LINEARIZATION OF RESONANT VECTOR FIELDS

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Abstract. A method allowing the linearization of a large class of vector fields with resonant eigenvalues is presented, the admissible nonlinearities being characterized by conditions that are easy to check. This method also gives information on the terms actually present in a nonlinear normal form of a given resonant vector field.

1. Introduction

Normal forms for vector fields, or (autonomous) differential equations, are very important from the theoretical point of view, and also from the point of view of applications; in particular they are the main technique in bifurcation theory, involving families of differential equations depending on parameters [2].

The critical points of generic vector fields are never resonant, but the study of resonances becomes fundamental when considering families of vector fields, depending even on only one parameter.

Given a nonlinear vector field:

$$X(x) = Ax + a(x), \quad a(0) = \frac{\partial a}{\partial x}(0) = 0$$

it follows from the classical results (section 2) that:

- If there are no resonance relations between the eigenvalues of A, the vector field is linearizable for any nonlinearity $a(x)$.
- If there are resonances, the vector field is reducible to a resonant normal form: its nonlinear part contains resonant monomials only.

Remark 1. If the nonlinear terms contain no resonant monomials, this does not mean that the corresponding vector field is linearizable (example 1).

Remark 2. If the matrix A is diagonalizable, and the nonlinear terms contain only resonant monomials, or start with a resonant monomial, the corresponding vector field is not linearizable; however, this is not true if A is not diagonalizable (example 2).

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If there are resonances, the linearizability of the vector field depends on the monomials that are actually present in its nonlinear part: the linearizability of X is not determined by its 1-jet (linear part) [9], in contrast to the classical linearization results and also those on topological equivalence [8, 6].

Our main objective here is, given a resonant matrix A, to present effective conditions on the nonlinearty $a(x)$ for the resonant vector field $X(x) = Ax + a(x)$ to be linearizable; and also a simple way of identifying the resonant monomials that have to appear in the normal form of a given resonant vector field, in particular those of smaller degree, when holomorphic or C^∞ linearization is impossible. This is specially important as in many cases it permits to establish the linearization with a finite degree of differentiability (example 3).

Sections 7 and 8 describe the applications to vector field in two and three dimensions, respectively.

We restrict our considerations to the linearization problem in the formal category: in the holomorphic category, if the Brjuno condition is verified, the existence of a formal linearizing change of variables implies the existence of a holomorphic one [5]; in the smooth case, assuming hyperbolicity, the existence of a formal linearizing change of variables implies the existence of a C^∞ one [7].

It is important to notice that all classical linearization results, in the analytic or in the C^∞ category, are applicable (in the real case) only to hyperbolic critical points, as the absence of resonances implies hyperbolicity; under this condition, it follows from the Hartman-Grobman theorem that the vector fields are topologically conjugated to their linear parts.

The topological equivalence to the linear part in the resonant case was considered by Guckenheimer [8] (Poincaré domain) and Camacho et al. [6] (Siegel domain, dimension 3), under the condition that no two eigenvalues lie in the same line through the origin.

Thus our results can be applied to situations for which even the topological situation was not previously determined.

We will consider our vector fields in complex variables, but the results are also valid for real vector fields; however, in that case they are effective essentially only when the eigenvalues are also real.

The approach presented here can be extended to maps instead of vector fields; this is done in section 6 to obtain an analogue of the main linearization result.

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2. Basic results and definitions

Let $X(x)$ be a vector field on a domain U in $\mathbb{C}^n$; X is assumed to be holomorphic as a map $X : U \longrightarrow \mathbb{C}^n$ and supposed to have a singular point at the origin in $\mathbb{C}^n$ with linear part A:

$$X(x) = Ax + a(x), \quad a(0) = \frac{\partial a}{\partial x}(0) = 0.$$

It will always be assumed that A is in the Jordan canonical form:

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & \cdots & 0 \\ \varepsilon_1 & \lambda_2 & \ddots & & & \vdots \\ 0 & \varepsilon_2 & \lambda_3 \vdots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \varepsilon_{n-1} & \lambda_n \end{bmatrix}$$

where $\varepsilon_i \in \{0, 1\}$ and $\varepsilon_i = 1 \Longrightarrow \lambda_i = \lambda_{i+1}$.

X is said to be biholomorphically equivalent to its linear part if there exists an holomorphic change of coordinates $z = \psi(x)$, preserving the origin, $\psi(0) = 0$, with inverse $x = \xi(z)$ also holomorphic, such that in the new coordinates the nonlinear part is zero:

$$\frac{\partial \psi}{\partial x}(\xi(z))X(\xi(z)) = Az.$$

Let $\lambda = (\lambda_1, \ldots \lambda_n) \in \mathbb{C}^n$ be the vector of the eigenvalues of the linear part A of X, which are not assumed to be distinct (see [2]).

The eigenvalues are said to be *resonant* if, for some i, there exists $I = (i_1, \ldots, i_n)$, with i_j nonnegative integers and $|I| = i_1 + \cdots + i_n = k \geq 2$, such that:

$$I \cdot \lambda - \lambda_i = 0.$$

Then $|I| = k$ is the order of this resonance.

A monomial $x^I e_i$ is said to be resonant if $I \cdot \lambda - \lambda_i = 0$.

The eigenvalues are said to satisfy the *strong eigenvalue condition* [9] if there exists no $I = (i_1, \ldots, i_n)$, with i_j nonnegative integers and $|I| = i_1 + \cdots + i_n = k \geq 1$, such that $I \cdot \lambda = 0$.

The vector λ belongs to the Poincaré domain if zero is not in the convex hull of the n points $\{\lambda_1, \ldots \lambda_n\}$ in the complex plane, and to the Siegel domain otherwise. If λ belongs to the Poincaré domain then it satisfies the strong eigenvalue condition.

Poincaré Theorem. [2] *If the vector λ belongs to the Poincaré domain and is nonresonant, X is biholomorphically equivalent to its linear part in the neighbourhood of the singular point.*

The vector λ is said to be of type (C,ν) if, for any i:

$$|I \cdot \lambda - \lambda_i| \geq \frac{C}{|I|^\nu}$$

with $I = (i_1, \dots, i_n)$, i_j nonnegative integers and $|I| \geq 2$, with C, $\nu > 0$.

Clearly if λ is of type (C,ν) then it is nonresonant; if λ belongs to the Poincaré domain and is nonresonant then it is of type (C,ν) for convenient C and ν. Every point in the Poincaré domain satisfies not more than a finite number of resonances and has a neighbourhood where no other resonance relation is satisfied [2].

Siegel Theorem. [2] *If λ is of type (C,ν), X is biholomorphically equivalent to its linear part in the neighbourhood of the singular point.*

The assumption of this theorem can be optimized: a vector field with nonlinearity

$$a(x) = \sum_{i=1}^{n} \sum_{|I|\geq 2} a_i^I x^I e_i$$

satisfies the Brjuno condition if the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \ln \omega_k, \qquad \omega_k = \min\{I \cdot \lambda - \lambda_i : \ |I \cdot \lambda - \lambda_i| > 0, \ \|I - e_i\| < 2^k\}$$

is convergent.

Brjuno Theorem. [5] *If X is formally equivalent to its linear part, and the Brjuno condition is verified, then X is biholomorphically equivalent to its linear part in the neighbourhood of the singular point.*

In the C^∞ case we have:

Sternberg Theorem. [11] *If λ is non-resonant, X is smoothly equivalent to its linear part in the neighbourhood of the singular point.*

For real vector fiels, non resonance implies hyperbolicity. It was proved by K. T. Chen [7] that, if the critical point is hyperbolic, the existence of a formal linearizing change of variables implies the existence of a C^∞ one:

Chen Theorem. [7] *If X is formally equivalent to its linear part, and the singular point is hyperbolic, then X is smoothly equivalent to its linear part in the neighbourhood of the singular point.*

This theorem was improved by [4], but the corresponding result is not applicable in our situation.

When there are resonances, the Poincaré-Dulac theorem allows the elimination of nonresonant terms by a formal change of variables, holomorphic under certain conditions:

Poincaré-Dulac Theorem. [2] *If the vector λ belongs to the Poincaré domain, X is biholomorphically equivalent, in the neighbourhood of the singular point, to a normal form consisting of its linear part and the resonant monomials.*

The theorem does not guarantee that the linearization can be performed if the nonlinearity a does not contain any resonant term, as the following example shows:

Example 1. Let $X(x,y) = (-x + y^3, y + x^4 y)$ be a vector field in $\mathbb{C}^2$; the eigenvalues are -1 and 1, therefore resonant, but the nonlinearity does not contain resonant monomials, of the form $x^{k+1}y^k e_1$ or $x^k y^{k+1} e_2$.

The first step in the Poincaré-Dulac method leads to the change of variables $\xi = x - y^3/4$, $\eta = y$, eliminating the lower order term of the nonlinearity, but in the new coordinates:

$$X(\xi,\eta) = \left(-\xi - \frac{3}{4}\left(\xi + \frac{\eta^3}{4}\right)^4 \eta^3,\ \eta + \left(\xi + \frac{\eta^3}{4}\right)^4 \eta \right)$$

and the resulting nonlinearity has now resonant monomials, $\xi^4\eta^3 e_1$ and $\xi^3\eta^4 e_2$.

As shown in theorem 2, it is also possible to linearize vector fields with resonant monomials in their non linear parts:

Example 2. Let $X(x,y,z) = (x, x+y, 5z + xy^4)$ be a vector field in $\mathbb{C}^3$; the eigenvalues are 1, with multiplicity 2, and 5, therefore resonant, and the nonlinearity contains the resonant monomial $xy^4 e_3$.

The first step in the Poincaré-Dulac method leads to the change of variables $\xi = x$, $\eta = y$, $\zeta = z - y^5/5$, which in fact linearizes the vector field.

3. Main result

Our aim is to obtain linearization results when there are resonances; this forces us to restrict the nonlinear terms: the linearizability of X at x_0 in general is not determined by its 1-jet (linear part) [9], but the allowed nonlinearities are characterized by conditions that are easy to check.

We restrict ourselves to formal linearization, the holomorphic and C^∞ versions being then a consequence of Brjuno teorem, in the holomorphic case, or Chen theorem, in the smooth case, whenever their respective assumptions are verified.

In general terms, our main result (theorem 1) can be described as follows: associated to the linear part Ax of the vector field we consider the linear operator L_A, the Lie derivative, defined on the vector space $\mathrm{D}(\mathcal{F})$ of formal vector fields; if $(\lambda_1, \dots \lambda_n) \in \mathbb{C}^n$ are the eigenvalues of A, then the eigenvalues of L_A are all possible values $I \cdot \lambda - \lambda_i$ with $I = (i_1, \dots , i_n)$ such that i_j are nonnegative integers and $i_1 + \cdots + i_n \geq 1$.

We construct subspaces of $\mathrm{D}(\mathcal{F})$, such that the restriction of L_A to them satisfies the strong eigenvalue condition [9], i.e. no integer (non negative) linear combination of the eigenvalues $I \cdot \lambda - \lambda_i$ of the restriction is zero unless all coefficients are zero; the vector fields in those subspaces are formally linearizable. This includes as a particular case the results obtained in [3]:

Theorem ([3])**.** *Let X be an holomorphic vector field on a neighbourhood U of the origin in $\mathbb{C}^n$ which, in coordinates x, can be written as:*

$$X(x) = Ax + a(x), \qquad A = \operatorname{diag}\{\lambda_1, \dots , \lambda_n\}.$$

If in the nonlinearity:

$$a(x) = \sum_{i=1}^{n} \sum_{|I|\geq 2} a_i^I x^I e_i$$

all $I \cdot \lambda - \lambda_i$, for which a_i^I is non zero, are positive integer multiples of a fixed non zero complex number, there exists a holomorphic change of coordinates $y = \psi(x)$ linearizing X.

Remark 3. In the real analytic category, all $I \cdot \lambda - \lambda_i$ should be positive integer multiples of a fixed non zero real number.

A geometric interpretation of the hypothesis above is that all $I \cdot \lambda - \lambda_i$, for which a_i^I is non zero, thought of as points in the complex domain, belong to a straight (open) half-line through the origin and correspond to segments whose length is an integer multiple of a fixed one.

Here a much larger class of nonlinearities will be considered:

Definition 1. The nonlinearity $a(x)$ is admissible if all linear combinations with non negative integers (not all zero) of $I \cdot \lambda - \lambda_i$, for which a_i^I is non zero, are non zero.

Let $\mathcal{M}$ be the set of points in $\mathbb{Z}^n$ such that at most one coordinate is -1 and all the others are non negative, and consider a representation of the monomial $x^I e_i$ by a point $P_i^I = I - e_i$ in $\mathcal{M}$.

Then, to a nonlinearity $a(x)$ there corresponds a set:

$$\mathcal{A} = \{P_i^I = I - e_i, \text{ such that } a_i^I \neq 0\} \subset \mathcal{M}$$

We define $\mathcal{C}$ as the set of all linear combinations with non negative integers (not all zero) of points in $\mathcal{A}$ that belong to $\mathcal{M}$. Then $a(x)$ being admissible is equivalent to:

$$0 \notin \lambda \cdot \mathcal{C}$$

Let $\mathbb{K}$ be the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, and denote by $\mathcal{F} = \mathbb{K}[[x_1, \ldots, x_n]]$ the formal power series algebra over $\mathbb{K}$. A formal vector field X can be seen as a linear operator on $\mathcal{F}$, in fact as a derivation:

$$X(fg) = X(f)g + fX(g), \quad f, g \in \mathcal{F}$$

As usual, we identify the set $\mathrm{D}(\mathcal{F})$ of derivations on $\mathcal{F}$ with $\mathcal{F}^n$ by writing:

$$X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}, \quad f_i \in \mathcal{F}$$

and, as we have been doing, we can consider $\frac{\partial}{\partial x_i} = e_i$.

To the linear vector field Ax we associate the linear operator L_A on $\mathcal{F}^n$; if the eigenvalues λ of A are resonant, L_A is not an isomophism.

Given a nonlinearity $a(x)$, we can construct a subspace of $\mathrm{D}(\mathcal{F})$, or $\mathcal{F}^n$, generated by its monomials:

$$\hat{\mathcal{C}} = \text{linear span } \{x^I e_i, \text{ such that } a_i^I \neq 0\} \subset \mathcal{F}^n$$

Then the nonlinearity $a(x)$ is admissible if the restriction of L_A to $\hat{\mathcal{C}}$ satisfies the strong eigenvalue condition.

Theorem 1. *Let X be a formal (holomorphic, C^∞) vector field on a neighbourhood U of the origin in $\mathbb{C}^n$ which, in coordinates x, can be written as:*

$$X(x) = Ax + a(x) \qquad a(0) = \frac{\partial a}{\partial x}(0) = 0.$$

If the nonlinearity $a(x)$ is admissible (the Brjuno condition is verified, the critical poit is hyperbolic), there exists a formal (holomorphic, C^∞) change of coordinates $y = \psi(x)$ linearizing the vector field X.

By preventing resonances, all classical linearization results are applicable (in the real case) only to hyperbolic critical points; under this condition, it follows from the Hartman-Grobman theorem that the vector fields are topologically conjugated to their linear parts. The above result can be applied to non hyperbolic critical points in the holomorpic and real analytic categories, for which even the topological situation was not previously determined, as discussed in section 5.

Proof of Theorem 1. We follow the scheme for the proof of the existence of a formal linearization in [2], with small adaptations; we need to prove that the changes do not affect the conclusion, and mainly that, if we begin with an admissible nonlinearity, we only have admissible nonlinearities ar every step.

The vector field X is written in the x coordinates as:

$$X(x) = Ax + a(x) = Ax + \sum_{i=1}^{n} \sum_{|I| \geq 2} a_i^I x^I e_i$$

where $a(x)$ is an admissible nonlinearity. By a convenient abuse of notation, we say that a monomial is admissible (for a given nonlinearity) if it is represented by a point in $\mathcal{C}$.

We decompose $a(x)$ as:

$$a(x) = v_r(x) + v_{r+1}(x) + \dots$$

where v_s are the terms of degree $s \geq 2$; futhermore we write each v_r as a sum of admissible monomials:

$$v_r(x) = \sum_{i=1}^{n} \sum_{|I|=r} a_i^I x^I e_i$$

and order these monomials according to any given order, that we take to be the lexicographical order on (i, I). Denote by m_r the first monomial in v_r, and let $w_r = v_r - m_r$; then:

$$a(x) = m_r(x) + w_r(x) + v_{r+1}(x) + \dots$$

If A is diagonal, $A = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, the solution h of the homological equation:

$$\text{(H)} \qquad L_A h(x) = m_r(x), \quad L_A h(x) = Ah(x) - \frac{\partial h}{\partial x}(x) Ax$$

is a multiple of m_r and therefore an admissible monomial; the change of coordinates $x = z + h(z)$ is also admissible ($h(z)$ is a monomial represented by a point in $\mathcal{C}$) and it is easy to see that it gives, in the new coordinates z:

$$X(z) = Az + w_r(z) + u_{r+1}(z) + u_{r+2}(z) + \dots$$

where again u_k stand for the terms of degree $k > r$.

The main point then is to prove that, in the new coordinates, the resulting nonlinearity is also admissible; this will follow from a sequence of lemmas, each corresponding to a step in the process of that change of coordinates $x = z + h(z)$, namely:

- Computing $a(z + h(z))$: lemma 1.
- Computing the inverse $z = x + g(x)$ of $x = z + h(z)$: lemma 2.
- Computing $\dfrac{\partial}{\partial x}(x + g(x))|_{x=z+h(z)}\, a(z + h(z))$: lemma 3.

Lemma 1. *If $x = z + az^I e_i$ is a change of coordinates and $x^J e_j$ a monomial, such that the respective points $P_i^I = I - e_i$ and $P_j^J = J - e_j$ belong to $\mathcal{C}$, then the points corresponding to all monomials in $(z + az^I e_i)^J e_j$ also belong to $\mathcal{C}$.*

Proof. In fact all monomials in $(z + az^I e_i)^J e_j$ are (ignoring coefficients) of the form:

$$z_1^{j_1} \dots z_{i-1}^{j_{i-1}} z_i^k z^{(j_i-k)I} z_{i+1}^{j_{i+1}} \dots z_n^{j_n} = z^K e_j, \quad K = J - j_i e_i + k e_i + (j_i - k) I$$

Thus:

$$K - e_j = J - e_j + (j_i - k)(I - e_i)$$

and therefore $K - e_j \in \mathcal{C}$. □

Lemma 2. *If $x = z + h(z) = z + az^I e_i$ is a change of coordinates such that $P_i^I = I - e_i \in \mathcal{C}$, and its inverse is $z = x + g(x)$, then the points corresponding to all monomials in $g(x)$ also belong to $\mathcal{C}$.*

Proof. Let:

$$g(x) = \sum_{i=1}^{n} \sum_{|I| \geq 2} g_i^I x^I e_i$$

Then, as $z = z + az^I e_i + g(z + az^I e_i)$, it follows that all monomials in $g(z + az^I e_i)$ cancel each other leaving only $-az^I e_i$:

$$\sum_{j=1}^{n} \sum_{|J| \geq 2} g_j^J (z + az^I e_i)^J e_j = -az^I e_i$$

Ordering the monomials by degree and lexicographic order on (i, I) for the same degree, let $g_j^J x^J e_j$ be the first non admissible monomial in $g(x)$. From the previous lemma, we know that the admissible monomials in $g(x)$ give only admissible monomials in the variables z; on the other hand $g_j^J x^J e_j$ gives rise to a non admissible monomial $g_j^J z^J e_j$, which cannot be cancelled: all monomials coming from lower order monomials are admissible, and all others are of higher order.

Therefore the set of non admissible monomials must be empty, and $g(x)$ is an admissible nonlinearity.

Alternatively, from $x = z + az^I e_i$, it follows that $z_j = x_j$ for all $j \neq i$, and the remaining equation can be put in the form:

$$\xi = \eta + \alpha\eta^k, \quad \text{where} \ \ \alpha = ax^{I-i_ie_i}, \quad \xi = x_i, \quad \eta = z_i, \quad k = i_i$$

It is easy to see that this equation has a formal solution given by:

$$\begin{aligned}\eta =& \xi - c_1(k)\alpha\xi^k + c_2(k)\alpha^2\xi^{2k-1} - c_3(k)\alpha^3\xi^{3k-2} + \dots \\ & c_1(k) = 1, \quad c_2(k) = k, \quad c_3(k) = k^2 + C_2^k, \quad \dots\end{aligned}$$

when $k > 0$, and $\xi - \alpha$ when $k = 0$ (this case is trivial). Thus we obtain $z = x + g(x)$, with:

$$\begin{aligned}g(x) =& -ax^{I-i_ie_i}x_i^{i_i} + ka^2x^{2(I-i_ie_i)}x_i^{2i_i-1} - c_3(k)a^3x^{3(I-i_ie_i)}x_i^{3i_i-2} + \dots \\ =& \sum_{r\geq 1} c_r(k)a^r x^{r(I-i_ie_i)+(ri_i-(r-1))e_i}e_i = \sum_{r\geq 1} c_r(k)a^r x^{r(I-e_i)+e_i}e_i\end{aligned}$$

The monomials in $g(x)$ are represented by points:

$$P_i^{r(I-e_i)+e_i} = r(I - e_i) = rP_i^I \in \mathcal{C}, \quad r \geq 1$$

$\square$

Lemma 3. *Let X be the vector field:*

$$X(x) = Ax + a(x) \qquad a(0) = \frac{\partial a}{\partial x}(0) = 0.$$

If the nonlinearity $a(x)$ is admissible, and if $x = z + \alpha z^I e_i$ is a change of coordinates such that $P_i^I = I - e_i \in \mathcal{C}$, then, in the new coordinates z:

$$X(z) = Az + b(z) \qquad b(0) = \frac{\partial b}{\partial z}(0) = 0$$

with $b(z)$ admissible.

Proof. We are assuming that $A = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. Let $z = x + g(x)$ be the inverse of $x = z + \alpha z^I e_i$; from the previous lemma 2, we know that $g(x)$ is admissible.

From $z = x + g(x)$ we obtain:

$$\begin{aligned}(1) \qquad \dot{z} =& \dot{x} + \sum_{j=1}^{n} \frac{\partial g(x)}{\partial x_j}\dot{x}_j = \\ =& Ax + a(x) + \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{|K|\geq 2} g_k^K k_j x^{K-e_j}(\lambda_j x_j + a_j(x))e_k\end{aligned}$$

The nonlinearity in the last member is admissible: $a(x)$ is admissible, as are the monomials $g_k^K k_j x^{K-e_j}\lambda_j x_j e_k = g_k^K k_j \lambda_j x^K e_k$. The monomials of $x^{K-e_j}a_j(x)e_k$ are also admissible, as they are of the form $x^{K-e_j}x^J e_k$, where $x^J e_j$ is admissible, and then:

$$(K - e_j + J) - e_k = (K - e_k) + (J - e_j)$$

We obtain $\dot{z} = Az + b(z)$ by using $x = z + \alpha z^I e_i$ in the last member of (1); the resulting nonlinearity $b(z)$ is admissible by lemma 1. $\square$

If A has a non zero nilpotent part, the solution of the homological equation (H) is not necessarily a monomial multiple of $m_r(x) = x^I e_i$, i.e. $m_r(x) = x^I e_i$ is not necessarily an eigenvector of L_A corresponding to the eigenvalue $\lambda \cdot I - \lambda_i$, it just belongs to its generalized eigenspace.

Thus, if A is not diagonal (over $\mathbb{C}$), the solution of the homological equation (H) is a polynomial $S_r(x)$, not necessarily a monomial; then we make successive changes of coordinates based on the monomials involved in that solution. It follows from lemma 4 that the end result is the same as far as the r-degree terms are concerned:

Lemma 4. *Let $x = \xi_1(z) = z + \alpha z^I e_i$ and $x = \xi_2(z) = z + \beta z^J e_j$ be two diffeomorhisms around the origin, with $|I| = |J| = r$. Then the changes of coordinates $x = z + \alpha z^I e_i + \beta z^J e_j$ and $x = \xi_1 \circ \xi_2(z)$ applied on the vector field $X(x) = Ax + v_r(x) + v_{r+1}(x) + \dots$, where v_s are the homogeneous terms of degree $s \geq 2$, give the same r-degree terms.*

Proof. The result follows from the fact that the r-degree terms obtained after a change of coordinates $x = z + h(z)$ are given by $v_r - L_A(h)$, and thus for the two changes of coordinates $x = z + \alpha z^I e_i + \beta z^J e_j$ and $x = \xi_1 \circ \xi_2(z)$ the new r-degree terms are $v_r - L_A(\alpha z^I e_i + \beta z^J e_j)$ and $v_r - L_A(\alpha z^I e_i) - L_A(\beta z^J e_j)$, respectively. $\square$

We need to characterize the monomials appearing in $S_r(x)$, no longer just a multiple of $v_r(x)$.

Lemma 5. *Let $z^I e_i$ and $z^J e_j$ be two monomials, with corresponding points $P_i^I,\ P_j^J \in \mathcal{M}$. Then the point corresponding to the monomials in $[z^I e_i, z^J e_j]$ is $P_i^I + P_j^J$.*

Proof. The two monomials in $[z^I e_i, z^J e_j]$ are, disregarding coefficients:

$$z^{J-e_i} z^I e_j, \quad z^{I-e_j} z^J e_i$$

and as:

$$P_j^{J-e_i+I} = P_i^{I-e_j+J} = P_i^I + P_j^J$$

the lemma is proved. $\square$

Specializing lemma 5 for the monomials in Ax, we see that the monomials in $L_A(z^I e_i)$ are $z^I e_i$ or obtained from $z^I e_i$ (always disregarding coefficients) by some permutation of the type

$$\begin{aligned} z^I e_i &\longrightarrow z^I e_{i+1} && \varepsilon_i = 1 \\ z^I e_i &\longrightarrow z^{I-e_{j+1}+e_j} e_i, && \varepsilon_j = 1 \end{aligned}$$

which, on the corresponding points, induce:

$$\begin{aligned} P_i^I =& I - e_i \longrightarrow P_{i+1}^I = I - e_{i+1} && \varepsilon_i = 1 \\ P_i^I =& I - e_i \longrightarrow P_i^{I-e_{j+1}+e_j} = I - e_{j+1} + e_j - e_i, && \varepsilon_j = 1 \end{aligned}$$

and it is easy to see that:

$$\lambda \cdot P_i^I = \lambda \cdot P_{i+1}^I, \quad \lambda \cdot P_i^I = \lambda \cdot P_i^{I-e_{j+1}+e_j}.$$

Thus the monomials in the solution $S_r(x)$ of the homological equation $L_A(x)h(x) = v_r(x)$ are such that the corresponding points are obtained from points in $\mathcal{C}$ by a permutation preserving the value of λ; note that if we extend $\mathcal{C}$ to contain all these permutations, it is still true that λ will be non zero in the extended set.

Lemma 6. *Lemma 3 remains valid for non semisimple linear parts.*

Proof. If A has a non zero nilpotent part, the only change in the argument for the proof of lemma 3 is that in (1) it appears:

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{|K|\geq 2} g_k^K k_j x^{K-e_j} (\epsilon_j x_{j-1} + \lambda_j x_j + a_j(x)) e_k \qquad \epsilon_j = 0, 1$$

This gives new monomials $x^{K-e_j} x_{j-1} e_k$ when $\epsilon_j = 1$, and therefore $\lambda_{j-1} = \lambda_j$; the points $P_k^{K-e_j+e_{j-1}}$ representing those monomials again are obtained from points in $\mathcal{C}$ by a permutation preserving the value of λ, i.e. $\lambda \cdot P_k^{K-e_j+e_{j-1}} = \lambda \cdot P_k^K$, and therefore the argument can be extended to this case. □

The only change made in the classical scheme of proof is that we solve the homological equation (H) for a monomial m_r and not for the homogeneous term v_r; as the corresponding change of variables, or each one of a sequence of them corresponding to polynomial solutions of the homological equation (H), preserves $w_r = v_r - m_r$, our proof is complete since, in view of the lemmas, each step in the process leads to an admissible nonlinearity. □

4. Vector fields with resonant monomials

When A is not semisimple, a monomial being resonant means that it belongs to the generalized eigenspace of the linear operator L_A corresponding to the zero eigenvalue, but that monomial can still be in the image of L_A; note that the possibility of solving the homological equation (H) for a given monomial depends only on that monomial belonging to the image of L_A. These resonant monomials can be dealt with as long as they do not subsequently generate monomials that do not belong to the image of the linear operator L_A.

We define $\mathcal{G}$ as a subset of the set $\mathcal{R}$ of resonant monomials belonging to the image of L_A and for which there exists another subset $\mathcal{U}$ of resonant monomials such that:

$$L_A(\mathcal{U}) = \mathcal{G}, \qquad \mathcal{G} + \mathcal{U} \subset \mathcal{G}$$

The complement of $\mathcal{G}$ in $\mathcal{R}$ will be denoted by $\mathcal{B}$.

Above, and subsequently, the sums involved do not refer to monomials but to their corresponding points.

Remark 4. The condition $L_A(\mathcal{U}) = \mathcal{G}$ means that all monomials in $\mathcal{G}$ can be obtained as the image by L_A of a polynomial whose monomials are in $\mathcal{U}$, and also that all monomials in $L_A(\mathcal{U})$ belong to $\mathcal{G}$.

Remark 5. It follows from lemma 5 that this last condition $\mathcal{G}+\mathcal{U} \subset \mathcal{G}$ is equivalent to $[\mathcal{G},\mathcal{U}] \subset \mathcal{G}$. The sets $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{U}$ are not unique.

As seen before, to a nonlinearity $a(x)$ there corresponds a set $\mathcal{A} = \{P_i^I = I - e_i, \text{ such that } a_i^I \neq 0\} \subset \mathcal{M}$. We extend $\mathcal{A}$ to a set $\mathcal{A}_{ext}$ so that:

- $\mathcal{A} \subset \mathcal{A}_{ext}$
- $\mathcal{A}_{ext} + \mathcal{U} \subset \mathcal{A}_{ext}$
- $\mathcal{A}_{ext}$ is closed for the following permutations:

$$\begin{aligned} &P_i^I = I - e_i \in \mathcal{A}_{ext},\ \varepsilon_i = 1 \Longrightarrow P_{i+1}^I \in \mathcal{A}_{ext} \\ &P_i^I = I - e_i \in \mathcal{A}_{ext},\ \varepsilon_k = 1 \Longrightarrow P_i^J \in \mathcal{A}_{ext}, J = I - e_{k+1} + e_k \end{aligned} \tag{2}$$

  whenever their result belongs to $\mathcal{M} - \mathcal{R}$ (i.e. correspond to some non-resonant monomial).

Again, we define $\mathcal{C}$ as the set of all linear combinations with non negative integers (not all zero) of points in $\mathcal{A}_{ext}$ that belong to $\mathcal{M}$.

Definition 2. A nonlinearity $a(x)$ is weakly admissible if all resonant monomials in $\mathcal{C}$ are in $\mathcal{G}$.

Theorem 2. *Let X be a formal (holomorphic, C^∞) vector field on a neighbourhood U of the origin in $\mathbb{C}^n$ which, in coordinates x, can be written as:*

$$X(x) = Ax + a(x) \qquad a(0) = \frac{\partial a}{\partial x}(0) = 0.$$

If the nonlinearity $a(x)$ is weakly admissible (the Brjuno condition is verified, the critical poit is hyperbolic), there exists a formal (holomorphic, C^∞) change of coordinates $y = \psi(x)$ linearizing the vector field X.

It is important to notice that this method also provides information on the normal form of the vector field X when it is not formally linearizable:

Corollary 1. *Let X be a formal (holomorphic, C^∞) vector field on a neighbourhood U of the origin in $\mathbb{C}^n$ which, in coordinates x, can be written as:*

$$X(x) = Ax + a(x) \qquad a(0) = \frac{\partial a}{\partial x}(0) = 0.$$

A resonant normal form for X can be obtained involving only the nonlinear resonant monomials corresponding to points in $\mathcal{C} \cap \mathcal{B}$.

Example 3. Let $X(x, y) = (x + ay^4, -y + bx^2)$ be a vector field in $\mathbb{C}^2$; the eigenvalues are -1 and 1, therefore resonant, the nonlinearity does not contain resonant monomials, but $\mathcal{C} \cap \mathcal{R}$ contains the points corresponding to the monomials of the form $x^{7k+1}y^{7k}e_1$ or $x^{7k}y^{7k+1}e_2$. Thus we see that in its normal form:

$$X(x, x) = (x - x^8y^7\varphi(x^7y^7), -y + x^7y^8\psi(x^7y^7))$$

the lowest nonlinear terms are at least of degree 15.

Then we can apply the results of Sell [10] (see also the result of Samovol [1]) to prove C^7-linearization: the change of coordinates to obtain the above normal form is holomorphic, in fact polynomial, and we can take $Q = 14$, noticing that there are no resonances of that order and all derivatives up to order 13 of the nonlinearity $F(x, y) = x^7y^7(x\varphi(x^7y^7), y\psi(x^7y^7))$ are zero at the origin, as required.

Remark 6. The resonant normal form can be further simplied in many cases [12, 13]. The changes of coordinates then do not necessarily correspond to monomials in the image of L_A, they can correspond to monomials in $[\mathcal{R}, \mathcal{R}]$.

Proof of Theorem 2. It follows from the lemmas that a monomial $z^Je_j \in \mathcal{C}$ gives rise, after a change of coordinates $x = z + z^Ie_i$, to monomials corresponding to $(J-e_j)+s(I-e_i)$ or $(J-e_j)+(s(I-e_i)-e_k+e_{k-1})$ when $\epsilon_k = 1$ in the canonical Jordan form.

Assume $v_r(x)$ is not resonant; then all the monomials $z^I e_i$ in the solution $S_r(x)$ of the corresponding homological equation are in $\mathcal{C}$, and the proof follows as for theorem 1.

Only the case $v_r(x) \in \mathcal{C} \cap \mathcal{R} \subset \mathcal{G}$ remains to be considered; then all monomials in $S_r(x)$ belong to $\mathcal{U}$. The corresponding changes of coordinates are of the type $x = z + z^I e_i$ with $z^I e_i \in \mathcal{U}$ but not necessarily $z^I e_i \in \mathcal{C}$. Even so, it follows from the definition of the set $\mathcal{C}$ that it contains $(J-e_j)+s(I-e_i)$, as $\mathcal{C}+\mathcal{U} \subset \mathcal{C}$, and also $(J-e_j)+(s(I-e_i)-e_k+e_{k-1})$, as $\mathcal{C}$ is closed for this type of permutation. $\square$

5. Finite determinacy

In this context, k-determinacy means that the conditions involve only the k-jet of the vector field; see [9] for a very careful discussion of the subject.

In he Poincaré domain the number of possible resonances, and thus of resonant monomials, is finite; denote by d the biggest degree of a resonant monomial.

Corollary 2. *Let X be a (holomorphic, C^∞ hyperbolic) vector field with the eigenvalues vector λ in the Poincaré domain, such that:*

$$j^d X(x) = Ax + h_d(x)$$

where $h_d(x)$ is an weakly admissible nonlinearity with terms of degree at most d. Then X is linearizable.

Proof. The linearization procedure we have been describing works for the vector field $Y(x) = j^d X(x) = Ax + h_d(x)$, giving a change of coordinates that linearizes $j^d X(x)$.

Making that change of coordinates in X results in a new vector field $Z(x) = Ax + a(x)$. All monomials in the nonlinear part $a(x)$ of Z are of the type $x^I e_i$ with $|I| > d$, therefore can never be resonant or generate resonant ones at subsequent steps. $\square$

In the Siegel domain the linearizable resonant vector fields have infinite codimension and are never finitely determined. But if we consider special classes of vector fields the situation can be completely different; in particular, if we consider polynomial vector fields of degree less than r, say, the codimension becomes finite, of course, and we can estimate it, as the next example shows:

Example 4. Consider a polynomial vector field of degree less than r:

$$X(x,y) = (x + a_1(x,y,x), x + y + a_2(x,y,x), -z + a_3(x,y,x))$$

From the results of 8.4, it follows that we need to avoid the monomials for whose corresponding points $\lambda < 0$ or $\lambda = 0$, $\mu = (0,-1,1) \leq 0$; of course, instead of $\lambda < 0$ we could choose to avoid $\lambda > 0$.

If we consider quadratic nonlinearities, $r = 3$, we see that even if there are no resonant monomials of degree 2, we have to avoid z^2e_1, yze_1, corresponding to the points $(-1,0,2)$ and $(-1,1,1)$, xze_1, yze_2 and z^2e_3 corresponding to the point $(0,0,1)$, z^2e_2 corresponding to the point $(0,-1,2)$, xze_2 corresponding to the point $(1,-1,1)$, as for these points $\lambda < 0$. Thus the linearizable nonlinearities have codimension at most 7 in the space of all quadratic nonlinearities, of dimension 18.

If we consider also cubic nonlinearities, $r = 4$, we see that we also have to avoid:

- z^3e_1, corresponding to the point $(-1,0,3)$, yz^2e_1, corresponding to the point $(-1,1,2)$, xz^2e_1, yz^2e_2, z^3e_3 corresponding to the point $(0,0,2)$, xz^2e_2, corresponding to the point $(1,-1,2)$, z^3e_2, corresponding to the point $(0,-1,3)$, as for these points $\lambda < 0$;
- y^2ze_1, corresponding to the point $(-1,2,1)$, x^2ze_2, corresponding to the point $(2,-1,1)$, and $xyze_1$, y^2ze_2, yz^2e_3 corresponding to the point $(0,1,1)$, as $\lambda = 0$, $\mu \leq 0$, for those points.

Thus the linearizable nonlinearities have codimension at most 19 in the space of all quadratic and cubic nonlinearities.

6. Linearization of maps

Let $F(x)$ be a holomorphic map on a domain $U \subset \mathbb{C}^n \longrightarrow \mathbb{C}^n$; F is supposed to have a singular point at the origin in $\mathbb{C}^n$ with linear part A:

$$F(x) = Ax + a(x), \quad a(0) = \frac{\partial a}{\partial x}(0) = 0.$$

It will always be assumed that A is in the Jordan canonical form.

F is said to be biholomorphically equivalent to its linear part if there exists an holomorphic change of coordinates $z = \psi(x)$, preserving the origin, $\psi(0) = 0$, with inverse $x = \xi(z)$ also holomorphic, such that in the new coordinates the nonlinear part is zero:

$$\psi(F(\xi(z))) = Az.$$

The eigenvalues $\lambda = (\lambda_1, \dots \lambda_n) \in \mathbb{C}^n$ are said to be *resonant of order k* if, for some i, there exists $I = (i_1, \dots, i_n)$, with i_j nonnegative integers and $|I| = i_1 + \cdots + i_n = k \geq 2$, such that:

$$\lambda^I - \lambda_i = 0.$$

A monomial $x^I e_i$ is said to be resonant if $\lambda^I - \lambda_i = 0$.

The linearization procedure now involves, at each step, the solution h of the homological equation:

$$\text{(H')} \qquad L_A h(x) = m_r(x), \quad L_A h(x) = h(Ax) - Ah(x)$$

and the change of coordinates $x = z + h(z)$. Writing F in the new coordinates involves only the inverse $z = x + g(x)$ and composition of maps:

$$G(z) = F(z + h(z)) + g \circ F(z + h(z))$$

If A is non singular, and therefore all eigenvalues are non zero, we define $\nu_i = \log \lambda_i$; then the condition $\lambda^I - \lambda_i = 0$ becomes:

$$I \cdot \nu - \nu_i = 0 \qquad \mod 2\pi i$$

Definition 3. If A is non singular, the nonlinearity $a(x)$ is admissible if all linear combinations with non negative integers (not all zero) of $I \cdot \nu - \nu_i$, for which a_i^I is non zero, are non zero modulo $2\pi i$.

Theorem 3. *Let F be a formal (holomorphic, C^∞) map on a neighbourhood U of the origin in $\mathbb{C}^n$ which, in coordinates x, can be written as:*

$$F(x) = Ax + a(x) \qquad a(0) = \frac{\partial a}{\partial x}(0) = 0.$$

If the nonlinearity $a(x)$ is admissible, there exists a formal change of coordinates $y = \psi(x)$ linearizing the map F.

Remark 7. If $\lambda_i = 0$ for some i, the situation is quite different as $\log 0$ is not well defined, but our method still gives a sufficient condition for linearizability: if all linear combinations with non negative integers (not all zero) of $P_i^I = I - e_i$, for which a_i^I is non zero, do not correspond to resonant monomials, then F is linearizable.

7. Applications: vector fields in $\mathbb{R}^2$

The Jordan canonical of the linear part of a resonant vector field in $\mathbb{R}^2$ is diagonal (over $\mathbb{C}$), unless the two eigenvalues are zero; that case is very similar to the case of vector fields in $\mathbb{R}^3$ with a Jordan block of dimension two, considered in the next section.

Here and in the next section we normalize the eigenvalues, using the fact that a linearizing change of coordinates for a vector field also linearizes any scalar multiple of that vector field.

The case of pure imaginary eigenvalues is then equivalent to the case 7.3 over $\mathbb{C}$, but over $\mathbb{R}$ it is much harder, and our results are not directly very useful.

The presence of resonances in the plane (real or complex) means that the condition for 1-determinacy of the topological situation [8, 6] is not verified, and our results are therefore useful even at that level; finite determinacy is valid only in the case 7.1, but in any case we get (holomorphic, analytic, smooth) conjugacy to the linear part, not just topological equivalence.

In the plane and in the presence of resonances, the points $\lambda \cdot I - \lambda_i$ always belong to some straight line and (after normalization) are integers on the real line; the difference between theorem 1 and the result of [3] is just this: there all those points have to be positive, or all negative, and now it is possible to allow the two signs as long as the linear combinations giving zero are not realizable as monomials, as shown in example 5.

7.1. $\lambda = (1, k \neq 1),\ k \in \mathbb{N}$.

The only resonant monomial is $x^k e_2$. Therefore a vector field $X(x, y) = (x, ky) + a(x, y)$ will be linearizable if:

$$(k, -1) \notin \mathcal{C}$$

otherwise its normal form will contain just one resonant monomial, exactly $x^k e_2$.

Example 5. Consider an arbitrary vector field with linear part $A = \operatorname{diag}(1, k)$:

$$X(x, y) = (x + a_1(x, y), ky + c_2 x^2 + \ldots + c_k x^k + a_2(x, y)y + a_3(x)x^{k+1})$$

To be able to apply the Poincaré-Dulac theorem to get a linear normal form we need:

$$j^k a_1(x, y) = j^{k-1} a_2(x, y) = 0, \quad c_2 = \cdots = c_k = 0$$

To apply the linearization result of [3], it is necessary that:

$$j^k a_1(x, y) = j^{k-1} a_2(x, y) = c_k = 0 \quad \text{or} \quad c_2 = \cdots = c_k = 0$$

On the other hand, it follows from theorem 1 that X is linarizable if:

$$c_2 = \cdots = c_k = 0$$

or

$$(1, r) \cdot (i_1 - 1, i_2) \leq k - r, \quad i_1 + i_2 \leq k \Longrightarrow a_1^{(i_1, i_2)} = 0$$

$$(1, r) \cdot (i_1, i_2 - 1) \leq k - r, \quad i_1 + i_2 \leq k \Longrightarrow a_2^{(i_1, i_2 - 1)} = 0$$

where r is the smallest i for which $c_i \neq 0$.

7.2. $\lambda = (1, 0)$.

The resonant monomials are $\{xy^i e_1,\ y^i e_2,\ i \in \mathbb{N}\}$. A vector field is linearizable (in the formal, real analytic, holomorphic categories) if it is of one the two following forms:

$$X(x,y) = (x + x^2\varphi(x,y), x\psi(x,y))$$
$$X(x,y) = (x + \varphi_1(y), 0)$$

7.3. $\lambda = (1, -k),\ k \in \mathbb{N}$.

The resonant monomials are $\{x^{sk+1}y^s e_1,\ x^{sk}y^{s+1}e_2,\ s \in \mathbb{N}\}$. A vector field X is linearizable (in the formal, real analytic, holomorphic categories) if, for instance, it is of one of the two following forms:

$$X(x,y) = (x + x^2\varphi(x, x^k y), -ky + \psi(x, x^k y))$$
$$X(x,y) = (x + \varphi(x^k y, y), -ky + y^2\psi(x^k y, y))$$

8. Applications: vector fields in $\mathbb{R}^3$

In all cases considered below, we take as $\mathcal{G}$ all resonant monomials that belong to the image of L_A; the monomials are represented by points in $\mathbb{R}^n$ and we construct a vector $\mu \in \mathbb{R}^n$ such that:

- $\mathcal{G}$ is exactly the subset of resonant monomials for which the inner product with μ is bigger than $c \geq 0$.
- $\mathcal{U}$ is a subset of resonant monomials for which the inner product with μ is bigger or equal to c.

In many cases $\mathcal{U}$ will be all the set of resonant monomials for which the inner product with $\mu \geq c$. It will be necessary to show that $[A, \mathcal{U}] = \mathcal{G}$ (it is a straightforward but often very laborious computation that we will generally omit) but the other property, $\mathcal{G} + \mathcal{U} \subset \mathcal{G}$, will follow immediatly:

$$\mu \cdot \mathcal{G} > c,\ \mu \cdot \mathcal{U} \geq c \Longrightarrow \mu \cdot (\mathcal{G} + \mathcal{U}) > c \Longrightarrow \mathcal{G} + \mathcal{U} \subset \mathcal{G}$$

Corollary 3. *If there exists $\mu \in \mathbb{R}^n$ such that $\mathcal{G}$ is exactly the subset of resonant monomials for which the inner product with μ is bigger than $c \geq 0$, and $\mu \cdot \mathcal{U} \geq c$, the vector field $X(x) = Ax + a(x)$ is linearizable if:*

$$P \in \mathcal{C},\ \lambda \cdot P = 0 \Longrightarrow \mu \cdot P > c$$

In particular, a vector field $X(x) = Ax + a(x)$ is linearizable if:

$$\lambda \cdot \mathcal{A} \geq 0 \quad (\text{or} \leq 0), \qquad P \in \mathcal{A},\ \lambda \cdot P = 0 \Longrightarrow \mu \cdot P > c$$

Remark 8. When dealing with complex eigenvalues, we should assume that all $\lambda \cdot \mathcal{A} \in \mathbb{C}$ are in a closed semiplane, instead of the condition $\lambda \cdot \mathcal{A} \geq 0$.

Here we will be concerned only with vector fields whose linear part (in the Jordan canonical form) is not diagonal, with one Jordan block of dimension two; the results for the diagonal case can be obtained as in the previous section.

8.1. $\lambda = (1, 1, k \neq 1),\ k \in \mathbb{N}$.

The resonant monomials are of the form $x^{k-i}y^i e_3$ with $i \leq k$, and since $L_A(x^{k-i}y^i e_3) = i x^{k-i+1} y^{i-1} e_3$, we see that:

$$\mu = (1, 0, 0) \qquad \mathcal{G} = \{x^{k-i}y^i e_3,\ i < k\} \qquad \mathcal{U} = \{x^{k-i}y^i e_3,\ i \leq k\}$$

Therefore a vector field $X(x, y, z) = (x, x + y, kz) + a(x, y, z)$ will be linearizable if:

$$(0, k, -1) \notin \mathcal{C}$$

otherwise its normal form will contain just one resonant monomial, exactly $y^k e_3$:

$$X(x, y, z) = (x, x + y, kz + ay^k)$$

8.2. $\lambda = (k \neq 1, k, 1),\ k \in \mathbb{N}$.

The resonant monomials are:

$$z^k e_1, \quad z^k e_2$$

and since $L_A(z^k e_1) = -z^k e_2$, we see that:

$$\mu = (0, -1, 0) \qquad \mathcal{G} = \{z^k e_2\} \qquad \mathcal{U} = \{z^k e_1,\ z^k e_2\}$$

Therefore a vector field $X(x, y, z) = (kx, x + ky, z) + a(x, y, z)$ will be linearizable if:

$$(-1, 0, k) \notin \mathcal{C}$$

otherwise its normal form will contain just one resonant monomial, exactly $z^k e_1$:

$$X(x, y, z) = (kx + az^k, x + ky, z)$$

Remark 9. In both previous cases, it is to verify if a given vector field is lineariarizable, but it is not easy to give a general formula for a linearizable vector field. The analysis of the details is similar to that of example 5

8.3. $\lambda = (1, 1, 0)$.
The resonant monomials are:

$$\{xz^i e_1,\ yz^i e_1,\ xz^i e_2,\ yz^i e_2,\ z^i e_3 \quad i \in \mathbb{N}\}$$

We can take:

$$\mu = (1, 0, 0) \qquad \mathcal{G} = \{xz^i e_2,\ i \in \mathbb{N}\} \qquad \mathcal{U} = \{xz^i e_1, yz^i e_2,\ i \in \mathbb{N}\}$$

Therefore a vector field $X(x, y, z) = (x, x + y, 0) + a(x, y, z)$ will be linearizable if:

$$\begin{aligned} &\frac{\partial}{\partial x} a_1(0, 0, z) \equiv 0, \quad \frac{\partial}{\partial y} a_1(0, 0, z) \equiv 0 \\ &\frac{\partial}{\partial y} a_2(0, 0, z) \equiv 0 \\ &a_3(0, 0, z) \equiv 0 \end{aligned}$$

The resonant normal form will be:

$$(x + x\varphi_1(z) + y\psi_1(z), x + y + y\psi_2(z), z\theta(z))$$

where $\varphi_1(0) = \psi_1(0) = \psi_2(0) = \theta(0) = 0$.

8.4. $\lambda = (1, 1, -k),\ k \in \mathbb{N}$.
The resonant monomials are:

$$\{x^{sk+1-i} y^i z^s e_1,\ i \leq sk{+}1,\ x^{sk+1-i} y^i z^s e_2,\ i \leq sk{+}1,\ x^{sk-i} y^i z^{s+1} e_3,\ i \leq sk\}$$

with $s \in \mathbb{N}$. From the computation of L_A restricted to the resonant monomials, we see that we can take $\mu = (0, -1, k)$ with:

$$\mathcal{G} = \{x^{sk+1-i} y^i z^s e_1,\ i \leq sk{-}1,\ x^{sk+1-i} y^i z^s e_2,\ i \leq sk,\ x^{sk-i} y^i z^{s+1} e_3,\ i \leq sk{-}1\}$$

and

$$\mathcal{U} = \mathcal{G} \cup \{xy^{sk} z^s e_1,\ y^{sk+1} z^s e_2,\ y^{sk} z^{s+1} e_3\}$$

The resonant normal form will be:

$$(x + x\varphi_1(y^k z) + y\psi_1(y^k z), x + y + y\psi_2(y^k z), -kz + z\theta(y^k z))$$

where $\varphi_1(0) = \psi_1(0) = \psi_2(0) = \theta(0) = 0$.

8.5. $\lambda = (k \neq 1, k, -1),\ k \in \mathbb{N}$.

The resonant monomials are:

$\{x^{s+1-i}y^i z^{sk}e_1,\ i \leq s{+}1,\ x^{s+1-i}y^i z^{sk}e_2,\ i \leq s{+}1,\ x^{s-i}y^i z^{sk+1}e_3,\ i \leq s\}$

with $s \in \mathbb{N}$. Again from the computation of L_A restricted to the resonant monomials, we see that we can take $\mu = (0, -k, 1)$ with:

$\mathcal{G} = \{x^{s+1-i}y^i z^{sk}e_1,\ i \leq s{-}1,\ x^{s+1-i}y^i z^{sk}e_2,\ i \leq s,\ x^{s-i}y^i z^{sk+1}e_3,\ i \leq s{-}1\}$

and

$$\mathcal{U} = \mathcal{G} \cup \{xy^s z^{sk}e_1,\ y^{s+1}z^{sk}e_2,\ y^s z^{sk+1}e_3\}$$

The resonant normal form will be:

$$(kx + x\varphi_1(yz^k) + y\psi_1(yz^k), x + ky + y\psi_2(yz^k), -z + z\theta(yz^k))$$

where $\varphi_1(0) = \psi_1(0) = \psi_2(0) = \theta(0) = 0$.

Remark 10. It is not easy, and perhaps not very useful, to give a general formula for a linearizable vector field in the two previous cases. But it is easy to see that a vector field $X(x, y, z) = (kx, x{+}ky, -z){+}a(x, y, z)$ will be linearizable if, for instance:

$$a_1(x, y, z) = P_2^1(x, y)\alpha_1(x, y, xz^k, yz^k) + x^2z^k\beta_1(xz^k, yz^k)$$
$$a_2(x, y, z) = P_2^1(x, y)\alpha_2(x, y, xz^k, yz^k) + x\beta_2(xz^k, yz^k)$$
$$a_3(x, y, z) = (ax + by)\alpha_3(x, y, xz^k, yz^k) + xz^{k+1}\beta_3(xz^k, yz^k)$$

where P_2^i are homogeneous polynomials of degree two. Note that it contains resonant monomials, all in $\mathcal{G}$.

8.6. $\lambda = (0, 0, 1)$.

The resonant monomials are:

$$\{x^iy^je_1,\ x^iy^je_2,\ x^iy^jze_3 \quad i, j \geq 0\}$$

We can take $\mu = (1, 0, 0)$ with:

$$\mathcal{G} = \{x^{2+i}y^je_1, \quad x^{1+i}y^je_2, \quad x^{1+i}y^jze_3,\ i, j \geq 0\}$$

and $\mathcal{U} = \mathcal{G} \cup \{xy^je_1,\ y^je_2,\ y^jze_3\}$ Therefore a vector field $X(x, y, z) = (0, x, z) + a(x, y, z)$ will be linearizable if:

$$a_1(x, y, z) = x^2\alpha_1(x, y) + z\beta_1(x, y, z)$$
$$a_2(x, y, z) = x\alpha_2(x, y) + z\beta_2(x, y, z)$$
$$a_3(x, y, z) = x\alpha_3(x, y) + z^2\beta_3(x, y, z)$$

The resonant normal form will be:

$$(x\varphi_1(y) + y\psi_1(y), x + y\psi_2(y), z + z\theta(y))$$

where $\varphi_1(0) = \psi_1(0) = \psi_2(0) = \theta(0) = 0$.

Remark 11. In $\mathbb{R}^2$, with zero a double eigenvalue and non zero linear part, the vector field $X(x,y) = (x^2\alpha_1(x,y), x + x\alpha_2(x,y))$ is linearizable, and the resonant normal form is $(x\varphi_1(y) + y\psi_1(y), x + y\psi_2(y))$ where $\varphi_1(0) = \psi_1(0) = \psi_2(0) = 0$.

8.7. $\lambda = (0,0,0)$.

All monomials are resonant; we assume there are two Jordan blocks, thus one of dimension two.

We can take $\mu = (1,0,0)$ with:

$$\mathcal{G} = \{x^{i+2}y^jz^ke_1,\ x^{i+1}y^jz^ke_2,\ x^{i+1}y^jz^ke_3,\ i,j,k \geq 0\}$$

and $\mathcal{U} = \{x^{i+1}y^jz^ke_1,\ x^iy^jz^ke_2,\ x^iy^jz^ke_3,\ i,j,k \geq 0\}$.

A vector field is linearizable if it is of the form:

$$X(x,y,z) = (x^2\alpha_1(x,y,z), x + x\alpha_2(x,y,z), x\alpha_3(x,y,z))$$

and the resonant normal form is:

$$X(x,y,z) = (x\varphi_1(y,z) + \psi_1(y,z), x + \psi_2(y,z), \psi_3(y,z))$$

with $\varphi_1(0) = \psi_1(0) = \psi_2(0) = \psi_3(0) = 0$

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