such a subvariety and the corresponding sheaf is called a brane. For each of the complex structures on $\mathbf{M}_X(n,d)$ a brane is classified as follows: it is of type A if it is a Lagrangian subvariety with respect to the corresponding Kähler form and the sheaf over it is a flat bundle, and it is of type B if it is a holomorphic subvariety and the sheaf over it is also holomorphic. Thus, for instance, a (BBB)-brane is a subvariety equipped with a sheaf, both holomorphic with respect to all three complex structures I_1, I_2 and I_3; in other words, it is a hyperholomorphic subvariety equipped with a hyperholomorphic sheaf. Similarly, a (BAA)-brane is a subvariety which is holomorphic with respect to I_1, and Lagrangian with respect to I_2 and I_3. There are only two other possible types of branes on $\mathbf{M}_X(n,d)$, namely (ABA)- and (AAB)-branes. Again all this holds for any complex Lie group and not just $\mathrm{GL}(n,\mathbb{C})$.

According to [KW], mirror symmetry conjecturally interchanges (BBB)-branes and (BAA)-branes, and mathematically this duality should again be realized via a Fourier-Mukai transform (in complex structure I_1). The support of the (BAA)-brane should depend not only on the support of the dual (BBB)-brane but also on the hyperholomophic sheaf over it (and vice-versa). A similar story holds for a pair of (ABA)- and (AAB)-branes.

Since Kapustin and Witten's paper—and because of it—an intense study of several kinds of branes on Higgs bundle moduli spaces has been carried out. Some examples may be found it [Hi4, BS1, BGP, HS, BCFG, Hi5, Ga, FJ, BS2, FP, B, HMDP] (see also [AFES] for a survey on this subject). Most of these works mainly focus either on the smooth locus of the Hitchin system (exceptions are [BS2, FP, B]) or only deal with the support of the branes and not with the sheaves on it (exceptions are [Hi4, Hi5, Ga, FJ, FP]).

In this paper, we introduce families of dual (BBB)-branes and (BAA)-branes on $\mathbf{M}_X(n,d)$, the moduli space for the self-dual group $\mathrm{GL}(n,\mathbb{C})$. Our construction is notable for two reasons. Firstly, our branes are supported on a subspace $B^\gamma \subset B_{X,n}$ of the singular locus of the Hitchin map. Secondly, as required in the general picture, our branes come equipped with natural hyperholomorphic/flat sheaves and, taking due account of these, we can explicitly prove (when $d = 0$) that the branes are dual under a Fourier-Mukai transformation over an open dense subspace B^γ_{red} of B^γ.

In the following we outline our construction in more detail. Let $\mathrm{Jac}(X)[n]$ denote the subgroup of the Jacobian $\mathrm{Jac}(X)$ of X of elements of order n. Let $\gamma \in \mathrm{Jac}(X)[n]$ be a non-trivial element, and let $L_\gamma \to X$ be the corresponding line bundle. Here we consider the subvariety $\mathbf{M}_X(n,d)^\gamma$ of $\mathbf{M}_X(n,d)$ of points (E,φ) fixed by tensorisation by L_γ, i.e. $(E,\varphi) \cong (E \otimes L_\gamma, \varphi)$. Using the description in [HT, NR] we prove that, when γ has order n, $\mathbf{M}_X(n,d)^\gamma$ is a hyperholomorphic subvariety and that it can be endowed with a family of hyperholomorphic bundles. In other words, we have a family of (BBB)-branes supported on $\mathbf{M}_X(n,d)^\gamma$. Let B^γ be the image of $\mathbf{M}_X(n,d)^\gamma$ under the Hitchin map h. Then $B^\gamma \subset B_{X,n}$ lies in the singular locus of h, in the sense that the corresponding spectral curves are always singular and possibly non-reduced (but always irreducible). Let B^γ_{red} be the subspace of B^γ whose associated spectral curves are reduced. Let $p_\gamma : X_\gamma \to X$ be the unramified n-cover naturally defined by L_γ, with Galois group isomorphic to $\mathbb{Z}_n$. Then B^γ_{red} can be described as the subspace of all points in the Hitchin base $B_{X,n}$ whose spectral curves have X_γ as a normalisation. It basically follows from [NR] that the pushforward by p_γ yields an isomorphism between $\mathbf{M}_X(n,d)^\gamma$ and $T^*\,\mathrm{Jac}^d(X_\gamma)/\mathbb{Z}_n$, with the Galois group acting by pullback. From this one defines a hyperholomorphic line bundle $\mathscr{L}$ over $\mathbf{M}_X(n,d)^\gamma$, naturally associated to a flat line bundle $\mathcal{L}$ on X. We call the pair $(\mathbf{M}_X(n,d)^\gamma, \mathscr{L})$ a *basic Narasimhan-Ramanan* (BBB)-*brane*, since $\mathcal{L}$ is a line bundle on the base X. We represent it by $(\mathbf{BBB})^\gamma_{\mathcal{L}}$ and write $(\mathbf{BBB})^{\gamma,\mathcal{L}}_{\mathrm{red}}$ for its restriction to B^γ_{red}. More generally, we can construct a hypeholomorphic sheaf $\mathscr{F}$ on $\mathbf{M}_X(n,d)^\gamma$, canonically associated to a flat line bundle $\mathcal{F}$ over X_γ, and we call the pair $(\mathbf{M}_X(n,d)^\gamma, \mathscr{F})$ a *non-basic Narasimhan-Ramanan* (BBB)-*brane* and represent it by $(\mathbf{BBB})^\gamma_{\mathcal{F}}$.

For a given $b \in B^\gamma_{\mathrm{red}}$, let X_b the corresponding spectral curve and $\nu_b : X_\gamma \to X_b$ the normalisation. The spectral data of the intersection $\mathbf{M}_X(n,d)^\gamma \cap h^{-1}(b)$ is given by $\mathrm{Jac}^d(X_\gamma)$, embedded in the compactified Jacobian $\overline{\mathrm{Jac}}^{\,\delta+d}(X_b)$ via pushforward by ν_b, where $\delta = n(n-1)(g-1)$. Hence it lies in $\overline{\mathrm{Jac}}^{\,\delta+d}(X_b) \setminus \mathrm{Jac}^{\delta+d}(X_b)$.

We then turn to the question of identifying the corresponding dual (BAA)-brane of $(\mathbf{BBB})^\gamma_{\mathcal{L}}$. First, we look at what happens on a fibre over $b \in B^\gamma_{\mathrm{red}}$, and consider the subspace of $\mathrm{Jac}^{\delta+d}(X_b)$ consisting of those line bundles over X_b whose pullback to X_γ is isomorphic to $p_\gamma^*(\mathcal{L} \otimes K^{(n-1)/2})$. Thus we are looking at the fibre over $p_\gamma^*(\mathcal{L} \otimes K^{(n-1)/2})$ of the map $\mathrm{Jac}^{\delta+d}(X_b) \to \mathrm{Jac}^{\delta+d}(X_\gamma)$ induced by pullback under ν_b. We prove that, upon varying b over B^γ_{red}, this defines a complex Lagrangian subvariety of $\mathbf{M}_X(n,d)$ on the locus of reduced spectral curves. With the trivial line bundle over it, it hence becomes a (BAA)-brane on $\mathbf{M}_X(n,d)$, which we denote by $(\mathbf{BAA})^{\gamma,\mathcal{L}}_{\mathrm{red}}$. A similar story holds for the non-basic case $(\mathbf{BBB})^{\gamma}_{\mathcal{F}}$. Moreover, the Higgs bundles over the intersection $(\mathbf{BAA})^{\gamma,\mathcal{L}}_{\mathrm{red}} \cap h^{-1}(b)$ are precisely described as the ones that can be obtained as Hecke transforms (associated to the divisor of singularities of X_b) of the γ-fixed Higgs bundle with underlying bundle $p_{\gamma,*}p_\gamma^*(\mathcal{L}\otimes K^{(n-1)/2}) \cong \mathcal{L}\otimes K^{(n-1)/2}\otimes\bigoplus_{i=0}^{n-1} L_\gamma^{-i}$, and with the Higgs field naturally determined by b. Hence, we can roughly say that the subvariety defining the support of $(\mathbf{BAA})^{\gamma,\mathcal{L}}_{\mathrm{red}}$ is obtained as a Hecke transformation of the subvariety of those Higgs bundles in $\mathbf{M}_X(n,\delta+d)^\gamma$ whose underlying bundle is $p_{\gamma,*}p_\gamma^*(\mathcal{L}\otimes K^{(n-1)/2})$. For this reason we refer to $(\mathbf{BAA})^{\gamma,\mathcal{L}}_{\mathrm{red}}$ as a *Hecke* (BAA)-*brane*. Hecke transformations in the context of Higgs bundles have previously appeared in several papers; see, for example, [Hi6, HR, Ra, Wi, W].

Our main result is that, for $d=0$, the branes $(\mathbf{BBB})^{\gamma,\mathcal{L}}_{\mathrm{red}}$ and $(\mathbf{BAA})^{\gamma,\mathcal{L}}_{\mathrm{red}}$ are dual when restricted to the locus of reduced spectral curves (the analogous result also holds in the non-basic case). Indeed, we explicitly describe a Fourier-Mukai transform over the fibres of B^γ_{red} interchanging them. This Fourier-Mukai transform is carried out using the autoduality of compactified Jacobians of integral curves with planar singularities, from the general results of D. Arinkin [Ar]. It uses a Hitchin section (which embeds B^γ_{red} as a subvariety of the support of $(\mathbf{BAA})^{\gamma,\mathcal{L}}_{\mathrm{red}}$), to identify $\overline{\mathrm{Jac}}^{\,\delta+d}(X_b)$ with the corresponding $\overline{\mathrm{Jac}}^{\,0}(X_b)$, which is then shown to be autodual.

The corresponding result for the non-basic case is true as well. For d non-multiple of n a similar result should hold, but the duality should require a gerbe to work out properly. We also note that the results in this paper provide evidence for the dualities suggested in [FP].

It is interesting to notice that the support of our (BBB)-branes play a central role in the proof by Hausel and Thaddeus [HT] of topological mirror symmetry for the moduli spaces of Higgs bundles for the Langlands dual groups $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{PGL}(n,\mathbb{C})$ for $n=2,3$ (the general case has recently been proved by Groechenig, Wiss and Ziegler [GWZ]). One might thus expect that further study of our dual branes in this setting would provide a better geometric undestanding of the calculation of Hausel and Thaddeus. We will come back to this question in a future article.

Here is a brief description of the organisation of the paper. In Section 2 we recall some background material on the Hitchin system, and also on the Fourier-Mukai for compactified Jacobians. Section 3 deals with the construction and description of the Narasimhan-Ramanan (BBB)-branes, including the corresponding spectral data over a subspace B^γ of the Hitchin base. In Section 4 we describe a family of (BAA)-branes, whose support maps to B^γ under h. In Section 5 we prove that, for $d=0$, these two families (when restricted to the open dense subspace B^γ_{red}) are dual under mirror symmetry, by explicitly proving a Fourier-Mukai interchanging them. Finally, in Section 6, we generalise the previous study to the case where γ has order strictly less than n in $\mathrm{Jac}(X)[n]$. We give a description of a hyperholomorphic subvariety which conjecturally admits a hyperholomophic sheaf, hence conjecturally a (BBB)-brane, and describe a Lagrangian subvariety which is the support of the conjecturally dual (BAA)-brane.

2. Preliminaries

2.1. **Higgs bundles and their moduli space.** Let X be a smooth projective curve over $\mathbb{C}$, of genus $g\geqslant 2$. A Higgs bundle over X is defined [Hi1, Si1, Si2, Si3] as a pair (E,φ) given by a holomorphic vector bundle $E\to X$, and a holomorphic section of the endomorphisms bundle, twisted by the canonical bundle K of X,

$$\varphi\in H^0(X,\mathrm{End}(E)\otimes K).$$

If the rank of E is n, then (E,φ) is also, more precisely, called a $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle.

Let $\mathbf{M}_X(n,d)$ denote the moduli space of rank n and degree d polystable Higgs bundles on X. It is a quasi-projective variety, whose complex dimension is given by

$$\dim\mathbf{M}_X(n,d) = 2n^2(g-1)+2.$$

The moduli space $\mathbf{M}_X(n,d)$ is also denoted, following Simpson's notation introduced in [Si2], by $\mathbf{M}^{\mathrm{Dol}}_{X,n}(d)$. The reason for this is that there is as a hyperkähler manifold [Hi1, Si3] $\mathbf{M}_{X,n}(d)$ with complex structures

$$I_1,\quad I_2\quad\text{and}\quad I_3=I_1I_2 \tag{2.1}$$

such that one can consider $\mathbf{M}^{\mathrm{Dol}}_{X,n}(d)$ as $(\mathbf{M}_{X,n}(d), I_1)$; it also called the *Dolbeault moduli space*. The *De Rham moduli space* $\mathbf{M}^{\mathrm{DR}}_{X,n}(d)$ is given by $(\mathbf{M}_{X,n}(d), I_2)$. It is the moduli space of connections on a fixed C^∞ vector bundle $\mathbb{E}$ over X of rank n and degree d, with constant central curvature equal to d/n (hence projectively flat, and actually flat if $d = 0$).

Denote by ω_j the Kähler form associated to I_j and by $\Omega_j = \omega_{j+1} + i\omega_{j-1}$ the corresponding holomorphic symplectic form.

Non-abelian Hodge theory establishes the existence of a homeomorphism [Hi1, Si2, Si3, Do, Co] between these spaces

$$\mathbf{M}^{\mathrm{DR}}_{X,n}(d) \overset{homeo}{\cong} \mathbf{M}^{\mathrm{Dol}}_{X,n}(d).$$

We shall mainly use the notation $\mathbf{M}_X(n,d)$ instead of $\mathbf{M}^{\mathrm{Dol}}_{X,n}(d)$ for the moduli of Higgs bundles.

Given a Higgs bundle (E,φ), we have the associated deformation complex

$$C^\bullet : \mathrm{End}(E) \xrightarrow{[-,\varphi]} \mathrm{End}(E) \otimes K,$$

with hypercohomology fits in the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbb{H}^0(C^\bullet) \longrightarrow H^0(\mathrm{End}(E)) \longrightarrow H^0(\mathrm{End}(E)\otimes K) \longrightarrow \mathbb{H}^1(C^\bullet) \longrightarrow \\ \longrightarrow H^1(\mathrm{End}(E)) \longrightarrow H^1(\mathrm{End}(E)\otimes K) \longrightarrow \mathbb{H}^2(C^\bullet) \longrightarrow 0. \end{aligned}$$

If (E,φ) is a stable Higgs bundle, it represents a smooth point of the moduli space $\mathbf{M}_X(n,d)$ with tangent space $T_{(E,\varphi)}\mathbf{M}_X(n,d) = \mathbb{H}^1(C^\bullet)$. Thanks to Serre duality, $\varphi \in H^0(\mathrm{End}(E)\otimes K)$ is an element of the dual space of $H^1(\mathrm{End}(E))$.

We define the 1-form θ as the contraction of φ with the map $\mathbb{H}^1(C^\bullet) \to H^1(\mathrm{End}(E))$. It can be check that $d\theta$ defines a holomorphic symplectic form on the smooth locus of $\mathbf{M}_X(n,d)$ which coincides with $\Omega_1 = \omega_2 + i\omega_3$.

2.2. **The Hitchin system.** We recall here the spectral construction given in [Hi2, BNR]. Let $(p_1,\dots,p_n)$ be a base of $\mathrm{GL}(n,\mathbb{C})$-invariant polynomials with $\deg(p_i) = i$; for instance, we could take $p_i(-) = (-1)^i \mathrm{tr}(\wedge^i -)$. The *Hitchin map* is the projection

$$\begin{array}{rccc} h_{X,n}: & \mathbf{M}_X(n,d) & \longrightarrow & B_{X,n} := \bigoplus_{i=0}^{n-1} H^0(X,K^i) \\ & (E,\varphi) & \longmapsto & (p_1(\varphi),\dots,p_n(\varphi)). \end{array}$$

Note that $\dim(B_{X,n}) = n^2(g-1)+1 = \dim(\mathbf{M}_X(n,d))/2$.

Consider the total space $|K|$ of the canonical bundle and the obvious algebraic surjection $\pi : |K| \to X$. The pullback bundle $\pi^*K \to |K|$ comes naturally equipped with a tautological section λ. Given an element $b \in B_{X,n}$, with $b = (b_1,\dots,b_n)$, we construct the *spectral curve* $X_b \subset |K|$ by considering the vanishing locus of the section of π^*K^n

$$\lambda^n + (\pi^*b_1)\lambda^{n-1} + \cdots + (\pi^*b_{n-1})\lambda + \pi^*b_n \in H^0(|K|, \pi^*K^n).$$

It follows that the restriction of $\pi : |K| \to X$ to X_b yields a ramified degree n cover that, by abuse of notation, we also denote with

$$\pi : X_b \longrightarrow X.$$

For generic b, the spectral curve X_b is smooth, but it can be singular, reductive and even non-reduced. Since the canonical divisor of the symplectic surface $|K|$ is zero and X_b belongs to the linear system $|nX|$, one can compute the genus of X_b, yielding

$$g(X_b) = 1 + n^2(g-1). \tag{2.2}$$

Furthermore, using Riemann-Roch, we see that $\pi_*\mathcal{O}_{X_b}$ is a rank n vector bundle of degree

$$\deg(\pi_*\mathcal{O}_{X_b}) = -n(n-1)(g-1).$$

Given a torsion-free rank 1 sheaf $\mathcal{F}$ over X_b of degree $\delta + d$, where

$$\delta := n(n-1)(g-1), \tag{2.3}$$

we have that $E_{\mathcal{F}} := \pi_* \mathcal{F}$ is a vector bundle on X of rank n and degree d. Since π is an affine morphism, the natural $\mathcal{O}_{|K|}$-module structure on $\mathcal{F}$, given by understanding $\mathcal{F}$ as a sheaf supported on $|K|$, corresponds to a $\pi_* \mathcal{O}_{|K|} = \mathrm{Sym}^\bullet(K^*)$-module structure on $E_{\mathcal{F}}$. Such structure on $E_{\mathcal{F}}$ is equivalent to a Higgs field

$$\varphi_{\mathcal{F}} : E_{\mathcal{F}} \longrightarrow E_{\mathcal{F}} \otimes K,$$

as we know from [BNR, Si2]. This stablishes a one-to-one correspondence between torsion-free rank 1 sheaves on X_b and Higgs bundles $(E_{\mathcal{F}}, \varphi_{\mathcal{F}})$ such that

$$h_{X,n}(E_{\mathcal{F}}, \varphi_{\mathcal{F}}) = b.$$

The pair $(X_b, \mathcal{F})$ is said to be the *spectral data* of the Higgs bundle (E, φ).

Furthermore, since this correspondence is done in [Si2] in a relative setting and semistability and stability are preserved under the spectral correspondence [Si2, Corollary 6.9], one has an isomorphism between the Hitchin fibre and the moduli space of rank 1 torsion free sheaves on X_b of degree δ. This moduli space is precisely the compactified Jacobian (of degree $\delta + d$) of X_b, which we denote by $\overline{\mathrm{Jac}}^{\delta+d}(X_b)$. Hence

$$h^{-1}_{X,n}(b) \cong \overline{\mathrm{Jac}}^{\delta+d}(X_b). \tag{2.4}$$

2.3. **Fourier–Mukai on compactified Jacobians of integral curves.** In this section we review autoduality of compactified Jacobians of integral curves with planar singularities and the associated Fourier-Mukai transform given by Arinkin in [Ar] thanks to the construction of a Poincaré sheaf.

By construction, the spectral curve X_b has planar singularities as it is contained in $|K|$. Therefore, when X_b is integral, Arinkin's duality becomes a duality of the corresponding Hitchin fibres $h^{-1}_{X,n}(b) \cong \overline{\mathrm{Jac}}^{\delta+d}(X_b)$.

Suppose that X_b is integral. Then every semistable rank 1 torsion free sheaf on X_b is indeed stable and $\overline{\mathrm{Jac}}^{\delta+d}(X_b)$ is a fine moduli space with universal family $\mathcal{U}_b \to X_b \times \overline{\mathrm{Jac}}^{\delta+d}(X_b)$. Denote by $\mathcal{U}_b^0$ its restriction to $X_b \times \mathrm{Jac}^{\delta}(X_b)$. Before constructing the Poincaré sheaf, we first construct the Poincaré bundle using $\mathcal{U}_b$ and $\mathcal{U}_b^0$.

Given a flat morphism $f : Y \to S$ whose geometric fibres are curves, we can define the determinant of cohomology (see [KM] and [Es, Section 6.1]) as follows. If $\mathcal{F}$ is an S-flat sheaf on Y, the determinant of cohomology $\mathcal{D}_f(\mathcal{E})$ is an invertible sheaf on S, constructed locally as the determinant of complexes of free sheaves, which is locally quasi-isomorphic to $Rf_*\mathcal{E}$. Consider the triple product $X_b \times \overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \mathrm{Jac}(X_b)$ and the projection $f_{23} : X_b \times \overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \mathrm{Jac}^{\delta+d}(X_b) \to \overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \mathrm{Jac}^{\delta+d}(X_b)$, which is flat and whose fibres are curves. Consider as well the corresponding obvious projections f_{12} and f_{13}. The Poincaré line bundle $\mathcal{P}_b \to \overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \mathrm{Jac}^{\delta}(X_b)$ is the invertible sheaf

$$\mathcal{P}_b := \mathcal{D}_{f_{23}}\left(f_{12}^*\mathcal{U}_b \otimes f_{13}^*\mathcal{U}_b^0\right)^{-1} \otimes \mathcal{D}_{f_{23}}\left(f_{13}^*\mathcal{U}_b^0\right) \otimes \mathcal{D}_{f_{23}}\left(f_{12}^*\mathcal{U}_b\right).$$

The restriction of the Poincaré bundle $\mathcal{P}_b$ to the point associated to $M \in \mathrm{Jac}^{\delta}(X_b)$, that is, $\mathcal{P}_{b,M} := \mathcal{P}_b|_{\overline{\mathrm{Jac}}^{\delta+d}(X_b)\times\{M\}}$, is the line bundle over $\overline{\mathrm{Jac}}^{\delta+d}(X_b)$ given by

$$\mathcal{P}_{b,M} = \mathcal{D}_{f_2}(\mathcal{U}_b \otimes f_1^*M)^{-1} \otimes \mathcal{D}_{f_2}(f_1^*M) \otimes \mathcal{D}_{f_2}(\mathcal{U}_b), \tag{2.5}$$

where we have considered the obvious projections $f_1 : X_b \times \overline{\mathrm{Jac}}^{\delta+d}(X_b) \to X_b$ and $f_2 : X_b \times \overline{\mathrm{Jac}}^{\delta+d}(X_b) \to \overline{\mathrm{Jac}}^{\delta+d}(X_b)$.

Our Poincaré bundle is constructed over $\overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \mathrm{Jac}^{\delta+d}(X_b)$. Of course, one can perform a similar construction over $\mathrm{Jac}^{\delta+d}(X_b) \times \overline{\mathrm{Jac}}^{\delta+d}(X_b)$, which coincides with $\mathcal{P}_b$ after restricting both to $\mathrm{Jac}^{\delta+d}(X_b) \times \mathrm{Jac}^{\delta+d}(X_b)$. Gluing both line bundles over $\mathrm{Jac}^{\delta+d}(X_b) \times \mathrm{Jac}^{\delta+d}(X_b)$, one can define the line bundle

$$\mathcal{P}_b^\sharp \to \left(\overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \overline{\mathrm{Jac}}^{\delta+d}(X_b)\right)^\sharp,$$

where

$$\left(\overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \overline{\mathrm{Jac}}^{\delta+d}(X_b)\right)^\sharp := \left(\mathrm{Jac}^{\delta+d}(X_b) \times \overline{\mathrm{Jac}}^{\delta+d}(X_b)\right) \cup \left(\overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \mathrm{Jac}^{\delta+d}(X_b)\right).$$

Consider the injection

$$j : \left(\overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \overline{\mathrm{Jac}}^{\delta+d}(X_b)\right)^\sharp \hookrightarrow \overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \overline{\mathrm{Jac}}^{\delta+d}(X_b),$$

and define the Poincaré sheaf, as

$$\overline{\mathcal{P}}_b := j_* \mathcal{P}_b^\sharp.$$

Denote by π_1 (resp. π_2) the projection $\overline{\mathrm{Jac}}^{\delta+d}(X_b) \times \overline{\mathrm{Jac}}^{\delta+d}(X_b) \to \overline{\mathrm{Jac}}^{\delta+d}(X_b)$ onto the first (resp. second) factor. Using $\overline{\mathcal{P}}_b$ as a kernel, one can consider the integral functor

$$\begin{array}{rccc} \Theta_b : & D^b\left(\overline{\mathrm{Jac}}^{\delta+d}(X_b)\right) & \longrightarrow & D^b\left(\overline{\mathrm{Jac}}^{\delta+d}(X_b)\right) \\ & \mathcal{E}^\bullet & \longmapsto & R\pi_{2,*}(\pi_1^*\mathcal{E}^\bullet \otimes \overline{\mathcal{P}}_b) \end{array}. \tag{2.6}$$

Theorem 2.1 ([Ar])**.** *The moduli space of rank* 1 *torsion free sheaves over* $\overline{\mathrm{Jac}}^{\delta+d}(X_b)$ *is* $\overline{\mathrm{Jac}}^{\delta+d}(X_b)$ *itself. Furthermore* Θ_b *is a derived equivalence.*

3. Narasimhan-Ramanan (BBB)-branes

3.1. **Construction of** (BBB)**-branes.** In this section we construct a (BBB)-brane on $\mathbf{M}_X(n,d)$. By definition, this is [KW] a pair $(\mathrm{N}, (\mathscr{F}, \nabla_{\mathscr{F}}))$ given by:

(1) A hyperholomorphic subvariety $\mathrm{N} \subset \mathbf{M}_X(n,d)$, *i.e.* a subvariety which is holomorphic with respect to the three complex structures I_1, I_2 and I_3 (cf. (2.1)).
(2) A hyperholomorphic sheaf $(\mathscr{F}, \nabla_{\mathscr{F}})$ supported on $\mathbf{N}$, *i.e.* a sheaf $\mathscr{F}$ equipped with a connection whose curvature $\nabla_{\mathscr{F}}$ is of type $(1,1)$ in the complex structures I_1, I_2 and I_3.

Remark 3.1. Notice that a flat connection is trivially of type $(1,1)$ in any complex structure.

We start with the support of our (BBB)-brane. Inside the Jacobian $\mathrm{Jac}(X) = \mathrm{Jac}^0(X)$ consider the subgroup $\mathrm{Jac}(X)[n]$ of n-torsion elements,

$$\mathrm{Jac}(X)[n] := \{L \in \mathrm{Jac}(X) \mid L^n \cong \mathcal{O}_X\}.$$

Recall that, as an abstract group, we have $\mathrm{Jac}(X)[n] \cong \mathbb{Z}_n^{2g}$. It acts on the moduli space of Higgs bundles by tensorisation, *i.e.* as

$$\begin{array}{rccc} \gamma : & \mathbf{M}_X(n,d) & \longrightarrow & \mathbf{M}_X(n,d) \\ & (E,\varphi) & \longmapsto & (E \otimes L_\gamma, \varphi). \end{array} \tag{3.1}$$

Choose, once and for all, a non-trivial element

$$\gamma \in \mathrm{Jac}(X)[n],$$

and denote the corresponding line bundle by L_γ. Every construction in this paper is carried out for such a choice of γ.

Denote by $\mathbf{M}_X(n,d)^\gamma$ the subvariety of points fixed by $\gamma \in \mathrm{Jac}(X)[n]$ under (3.1). It is a hyperholomorphic subvariety since the tensorisation by line bundles is holomorphic in the three complex structures of $\mathbf{M}_X(n,d)$ (see [GR] for a proof in the case of $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles, that can be easily adapted to the case of $\mathrm{GL}(n,\mathbb{C})$).

Remark 3.2. Since we are working with $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles (not fixing the determinant) one can ask why don't we consider the same action, but of $\mathrm{Jac}(X)$ in $\mathbf{M}_X(n,d)$. In fact, it is straightforward to see that there are fixed points under tensorisation by $L \in \mathrm{Jac}(X)$ if and only if L has finite order. Hence the description of $\mathbf{M}_X(n,d)^\gamma$ is precisely the same.

Let $n = n' \cdot m$, and suppose $\gamma \in \mathbb{Z}_n$ has order m. Associated to L_γ there is a unique smooth projective curve X_γ, defined as the spectral cover of X given as the vanishing locus in the total space $|L_\gamma|$ of L_γ of the section $\theta^m - 1 = 0$, where $\theta \in H^0(|L_\gamma|, p_\gamma^* L_\gamma)$ is the tautological section, being $p_\gamma : |L_\gamma| \to X$ the projection.Let p_γ still denote the restriction to X_γ of the projection $|L_\gamma| \to X$, by

$$p_\gamma : X_\gamma \to X.$$

Observe that it is a unramified regular m-cover of X. Notice the difference between these covers and the ones appearing in the Hitchin system: the later ones are always ramified and the corresponding spectral curves are always subvarieties of the total space of K. Notice also that $p_\gamma^* L_\gamma$ is trivial over X_γ since the nowhere vanishing section $\theta : \mathcal{O}_{X_\gamma} \to p_\gamma^* L_\gamma$ gives a canonical trivialisation.

The following is the fundamental result describing the fixed point subvariety $\mathbf{M}_X(n,d)^\gamma$. The proof for vector bundles is done in [NR] for the locus of simple bundles, and then extended to all moduli space in Proposition 3.46 of [Na]. The adaptation for Higgs bundles is in [HT].

Theorem 3.3 ([NR, Na, HT])**.** *Let (E,φ) represent a point in $\mathbf{M}_X(n,d)$. Assume $n = n'\cdot m$ and let $\gamma \in \mathrm{Jac}(X)[n]$ be a non-trivial element of order m. Then (E,φ) is fixed by γ if and only if it is the pushforward of a semistable rank n' Higgs bundle (F,ϕ) over X_γ, that is,*

$$(E,\varphi) \cong (p_{\gamma,*}F, p_{\gamma,*}\phi).$$

Consider the moduli space $\mathbf{M}_{X_\gamma}(n',d)$ of rank n' and degree d Higgs bundles over X_γ. Since it is compatible with the other complex structures, the pushforward under p_γ gives rise to a hyperholomorphic morphism which is surjective onto $\mathbf{M}_X(n,d)^\gamma$ by Theorem 3.3,

$$\begin{array}{rccc} \check{p}_\gamma : & \mathbf{M}_{X_\gamma}(n',d) & \longrightarrow & \mathbf{M}_X(n,d)^\gamma \\ & (F,\phi) & \longmapsto & (p_{\gamma,*}F, p_{\gamma,*}\phi). \end{array} \tag{3.2}$$

By [NR], two points in semistable vector bundles over X_γ are pushed forward to isomorphic vector bundles on X if and only if they are in the same $\mathbb{Z}_m$-orbit. A analogous result holds for Higgs bundles, as well, hence we have an isomorphism

$$\mathbf{M}_X(n,d)^\gamma \cong \mathbf{M}_{X_\gamma}(n',d)/\mathbb{Z}_m.$$

Assumption 1. *From now on, until the end of Section 5, we will be assuming that $\gamma \in \mathrm{Jac}(X)[n]$ has maximal order n.*

In particular, we have the following corollary.

Corollary 3.4. *If $\gamma \in \mathrm{Jac}(X)[n]$ has order n, then we have an isomorphism*

$$\mathbf{M}_X(n,d)^\gamma \cong \mathbf{M}_{X_\gamma}(1,d)/\mathbb{Z}_n.$$

Since $\mathbf{M}_{X_\gamma}(1,d) \cong T^*\mathrm{Jac}^d(X_\gamma)$, it naturally fibres over the Jacobian,

$$\mathbf{M}_{X_\gamma}(1,d) \longrightarrow \mathrm{Jac}^d(X_\gamma).$$

Recall the norm map $\mathrm{Nm} : \mathrm{Jac}^d(X_\gamma) \to \mathrm{Jac}^d(X)$, given by $\mathrm{Nm}(\mathcal{O}(D)) = \mathcal{O}(p_\gamma(D))$. It is obviously invariant under the action of the Galois group, so it factors through the quotient by $\mathbb{Z}_n$. Combined with the previous projection, we get

$$g : \mathbf{M}_{X_\gamma}(1,d)/\mathbb{Z}_n \longrightarrow \mathrm{Jac}^d(X).$$

We will use these maps to construct flat (hence hyperholomorphic) bundles in $\mathbf{M}_{X_\gamma}(1,d)$ and $\mathbf{M}_{X_\gamma}(1,d)/\mathbb{Z}_n$.

There are natural embeddings $X \hookrightarrow \mathrm{Jac}^1(X)$ and $X_\gamma \hookrightarrow \mathrm{Jac}^1(X_\gamma)$. For $d \neq 1$, fix a point $x_0 \in X$, and $\widetilde{x}_0 \in X_\gamma$ with $p_\gamma(\widetilde{x}_0) = x_0$. This allows us to generalize the embeddings for all d, $X \hookrightarrow \mathrm{Jac}^d(X)$ and $X_\gamma \hookrightarrow \mathrm{Jac}^d(X_\gamma)$.

Let $\mathcal{L} \to X$ be a line bundle on the base curve with a flat connection $\nabla_\mathcal{L}$. Since $\pi_1(\mathrm{Jac}^d(X))$ is the abelianization of $\pi_1(X)$, there exists a unique (up to isomorphism) flat line bundle $(\check{\mathcal{L}}, \check{\nabla}_\mathcal{L})$ that restricts to $(\mathcal{L},\nabla_\mathcal{L})$ in $X \subset \mathrm{Jac}^d(X)$. Consider the flat (thus hyperholomorphic by Remark 3.1) line bundle $g^*(\check{\mathcal{L}}, \check{\nabla}_\mathcal{L})$ on $\mathbf{M}_{X_\gamma}(1,d)/\mathbb{Z}_n$ and denote by $(\mathscr{L}, \nabla_\mathscr{L})$ its pushforward under the isomorphism of Corollary 3.4, which is a hyperholomorphic sheaf supported on $\mathbf{M}_X(n,d)^\gamma$. This pair is therefore a (BBB)-brane on $\mathbf{M}_X(n,d)$, which we call a *basic Narasimhan-Ramanan* (BBB)*-brane associated to $\gamma \in \mathrm{Jac}(X)[n]$ and to $\mathcal{L} \to X$*. We denote it by

$$(\mathbf{BBB})^\gamma_\mathcal{L} := (\mathbf{M}_X(n,d)^\gamma, (\mathscr{L}, \nabla_\mathscr{L}))\,.$$

Analogously, given a line bundle $\mathcal{F} \to X_\gamma$ with a flat connection $\nabla_\mathcal{F}$, let $(\check{\mathcal{F}}, \check{\nabla}_\mathcal{F})$ be the (unique up to isomorphism) flat line bundle on $\mathrm{Jac}^d(X_\gamma)$ that restricts to $(\mathcal{F},\nabla_\mathcal{F})$ in $X \hookrightarrow \mathrm{Jac}^d(X_\gamma)$. Note that $f^*(\check{\mathcal{F}}, \check{\nabla}_\mathcal{F})$ is a flat line bundle on $\mathbf{M}_{X_\gamma}(1,d)$, hence hyperholomorphic by Remark 3.1. Taking the pushforward under the hyperholomorphic morphism $\check{p}_\gamma$, we obtain the hyperholomorphic sheaf $(\mathscr{F}, \nabla_\mathscr{F}) := p_{\gamma,*}f^*(\check{\mathcal{F}}, \check{\nabla}_\mathcal{F})$ supported on $\mathbf{M}_X(n,d)^\gamma$. As before, this defines a (BBB)-brane, the *non-basic Narasimhan-Ramanan* (BBB)*-brane* associated to $\gamma \in \mathrm{Jac}(X)[n]$ and to $\mathcal{F} \to X_\gamma$, which we denote by

$$(\mathbf{BBB})^\gamma_\mathcal{F} := (\mathbf{M}_X(n,d)^\gamma, (\mathscr{F}, \nabla_\mathscr{F}))\,.$$

3.2. **Spectral data of γ-invariant Higgs bundles.** Recalling the Hitchin fibration, $h_{X,n} : \mathbf{M}_X(n,d) \to B_{X,n}$, we denote its restriction to the fixed point set $\mathbf{M}_X(n,d)$ and the image of the later by

$$h_\gamma : \mathbf{M}_X(n,d)^\gamma \to B^\gamma_{X,n} := h_{X,n}\left(\mathbf{M}_X(n,d)^\gamma\right) \subset B_{X,n}.$$

Also, we denote by B^γ_{red} the subscheme of $B^\gamma_{X,n}$ given by those b such that the associated spectral curve, X_b, is reduced.

Denoting by K_γ the canonical bundle of X_γ, one can consider the Hitchin base for the curve X_γ,

$$B_{X_\gamma,n} = \bigoplus_{i=0}^{n-1} H^0(X_\gamma, K^i_\gamma),$$

Since the n-covering $p_\gamma : X_\gamma \to X$ is étale, we have that $p^*_\gamma K \cong K_\gamma$ and, then, this yields

$$\begin{array}{ccc} B_{X,n} & \longrightarrow & B_{X_\gamma,n} \\ b & \longmapsto & p^*_\gamma b. \end{array} \tag{3.3}$$

Lemma 3.5. *The map (3.3) is injective and its image is $B^{\mathbb{Z}_n}_{X_\gamma,n}$, the fixed point locus under the action of the Galois group $\mathbb{Z}_n$ of p_γ.*

Proof. This is clear since p_γ is an étale covering, thus is a local diffeomorphism. Hence for every $x \in X$, $dp_\gamma(x) : T_x X_\gamma \to T_{p_\gamma(x)}X$ is an isomorphism. Therefore the map $dp_\gamma : TX_\gamma \to TX$ between the tangent bundles is surjective. Hence, for each i, $p^*_\gamma : K^i = \Lambda^i TX^* \to \Lambda^i TX^*_\gamma = K^i_\gamma$ is basically given by dual of dp_γ, it follows that $p^*_\gamma : K^i \to K^i_\gamma$ is injective. From this it follows (again using that p_γ is surjective) that $p^*_\gamma : H^0(X, K^i) \to H^0(X_\gamma, K^i_\gamma)$ is injective for every i, and therefore (3.3) is injective as well. □

Let $\xi = \exp(2\pi \mathrm{i}/n) \in \mathbb{Z}_n$ be the standard generator of the Galois group $\mathbb{Z}_n$ of the cover $p_\gamma : X_\gamma \to X$. We will use the following notation repeatedly: given a section ϕ of the canonical line bundle K_γ of X_γ (i.e., ϕ is a Higgs field for a line bundle in X_γ), write

$$\phi_k := \xi^{k,*}\phi,$$

for $k = 0, \dots, n-1$. Also, we denote the induced morphism the Jacobian by

$$\begin{array}{cccc} \hat{\xi}^k : & \mathrm{Jac}^d(X_\gamma) & \longrightarrow & \mathrm{Jac}^d(X_\gamma) \\ & L & \longmapsto & \xi^{k,*}L. \end{array}$$

Define

$$\begin{array}{cccc} \chi : & H^0(X_\gamma, K_\gamma) & \longrightarrow & B_{X_\gamma,n} \\ & \phi & \longmapsto & \left(p_1(\bigoplus_{k=0}^{n-1}\phi_k), \dots, p_n(\bigoplus_{k=0}^{n-1}\phi_k)\right) \end{array}.$$

Notice that two sections $\phi, \phi' \in H^0(X_\gamma, K_\gamma)$ map to the same point under χ if and only if $\phi' = \phi_k$ for some $k = 0, \dots, n-1$. It follows that

$$\mathrm{Im}(\chi) \cong H^0(X_\gamma, K_\gamma)/\mathbb{Z}_n, \tag{3.4}$$

with the Galois group $\mathbb{Z}_n$ acting by pullback.

Since $X_\gamma \to X$ is an unramified cover, one naturally has $|K_\gamma| \cong |K| \times_X X_\gamma$. Denote by

$$q : |K_\gamma| \to |K|$$

the obvious projection.

Proposition 3.6. *Let $\gamma \in \mathrm{Jac}(X)[n]$ be of maximal order. Then:*

(i) For every $b \in B^\gamma_{X,n}$, the curve $X_{\widetilde{b}} \subset |K_\gamma|$ determined by $\widetilde{b} = p^*_\gamma b$ is the spectral curve associated with $(p^*_\gamma E, p^*_\gamma \varphi)$, where $h_{X,n}(E,\varphi) = b$.

(ii) The following diagram is Cartesian

$$\begin{array}{ccc} X_{\widetilde{b}} & \xrightarrow{\pi_{\widetilde{b}}} & X_\gamma \\ {\scriptstyle q_b}\downarrow & & \downarrow{\scriptstyle p_\gamma} \\ X_b & \xrightarrow[\pi]{} & X, \end{array} \tag{3.5}$$

where q_b coincides with the restriction of q.

(iii) *Let* $\phi \in H^0(X_\gamma, K_\gamma)$, $b = h_{X,n}(p_{\gamma,*}\phi)$ *and* $\widetilde{b} = p_\gamma^* b$. *Let also* $\tau = q^*\lambda$ *be the tautological section of* $\pi_{\widetilde{b}}^* K_\gamma$ *over* $|K_\gamma|$. *Then* $X_{\widetilde{b}}$ *is given by the vanishing of the section*

$$\prod_{k=0}^{n-1}(\tau - \phi_k)$$

in $|K_\gamma|$. *Hence*

$$X_{\widetilde{b}} = \bigcup_{k=0}^{n-1} \phi_k(X_\gamma). \tag{3.6}$$

and it is singular and reducible. Moreover, $X_{\widetilde{b}}$ *is reduced if and only if* ϕ *is not fixed by any element of the Galois group.*

(iv) *For every* $b \in B_{X,n}^\gamma$ *given by* $\phi \in H^0(X_\gamma, K_\gamma)$ *as in* (iii) *and such that* ϕ *is not fixed by any element o the Galois group* $\mathbb{Z}_n$, *one has that*

$$X_b = q_b \circ \phi(X_\gamma), \tag{3.7}$$

so one naturally has a morphism

$$\nu_\phi := q \circ \phi : X_\gamma \to X_b. \tag{3.8}$$

which constitutes a normalisation map. Hence the corresponding spectral curve X_b *is integral and singular, with singular divisor* $\operatorname{sing}(X_b)$ *of length* $n(n-1)(g-1)$.

(v) *For* $\phi \in H^0(X_\gamma, K_\gamma)$ *not fixed by any element of* $\mathbb{Z}_n$, *the curve in* $|K_\gamma| \times H^0(X_\gamma, K_\gamma)$ *given by*

$$\widetilde{X} = \bigsqcup_{k=0}^{n-1} \phi_i(X_\gamma) \times \{\phi_k\} \tag{3.9}$$

is the normalisation of $X_{\widetilde{b}}$, *where the normalisation morphism is induced by* $t : |K_\gamma| \times H^0(X_\gamma, K_\gamma) \to |K_\gamma|$. *Furthermore, there exists a morphism* $\widetilde{q} : \widetilde{X} \to X_\gamma$ *making the following diagram*

$$\begin{CD} \widetilde{X} @>{t}>> X_{\widetilde{b}} \\ @V{\widetilde{q}}VV @VV{q_b}V \\ X_\gamma @>>{\nu_\phi}> X_b \end{CD}$$

Cartesian.

Proof. (i) Let $\gamma \in \operatorname{Jac}(X)[n]$ be of maximal order and $(E, \varphi) \in \mathbf{M}_X(n, d)$, so that

$$(E, \varphi) \cong (p_{\gamma,*}F, p_{\gamma,*}\phi), \tag{3.10}$$

where (F, ϕ) is a Higgs line bundle over X_γ. Let $b = h_{X,n}(E, \varphi) \in B_{X,n}$, so that $b = (p_1(\varphi), \ldots, p_n(\varphi))$, and let $X_b \subset |K|$ be the corresponding spectral curve, defined by

$$\lambda^n + \pi^* p_1(\varphi)\lambda^{n-1} + \cdots + \pi^* p_n(\varphi) = 0.$$

Let now

$$p_\gamma^* b = (p_\gamma^* p_1(\varphi), \ldots, p_\gamma^* p_n(\varphi)) \in B_{X_\gamma, n}.$$

Since $p_\gamma^* p_i(\varphi) = p_i(p_\gamma^* \varphi)$, it is clear that

$$h_{X_\gamma, n}(p_\gamma^* E, p_\gamma^* \varphi) = p_\gamma^* b. \tag{3.11}$$

This proves the statement.

To see (ii), note that $X_{\widetilde{b}}$ is defined by

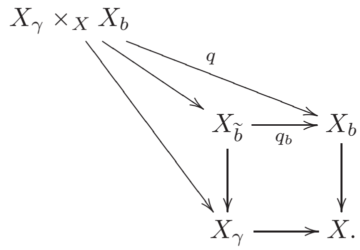
$$\tau^n + p_\gamma^* \pi^* p_1(\varphi)\tau^{n-1} + \cdots + p_\gamma^* \pi^* p_n(\varphi) = 0,$$

where $\tau = p_\gamma^* \lambda$. Note that τ is indeed the tautological section of $\tilde{p}^* K_\gamma$ in $|K_\gamma| = |p_\gamma^* K|$.

So, the image of $X_{\widetilde{b}}$ under $d{p_\gamma}^t : K_\gamma \to K$ is X_b, and we call this map q_b. Hence, we have maps $\pi_{\widetilde{b}} : X_{\widetilde{b}} \to X_\gamma$ and $q_b : X_{\widetilde{b}} \to X_b$, which moreover induce a morphism $(\pi_{\widetilde{b}}, \widetilde{p}) : X_{\widetilde{b}} \to X_\gamma \times_X X_b$. By the universal property of fibred products, it is enough to define a morphism

$$X_\gamma \times_X X_b \to X_{\widetilde{b}}$$

making the diagram below commutative



To do this, embed $X_\gamma \times_X X_b \hookrightarrow |K_\gamma| = X_\gamma \times_X |K|$ and check that the image is contained in $X_{\tilde{b}}$. It follows from this commutative diagram that q_b coincides with the restriction of q under the identification $X_{\tilde{b}} = X_\gamma \times_X X_b$.

(iii) From (3.10), we have that $(p_\gamma^* E, p_\gamma^* \varphi) \cong \bigoplus_{k=0}^{n-1} (\xi^{k,*} F, \phi_k)$, hence from (3.11), we must have (3.6).

From (3.6) and (ii) proven above we conclude the following. If ϕ is not fixed by any element of the Galois group, all the ϕ_k are different and then $X_{\tilde{b}}$ is reduced and has precisely n irreducible components, each of which is mapped isomorphically to X_γ by $\pi_{\tilde{b}}$ (which is smooth and irreducible by maximality of the order of γ).

On the other hand, if there is an element of the Galois group fixing ϕ, then $\phi_k = \phi_{k'}$ for some k and k', and $X_{\tilde{b}}$ is a non-reduced curve.

(iv) Recall from (3.6) that the $\phi_i(X_\gamma)$ are the irreducible components of $X_{\tilde{b}}$, and note that they all have the same image under $q_b : X_{\tilde{b}} \to X_b$, as

$$\phi_i(y) = \phi(\xi^i(y))$$

for any $y \in X_\gamma$. Then, since the image of q_b is X_b, so is the image of one irreducible component

$$\phi(X_\gamma) \hookrightarrow X_{\tilde{b}} \xrightarrow{q_b} X_b,$$

and (3.7) follows.

From (iii) we have that when ϕ is not fixed by any element of the Galois group, then $X_{\tilde{b}}$ is reduced and so is X_b, by Cartesianity of (3.5). Since $X_{\tilde{b}}$ is an $n:1$ cover, the projection of each irredicible component, as in (3.8), is generically an isomorphism. It is not an isomorphism because otherwise $X_b \longrightarrow X$ would be unramified. We conclude that X_γ normalizes X_b. Since X_γ is irreducible and the normalisation of X_b, then X_b is irreducible as well. Since X_γ is smooth with genus $n(g-1)+1$ as it is an unramified n-cover of X, while the genus of X_b is $n^2(g-1)+1$, by (2.2). It follows that the singular divisor $\mathrm{sing}(X_b)$ of X_b has degree δ, that is, the difference of the genus of X_b and X_γ:

$$\deg(\mathrm{sing}(X_b)) = n(n-1)(g-1) = \delta.$$

(v) The description (3.9) of the normalisation of $X_{\tilde{b}}$ follows from the description of $X_{\tilde{b}} = \bigcup_i \phi_k(X_\gamma)$.

Note that the composition

$$\phi_k(X_\gamma) \times \{\phi_k\} \xrightarrow{t} \phi_k(X_\gamma) \xrightarrow{q_b} X_b$$

coincide with ν_{ϕ_k}. Observe now that $\nu_{\phi_k} = q \circ \phi_k = \nu_\phi \circ \xi^k$. Then, setting $\tilde{q} = (q_0', q_1', \ldots, q_{n-1}')$ for $q_i' = \xi^i \circ \pi_\gamma$, we have that (6.6) commutes, □

After studying the spectral curves in Proposition 3.6, one can describe their loci in the Hitchin base.

Proposition 3.7. *Let $\gamma \in \mathrm{Jac}(X)[n]$ have maximal order n.*

(i) One has the isomorphism

$$B_{X,n}^\gamma \cong H^0(X_\gamma, K_\gamma)/\mathbb{Z}_n \cong \left(\bigoplus_{i=0}^{n-1} H^0(X, KL_\gamma^i) \right) / \mathbb{Z}_n,$$

where the Galois group acts by pullback in $H^0(X_\gamma, K_\gamma)$, and by multiplication under ξ^i on $H^0(X, KL_\gamma^i)$. In particular,

$$\dim(B_{X,n}^\gamma) = n(g-1)+1.$$

(ii) *X_b is non-reduced if and only if ϕ is fixed by an element of the Galois group. Hence, under the previous isomorphism B^γ_{red} is the dense subscheme given by the image of the points of $H^0(X_\gamma, K_\gamma)$ not fixed by any element of the Galois group.*

Proof. (i) By (3.11) and $p_\gamma^* p_i(\varphi) = p_i(p_\gamma^*\varphi) = p_i(\bigoplus_{i=0}^{n-1}\phi_i)$, we conclude that p_γ^* maps $B^\gamma_{X,n}$ to $\mathrm{Im}(\chi)$. This is an isomorphism by Lemma 3.5, since the elements in $\mathrm{Im}(\chi)$ are $\mathbb{Z}_n$-invariant. Thus, the first description of $B^\gamma_{X,n}$ follows from (3.4). The second description follows from this and the fact that, considering the pushforward under p_γ (which being a finite morphism, does not change H^0),

$$H^0(X_\gamma, K_\gamma) \cong \bigoplus_{i=0}^{n-1} H^0(X, KL_\gamma^i).$$

The dimension of $B^\gamma_{X,n}$ follows from $\dim H^0(X_\gamma, K_\gamma) = g(X_\gamma) = n(g-1)+1$

(ii) By (3.5), X_b is reduced if and only if $X_{\tilde{b}}$ is, and the later is reduced whenever $\phi_i = \phi_j$ for some $i \neq j$, i.e., whenever ϕ is fixed by some element of the Galois group. Then, (ii) follows easily from (i). □

From all of the above, we deduce the following. Let

$$h_{X_\gamma,1} : \mathbf{M}_{X_\gamma}(1,d) \longrightarrow H^0(X_\gamma, K_\gamma)$$

be the Hitchin map on $\mathbf{M}_{X_\gamma}(1,d)$ *i.e.* the projection of $T^*\mathrm{Jac}^d(X_\gamma) \to H^0(X_\gamma, K_\gamma)$, and let

$$h_\gamma : \mathbf{M}_X(n,d)^\gamma \longrightarrow B^\gamma_{X,n}$$

be the restriction of the Hitchin map of $\mathbf{M}_X(n,d)$ to $\mathbf{M}_X(n,d)^\gamma$.

Corollary 3.8. *There is a commutative and $\mathbb{Z}_n$-equivariant diagram*

$$\begin{array}{ccc} \mathbf{M}_{X_\gamma}(1,d) & \xrightarrow{\check{p}_\gamma} & \mathbf{M}_X(n,d)^\gamma \\ \downarrow{\scriptstyle h_{X_\gamma,1}} & & \downarrow{\scriptstyle h_\gamma} \\ H^0(X_\gamma,K_\gamma) & \xrightarrow[(p_\gamma^*)^{-1}\circ\chi]{} & B^\gamma_{X,n}. \end{array} \tag{3.12}$$

Let $b \in B^\gamma_{\mathrm{red}}$, and take $\phi \in H^0(X_\gamma, K_\gamma)$ such that $p_\gamma^* b = \chi(\phi)$. Then the diagram above restricts to

$$\bigsqcup_{i=0}^{n-1} (\mathrm{Jac}^d(X_\gamma) \times \{\phi_i\}) \xrightarrow{\check{p}_\gamma} h_\gamma^{-1}(b). \tag{3.13}$$

The Galois group $\mathbb{Z}_n$ permutes the components of the source. Hence

$$h_\gamma^{-1}(b) \cong \mathrm{Jac}^d(X_\gamma), \tag{3.14}$$

where the identifications between the different components in the source of (3.13) the Galois group action.

Proof. The top arrow is defined in (3.2), where we recall that we are considering γ of maximal order. By Lemma 3.5, the bottom arrow is well defined. Recalling that (3.3) is injective, commutativity follows since, for any $\phi \in H^0(X_\gamma, K_\gamma)$, one has for every invariant polynomial p_i, that

$$p_\gamma^* p_i(p_{\gamma,*}\phi) = p_i(p_\gamma^* p_{\gamma,*}\phi) = p_i\Big(\bigoplus_i \phi_i\Big).$$

Equivariance by the action of $\mathbb{Z}_n$ follows form the $\mathbb{Z}_n$-equivariance of the vertical arrows and the $\mathbb{Z}_n$-invariance of the horizontal arrows. The preimage in (3.12) of the fibre of $b \in B^\gamma_{\mathrm{red}}$ is $\bigcup_i h^{-1}_{X_\gamma,1}(\xi^{i,*}\phi)$, which are disjoint as the points in B^γ_{red} correspond to those ϕ not fixed by $\mathbb{Z}_n$. Finally, (3.14) follows from commutativity, equivariance, Corollary 3.4 and (i) of Proposition 3.7. □

Associated to $\phi \in H^0(X_\gamma, K_\gamma)$ projecting to $b \in B^\gamma_{\mathrm{red}}$ we have constructed in (3.8) the normalisation morphism ν_ϕ. By construction, we have the following commutative diagram

$$\begin{array}{ccc} X_\gamma & & \\ \Big\downarrow p_\gamma & \searrow^{\nu_\phi} & \\ & & X_b \\ & \swarrow_{\pi_b} & \\ X & & \end{array} \tag{3.15}$$

It follows from (3.8), that for every $\phi_i = \xi^{i,*}\phi$ projecting to the same $b \in B^\gamma_{\mathrm{red}}$, one has

$$\nu_{\phi_i} = \nu_\phi \circ \xi^i. \tag{3.16}$$

Consider the pushforward morphism

$$\begin{array}{rccc} \check{\nu}_\phi : & \mathrm{Jac}^d(X_\gamma) & \longrightarrow & \overline{\mathrm{Jac}}^{\,\delta+d}(X_b) \\ & \mathcal{E} & \longmapsto & \nu_{\phi,*}\mathcal{E} \end{array} \tag{3.17}$$

where we recall that δ is given by (2.3). If $\phi, \phi_k \in H^0(X_\gamma, K_\gamma)$, with $\phi_k = \phi \circ \xi^k$, are any two representatives of the class $b \in B^\gamma \cong H^0(X_\gamma, K_\gamma)/\mathbb{Z}_n$, then the corresponding morphisms $\check{\nu}_\phi$ and $\check{\nu}_{\phi_k}$ have the same image

$$\mathrm{Im}(\check{\nu}_\phi) = \mathrm{Im}(\check{\nu}_{\phi_k})$$

due to (3.16).

Proposition 3.9. *Let $b \in B^\gamma_{\mathrm{red}}$. Then,*

(i) the intersection of the fixed point subvariety $\mathbf{M}_X(n,d)^\gamma$ with the Hitchin fibre is

$$h_\gamma^{-1}(b) = \mathrm{Im}(\check{\nu}_\phi) \cong \mathrm{Jac}^d(X_\gamma).$$

(ii) the restrictions to $h_\gamma^{-1}(b)$ of the line bundle $\mathscr{L}$ and the sheaf $\mathscr{F}$ are respectively identified, under the isomorphim (3.14), with the bundles $\mathrm{Nm}^*\check{\mathcal{L}}$ and $\bigoplus_{i=0}^{n-1} \hat{\xi}^i_* \check{\mathcal{F}}$.

Proof. (i) The identification $h_\gamma^{-1}(b) = \mathrm{Im}(\check{\nu}_\phi)$ follows easily from the commutativity of (3.15) and the fact that, after Corollary 3.4, every point in $h_\gamma^{-1}(b)$ corresponds with the pushforward under p_γ of a line bundle on X_γ. Then, recall (3.14) which agrees with the fact that $\mathrm{Im}(\check{\nu}_\phi) \cong \mathrm{Jac}^d(X_\gamma)$ since $\check{\nu}_\phi$ is injective as it is the pushforward under the normalisation map of a curve. Note that Corollary 3.8 implies that the identification between $\mathrm{Jac}^d(X_\gamma) \times \{\phi\}$ and $\mathrm{Jac}^d(X_\gamma) \times \{\xi^{i,*}\phi\}$ made by $\hat{\xi}^i$, corresponds with the identification between $\mathrm{Im}(\nu_\phi)$ and $\mathrm{Im}(\nu_{\phi_i})$.

(ii) Recall from (3.13) and (3.14) that the Hitchin fibre $h_{X,n}^{-1}(b)$ is given by the quotient

$$\left(\bigsqcup_{i=0}^{n-1} \mathrm{Jac}^d(X_\gamma) \times \{\xi^{i,*}\phi\} \right) / \mathbb{Z}_n \cong \mathrm{Jac}^d(X_\gamma).$$

Recall as well that $\mathscr{L}$ is given by $g^*\check{\mathcal{L}}$ where g is given by the factorization of the norm map Nm through the quotient by $\mathbb{Z}_n$ and the trivial projection $T^*\mathrm{Jac}^d(X_\gamma) \to \mathrm{Jac}(X_\gamma)$. Since $\mathscr{L}$ is a line bundle, note that $\mathscr{L}|_{h_{X,n}^{-1}(b)}$ is a line bundle as well, it is indeed the descent line bundle given by $\mathbb{Z}_n$-invariant line bundle in $\bigsqcup_i \mathrm{Jac}^d(X_\gamma) \times \{\xi^{i,*}\phi\}$ corresponding to $\mathrm{Nm}^*\check{\mathcal{L}}$ on each component. This proves the statement for $\mathscr{L}$.

Recall that $\mathscr{F}$ is given by $\check{p}_{\gamma,*} f^*\mathcal{F}$ and recall from (3.13) that $h_{X,n}^{-1}(b)$ is the image under $\check{p}_\gamma$ of $\bigsqcup_i \mathrm{Jac}^d(X_\gamma) \times \{\xi^{i,*}\phi\}$. Since f is the trivial projection $T^*\mathrm{Jac}^d(X_\gamma) \to \mathrm{Jac}^d(X_\gamma)$, the restriction of $f^*\check{\mathcal{F}}$ to each $\mathrm{Jac}^d(X_\gamma) \times \{\xi^{i,*}\phi\}$ is simply $\check{\mathcal{F}}$. Then, the restriction of $\mathscr{F}$ to the image under $\check{p}_\gamma$ of $\bigsqcup_i \mathrm{Jac}^d(X_\gamma) \times \{\xi^{i,*}\phi\}$, corresponds with the direct sum of the pushforward of $\check{\mathcal{F}}$ under the isomorphisms

$$\hat{\xi}^i : \mathrm{Jac}^d(X_\gamma) \times \{\xi^{i,*}\phi\} \to \mathrm{Jac}^d(X_\gamma) \times \{\phi\}.$$

□

Remark 3.10. Since, for any i, the $\hat{\xi}^i$ are automorphisms of $\mathrm{Jac}^d(X_\gamma)$ with inverse $\hat{\xi}^{-i}$, we have that the pushforward under $\hat{\xi}^i$ corresponds with the pullback under $\hat{\xi}^{-i}$. Also, note that these constructions commute with dualization

$$\hat{\xi}^i_* \check{\mathcal{F}} \cong \widecheck{\xi^i_* \mathcal{F}}.$$

Remark 3.11. Since the dual of the norm map corresponds to $\hat{p}_\gamma$ under dualization, we have that

$$\mathrm{Nm}^* \check{\mathcal{L}} \cong \widecheck{p_\gamma^* \mathcal{L}}.$$

We observe as well that this bundle is invariant under the action of $\hat{\xi}^i$, as

$$\hat{\xi}^i_* \widecheck{p_\gamma^* \mathcal{L}} \cong \widecheck{\xi^i_* p_\gamma^* \mathcal{L}} \cong \widecheck{p_\gamma^* \mathcal{L}}.$$

4. Hecke (BAA)-branes

In a hyperkähler variety with Kähler structures $((I_1,\omega_1),(I_2,\omega_2),(I_3,\omega_3))$. By definition [KW], a (BAA)-brane consists of a pair $(\Sigma,(W,\nabla_W))$, where

(1) Σ is a complex Lagrangian subvariety for the holomorphic 2-form $\Omega = \omega_2 + \mathrm{i}\omega_3$;
(2) (W,∇_W) is a flat bundle supported on Σ.

In this section we construct a family of (BAA)-branes over the open subset determined by reduced spectral curves, which are mapped under the Hitchin fibration to the same locus of the Hitchin base as our (BBB)-branes $(\mathbf{BBB})^\gamma_{\mathcal{L}}$ and $(\mathbf{BBB})^\gamma_{\mathcal{F}}$. Its support is a subvariety of $\mathbf{M}_X(n,d)$, depending on $\gamma \in \mathrm{Jac}(X)[n]$ and on a holomorphic line bundle $\mathcal{J} \in \mathrm{Jac}^{\delta+d}(X_\gamma)$. Denote it by $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$; see (4.5). The basic idea of its fibrewise description is the following (cf. Theorem 4.6). Take a point on $b \in B^\gamma_{\mathrm{red}}$. As we know, the Hitchin fibre $h^{-1}_{X,n}(b)$ is the compactified Jacobian $\overline{\mathrm{Jac}}^{\delta+d}(X_b)$. Consider a normalisation $\nu_\phi : X_\gamma \to X_b$, where $\chi(\phi) = p_\gamma^* b$, and the induced pullback map $\hat{\nu}_\phi : \mathrm{Jac}^{\delta+d}(X_b) \to \mathrm{Jac}^{\delta+d}(X_\gamma)$. Suppose $\mathcal{J}$ is in the image of the pullback induced by $p_\gamma : X_\gamma \to X$, hence it is fixed by the Galois group action on $\mathrm{Jac}^{\delta+d}(X_\gamma)$. Then the intersection of $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ with $h^{-1}_{X,n}(b)$ will be the fibre of $\hat{\nu}_\phi$ over $\mathcal{J}$. By (3.16), and since $\xi^*\mathcal{J} \cong \mathcal{J}$, this fibre is independent of the choice of the representative ϕ of the class defined by b. If $\mathcal{J}$ does not descend to X, then it is not fixed by the Galois group, and the intersection of $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ with $h^{-1}_{X,n}(b)$ will be the union of the fibres of $\hat{\nu}_\phi$ over the orbit of $\mathcal{J}$ under the Galois group. Again by (3.16), this union is independent of the choice of ϕ.

We will also see that the Higgs bundles lying in $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ can be constructed as Hecke transformations of naturally associated Higgs bundles lying in $\mathbf{M}_X(n,\delta+d)^\gamma$. Hence, roughly speaking, our subvarieties $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ can be obtain as global "Hecke transformations" of the support of the Narasimhan-Ramanan (BBB)-branes over $\mathbf{M}_X(n,\delta+d)$. This justifies the notation for these subvarieties, as well as the name we have given to the corresponding branes, as *Hecke* (BAA)*-branes*. In fact, we shall also prove that each $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ is a Lagrangian subvariety, hence, when equipped with the trivial line bundle, becomes a (BAA)-brane, which we denote by $(\mathbf{BAA})^{\gamma,\mathcal{J}}_{\mathrm{red}}$.

Notice here a kind of duality on a fibre of the Hitchin map over a point on B^γ_{red}. For the Narasimhan-Ramanan (BBB)-branes, we considered the spectral data embedded in $\overline{\mathrm{Jac}}^{\delta+d}(X_b)$ under the pushforward morphism (3.17) induced by ν_ϕ. For the Hecke (BAA)-branes we consider the fibre of the pullback morphism again induced by ν_ϕ. Indeed, for degree $d = 0$ (or multiple of n), we will associate, in Section 5, to each brane $(\mathbf{BBB})^\gamma_{\mathcal{L}}$ or $(\mathbf{BBB})^\gamma_{\mathcal{F}}$, a (BAA)-brane $(\mathbf{BAA})^{\gamma,\mathcal{J}}_{\mathrm{red}}$ where $\mathcal{J}$ is naturally associated (via the Hitchin section) to $\mathcal{L}$ or $\mathcal{F}$. Moreover, we will prove there that these pair branes are indeed dual branes in the sense of mirror symmetry.

4.1. **Construction of the subvarieties.** In this subsection we construct the subvarieties which will support our (BAA)-branes. As always, let $\gamma \in \mathrm{Jac}(X)[n]$ be of maximal order. Recalling that $B^\gamma_{X,n} \cong H^0(X_\gamma, K_\gamma)/\mathbb{Z}_n$, we denote by H^γ_{red} the preimage of B^γ_{red}. This coincides with

$$H^\gamma_{\mathrm{red}} = \{\phi \in H^0(X_\gamma, K_\gamma) \,|\, \phi \text{ is not fixed by any element of } \mathbb{Z}_n\}.$$

Define the tautological morphism constructed with the sections of K_γ

$$\begin{array}{rccc} \Phi: & X_\gamma \times H^\gamma_{\mathrm{red}} & \longrightarrow & \mathbb{P}(K_\gamma) \times H^\gamma_{\mathrm{red}} \\ & (y,\phi) & \longmapsto & ([\phi(y):1],\phi), \end{array}$$

and note that the closed subset $\Phi(X_\gamma \times H^\gamma_{\mathrm{red}}) \subset \mathbb{P}(K_\gamma) \times H^\gamma_{\mathrm{red}}$ is the family of curves in $\mathbb{P}(K_\gamma)$ parametrized by H^γ_{red}, where the curve parametrized by ϕ is precisely $\phi(X_\gamma)$. Recalling that $K_\gamma \cong X_\gamma \times_X K$, so $\mathbb{P}(K_\gamma) \cong X_\gamma \times_X \mathbb{P}(K)$, we denote by q the obvious projection $\mathbb{P}(K_\gamma) \to \mathbb{P}(K)$. Consider the closed subset of $\mathbb{P}(K) \times H^\gamma_{\mathrm{red}}$,

$$\Sigma^\gamma := \left(q \times \mathbf{1}_{H^\gamma_{\mathrm{red}}}\right) \circ \Phi(X_\gamma \times H^\gamma_{\mathrm{red}}),$$

and note that this defines a family of curves $\Sigma^\gamma \to H^\gamma_{\mathrm{red}}$, which is flat since both Φ and q are flat morphisms. Furthermore, (3.7) implies that the restriction to $\phi \in H^\gamma_{\mathrm{red}} \subset H^0(K_\gamma)$ is the spectral curve X_b, where $p^*_\gamma b = \chi(\phi)$,

$$\Sigma^\gamma|_\phi = q(\phi(X_\gamma)) = X_b. \tag{4.1}$$

Hence, the geometric fibres of Σ^γ are reduced by (iv) Proposition 3.6. Since Σ^γ is a flat, finitely presented family of curves, with geometrically reduced fibres, it follows by an unpublished result of Mumford [BLR, Theorem 2, Section 8.2] that the associated relative Jacobian $\mathrm{Jac}^{\delta+d}_{H^\gamma_{\mathrm{red}}}(\Sigma^\gamma) \to H^\gamma_{\mathrm{red}}$ exists and it is fine *i.e.* it is the parametrization space of a universal family of line bundles $\mathcal{U}^\gamma_{\mathrm{red}} \to \Sigma^\gamma \times_{H^\gamma_{\mathrm{red}}} \mathrm{Jac}_{H^\gamma_{\mathrm{red}}}(\Sigma^\gamma)$. Note that the trivial family $X_\gamma \times H^\gamma_{\mathrm{red}}$ satisfies as well the previous conditions, so the relative Jacobian $\mathrm{Jac}^{\delta+d}_{H^\gamma_{\mathrm{red}}}(X_\gamma \times H^\gamma_{\mathrm{red}})$ exists as well. Since the family $X_\gamma \times H^\gamma_{\mathrm{red}}$ is trivial, we have

$$\mathrm{Jac}^{\delta+d}_{H^\gamma_{\mathrm{red}}}(X_\gamma \times H^\gamma_{\mathrm{red}}) \cong \mathrm{Jac}^{\delta+d}(X_\gamma) \times H^\gamma_{\mathrm{red}},$$

and it is trivially an open subset of $\mathrm{Jac}^{\delta+d}(X_\gamma) \times H^0(X_\gamma, K_\gamma) \cong \mathbf{M}_{X_\gamma}(1, \delta+d)$. Later on, the choice of the notation $\delta + d$ for the degree will become clear.

By construction, Σ^γ comes equipped with the morphism $\nu := (q \times \mathbf{1}_{H^\gamma_{\mathrm{red}}}) \circ \Phi$ of $H^0(K_\gamma)$-schemes

$$\nu : (X_\gamma \times H^\gamma_{\mathrm{red}}) \longrightarrow \Sigma^\gamma, \tag{4.2}$$

which coincides fibrewise with (3.8),

$$\nu_\phi : X_\gamma \times \{\phi\} \longrightarrow \Sigma^\gamma|_\phi = X_b.$$

Since the pullback morphism is functorial for line bundles, associated to (4.2) one can define

$$\begin{array}{cccc} \hat{\nu} : & \mathrm{Jac}^{\delta+d}_{H^\gamma_{\mathrm{red}}}(\Sigma^\gamma) & \longrightarrow & \mathrm{Jac}^{\delta+d}_{H^\gamma_{\mathrm{red}}}(X_\gamma \times H^\gamma_{\mathrm{red}}) \cong \mathrm{Jac}^{\delta+d}(X_\gamma) \times H^\gamma_{\mathrm{red}} \\ & L \to \Sigma^\gamma|_\phi & \longmapsto & \nu^*_\phi L \to (X_\gamma \times \{\phi\}). \end{array} \tag{4.3}$$

Consider the projection $\pi : \mathbb{P}(K) \to X$ given by the structural morphism of $K \to X$. Note that, $\Sigma^\gamma \subset \mathbb{P}(K) \times H^\gamma_{\mathrm{red}}$ comes naturally equipped with the projection

$$\Pi = (\pi \times \mathbf{1}_{H^\gamma_{\mathrm{red}}}) : \Sigma^\gamma \to X \times H^\gamma_{\mathrm{red}}.$$

It follows from the fibrewise description of Σ^γ (4.1) that Π is a ramified n-cover. Observe that $(\pi \times \mathbf{1}_{H^\gamma_{\mathrm{red}}})^* K \to \mathbb{P}(K) \times H^\gamma_{\mathrm{red}}$ has a tautological section λ and let us abuse of notation to denote by λ the tautological secion restricted to $\Sigma^\gamma \subset \mathbb{P}(K) \times H^\gamma_{\mathrm{red}}$. Recall the universal family of line bundles $\mathcal{U}^\gamma_{\mathrm{red}} \to \Sigma^\gamma \times_{H^\gamma_{\mathrm{red}}} \mathrm{Jac}_{H^\gamma_{\mathrm{red}}}(\Sigma^\gamma)$. It follows from the spectral correspondence that

$$(\mathcal{V}^\gamma, \Psi^\gamma) := \left(\Pi \times_{H^\gamma_{\mathrm{red}}} \mathbf{1}_{\mathrm{Jac}}\right)_* (\mathcal{U}^\gamma_{\mathrm{red}}, \lambda) \longrightarrow (X \times H^\gamma_{\mathrm{red}}) \times_{H^\gamma_{\mathrm{red}}} \mathrm{Jac}^{\delta+d}_{H^\gamma_{\mathrm{red}}}(\Sigma^\gamma),$$

is a family of stable rank n degree d Higgs bundles parametrized by $\mathrm{Jac}_{H^\gamma_{\mathrm{red}}}(\Sigma^\gamma)$. From moduli theory, this provides a morphism to the moduli space of Higgs bundles

$$s : \mathrm{Jac}^{\delta+d}_{H^\gamma_{\mathrm{red}}}(\Sigma^\gamma) \longrightarrow \mathbf{M}_X(n,d), \tag{4.4}$$

defined over Σ^γ via the spectral correspondence. Observe that the image of s is contained in $\mathbf{M}_X(n,d) \times_B B^\gamma$ by construction, and that it is an n to 1 map onto its image since for every $\phi_k = \xi^{k,*}\phi$, one has that

$$s(\mathrm{Jac}^{\delta+d}(\Sigma^\gamma|_\phi)) = s(\mathrm{Jac}^{\delta+d}(\Sigma^\gamma|_{\phi_k})).$$

Associated to every line bundle $\mathcal{J} \to X_\gamma$ of degree $d + \delta$, we define, using (4.4) and (4.3), the subvariety of $\mathbf{M}_X(n,d)$ given by

$$\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}} := s\left(\hat{\nu}^{-1}(\{\mathcal{J}\} \times H^\gamma_{\mathrm{red}})\right). \tag{4.5}$$

By construction, $\mathrm{Hec}^{\gamma,\mathcal{J}}$ lies inside $\mathbf{M}_X(n,d) \times_B B^\gamma$, so it fibres naturally over B^γ under the Hitchin fibration, whose restriction we denote by $h^\gamma_{\mathcal{J}}$,

$$h^\gamma_{\mathcal{J}} : \mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}} \longrightarrow B^\gamma_{\mathrm{red}}. \tag{4.6}$$

One has the following fibrewise description of $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$, consider $b \in B^\gamma_{\mathrm{red}}$ given by $\phi \in H^0(K_\gamma)$, and let $\phi_k = \xi^{k,*}\phi$, then

$$\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}|_b = \left\{ (E,\Phi) \in \mathbf{M}_X(n,d) \times_B B^\gamma \;\middle|\; \begin{array}{l} (E,\Phi) = \pi_{b,*}L \text{ for any line} \\ \text{bundle } L \to X_b \text{ satisfying} \\ \nu^*_\phi L \cong \xi^{i,*}\mathcal{J} \text{ for some } i. \end{array} \right\}.$$

Notice that if $\mathcal{J}$ descends to X, then $\xi^{i,*}\mathcal{J} \cong \mathcal{J}$ is independent of i.

In fact, locally on B^γ_{red}, $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ can be realised in terms of Hecke transforms of families of bundles as follows. Consider the family

$$\tilde{\mathscr{X}} = (X_\gamma \times H^\gamma_{\mathrm{red}})/\mathbb{Z}_n \longrightarrow B^\gamma_{\mathrm{red}}.$$

Let $\mathscr{X} \longrightarrow B^\gamma_{\mathrm{red}}$ be the family of spectral curves over B^γ_{red}. Note that $\mathscr{X} = \Sigma^\gamma/\mathbb{Z}_n$. Moreover, we have a Cartesian diagram:

$$\begin{array}{ccc} X_\gamma \times H^\gamma_{\mathrm{red}} & \xrightarrow{\nu} & \Sigma^\gamma \\ \downarrow & & \downarrow \\ \tilde{\mathscr{X}} & \xrightarrow[\mathbf{n}]{} & \mathscr{X} \end{array} \tag{4.7}$$

where $\mathbf{n}$ is the normalisation morphism.

The following two results follow from the preceding considerations.

Lemma 4.1. *$X_\gamma \times H^\gamma_{\mathrm{red}}$ is étale over $\tilde{\mathscr{X}}$.*

Proposition 4.2. *Let $\mathcal{J} \in \mathrm{Jac}(X_\gamma)$. Then, there exists a cover $\mathcal{U}$ of B^γ_{red} such that for any open set $U \in \mathcal{U}$*

$$\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}} \times_{B^\gamma_{\mathrm{red}}} U \cong \bigcup_{i=0}^{n-1} \hat{\mathbf{n}}_U^{-1}(\xi^{i,*}\mathcal{J} \times U).$$

In the above

$$\hat{\mathbf{n}} : \mathrm{Jac}^{\delta+d}_{B^\gamma_{\mathrm{red}}}(\mathscr{X}) \longrightarrow \mathrm{Jac}^{\delta+d}_{B^\gamma_{\mathrm{red}}}(\tilde{\mathscr{X}})$$

is the pullback morphism, and $\hat{\mathbf{n}}_U$ is its restriction over U.

4.2. **Properties of $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$.** We now study the properties of $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$. Particularly relevant are the proofs that $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ are indeed Lagrangian subvarieties of $\mathbf{M}_X(n,d)$ and the proof that the Higgs bundles lying on them arise as Hecke transformations of certain associated Higgs bundles representing points in $\mathbf{M}_X(n,\delta+d)^\gamma$.

Proposition 4.3. *The subvariety $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ is contained in the stable locus of $\mathbf{M}_X(n,d)$.*

Proof. It follows by construction, since we are considering only spectral data given by reduced and irreducible spectral curves. □

Consider the pullback map between the moduli spaces of Higgs bundles over X and over X_γ,

$$\begin{array}{rccc} \hat{p}_\gamma : & \mathbf{M}_X(n,d) & \longrightarrow & \mathbf{M}_{X_\gamma}(n,nd) \\ & (E,\Phi) & \longmapsto & (p_\gamma^*E, p_\gamma^*\Phi), \end{array} \tag{4.8}$$

which is well defined since pullback by finite étale maps of solutions to Hitchin equations are solutions as well.

Recall the unipotent locus described in [FP, Section 4]. We now study its relation with $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$.

Proposition 4.4. *We have*

$$\hat{p}_\gamma\left(\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}\right) \subset \bigcup_{i=0}^{n-1} \mathrm{Uni}^{\xi^{i,*}\mathcal{J}}_{X_\gamma}(n,nd).$$

In particular, if $\mathcal{J}$ descends to X, then

$$\hat{p}_\gamma\left(\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}\right) \subset \mathrm{Uni}^{\mathcal{J}}_{X_\gamma}(n,nd).$$

Proof. This can be seen from the spectral data, as by [FP, Proposition 4.5]

$$\mathrm{Uni}^{\mathcal{J}}_{X_\gamma}(n,nd) \cap \tilde{h}^{-1}(\tilde{b}) = \{\tilde{L} \to X_{\tilde{b}} \;:\; \tilde{\nu}^*\tilde{L} = q^*\mathcal{J}\}.$$

Now, by Cartesianity of eqrefeq cartesian diagram spectral in Proposition 3.6, the spectral data of $p_\gamma^*(E,\phi)$ is the pullback of the spectral data of (E,ϕ). Thus

$$\hat{p}_\gamma\left(\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}\right) \cap \tilde{h}^{-1}_{X_\gamma,1}(\tilde{b}) = \left\{\tilde{L} \to X_{\tilde{b}} \;:\; \tilde{L} = \tilde{p}^*L, L \to X_b, \nu_\phi^* L = \mathcal{J}\right\}.$$

But, thanks to (v) Proposition 3.6, we have

$$\tilde{\nu}^*\tilde{p}^*L = q^*\xi^{k_{ij},*}\mathcal{J}.$$

We have thus proven

$$\hat{p}_\gamma\left(\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}\right) \cap \tilde{h}^{-1}_{X_\gamma,n}(\tilde{b}) \subset \mathrm{Uni}^{\xi_k^*\mathcal{J}}_{X_\gamma}(n,nd) \cap \tilde{h}^{-1}_{X_\gamma,n}(\tilde{b}).$$

□

Proposition 4.4 has important consequences, as the holomorphic 2-form vanishes in $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$.

Proposition 4.5. *The subvariety* $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ *is isotropic.*

Proof. Let $\omega_X \in \Omega^0(\bigwedge^2 T^*\mathbf{M}_X(n,d))$ be the symplectic form on $\mathbf{M}_X(n,d)$ associated with the complex structure $\mathrm{Jac}(X)[n]_1$, and likewise for $\omega_{X_\gamma} \in \Omega^0(\bigwedge^2 T^*\mathbf{M}_{X_\gamma}(n,nd))$. We need to check that the stalks of ω_X on all stable points $(E,\phi) \in \mathbf{M}_X(n,d))$ vanish, which by Proposition 4.3 are all points. Recall that ω_X is defined on the tangent space $T_{(E,\varphi)}\mathbf{M}_X(n,d) \cong \mathbb{H}^1(C^\bullet)$ by the pairing

$$\langle\cdot,\cdot\rangle_X : H^0(X,\mathrm{End}E \otimes K_X) \otimes H^1(X,\mathrm{End}E) \longrightarrow H^0(X,\mathbb{C}) \cong \mathbb{C}.$$

In a similar fashion, one defines ω_{X_γ} by pairing the spaces $H^0(X_\gamma,\mathrm{End}\tilde{E} \otimes K_\gamma)$ and $H^1(X_\gamma,\mathrm{End}\tilde{E})$. Let recall from (4.8) that the pullback map $\hat{p}_\gamma$. By [FP, Proposition 4.5], $\hat{p}_\gamma(E,\varphi) = (\tilde{E},\tilde{\varphi})$ is stable. Thus for every $v,w \in T_{(E,\varphi)}\mathbf{M}_X(n,d)$ one may consider

$$\omega_{X_\gamma}(d\check{p}_\gamma v, d\check{p}_\gamma w) = \langle p_\gamma^* v, p_\gamma^* w\rangle_{X_\gamma} = p_\gamma^*\langle v,w\rangle_X = (d\check{p}_\gamma)^t\omega_X(v,w),$$

where the first and last equality follow from the identification $T_{(E,\varphi)}\mathbf{M}_X(n,d) \cong \mathbb{H}^1(C^\bullet)$. By Proposition 4.4 and [FP, Proposition 4.2], $(d\check{p}_\gamma)^t\omega_X(v,w) = 0 = p_\gamma^*\langle v,w\rangle_X$. Since $\langle v,w\rangle_X \in \mathbb{C} = H^0(X,\mathbb{C})$, and p_γ is a local isomorphism, it follows that the globally constant function $\langle v,w\rangle_X$ vanishes locally, so it must be globally zero. Hence isotropicity follows. □

We now describe the fibres of the fibration (4.6) restricted to the locus of reduced spectral curves, showing that the points in $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ are Hecke transforms of certain γ-invariant Higgs bundles determined by $\mathcal{J}$, what justifies the choice of the notation Hec. For every $b \in B^\gamma_{\mathrm{red}}$ given by $\phi \in H^0(X_\gamma,K)$, define

$$(E_{\mathcal{J}},\varphi_\phi) := \left(p_{\gamma,*}\mathcal{J}, p_{\gamma,*}\nu_\phi^*\lambda\right), \tag{4.9}$$

where λ is the tautological section restricted to X_b. By Theorem 3.3 these bundles are γ-invariant.

Theorem 4.6. *The fibre of* (4.6) *over* $b \in B^\gamma_{\mathrm{red}}$ *is a union of torsors over* $H^0(X_b,\mathcal{O}^*_{\mathrm{sing}(X_b)})$. *In fact*

$$(h^\gamma_{\mathcal{J}})^{-1}(b) \cong \bigsqcup_{i=0}^{n_{\mathcal{J}}-1} H^0(X_b,\mathcal{O}^*_{\mathrm{sing}(X_b)}),$$

where $n_{\mathcal{J}}$ *is number of distinct points in the orbit of* $\mathcal{J}$ *under the Galois group* $\mathbb{Z}_n$.

Moreover, the Higgs bundles $(E,\psi) \in (h^\gamma_{\mathcal{J}})^{-1}(b)$ *are Hecke transforms, at the divisor* $\pi_b(\mathrm{sing}(X_b))$ *on* X, *of the* γ*-invariant Higgs bundles* $(E_{\mathcal{J}},\varphi_\phi)$, *i.e. one has the short exact sequence*

$$0 \longrightarrow E \longrightarrow E_{\mathcal{J}} \longrightarrow \mathcal{O}_{\pi_b(\mathrm{sing}(X_b))} \longrightarrow 0, \tag{4.10}$$

and

$$\psi = \varphi_\phi|_E. \tag{4.11}$$

Proof. Recall from (3.8) that $\nu_\phi : X_\gamma \to X_b$ is a normalisation map. Hence, by [Gr, Proposition 21.5.8], we consider the following short exact sequence of sheaves on X_b.

$$0 \longrightarrow \mathcal{O}^*_{X_b} \longrightarrow \nu_{\phi,*}\mathcal{O}^*_{X_\gamma} \longrightarrow \mathcal{O}^*_{\mathrm{sing}(X_b)} \longrightarrow 0$$

and the induced long exact sequence in cohomology, which is

$$0 \longrightarrow H^0(X_b, \mathcal{O}^*_{X_b}) \longrightarrow H^0(X_b, \nu_{\phi,*}\mathcal{O}^*_{X_\gamma}) \longrightarrow H^0(X_b, \mathcal{O}^*_{\mathrm{sing}(X_b)}) \longrightarrow H^1(X_b, \mathcal{O}^*_{X_b}) \longrightarrow H^1(X_b, \nu_{\phi,*}\mathcal{O}^*_{X_\gamma}) \longrightarrow 0,$$

as $H^1(X_b, \mathcal{O}^*_{\mathrm{sing}(X_b)}) = 0$, since $\mathcal{O}^*_{\mathrm{sing}(X_b)}$ is a torsion sheaf.

Since ν_ϕ is a finite morphism, $H^0(X_b, \nu_{\phi,*}\mathcal{O}^*_{X_\gamma}) \cong H^0(X_\gamma, \mathcal{O}^*_{X_\gamma}) \cong \mathbb{C}^* \cong H^0(\mathcal{O}^*_{X_b})$ and also $H^1(X_b, \nu_{\phi,*}\mathcal{O}^*_{X_\gamma}) \cong H^1(X_\gamma, \mathcal{O}^*_{X_\gamma})$. Therefore, we get the short exact sequence

$$0 \longrightarrow H^0(X_b, \mathcal{O}^*_{\mathrm{sing}(X_b)}) \longrightarrow H^1(X_b, \mathcal{O}^*_{X_b}) \longrightarrow H^1(X_\gamma, \mathcal{O}^*_{X_\gamma}) \longrightarrow 0, \tag{4.12}$$

where the second morphism corresponds to the pullback map $\hat{\nu}_\phi : \mathrm{Jac}(X_b) \to \mathrm{Jac}(X_\gamma)$ for via ν_ϕ. By (4.12), the kernel of $\hat{\nu}_\phi$ is isomorphic to $H^0(X_b, \mathcal{O}^*_{\mathrm{sing}(X_b)})$. Since the preimage under $\hat{\nu}_\phi$ of any line bundle on $\mathrm{Jac}^{\delta+d}(X_\gamma)$ is (non-canonically) isomorphic to $\ker \hat{\nu}_\phi$. After (3.16) we have $\hat{\nu}_\phi^{-1}(\mathcal{J}) = \hat{\nu}_{b,j}^{-1}(\xi^{(i-j),*}\mathcal{J})$, so they are exactly $n_{\mathcal{J}}$ of them. Then first statement follows.

Since every $(E, \varphi) \in (h^\gamma_{\mathcal{J}})^{-1}(b)$ is given by a spectral data $L \to X_b$ such that $\nu_\phi^* L \cong \mathcal{J}$, one has that

$$0 \longrightarrow L \longrightarrow \nu_{\phi,*}\mathcal{J} \longrightarrow \mathcal{O}_{\mathrm{sing}(X_b)} \longrightarrow 0. \tag{4.13}$$

Then, (4.10) holds after taking the pushforward under π_b of (4.13), were we observe that $\pi_{b,*}\nu_{\phi,*}\mathcal{J} = p_{\gamma,*}\mathcal{J} = E_{\mathcal{J}}$ since (3.15) commutes. Finally, (4.11) is a consequence of the inclusion $L \subset \nu_{\phi,*}\mathcal{J}$ of sheaves in $|K|$. □

Remark 4.7. Proposition 4.2 is the version of (4.10) in terms of families of spectral data.

Remark 4.8. If $\mathcal{J}$ is the pullback of a line bundle on X, then $n_{\mathcal{J}} = 1$ since it is invariant under the Galois group. Then $(h^\gamma_{\mathcal{J}})^{-1}(b)$ is a torsor for $H^0(X_b, \mathcal{O}^*_{\mathrm{sing}(X_b)})$. Notice that this case only occurs when d is a multiple of n (hence can assume $d = 0$), since $\deg(\mathcal{J}) = \delta + d$.

The next result shows that, again if $\mathcal{J}$ descends to X, then the Hitchin section restricted to B^γ_{red} maps to $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$, thus providing a section of (4.6).

Proposition 4.9. *Suppose d is a multiple of n. Consider a line bundle $\mathcal{L} \to X$ of degree $(\delta + d)/n$. The Hitchin section associated to $\mathcal{L}$ intersects every fibre of* $\mathrm{Hec}^{\gamma, p_\gamma^*\mathcal{L}}_{\mathrm{red}} \longrightarrow B^\gamma_{\mathrm{red}}$, *providing a section of this fibration.*

Proof. One can consider, for every $b \in B^\gamma_{\mathrm{red}}$, the Higgs bundle obtained from the spectral data $\pi_b^*\mathcal{L} \to X_b$. By the commutativity of (3.15), it is contained in $\mathrm{Hec}^{\gamma, p_\gamma^*\mathcal{L}}_{\mathrm{red}}$. □

Since we already know, after Proposition 4.5, that $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ is isotropic, Lagrangianity follows after showing that is is mid-dimensional.

Theorem 4.10. $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ *is a Lagrangian subvariety.*

Proof. Thanks to Theorem 4.6 the dimension of the fibres of (4.6) is equal to the dimension of $H^0(X_b, \mathcal{O}^*_{\mathrm{sing}(X_b)})$. We have that $\dim H^0(X_b, \mathcal{O}^*_{\mathrm{sing}(X_b)})$ equal to the degree of the singular divisor $\mathrm{sing}(X_b)$ of X_b, and by (iv) of Proposition 3.6 this is $n(n-1)(g-1)$. From (i) and (ii) Proposition 3.7 $\dim B^\gamma_{\mathrm{red}} = \dim B^\gamma_{X,n} = n(g-1)+1$. So we have a mid dimensional subvariety, as the dimension of the fibers and the base of (4.6) are additive since we lie in the stable, and therefore smooth, locus. It is moreover isotropic by Proposition 4.5, thus Lagrangian. □

For every $\gamma \in \mathrm{Jac}(X)[n]$ of maximal order and every line bundle $\mathcal{J} \to X_\gamma$ of degree $\delta + d$, we have thus constructed a Lagrangian subvariety $\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}$ of the moduli space of Higgs bundles $\mathbf{M}_X(n, d)$, which lies in the locus given by reduced spectral curves. Taking the trivial flat bundle on it, we obtain a (BAA)-brane,

$$(\mathbf{BAA})^{\gamma,\mathcal{J}}_{\mathrm{red}} := \left(\mathrm{Hec}^{\gamma,\mathcal{J}}_{\mathrm{red}}, (\mathcal{O}, \nabla_{\mathcal{O}})\right),$$

which we call the *Hecke* (BAA)-*brane associated to* $\gamma \in \mathrm{Jac}(X)[n]$ *and* $\mathcal{J} \to X_\gamma$ (restricted to the locus of reduced spectral curves).

5. Mirror symmetry in the locus of reduced spectral curves

In this section we study the duality between the Narasimhan-Ramanan (BBB)-branes and the Hecke (BAA)-branes that we have constructed in the preceding sections. For a choice of Lagrangian section of the Hitchin fibration, one can construct a relative Poincaré sheaf over the locus of reduced spectral curves, and mirror symmetry is expected to be realised by a Fourier–Mukai transform associated to it. Since the duality is not yet well understood in the Hitchin fibres corresponding to non-reduced curves, we will restrict our study to the locus of reduced spectral curves.

In this section, we perform a Fourier–Mukai transform of the restriction of the Narasimhan-Ramanan (BBB)-branes $(\mathbf{BBB})^{\gamma}_{\mathcal{L}}$ and $(\mathbf{BBB})^{\gamma}_{\mathcal{F}}$ to a Hitchin fibre associated to a reduced curve. We will see that the support of the transformed sheaf is precisely the intersection of the Hecke (BAA)-brane with the corresponding Hitchin fibre. By functoriality of the relative Fourier–Mukai transform, the fibrewise Fourier–Mukai transform that we perform corresponds to the restriction to a given fibre of the relative one.

In this section we address only the case of trivial degree $d = 0$. We do so because, in this case there exist global Lagrangian sections of the Hitchin fibration $h_{X,n} : \mathbf{M}_X(n, 0) \to B_{X,n}$, the so-called Hitchin sections. This allow us to perform the relative Fourier–Mukai transform without using a gerbe (or using a trivial one). For d non-multiple of n, there is non such global Lagrangian section, hence a gerbe is required to perform the relative Fourier-Mukai (cf. [HT]). Another option for such d is to consider instead the moduli spaces of parabolic Higgs bundles, since, for an appropriate choice of weights, there is a Hitchin section, hence the duality can be performed without the gerbe (cf. [GO]).

Consider the Hitchin section associated to a square root of the canonical bundle $K^{1/2}$, *i.e.* take for every spectral curve $\pi_b : X_b \to X$, the line bundle $\pi_b^* K^{(n-1)/2}$ on X_b. We consider here the Fourier–Mukai associated to this Hitchin section, so $\pi_b^* K^{(n-1)/2}$ will be the distinguished point in our Hitchin fibre. Note that $p_\gamma^* K \cong K_\gamma$, so $p_\gamma^* K_K^{1/2}$ is a square root of K_γ, giving rise to an identification between the abelian variety $\mathrm{Jac}^0(X_\gamma)$ and the torsor $\mathrm{Jac}^\delta(X_\gamma)$.

From (iv) Proposition 3.6, we have that, for any $b \in B^{\gamma}_{\mathrm{red}}$, the spectral curve X_b is reduced, irreducible and has only planar singularities. We find ourselves in the situation described in Section 2.3, and, thanks to [Ar], there exists a Poincaré sheaf $\overline{\mathcal{P}}_b \to \overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b)$ giving rise to the derived equivalence Θ_b of (2.6).

We recall that the Poincaré sheaf $\overline{\mathcal{P}}_b$ is constructed from the Poincaré bundle $\mathcal{P}_b \to \overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b)$. We provide first some technical results concerning $\mathcal{P}_b$. Since X_γ is smooth, the jacobian $\mathrm{Jac}^0(X_\gamma)$ is a smooth abelian variety known to be self-dual and one can naturally define a Poincaré bundle over the product $\mathrm{Jac}^0(X_\gamma) \times \mathrm{Jac}^\delta(X_\gamma)$. As $\delta = n(n-1)(g-1)$ is a multiple of n, the choice of a point $x_0 \in X$ defines an isomorphism between $\mathrm{Jac}^0(X_\gamma)$ and $\mathrm{Jac}^\delta(X_\gamma)$ defined via tensoring by $\mathcal{O}_{X_\gamma}(p_\gamma^{-1}(x_0))$, where we recall that $p_\gamma : X_\gamma \to X$ is an n-th cover.

Denote by $\widetilde{\mathcal{P}}_\gamma$ the Poincaré bundle over $\mathrm{Jac}^0(X_\gamma) \times \mathrm{Jac}^\delta(X_\gamma)$. Recall from (3.17) the pushforward morphism $\check{\nu}_\phi$ induced from the normalisation map $\nu_\phi : X_\gamma \to X_b$, and consider as well the pullback map

$$\hat{\nu}_\phi : \mathrm{Jac}^\delta(X_b) \to \mathrm{Jac}^\delta(X_\gamma).$$

One sees that both $(\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}})^* \mathcal{P}_b$ and $(\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu}_\phi)^* \widetilde{\mathcal{P}}_\gamma$ are bundles over $\mathrm{Jac}^0(X_\gamma) \times \mathrm{Jac}^\delta(X_b)$ where $\mathbf{1}_{\widetilde{\mathrm{Jac}}}$ and $\mathbf{1}_{\mathrm{Jac}}$ are the identity morphisms in $\mathrm{Jac}^0(X_\gamma)$ and $\mathrm{Jac}^\delta(X_b)$. Similarly to [FP, Lemma 5.2], one can prove that they are indeed isomorphic.

Proposition 5.1. *One has that*

$$(\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}})^* \mathcal{P}_b \cong (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu}_\phi)^* \widetilde{\mathcal{P}}_\gamma.$$

Proof. After a certain adaptation, the proof is analogous to that of [FP, Lemma 5.2]. We include it in the present paper for the sake of clarity.

Note that $(\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}})^* \mathcal{P}_b$ is a family of topologically trivial line bundles over $\mathrm{Jac}^0(X_\gamma)$ parametrized by $\mathrm{Jac}^\delta(X_b)$. Since $\widetilde{\mathcal{P}}_\gamma \to \mathrm{Jac}^0(X_\gamma) \times \mathrm{Jac}^0(X_\gamma)$ is a universal family for these objects, there exists a map

$$g : \mathrm{Jac}^\delta(X_b) \to \mathrm{Jac}^0(X_\gamma),$$

such that

$$(\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}})^* \mathcal{P}_b \cong (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times g)^* \widetilde{\mathcal{P}}_\gamma.$$

We claim that this map is $g = \hat{\nu}_\phi$ but to see it, we need some preliminary statements.

For each $M \in \mathrm{Jac}^\delta(X_b)$, recall the description of $\mathcal{P}_{b,M}$ given in (2.5). Recall as well the projections $f_1 : X_b \times \overline{\mathrm{Jac}}^\delta(X_b) \to X_b$ and $f_2 : X_b \times \overline{\mathrm{Jac}}^\delta(X_b) \to \overline{\mathrm{Jac}}^\delta(X_b)$, and take the obvious projections $\widetilde{f}_1 : X_\gamma \times \mathrm{Jac}^0(X_\gamma) \to X_\gamma$ and $\widetilde{f}_2 : X_\gamma \times \mathrm{Jac}^0(X_\gamma) \to \mathrm{Jac}^0(X_\gamma)$. One has the following commuting diagrams,

$$\begin{array}{ccc} X_\gamma \times \mathrm{Jac}^0(X_\gamma) & \xrightarrow{\nu_\phi \times \check{\nu}_\phi} & X_b \times \overline{\mathrm{Jac}}^\delta(X_b) \\ \Big\downarrow{\scriptstyle \tilde{f}_1} & & \Big\downarrow{\scriptstyle f_1} \\ X_\gamma & \xrightarrow{\nu_\phi} & X_b, \end{array}$$

and

$$\begin{array}{ccc} X_\gamma \times \mathrm{Jac}^0(X_\gamma) & \xrightarrow{\nu_\phi \times \check{\nu}_\phi} & X_b \times \overline{\mathrm{Jac}}^\delta(X_b) \\ \Big\downarrow{\scriptstyle \tilde{f}_2} & & \Big\downarrow{\scriptstyle f_2} \\ \mathrm{Jac}^0(X_\gamma) & \xrightarrow{\check{\nu}_\phi} & \overline{\mathrm{Jac}}^\delta(X_b). \end{array}$$

We know from [Es, Proposition 44 (1)] that the determinant of cohomology commutes with base change, *i.e.*

$$\check{\nu}_\phi^* \mathcal{D}_{f_2} = \mathcal{D}_{\tilde{f}_2} (\nu_\phi \times \check{\nu}_\phi)^*.$$

Let us denote by $\widetilde{\mathcal{U}}_\gamma \to X_\gamma \times \mathrm{Jac}^0(X_\gamma)$ the universal bundle of topologically trivial line bundles over X_γ. Observe that, after Section 2.3, the Poincaré bundle $\widetilde{\mathcal{P}}_\gamma$ satisfies

$$\widetilde{\mathcal{P}}_{\gamma,N} = \mathcal{D}_{\tilde{f}_2}(\widetilde{\mathcal{U}}_\gamma \otimes \widetilde{f}_1^* N)^{-1} \otimes \mathcal{D}_{\tilde{f}_2}(\widetilde{f}_1^* N) \otimes \mathcal{D}_{\tilde{f}_2}(\widetilde{\mathcal{U}}_\gamma).$$

Recall that $\mathcal{U}_b \to X_b \times \overline{\mathrm{Jac}}^\delta$ is the universal sheaf of degree δ torsion free sheaves on X_b and consider the pullback $(\nu_\phi \times \check{\nu}_\phi)^* \mathcal{U}_b$ which is a sheaf over $X_\gamma \times \mathrm{Jac}^0(X_\gamma)$. Given $G \in \mathrm{Jac}^0(X_\gamma)$, observe that the restriction of $(\nu_\phi \times \check{\nu}_\phi)^* \mathcal{U}_b$ to $X_\gamma \times \{L\}$ coincides with

$$\begin{aligned} ((\nu_\phi \times \check{\nu}_\phi)^* \mathcal{U}_b)\,|_{X_\gamma \times \{G\}} &\cong \left((\mathbf{1}_{X_\gamma} \times \check{\nu}_\phi)^* (\nu_\phi \times \mathbf{1}_{\mathrm{Jac}})^* \mathcal{U}_b \right) |_{X_\gamma \times \{G\}} \\ &\cong ((\nu_\phi \times \mathbf{1}_{\mathrm{Jac}})^* \mathcal{U}_b)\,|_{X_\gamma \times \{\check{\nu}_\phi(G)\}} \\ &\cong \nu_\phi^* \left(\mathcal{U}_b|_{X_b \times \{\check{\nu}_\phi(G)\}} \right) \\ &\cong \nu_\phi^* \left(\nu_{\phi,*} (\widetilde{\mathcal{U}}_\gamma|_{X_\gamma \times \{G\}} \right). \end{aligned}$$

Then, we see that $\widetilde{\mathcal{U}}_\gamma$ is a subfamily of $(\nu_\phi \times \check{\nu}_\phi)^* \mathcal{U}_b$, fitting in the exact sequence

$$0 \longrightarrow \widetilde{\mathcal{U}}_\gamma \longrightarrow (\nu_\phi \times \check{\nu}_\phi)^* \mathcal{U}_b \longrightarrow \mathcal{Q}_b \longrightarrow 0, \tag{5.1}$$

where $\mathcal{Q}_b$ is the (constant) family of sky-scraper sheaves supported on $\nu_\phi^{-1}(\mathrm{sing}(X_b))$.

The additive property of the determinant of cohomology [Es, Proposition 44 (4)] applied to (5.1) gives

$$\mathcal{D}_{\tilde{f}_2}((\nu_\phi \times \check{\nu}_\phi)^* \mathcal{U}_b) \cong \mathcal{D}_{\tilde{f}_2}(\widetilde{\mathcal{U}}_\gamma) \otimes \mathcal{D}_{\tilde{f}_2}(\mathcal{Q}_b).$$

Given any line bundle N over X_γ, we have that $\mathcal{Q}_b \otimes \widetilde{f}_1^* N$ is isomorphic to $\mathcal{Q}_b$ itself, hence

$$\mathcal{D}_{\tilde{f}_2} \left(\mathcal{Q}_b \otimes \widetilde{f}_1^* N \right) \cong \mathcal{D}_{\tilde{f}_2} \left(\mathcal{Q}_b \right).$$

Also, one has the following commutative diagram

Using these properties, we can show for any $M \in \mathrm{Jac}^\delta(X_b)$ that

$$\begin{aligned}
\widetilde{\mathcal{P}}_{\gamma,g(M)} &\cong \check{\nu}_\phi^* \mathcal{P}_{b,M} \\
&\cong \check{\nu}_\phi^* \left(\mathcal{D}_{f_2}\left(\mathcal{U}_b \otimes f_1^* M\right)^{-1} \otimes \mathcal{D}_{f_2}(f_1^*M) \otimes \mathcal{D}_{f_2}\left(\mathcal{U}_b\right) \right) \\
&\cong \check{\nu}_\phi^* \mathcal{D}_{f_2}\left(\mathcal{U}_b \otimes f_1^* M\right)^{-1} \otimes \check{\nu}_\phi^* \mathcal{D}_{f_2}(f_1^*M) \otimes \check{\nu}_\phi^* \mathcal{D}_{f_2}\left(\mathcal{U}_b\right) \\
&\cong \mathcal{D}_{\tilde{f}_2}\left((\nu_\phi \times \check{\nu}_\phi)^* \left(\mathcal{U}_b \otimes f_1^* M\right)\right)^{-1} \otimes \mathcal{D}_{\tilde{f}_2}(\nu_\phi \times \check{\nu}_\phi)^*(f_1^*M) \otimes \mathcal{D}_{\tilde{f}_2}\left((\nu_\phi \times \check{\nu}_\phi)^* \mathcal{U}_b\right) \\
&\cong \mathcal{D}_{\tilde{f}_2}\left((\nu_\phi \times \check{\nu}_\phi)^* \mathcal{U}_b \otimes \widetilde{f}_1^* \nu_\phi^* M\right)^{-1} \otimes \mathcal{D}_{\tilde{f}_2}\left(\widetilde{f}_1^* \nu_\phi^* M\right) \otimes \mathcal{D}_{\tilde{f}_2}\left((\nu_\phi \times \check{\nu}_\phi)^* \mathcal{U}_b\right) \\
&\cong \mathcal{D}_{\tilde{f}_2}(\widetilde{\mathcal{U}}_\gamma \otimes \widetilde{f}_1^* \nu_\phi^* M)^{-1} \otimes \mathcal{D}_{\tilde{f}_2}(\mathcal{Q}_b \otimes \widetilde{f}_1^* \nu_\phi^* M)^{-1} \otimes \mathcal{D}_{\tilde{f}_2}\left(\widetilde{f}_1^* \nu_\phi^* M\right) \\
&\qquad \otimes \mathcal{D}_{\tilde{f}_2}(\widetilde{\mathcal{U}}_\gamma) \otimes \mathcal{D}_{\tilde{f}_2}(\mathcal{Q}_b) \\
&\cong \mathcal{D}_{\tilde{f}_2}(\widetilde{\mathcal{U}}_\gamma \otimes \widetilde{f}_1^* \nu_\phi^* M)^{-1} \otimes \mathcal{D}_{\tilde{f}_2}(\mathcal{Q}_b)^{-1} \otimes \mathcal{D}_{\tilde{f}_2}\left(\widetilde{f}_1^* \nu_\phi^* M\right) \otimes \mathcal{D}_{\tilde{f}_2}(\widetilde{\mathcal{U}}_\gamma) \otimes \mathcal{D}_{\tilde{f}_2}(\mathcal{Q}_b) \\
&\cong \mathcal{D}_{\tilde{f}_2}(\widetilde{\mathcal{U}}_\gamma \otimes \widetilde{f}_1^* \nu_\phi^* M)^{-1} \otimes \mathcal{D}_{\tilde{f}_2}\left(\widetilde{f}_1^* \nu_\phi^* M\right) \otimes \mathcal{D}_{\tilde{f}_2}(\widetilde{\mathcal{U}}_\gamma) \\
&\cong \widetilde{\mathcal{P}}_{\gamma,\nu_\phi^* M} \\
&\cong \widetilde{\mathcal{P}}_{\gamma,\hat{\nu}_\phi(M)}.
\end{aligned}$$

Then, $g = \hat{\nu}_\phi$ and this finish the proof. □

Consider the projections to the first and second factors

$$\begin{array}{ccccc}
 & & \mathrm{Jac}^0(X_\gamma) \times \mathrm{Jac}^\delta(X_\gamma) & & \\
 & \stackrel{\tilde{\pi}_1}{\swarrow} & & \stackrel{\tilde{\pi}_2}{\searrow} & \\
\mathrm{Jac}^0(X_\gamma) & & & & \mathrm{Jac}^\delta(X_\gamma),
\end{array}$$

and, using $\widetilde{\mathcal{P}}_\gamma \to \mathrm{Jac}^0(X_\gamma) \times \mathrm{Jac}^\delta(X_\gamma)$, define the integral functor

$$\begin{array}{rccc}
\Theta_\gamma : & D^b(\mathrm{Jac}^0(X_\gamma)) & \longrightarrow & D^b(\mathrm{Jac}^\delta(X_\gamma)) \\
 & \mathcal{E}^\bullet & \longmapsto & R\tilde{\pi}_{2,*}(\tilde{\pi}_1^* \mathcal{E}^\bullet \otimes \widetilde{\mathcal{P}}_u).
\end{array}$$

One can now study the image under Θ_b of the complexes constructed by pushforward via the embedding $\check{\nu}_\phi : \mathrm{Jac}^0(X_\gamma)) \to \overline{\mathrm{Jac}}^\delta(X_b)$. Let us denote the inclusion of the Jacobian into the compactified Jacobian by $h : \mathrm{Jac}^\delta(X_b) \hookrightarrow \overline{\mathrm{Jac}}^\delta(X_b)$.

Theorem 5.2. *There exists an isomorphism*

$$\Theta_b(R\check{\nu}_{\phi,*}\mathcal{E}^\bullet) \cong Rh_* \hat{\nu}_\phi^* \Theta_\gamma(\mathcal{E}^\bullet).$$

Proof. First of all, note that $R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet$ is a complex whose sheaves are all supported in $\mathrm{Im}(\hat{\nu}_\phi)$, and this does not intersect $\mathrm{Jac}^\delta(X_b)$. Therefore,

$$\mathrm{supp}(\pi_1^* R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet) \cap \left(\overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b)\right)^\sharp = \mathrm{Im}(\hat{\nu}_\phi) \times \mathrm{Jac}^\delta(X_b),$$

which is contained in $\overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b) \subset \left(\overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b)\right)^\sharp$.

Recall from Section 2.3 the injection j and note that the following diagram commutes,

$$\begin{array}{ccc}
\left(\overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b)\right)^\sharp & \stackrel{j}{\hookrightarrow} & \overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b) \\
\uparrow & \stackrel{i}{\nearrow} & \\
\overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b) & &
\end{array}$$

Since $\overline{\mathcal{P}}_b = j_*\mathcal{P}_b^\sharp$, one can see that

$$\overline{\mathcal{P}}_b \otimes \pi_1^* R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet \cong i_*\mathcal{P}_b \otimes \pi_1^* R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet.$$

Applying the projection formula yields

$$\overline{\mathcal{P}}_b \otimes \pi_1^* R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet \cong Ri_*(\mathcal{P}_b \otimes i^*\pi_1^* R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet).$$

Consider the obvious projections

$$\begin{array}{ccc} & \overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b) & \\ {}^{\pi_1'}\swarrow & & \searrow^{\pi_2'} \\ \overline{\mathrm{Jac}}^\delta(X_b) & & \mathrm{Jac}^\delta(X_b). \end{array}$$

Recalling the inclusion $h : \mathrm{Jac}^\delta(X_b) \hookrightarrow \overline{\mathrm{Jac}}^\delta(X_b)$, one has that

- $h \circ \pi_2' = \pi_2 \circ i$,
- $\pi_1' = \pi_1 \circ i$.

As a consequence of this,

$$\begin{aligned} \Theta_b(R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet) =& R\pi_{2,*}(\overline{\mathcal{P}}_b \otimes \pi_1^* R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet) \\ \cong& R\pi_{2,*}Ri_*(\mathcal{P}_b \otimes i^*\pi_1^* R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet) \\ \cong& Rh_*R\pi'_{2,*}(\mathcal{P}_b \otimes (\pi_1')^* R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet). \end{aligned}$$

Consider also

$$\begin{array}{ccc} & \mathrm{Jac}^0(X_\gamma) \times \mathrm{Jac}^\delta(X_b) & \\ {}^{\widetilde{\pi}_1'}\swarrow & & \searrow^{\widetilde{\pi}_2'} \\ \mathrm{Jac}^0(X_\gamma) & & \mathrm{Jac}^\delta(X_b), \end{array}$$

and, recalling the projections $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$, observe that

- $\widetilde{\pi}_2' = \pi_2' \circ (\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}})$,
- $\widetilde{\pi}_1' = \widetilde{\pi}_1 \circ (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu}_\phi)$,
- $\pi_1' \circ (\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}}) = \check{\nu}_\phi \circ \widetilde{\pi}_1'$, and
- $\widetilde{\pi}_2 \circ (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu}_\phi) = \hat{\nu}_\phi \circ \widetilde{\pi}_2'$.

Finally, thanks to Proposition 5.1 and that all the maps involved are flat, one has the following,

$$\begin{aligned} \Theta_b(R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet) =& Rh_*R\pi'_{2,*}((\pi_1')^* R\hat{\nu}_{\phi,*}\mathcal{E}^\bullet \otimes \mathcal{P}_b) \\ \cong& Rh_*R\pi'_{2,*}(R(\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}})_*(\widetilde{\pi}_1')^*\mathcal{E}^\bullet \otimes \mathcal{P}_b) \\ \cong& Rh_*R\pi'_{2,*}R(\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}})_*((\widetilde{\pi}_1')^*\mathcal{E}^\bullet \otimes (\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}})^*\mathcal{P}_b) \\ \cong& Rh_*R\pi'_{2,*}R(\check{\nu}_\phi \times \mathbf{1}_{\mathrm{Jac}})_*((\widetilde{\pi}_1')^*\mathcal{E}^\bullet \otimes (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu}_\phi)^*\widetilde{\mathcal{P}}_\gamma) \\ \cong& Rh_*R\widetilde{\pi}'_{2,*}((\widetilde{\pi}_1')^*\mathcal{E}^\bullet \otimes (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu}_\phi)^*\widetilde{\mathcal{P}}_\gamma) \\ \cong& Rh_*R\widetilde{\pi}'_{2,*}(\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \hat{\nu}_\phi)^*(\widetilde{\pi}_1^*\mathcal{E}^\bullet \otimes \widetilde{\mathcal{P}}_\gamma) \\ \cong& Rh_*\hat{\nu}_\phi^* R\widetilde{\pi}_{2,*}(\widetilde{\pi}_1^*\mathcal{E}^\bullet \otimes \widetilde{\mathcal{P}}_\gamma) \\ \cong& Rh_*\hat{\nu}_\phi^* R\widetilde{\pi}_{2,*}(\widetilde{\pi}_1^*\mathcal{E}^\bullet \otimes \widetilde{\mathcal{P}}_\gamma) \\ \cong& Rh_*\hat{\nu}_\phi^*\Theta_\gamma(\mathcal{E}^\bullet), \end{aligned}$$

and the proof is complete. □

In the notation of [Mu2, Definition 2.3]), a sheaf is WIT_n if its Fourier–Mukai transform is a complex supported in degree n. After Theorem 5.2 one has the following.

Corollary 5.3. *Let $\mathcal{E}$ be a sheaf on $\mathrm{Jac}^0(X_\gamma)$ which is WIT_n with respect to Θ_γ. Then $\check{\nu}_{\phi,*}\mathcal{E}$ is a WIT_n sheaf with respect to Θ_b.*

We recall from [Mu2] that topologically trivial line bundles are WIT_{-m}, where m is the dimension of the abelian variety. In fact, as complexes supported in degree $-m$, a the fourier–Mukai transform of a topologically trivial line bundle $\check{\mathcal{J}}$ is the sky-scraper sheaf at the point given by $\check{\mathcal{J}}$.

Given $\mathcal{L}$ and $\mathcal{F}$, respectively, degree 0 line bundles on the base X and on X_γ, the branes $(\mathbf{BBB})^\gamma_{\mathcal{L}}$ and $(\mathbf{BBB})^\gamma_{\mathcal{F}}$ consist of a hyperholomorphic line bundle $\mathscr{L}$ and a hyperholomorphic vector bundle $\mathscr{F}$ supported on the fixed point locus $\mathbf{M}_X(n,0)^\gamma$. We know from Proposition 3.9 that $\mathscr{L}$ restricted to a Hitchin fibre associated to a reduced spectral curve, is $\mathrm{Nm}^*\check{\mathcal{L}}$ while the restriction of $\mathscr{F}$ is $\bigoplus_i \xi^i_*\check{\mathcal{F}}$. As sheaves over $\mathbf{M}_X(n,0)$, they correspond, respectively, with $\check{\nu}_{\phi,*}\check{\mathcal{L}}$ and $\bigoplus_i \check{\nu}_{\phi,*}\xi^i_*\check{\mathcal{F}}$.

Recall that the identification between $\mathrm{Jac}^0(X_\gamma)$ and $\mathrm{Jac}^\delta(X_\gamma)$ is determined by our choice of the Hitchin section that determines the relative Poincaré sheaf. In that case, we see that $\Theta_\gamma(\check{\mathcal{F}})$ is the complex supported in degree $\dim \mathrm{Jac}(X_\gamma) = n(g-1)+1$ given by the skyscraper sheaf supported in $\hat{\mathcal{F}} := \mathcal{F} \otimes p_\gamma^* K^{(n-1)/2}$.

Since the dual of the norm map Nm corresponds to the pullback under p_γ, then $\Theta_\gamma(\mathrm{Nm}^*\check{\mathcal{L}})$ is the complex supported in degree $\dim \mathrm{Jac}(X_\gamma)$ given by sky-scraper sheaf at $\hat{\mathcal{L}} := p_\gamma^*\mathcal{L} \otimes p_\gamma^* K^{(n-1)/2}$.

We have thus proved the following.

Theorem 5.4. *Let $b \in B^\gamma_{\mathrm{red}}$. The Fourier–Mukai transform of the restriction of $(\mathbf{BBB})^\gamma_{\mathcal{L}}$ to the Hitchin fibre $h^{-1}_{X,n}(b)$ is the trivial sheaf supported on $\mathrm{Hec}^{\gamma,\hat{\mathcal{L}}}_{\mathrm{red}} \cap h^{-1}_{X,n}(b)$,*

$$\Theta_b(\check{\nu}_{\phi,*}\check{\mathcal{L}}) \cong \hat{\nu}_\phi^*\mathcal{O}_{\hat{\mathcal{L}}} \cong \mathcal{O}_{\hat{\nu}_\phi^{-1}(\hat{\mathcal{L}})}.$$

Analogously, the transform of $(\mathbf{BBB})^\gamma_{\mathcal{F}}$ restricted to $h^{-1}_{X,n}(b)$ is is the trivial sheaf supported on $\mathrm{Hec}^{\gamma,\hat{\mathcal{F}}}_{\mathrm{red}} \cap h^{-1}_{X,n}(b)$,

$$\Theta_b\left(\bigoplus_i \check{\nu}_{\phi,*}\xi^i_*\check{\mathcal{F}}\right) \cong \bigoplus_i \hat{\nu}_\phi^*\mathcal{O}_{\xi^{i,*}\hat{\mathcal{F}}} \cong \bigoplus_i \mathcal{O}_{\hat{\nu}_\phi^{-1}(\xi^{i,*}\hat{\mathcal{F}})}.$$

6. The case of non maximal order

In the previous sections we have been dealing with branes corresponding to $\gamma \in \mathrm{Jac}(X)[n]$ of maximal order n. In this final section we shall briefly consider the cases where γ has order m. The main interest in hence when $1 < m < n$, so that n is not a prime number. Different kinds of (BBB)-branes appear depending on m, and we analise their support and argue what their duals should be. The main difficulty to carry out the full analysis is the lack of a natural hyperholomorphic bundle. The strategy is very similar to the maximal rank case, and crucially uses the construction of branes associated with parabolic subgroups from [FP, Section 6].

6.1. **The (BBB)-branes.** Let $n = n' \cdot m$, and let $\gamma \in \mathrm{Jac}(X)[n]$ be an order m element. With the same notation as in Section 3 we have the corresponding line bundle L_γ, and the associated étale m-cover $p_\gamma : X_\gamma \to X$, with Galois group isomorphic to $\mathbb{Z}_m$. Let $\xi = \exp(2\pi i/m)$ be the standard generator.

Recall that, by Theorem 3.3, pushforward defines a surjective morphism

$$\check{p}_\gamma : \mathbf{M}_{X_\gamma}(n',d) \longrightarrow \mathbf{M}_X(n,d)^\gamma$$

where $\mathbf{M}_{X_\gamma}(n',d)$ is the moduli space of rank n' degree d Higgs bundles on X_γ.

We next investigate how the Hitchin maps of both moduli spaces relate to one another. Recall the Hitchin map $h_{X_\gamma,n} : \mathbf{M}_{X_\gamma}(n,d) \to B_{X_\gamma,n}$ on the moduli space of Higgs bundles over X_γ.

Lemma 6.1. *Then there is a commutative diagram*

$$\begin{array}{ccc} \mathbf{M}_{X_\gamma}(n',d) & \xrightarrow{\ \epsilon\ } & \mathbf{M}_{X_\gamma}(n,d') \\ \downarrow{\scriptstyle h_{X_\gamma,n}} & & \downarrow{\scriptstyle h_{X_\gamma,n}} \\ B_{X_\gamma,n'} & \xrightarrow[\ \epsilon'\ ]{} & B_{X_\gamma,n} \end{array} \tag{6.1}$$

where ϵ is extension of the structure group and d' is determined by the latter.

Proof. This is due to the following fact: if $i : G_1 \hookrightarrow G_2$ is a subgroup of a complex reductive Lie group, then, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{g}_2]^{G_2} & \xrightarrow{i^*} & \mathbb{C}[\mathfrak{g}_1]^{G_1} \\ \downarrow & & \downarrow \\ \mathbb{C}[\mathfrak{g}_2] & \xrightarrow{i^*} & \mathbb{C}[\mathfrak{g}_1]. \end{array}$$

□

Consider the embedding $e : \mathrm{GL}(n', \mathbb{C}) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$

$$Y \overset{e}{\mapsto} \begin{pmatrix} Y & 0 & \dots & \dots & 0 \\ 0 & \xi \cdot Y & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & & & & \xi^{m-1} Y \end{pmatrix}. \tag{6.2}$$

Remark 6.2. Note that the image of the above map is contained in the Levi subgroup L of (for example) the parabolic subgroup P consisting of matrices with non-zero block diagonal entries and all remaining upper triangular entries. We fix L and P, and let $\mathrm{P} = L \ltimes \mathrm{U}$ with U the unipotent radical of P.

Let $\xi = \exp(2\pi i/m) \in \mathbb{Z}_m$ be the standard generator of the Galois group $\mathbb{Z}_m$ of the cover $p_\gamma : \widetilde{X}_\gamma \to X$. Define

$$\begin{array}{cccc} \epsilon : & \mathbf{M}_{X_\gamma}(n', d) & \longrightarrow & \mathbf{M}_{X_\gamma}(n', md) \\ & (F, \phi) & \longmapsto & \left(\bigoplus_{k=0}^{m-1} \xi^{k,*} E, \bigoplus_{k=0}^{m-1} \xi^{k,*} \phi \right) \end{array}$$

Note that ϵ is the morphism induced by extension of the structure group via (6.2). Then, by Lemma 6.1, there is a commutative diagram

$$\begin{array}{ccc} \mathbf{M}_{X_\gamma}(n', d) & \xrightarrow{\epsilon} & \mathbf{M}_{X_\gamma}(n, md) \\ \downarrow h_{X_\gamma, n'} & & \downarrow h_{X_\gamma, n} \\ B_{X_\gamma, n'} & \xrightarrow[\epsilon']{} & B_{X_\gamma, n}. \end{array} \tag{6.3}$$

The elements of $B_{X_\gamma,n'}$ will be denoted $\overline{b}$, and those of $B_{X_\gamma,n}$ by $\tilde{b}$. The following proposition is proved similarly to Proposition 3.6.

Proposition 6.3. *Let $\gamma \in \mathrm{Jac}(X)[n]$ be of order m. Then:*

1. *For every $b \in B^\gamma_{X,n} \backslash \{0\}$, there exists $\overline{b} \in B_{X_\gamma,n'}$ such that $\tilde{b} = p_\gamma^* b \in B_{\widetilde{X}_\gamma,n}$ satisfies $\tilde{b} = \epsilon'(\overline{b})$. Moreover $\epsilon'(\overline{b}') = \epsilon'(\overline{b})$ if and only if $\overline{b}' = \xi^{i,*}\overline{b}$ for some $i = 0, \dots, m-1$.*
2. *The curve $X_{\overline{b}}$ is the spectral curve of a given rank n' Higgs bundle on X_γ (F, ϕ) such that $(E, \varphi) = p_{\gamma,*}(F, \phi)$, and accordingly, $X_{\xi^{i,*}\overline{b}}$ is the spectral curve corresponding to $\xi^{i,*}(F, \phi)$.*
3. *The curve $X_{\tilde{b}} \subset |K_{X_\gamma}|$ determined by $\widetilde{b} = p_\gamma^* b$ is the spectral curve associated with $(p_\gamma^* E, p_\gamma^* \varphi)$, where $h(E, \varphi) = b$. It is a generically reduced curve with m irreducible components X_i $i = 1, \dots, m$ defined by*

   $$X_i = X_{\xi^{i-1,*}\overline{b}}$$

   isomorphic to $X_{\overline{b}}$. The normalisation

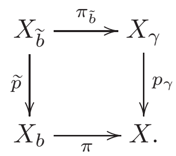
   $$\tilde{X}_{\tilde{b}} \xrightarrow{\tilde{\nu_b}} X_{\tilde{b}}$$

   satisfies

   $$\tilde{X}_{\tilde{b}} \cong \coprod_{i=0}^{m-1} X_i \cong \coprod_{i=0}^{m-1} X_{\xi^{i,*}\overline{b}}.$$

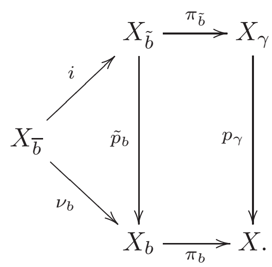
(4) the following diagram is Cartesian

(6.4)



(5) For every $b \in B^\gamma_{X,n} \backslash \{0\}$, the corresponding spectral curve X_b is singular and its normalisation is $X_{\overline{b}}$. It fits in a commutative diagram

(6.5)

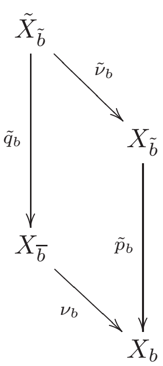


X_b is generically integral, and the genericity condition is the same as the one for $X_{\overline{b}}$ to be smooth. Its singular locus has length

$$\deg(\mathrm{sing}(X_b)) = n(n - n')(g - 1).$$

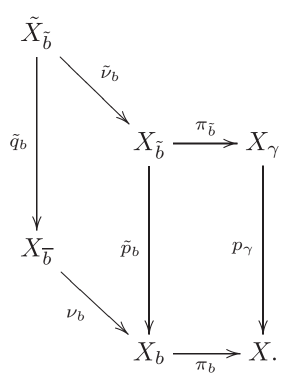
(6) the following diagram is Cartesian

(6.6)



(7) $\dim B^\gamma_{X,n} = (n')^2 m(g - 1) + 1.$

Proof. The proof is very similar to the one of Proposition 3.6. Let us discuss cartesianity of (6.6). Consider the larger diagram



The rightmost diagram is Cartesian and the leftmost is easily seen to be commutative. Moreover, it is easy to see that $\tilde{q}_b$ is a $\mathbb{Z}_m$-Galois cover. Hence it is enough to see that $\nu_b^* \pi_b^* L_\gamma$ trivializes over $X_{\overline{b}}$. This is the case, as by commutativity

$$\nu_b^* \pi_b^* L_\gamma = i^* \tilde{p}_b^* \pi_b^* L_\gamma = i^* \pi_{\tilde{b}}^* p_\gamma^* L_\gamma = \mathcal{O}_{X_{\overline{b}}}.$$

□

Let $B^{\gamma}_{\mathrm{red}}$ be the locus corresponding to integral curves.

Proposition 6.4. *Let $b \in B^{\gamma}_{\mathrm{red}}$. Then the intersection of the fixed point subvariety $\mathbf{M}_X(n,d)^{\gamma}$ with the Hitchin fibre is*

$$h_{\gamma}^{-1}(b) \cong \nu_{b,*}\,\mathrm{Jac}^{\kappa}\left(X_{\overline{b}}\right),$$

where $\kappa = d + ((n')^2 - n')m(g-1)$.

Proof. By Proposition 6.3 (2), the spectral data of elements in $\mathbf{M}_{X_{\gamma}}(n', d)$ is given by

$$\mathrm{Jac}^{n(n'-1)(g-1)+d}(X_{\overline{b}}).$$

By commutativity of diagram (6.5), $\nu_{b,*}\,\mathrm{Jac}^{n(n'-1)(g-1)+d}(X_{\overline{b}})$ is the spectral data for $p_{\gamma,*}\mathbf{M}_{X_{\gamma}}(n', d)$. By Theorem 3.3, we need only check that for $L \in \mathrm{Jac}^{n(n'-1)(g-1)+d}(X_{\overline{b}})$

$$\deg \nu_{b,*}L = (n^2 - n)(g-1) + d.$$

Since

$$\deg \nu_{b,*}L = \deg L + \deg(\mathrm{sing}(X_b)),$$

we may conclude by Proposition 6.3 (5) □

6.2. **The** (BAA)**-brane.** In this subsection we construct a Lagrangian submanifold of $\mathbf{M}_X(n,d)$, which is conjecturally the support of a (BAA) brane dual to a (BBB) brane whose support is $\mathbf{M}_X(n,d)^{\gamma}$.

Let $\mathrm{P} = \mathrm{L} \ltimes \mathrm{U}$ be the parabolic subgroup defined in Remark 6.2. Fix $W \to X$ a rank n degree d bundle.

For $b \in B^{\gamma}_{\mathrm{red}}$, denote by

$$n_b : \widetilde{X}_b \longrightarrow X_b$$

the normalisation of the spectral curve. By Proposition 6.3, we have isomorphisms $\widetilde{X}_b \cong X_{\xi^{i,*}\overline{b}}$ for all $i = 1, \ldots, m$. Moreover, if we denote by $\tilde{\xi}$ the generator of the Galois group of $\tilde{X}_{\tilde{b}} \longrightarrow X_{\overline{b}}$, we have a commutative diagram

(6.7)
$$\begin{array}{ccc} \widetilde{X}_b & \xrightarrow{\ \cong\ } & X_{\xi^{i,*}\overline{b}} \\ & \underset{\cong}{\searrow} & \big\downarrow{\scriptstyle \tilde{\xi}^{i}} \\ & & X_{\overline{b}}. \end{array}$$

From now on we will identify $\widetilde{X}_b$ and $X_{\overline{b}}$, and use the identifications in (6.7).

Assumption 2. *Assume that for all $b \in B^{\gamma}_{n}$, there exists*

(6.8)
$$\mathcal{L}_b \in \mathrm{Jac}^{\delta+d}(X_{\overline{b}})$$

such that $\pi_{b,*}\nu_{b,*}\mathcal{L}_b = W$, where $\delta + d = n(n-1)(g-1) + d$.

Define

(6.9)
$$\mathrm{Hec}^{\gamma,W} = \left\{ (E,\Phi) \in M^{\mathrm{Dol}}_{X,n} \times_B B^{\gamma}_{X,n} \;\middle|\; \begin{array}{l} \text{If } (E,\phi) = \pi_{b,*}L \\ \text{for some } L \to X_b \\ \text{then } n_b^{*}L = \mathcal{L}_b \end{array} \right\}$$

Remark 6.5. To understand the above assumption, let us compare with the maximal order case. In this case $W = p_{\gamma,*}\xi^{j,*}\mathcal{F}$ for some $j = 1, \ldots, n$, and some $\mathcal{F} \in \mathrm{Jac}(X_{\gamma})$ (possibly descending to X, but not necessarily).

Proposition 6.6. $\mathrm{Hec}^{\gamma,W}$ *is a manifold.*

Proof. Let $B^{\mathrm{red}}_{X_{\gamma},n} := p_{\gamma}^{*}B^{\gamma}_{\mathrm{red}}$, and let $B^{\mathrm{red}}_{X_{\gamma},n'}$ be its preimage under ϵ' defined in (6.1).

Over $B^{\mathrm{red}}_{X_{\gamma},n'}$, consider the families $\tilde{\Sigma} \subset \mathbb{P}(K_{\gamma}) \times B^{\mathrm{red}}_{X_{\gamma},n'}$ of spectral curves for $\mathbf{M}_{X_{\gamma}}(n', d)$, and $\Sigma \subset \mathbb{P}(K_X) \times B^{\mathrm{red}}_{X_{\gamma},n'}$ of spectral curves for $\mathbf{M}_X(n,d)$ with $\Sigma_{\overline{b}} = X_b$ for $b \in B^{\gamma}_{\mathrm{red}}$ such that $p_{\gamma}^{*}b = \oplus_i \xi^{i,*}\overline{b}$. Then, we have a normalisations morphism

$$\nu : \mathrm{Jac}_{B^{\mathrm{red}}_{X_{\gamma},n'}}(\tilde{\Sigma}) \longrightarrow \mathrm{Jac}_{B^{\mathrm{red}}_{X_{\gamma},n'}}(\Sigma).$$

Moreover, Assumption 2 is equivalent to assuming that there exists an element

$$\mathcal{L} \in \mathrm{Jac}_{B^{\mathrm{red}}_{X_\gamma,n'}}(\tilde{\Sigma})$$

such that under the spectral correspondence composed with pushforward

$$\begin{array}{ccc} s : \mathrm{Jac}_{B^{\mathrm{red}}_{X_\gamma,n'}}(\tilde{\Sigma}) & \xrightarrow{s} & \mathbf{M}_{X_\gamma}(n',d) \\ & \searrow & \downarrow{\scriptstyle \check{p}_\gamma} \\ & & \mathbf{M}_X(n,d) \end{array}$$

we have that $\check{p}_\gamma \circ s(\mathcal{L})$ parametrizes a family of Higgs bundles with underlying bundle W.

So

$$\mathrm{Hec}^{\gamma,W} = \check{p}_\gamma \circ s(\hat{\nu}^{-1}\mathcal{L}).$$

□

In order to prove that the above manifold is isotropic, we compare it with some branes inside $\mathbf{M}_{X_\gamma}(n,dm)$. We first need a lemma:

Lemma 6.7. *Let*

$$\mathcal{L}_i = (\xi^{-i})^*\mathcal{L}_b \in \mathrm{Jac}(X_{\xi^{i,*}\overline{b}})$$

be the pullback of $\mathcal{L}_1 := \mathcal{L}_b$ to $X_{\xi^{i,*}\overline{b}}$ under the isomorphisms defined in (6.7). There is an equality

$$p_\gamma^* W = \bigoplus_{i=0}^{m-1} \xi^{i,*} V$$

where $V = \pi_{\overline{b},*}\mathcal{L}_{\overline{b}} \in \mathbf{M}_{X_\gamma}(n',d)$.

Let $E_i = \xi^{i,*}V \otimes K_{X_\gamma}^{(m-i)(n')^2}$; on X_γ, and consider the variety

$$\mathrm{Uni}_{X_\gamma}^{(n',\dots,n')}(\overline{E}) = \left\{ (E,\varphi) \;\middle|\; \begin{array}{l} \exists\, \sigma \in H^0(X, E/\mathrm{P}) : \\ \varphi \in H^0(X, E_\sigma(\mathfrak{p}) \otimes K); \\ E_\sigma/\mathrm{U} := E_{\mathrm{L}} \cong \bigoplus_{i=1}^{s} E_i. \end{array} \right\}. \tag{6.10}$$

These varieties were studied in [FP, Section 6].

Lemma 6.8. *Let $b \in B_X^\gamma$. Let $\mathcal{L}_i$ be as in Lemma 6.7, and define*

$$\hat{\mathcal{L}}_i = \mathcal{L}_i \otimes \pi_{\overline{b}}^* K_{X_\gamma}^{(m-i)(n')^2} \in \mathrm{Jac}\left(X_{\xi^{i,*}\overline{b}}\right).$$

For each ordering $J = (j_1, \dots, J_m)$ of $\{1,\dots,m\}$, let

$$\hat{\mathcal{L}}_b^J = (\hat{\mathcal{L}}_1 \otimes K^{(n')^2(j_1-m)}, \dots, \hat{\mathcal{L}}_m \otimes K^{(n')^2(j_m-m)}) \in \mathrm{Jac}(\tilde{X}_{\tilde{b}}).$$

Then

$$\mathrm{Uni}_{X_\gamma}^{(n',\dots,n')}(\overline{E}) \cap \mathrm{Jac}(X_{\tilde{b}}) = \bigcup_{J \in \mathcal{O}rd_m} \hat{\hat{\nu}}^{-1}\hat{\mathcal{L}}_b^J.$$

Proof. We just need to note that the bundles $\hat{\mathcal{L}}_i$ satisfy condition [FP, Assumption 1] for the manifold $\mathrm{Uni}_{X_\gamma}^{n',\dots,n'}(\overline{E})$. With that, we may apply [FP, Proposition 6.6] to conclude. □

Proposition 6.9. *Let*

$$\hat{p}_\gamma : \mathbf{M}_X(n,d) \mapsto \mathbf{M}_{X_\gamma}(n,md)$$

be the morphism defined by pullback. Identify

$$(\mathrm{Hec}^{\gamma,W}) \cap \mathrm{Jac}(X_b) = \hat{\nu}^{-1}(\mathcal{L}_b).$$

and

$$\mathrm{Uni}_{X_\gamma}^{(n',\dots,n')}(\overline{E}) \cap \mathrm{Jac}(X_{\tilde{b}}) = \bigcup_{J \in \mathcal{O}rd_m} \hat{\hat{\nu}}^{-1}\hat{\mathcal{L}}_b^J.$$

Let $J_0 = (m, m-1, \dots, 1)$*. Then*

$$\begin{array}{ccc} \hat{\nu}^{-1}(\mathcal{L}_b) & \xhookrightarrow{\hat{p}_\gamma} & \hat{\hat{\nu}}^{-1}\hat{\mathcal{L}}_b^{J_0} \\ & \searrow \quad \nearrow & \\ & \hat{\nu}^{-1}(\mathcal{L}_b)/\mathbb{Z}_m & \end{array}$$

In particular, $\mathrm{Hec}^{\gamma,W}$ *is isotropic.*

Proof. Let $L \in \mathrm{Jac}(X_b)$ be a spectral data for $(E,\varphi) \in \mathrm{Hec}^{\gamma,W}$. By Cartesianity of (6.4), we know that $\tilde{L} = \tilde{p}_b^* L \in \mathrm{Jac}(X_{\tilde{b}})$ is the spectral data for $(\tilde{E},\tilde{\varphi}) := p_\gamma^*(E,\varphi)$. Also, Cartesianity of (6.6) implies that

$$\tilde{\nu}^*\tilde{L} = \tilde{q}_b^*\nu_b^*\mathcal{L}_b = (\mathcal{L}_1, \dots, \mathcal{L}_m),$$

for $\mathcal{L}_i$ as in Lemma 6.7. Note that

$$(\mathcal{L}_1, \dots, \mathcal{L}_m) = \hat{\mathcal{L}}_b^{J_0}, \qquad (\mathcal{L}_1, \dots, \mathcal{L}_m) \neq \hat{\mathcal{L}}_b^{J} \quad \text{for all } J \in \mathcal{O}rd_m \backslash J_0.$$

By Lemma 6.8, we thus have that

$$\hat{p}_\gamma(\mathrm{Hec}^{\gamma,W}) \subset \hat{\nu}_b^{-1}\mathcal{L}_b^{J_0} \subset \mathrm{Uni}_{X_\gamma}^{(n',\dots,n')}(\overline{E}).$$

Note that by Proposition 6.3 (4) $L_1, L_2 \in \mathrm{Jac}(X_b)$ satisfy $\tilde{L_1} = \tilde{L_2}$ if and only if $L_1 = L_2 \otimes \pi_b^* L_\gamma$. Since X_b/X is a ramified cover, it follows that $\hat{p}_\gamma$ factors through the quotient map.

Isotropicity is proved as in Proposition 4.5. □

We can finally prove the main theorem of this section, whose proof mimics that of Theorem 4.10 and is thus omitted.

Theorem 6.10. *The manifold* $\mathrm{Hec}^{\gamma,W}$ *is Lagrangian.*

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