

# Multiple-scale expansions for incompressible MHD systems

## Part I

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### **Abstract**

We extend the dynamo problem allowing interaction with the Navier-Stokes equation and including convection via the Boussinesq approximation. We want to find solutions that are stable at the short (fast) scales, allowing for instabilities to appear at the long (slow) scales. The latter are responsible for the appearance of a magnetic field. We expand the fields in a power series of the scale separation parameter and derive a closed set of equations for the dominant coefficients at the large scales. We assume space periodicity of the fields.

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# 1 Introduction

Lanotte et al. (1999) have considered the dynamo problem

$$\partial_t \tilde{\mathbf{H}} = \partial \times (\tilde{\mathbf{V}} \times \tilde{\mathbf{H}}) + \eta \partial^2 \tilde{\mathbf{H}}, \quad (1)$$

where  $\partial_t$  is the time derivative and  $\partial = (\partial_1, \partial_2, \partial_3)$  is the gradient. The solenoidal vector field  $\tilde{\mathbf{H}}$  is the magnetic field which involves large spatial scale,  $\eta$  is the magnetic diffusivity and  $\tilde{\mathbf{V}}$  is a prescribed short scale incompressible velocity field.

In this paper we want to take into account the evolution of  $\tilde{\mathbf{V}}$  and the effect of temperature  $\tilde{T}$ . Magnetic field generation by thermal convection is governed by the following system of equations

$$\begin{cases} \partial_t \tilde{\mathbf{V}} + (\tilde{\mathbf{V}} \cdot \partial) \tilde{\mathbf{V}} + \boldsymbol{\Omega} \times \tilde{\mathbf{V}} = -\partial \tilde{p} + \nu \partial^2 \tilde{\mathbf{V}} + (\tilde{\mathbf{H}} \cdot \partial) \tilde{\mathbf{H}} + \alpha (\tilde{T} - \tilde{T}_0) \mathbf{G} + \tilde{\mathbf{F}} \\ \partial \cdot \tilde{\mathbf{V}} = 0 \\ \partial_t \tilde{\mathbf{H}} = \partial \times (\tilde{\mathbf{V}} \times \tilde{\mathbf{H}}) + \eta \partial^2 \tilde{\mathbf{H}} + \tilde{\mathbf{R}} \\ \partial \cdot \tilde{\mathbf{H}} = 0 \\ \partial_t \tilde{T} + (\tilde{\mathbf{V}} \cdot \partial) \tilde{T} = k \partial^2 \tilde{T} + \frac{\sigma}{2} |\partial \times \tilde{\mathbf{H}}|^2 + \tilde{S} \end{cases} \quad (2)$$

The Navier-Stokes equation involves the Lorentz, Coriolis and buoyancy forces. The evolution of temperature is given by the heat balance equation.  $\tilde{\mathbf{F}}$ ,  $\tilde{\mathbf{R}}$ ,  $\tilde{S}$  are forcing terms.  $\tilde{S}$  results from external heat sources.  $\tilde{\mathbf{R}}$  results from imposed external currents or magnetic fields.  $\tilde{\mathbf{F}}$  represents body forces acting on the fluid.  $\mathbf{G}$  is usually the gravity force, i.e.  $\mathbf{G} = (0, 0, -g)$ . The term involving  $\mathbf{G}$  represents the buoyancy force due to temperature variation (Boussinesq approximation) (Chandrasekhar (1981)). From Maxwell equations,  $\partial \cdot \tilde{\mathbf{H}} = 0$  ( $\tilde{\mathbf{H}}$  is solenoidal). We restrict ourselves to the case of incompressible flows, i.e.  $\partial \cdot \tilde{\mathbf{V}} = 0$ . The vector  $\boldsymbol{\Omega}$  is the angular velocity associated with the Earth rotation,  $\nu$  is the kinematic viscosity and  $\alpha$ ,  $k$ ,  $\sigma$  are parameters related to compressibility, thermal conductivity and electrical conductivity, respectively.

We consider basic states in the periodic domain  $[0, L_1] \times [0, L_2] \times [0, L_3]$ .

## 2 Linearised equations

Considering small perturbations to the basic fields  $\tilde{p}$ ,  $\tilde{\mathbf{V}}$ ,  $\tilde{\mathbf{H}}$ ,  $\tilde{T}$ , i. e.

$$\tilde{p} \rightarrow \tilde{p} + p e^{\lambda t}, \quad \tilde{\mathbf{V}} \rightarrow \tilde{\mathbf{V}} + \mathbf{V} e^{\lambda t}, \quad \tilde{\mathbf{H}} \rightarrow \tilde{\mathbf{H}} + \mathbf{H} e^{\lambda t}, \quad \tilde{T} \rightarrow \tilde{T} + T e^{\lambda t},$$

we obtain, after neglecting second order terms in the perturbations  $p$ ,  $\mathbf{V}$ ,  $\mathbf{H}$ ,  $T$ , the following linearised system of equations:

$$\begin{cases} \partial \cdot \mathbf{V} = 0 \\ \partial p + \left( \lambda - \nu \partial^2 + (\tilde{\mathbf{V}} \cdot \partial) + (\bullet \cdot \partial) \tilde{\mathbf{V}} + \boldsymbol{\Omega} \times \right) \mathbf{V} - \left( (\tilde{\mathbf{H}} \cdot \partial) + (\bullet \cdot \partial) \tilde{\mathbf{H}} \right) \mathbf{H} - \alpha \mathbf{G} T = 0 \\ \left( \partial \times (\tilde{\mathbf{H}} \times \bullet) \right) \mathbf{V} + \left( \lambda - \eta \partial^2 - \partial \times (\tilde{\mathbf{V}} \times \bullet) \right) \mathbf{H} = 0 \\ \left( (\bullet \cdot \partial) \tilde{T} \right) \mathbf{V} - \left( \sigma (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \bullet) \right) \mathbf{H} + \left( \lambda - k \partial^2 + (\tilde{\mathbf{V}} \cdot \partial) \right) T = 0 \end{cases} \quad (3)$$

Here, the bullet symbol  $\bullet$  must be replaced by the respective field. From  $\partial \cdot \tilde{\mathbf{H}} = 0$ , we get  $\partial \cdot \mathbf{H} = 0$ .

Extending the block formalism introduced by Dubrulle and Frisch (1991) to the present case, we represent the system of equations (3) in the following compact form

$$\begin{cases} \mathbf{A} \mathbf{W} = 0 \\ \partial \cdot \mathbf{H} = 0 \end{cases} \quad (4)$$

where  $\mathbf{W}$  is a block column vector

$$\mathbf{W} = \begin{bmatrix} p \\ \mathbf{V} \\ \mathbf{H} \\ T \end{bmatrix}, \quad (5)$$

and  $\mathbf{A}$  is a block matrix

$$\mathbf{A} = \begin{bmatrix} 0 & \partial \cdot & 0 & 0 \\ \partial & \lambda - \nu \partial^2 + \tilde{\mathbf{V}} \cdot \partial + (\bullet \cdot \partial) \tilde{\mathbf{V}} + \Omega \times & -\tilde{\mathbf{H}} \cdot \partial - (\bullet \cdot \partial) \tilde{\mathbf{H}} & -\alpha \mathbf{G} \\ 0 & \partial \times (\tilde{\mathbf{H}} \times \bullet) & \lambda - \eta \partial^2 - \partial \times (\tilde{\mathbf{V}} \times \bullet) & 0 \\ 0 & (\bullet \cdot \partial) \tilde{T} & -\sigma (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \bullet) & \lambda - k \partial^2 + \tilde{\mathbf{V}} \cdot \partial \end{bmatrix}. \quad (6)$$

### 3 The multiple-scale expansion

Following the ideas presented by Lanotte et al. (1999); Zheligovsky et al. (2001); Zheligovsky and Podvigina (2003); Zheligovsky (2003), we want to find a solution for the system of equations (4), showing the behaviour at two different scales: the fast variables ( $\mathbf{x}$ ) scale or small scale and the slow variables ( $\mathbf{X} = \varepsilon \mathbf{x}$ ) scale or large scale. We suppose that the fields depend both on fast and slow variables,  $\mathbf{W} = \mathbf{W}(\mathbf{x}, \mathbf{X})$ . We then seek a power series solution, expanding the fields,  $\mathbf{W}$ , and the eigenvalue,  $\lambda$ , in a power series of  $\varepsilon$ , the scaling parameter:

$$\mathbf{W} = \mathbf{W}^{(0)} + \varepsilon \mathbf{W}^{(1)} + \varepsilon^2 \mathbf{W}^{(2)} + \dots + \varepsilon^{n+2} \mathbf{W}^{(n+2)} + O(\varepsilon^{n+3}), \quad (7)$$

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \dots + \varepsilon^{n+2} \lambda_{n+2} + O(\varepsilon^{n+3}). \quad (8)$$

Note that each coefficient  $\mathbf{W}^{(i)}$  in the fields expansion is a function of both  $\mathbf{x}$  and  $\mathbf{X}$ . We must now use the chain rule, replacing each  $\partial$  operator in  $\mathbf{A}$  by  $\partial + \varepsilon \nabla$ . The symbol  $\partial$  is now called the fast gradient and  $\nabla = (\nabla_1, \nabla_2, \nabla_3)$  is called the slow gradient. We obtain:

$$\mathbf{A} = \mathbf{A}^{(0)} + \varepsilon \mathbf{A}^{(1)} + \varepsilon^2 \mathbf{A}^{(2)} + \dots + \varepsilon^{n+2} \mathbf{A}^{(n+2)} + O(\varepsilon^{n+3}), \quad (9)$$

where

$$\mathbf{A}^{(0)} = \begin{bmatrix} 0 & \partial \cdot & 0 & 0 \\ \partial & \lambda_0 - \nu \partial^2 + \tilde{\mathbf{V}} \cdot \partial + (\bullet \cdot \partial) \tilde{\mathbf{V}} + \Omega \times & -\tilde{\mathbf{H}} \cdot \partial - (\bullet \cdot \partial) \tilde{\mathbf{H}} & -\alpha \mathbf{G} \\ 0 & \partial \times (\tilde{\mathbf{H}} \times \bullet) & \lambda_0 - \eta \partial^2 - \partial \times (\tilde{\mathbf{V}} \times \bullet) & 0 \\ 0 & (\bullet \cdot \partial) \tilde{T} & -\sigma (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \bullet) & \lambda_0 - k \partial^2 + \tilde{\mathbf{V}} \cdot \partial \end{bmatrix}, \quad (10)$$

$$\mathbf{A}^{(1)} = \begin{bmatrix} 0 & \nabla \cdot & 0 & 0 \\ \nabla & \lambda_1 - 2\nu \partial \cdot \nabla + \tilde{\mathbf{V}} \cdot \nabla & -\tilde{\mathbf{H}} \cdot \nabla & 0 \\ 0 & \nabla \times (\tilde{\mathbf{H}} \times \bullet) & \lambda_1 - 2\eta \partial \cdot \nabla - \nabla \times (\tilde{\mathbf{V}} \times \bullet) & 0 \\ 0 & 0 & -\sigma (\partial \times \tilde{\mathbf{H}}) \cdot (\nabla \times \bullet) & \lambda_1 - 2k \partial \cdot \nabla + \tilde{\mathbf{V}} \cdot \nabla \end{bmatrix}, \quad (11)$$

$$\mathbf{A}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 - \nu \nabla^2 & 0 & 0 \\ 0 & 0 & \lambda_2 - \eta \nabla^2 & 0 \\ 0 & 0 & 0 & \lambda_2 - k \nabla^2 \end{bmatrix}, \quad (12)$$

$$\mathbf{A}^{(n)} = \lambda_n \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad \forall n > 2. \quad (13)$$

Replacing the expansions above in the equation  $\mathbf{A}\mathbf{W} = 0$  and equating the terms in  $\varepsilon^n$  for each order  $n$ , we obtain

$$\text{Order } 0 : \quad \mathbf{A}^{(0)}\mathbf{W}^{(0)} = 0, \quad (14)$$

$$\partial \cdot \mathbf{H}^{(0)} = 0, \quad (15)$$

$$\text{Order } 1 : \quad \mathbf{A}^{(0)}\mathbf{W}^{(1)} + \mathbf{A}^{(1)}\mathbf{W}^{(0)} = 0, \quad (16)$$

$$\partial \cdot \mathbf{H}^{(1)} + \nabla \cdot \mathbf{H}^{(0)} = 0, \quad (17)$$

$$\text{Order } 2 : \quad \mathbf{A}^{(0)}\mathbf{W}^{(2)} + \mathbf{A}^{(1)}\mathbf{W}^{(1)} + \mathbf{A}^{(2)}\mathbf{W}^{(0)} = 0, \quad (18)$$

$$\partial \cdot \mathbf{H}^{(2)} + \nabla \cdot \mathbf{H}^{(1)} = 0, \quad (19)$$

$\vdots$

$$\text{Order } n : \quad \mathbf{A}^{(0)}\mathbf{W}^{(n)} + \mathbf{A}^{(1)}\mathbf{W}^{(n-1)} + \mathbf{A}^{(2)}\mathbf{W}^{(n-2)} + \sum_{i=3}^n \mathbf{A}^{(i)}\mathbf{W}^{(n-i)} = 0, \quad (20)$$

$$\partial \cdot \mathbf{H}^{(n)} + \nabla \cdot \mathbf{H}^{(n-1)} = 0. \quad (21)$$

Let's use the symbol  $\langle \bullet \rangle$  to denote the spatial average of the respective set of components over the spatial domain  $Q = [0, L_1] \times [0, L_2] \times [0, L_3]$

$$\langle \bullet \rangle = \frac{1}{\text{vol}(Q)} \int_Q \bullet dx^3, \quad (22)$$

Averaging, as defined above (22), is a necessary solvability condition. At each order, we obtain:

$$\text{Order } 0 : \quad \langle \mathbf{A}^{(0)}\mathbf{W}^{(0)} \rangle = 0, \quad (23)$$

$$\langle \partial \cdot \mathbf{H}^{(0)} \rangle = 0, \quad (24)$$

$$\text{Order } 1 : \quad \langle \mathbf{A}^{(0)}\mathbf{W}^{(1)} \rangle + \langle \mathbf{A}^{(1)}\mathbf{W}^{(0)} \rangle = 0, \quad (25)$$

$$\langle \partial \cdot \mathbf{H}^{(1)} \rangle + \langle \nabla \cdot \mathbf{H}^{(0)} \rangle = 0, \quad (26)$$

$$\text{Order } 2 : \quad \langle \mathbf{A}^{(0)}\mathbf{W}^{(2)} \rangle + \langle \mathbf{A}^{(1)}\mathbf{W}^{(1)} \rangle + \langle \mathbf{A}^{(2)}\mathbf{W}^{(0)} \rangle = 0, \quad (27)$$

$$\langle \partial \cdot \mathbf{H}^{(2)} \rangle + \langle \nabla \cdot \mathbf{H}^{(1)} \rangle = 0, \quad (28)$$

$\vdots$

$$\text{Order } n : \quad \langle \mathbf{A}^{(0)}\mathbf{W}^{(n)} \rangle + \langle \mathbf{A}^{(1)}\mathbf{W}^{(n-1)} \rangle + \langle \mathbf{A}^{(2)}\mathbf{W}^{(n-2)} \rangle + \sum_{i=3}^n \langle \mathbf{A}^{(i)}\mathbf{W}^{(n-i)} \rangle = 0, \quad (29)$$

$$\langle \partial \cdot \mathbf{H}^{(n)} \rangle + \langle \nabla \cdot \mathbf{H}^{(n-1)} \rangle = 0. \quad (30)$$

These extra conditions, or similar ones, are required to deal with the additional degrees of freedom introduced by the slow variables.

## 4 Formal solution

Let us define the fluctuation of  $\mathbf{W}^{(n)}$ ,  $\{\mathbf{W}^{(n)}\}$  by

$$\mathbf{W}^{(n)} = \langle \mathbf{W}^{(n)} \rangle + \{\mathbf{W}^{(n)}\}.$$

The fields,  $\mathbf{W}^{(n)} = \mathbf{W}^{(n)}(\mathbf{x}, \mathbf{X})$ , and the fluctuation  $\{\mathbf{W}^{(n)}\} = \{\mathbf{W}^{(n)}\}(\mathbf{x}, \mathbf{X})$  depend both on fast and slow variables. The average part or mean field,  $\langle \mathbf{W}^{(n)} \rangle = \langle \mathbf{W}^{(n)} \rangle(\mathbf{X})$ , depends only on slow variables. At each order, the fluctuation can be formally written, in terms of the mean fields (average part) and the fluctuations at lower orders, as

$$\{\mathbf{W}^{(n)}\} = -(\tilde{\mathbf{A}}^{(0)})^{-1} \left[ \sum_{i=0}^n \mathbf{A}^{(i)} \langle \mathbf{W}^{(n-i)} \rangle + \sum_{i=1}^n \mathbf{A}^{(i)} \{\mathbf{W}^{(n-i)}\} \right], \quad (31)$$

and the solution as

$$\mathbf{W}^{(n)} = \langle \mathbf{W}^{(n)} \rangle - (\tilde{\mathbf{A}}^{(0)})^{-1} \left[ \sum_{i=0}^n \mathbf{A}^{(i)} \langle \mathbf{W}^{(n-i)} \rangle + \sum_{i=1}^n \mathbf{A}^{(i)} \{\mathbf{W}^{(n-i)}\} \right]. \quad (32)$$

The operator  $\tilde{\mathbf{A}}^{(0)}$  is the restriction of  $\mathbf{A}^{(0)}$  to the subspace where it is invertible (Dubrulle and Frisch (1991); Gama et al. (1994)). It is still necessary to establish conditions that define this subspace, i.e. we must find sufficient conditions for the solvability of these equations.

To have a solution, we must then obtain a closed system of equations for  $\langle \mathbf{W}^{(n)} \rangle$ , i.e. to remove the extra degrees of freedom introduced by the large scale variable.

Below we write the equations for the fluctuation explicitly at all orders.

$$\text{Order 0 : } \quad \mathbf{A}^{(0)} \{\mathbf{W}^{(0)}\} = -\mathbf{A}^{(0)} \langle \mathbf{W}^{(0)} \rangle, \quad (33)$$

$$\partial \cdot \{\mathbf{H}^{(0)}\} = 0, \quad (34)$$

$$\text{Order 1 : } \quad \mathbf{A}^{(0)} \{\mathbf{W}^{(1)}\} = -\mathbf{A}^{(0)} \langle \mathbf{W}^{(1)} \rangle - \mathbf{A}^{(1)} \langle \mathbf{W}^{(0)} \rangle - \mathbf{A}^{(1)} \{\mathbf{W}^{(0)}\}, \quad (35)$$

$$\partial \cdot \{\mathbf{H}^{(1)}\} = -\nabla \cdot \langle \mathbf{H}^{(0)} \rangle - \nabla \cdot \{\mathbf{H}^{(0)}\}, \quad (36)$$

$$\text{Order 2 : } \quad \mathbf{A}^{(0)} \{\mathbf{W}^{(2)}\} = -\mathbf{A}^{(0)} \langle \mathbf{W}^{(2)} \rangle - \mathbf{A}^{(1)} \langle \mathbf{W}^{(1)} \rangle \\ - \mathbf{A}^{(2)} \langle \mathbf{W}^{(0)} \rangle - \mathbf{A}^{(1)} \{\mathbf{W}^{(1)}\} - \mathbf{A}^{(2)} \{\mathbf{W}^{(0)}\}, \quad (37)$$

$$\partial \cdot \{\mathbf{H}^{(2)}\} = -\nabla \cdot \langle \mathbf{H}^{(1)} \rangle - \nabla \cdot \{\mathbf{H}^{(1)}\}, \quad (38)$$

⋮

$$\text{Order } n : \quad \mathbf{A}^{(0)} \{\mathbf{W}^{(n)}\} = -\sum_{i=0}^n \mathbf{A}^{(i)} \langle \mathbf{W}^{(n-i)} \rangle - \sum_{i=1}^n \mathbf{A}^{(i)} \{\mathbf{W}^{(n-i)}\}, \quad (39)$$

$$\partial \cdot \{\mathbf{H}^{(n)}\} = -\nabla \cdot \langle \mathbf{H}^{(n-1)} \rangle - \nabla \cdot \{\mathbf{H}^{(n-1)}\}. \quad (40)$$

## 5 Some algebraic simplifications

The operator  $\nabla$  and the constants commute with the averaging operation (22). The basic fields depend only on fast variables, so the slow derivative of these fields vanishes. The fast derivative of spatial averages is also

zero. These considerations were already used to simplify the Joule effect term in (11) and (12).

The first simplification we would like to do is in the curls in the third lines of (10) and (11). We may write

$$\partial \times (\diamond \times \star) = \diamond(\partial \cdot \star) - \star(\partial \cdot \diamond) + (\star \cdot \partial) \diamond - (\diamond \cdot \partial) \star = \diamond(\partial \cdot \star) + (\star \cdot \partial) \diamond - (\diamond \cdot \partial) \star$$

$$\nabla \times (\diamond \times \star) = \diamond(\nabla \cdot \star) - \star(\nabla \cdot \diamond) + (\star \cdot \nabla) \diamond - (\diamond \cdot \nabla) \star = \diamond(\nabla \cdot \star) - (\diamond \cdot \nabla) \star$$

where  $\diamond$  represents either  $\tilde{\mathbf{H}}$  or  $\tilde{\mathbf{V}}$  and  $\star$  either  $\mathbf{H}$  or  $\mathbf{V}$ . In the first equation, the second term in the triple product expansion vanishes due to the solenoidality of the basic flow fields. In the second equation, the two terms in the middle vanish, since they involve the slow derivative of basic flow fields. Therefore, we can rewrite (10) and (11) as

$$\mathbf{A}^{(0)} = \begin{bmatrix} 0 & \partial \cdot & 0 & 0 \\ \partial & \lambda_0 - \nu \partial^2 + \tilde{\mathbf{V}} \cdot \partial + (\bullet \cdot \partial) \tilde{\mathbf{V}} + \Omega \times & -\tilde{\mathbf{H}} \cdot \partial - (\bullet \cdot \partial) \tilde{\mathbf{H}} & -\alpha \mathbf{G} \\ 0 & \tilde{\mathbf{H}}(\partial \cdot \bullet) + (\bullet \cdot \partial) \tilde{\mathbf{H}} - (\tilde{\mathbf{H}} \cdot \partial) & \lambda_0 - \eta \partial^2 - \tilde{\mathbf{V}}(\partial \cdot \bullet) - (\bullet \cdot \partial) \tilde{\mathbf{V}} + (\tilde{\mathbf{V}} \cdot \partial) & 0 \\ 0 & (\bullet \cdot \partial) \tilde{T} & -\sigma(\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \bullet) & \lambda_0 - k \partial^2 + \tilde{\mathbf{V}} \cdot \partial \end{bmatrix}, \quad (41)$$

$$\mathbf{A}^{(1)} = \begin{bmatrix} 0 & \nabla \cdot & 0 & 0 \\ \nabla & \lambda_1 - 2\nu \partial \cdot \nabla + \tilde{\mathbf{V}} \cdot \nabla & -\tilde{\mathbf{H}} \cdot \nabla & 0 \\ 0 & \tilde{\mathbf{H}}(\nabla \cdot \bullet) - (\tilde{\mathbf{H}} \cdot \nabla) \bullet & \lambda_1 - 2\eta \partial \cdot \nabla - \tilde{\mathbf{V}}(\nabla \cdot \bullet) + (\tilde{\mathbf{V}} \cdot \nabla) \bullet & 0 \\ 0 & 0 & -\sigma(\partial \times \tilde{\mathbf{H}}) \cdot (\nabla \times \bullet) & \lambda_1 - 2k \partial \cdot \nabla + \tilde{\mathbf{V}} \cdot \nabla \end{bmatrix}, \quad (42)$$

This is a more suitable form for some of the further calculations.

We may also write

$$(\partial \times \diamond) \cdot (\partial \times \star) = (\partial_i \diamond_j - \partial_j \diamond_i) \partial_i \star$$

and

$$(\partial \times \diamond) \cdot (\nabla \times \star) = (\partial_i \diamond_j - \partial_j \diamond_i) \nabla_i \star_j,$$

where the Einstein's dummy index notation has been used. This will be used to simplify the Joule term where convenient.

## 6 Simplifications due to the periodic boundary conditions

Due to space periodicity, averages that have spatial fast derivatives on the left vanish. This is not generally true for general boundary conditions and the possible simplification must be studied in each particular case. For the three-dimensional periodic case, the following equalities can be easily verified:

$$\langle \partial \psi \rangle = \langle \partial \cdot \mathbf{A} \rangle = \langle \partial \times \mathbf{A} \rangle = \langle \partial^2 \mathbf{A} \rangle = \langle \partial \times (\mathbf{A} \times \mathbf{B}) \rangle = 0,$$

where  $\psi$  is a scalar and  $\mathbf{A}$ ,  $\mathbf{B}$  are vector periodic functions. The equation

$$\langle \mathbf{A} \cdot \partial \mathbf{B} \rangle = 0$$

is satisfied if  $\mathbf{A}$  is solenoidal.

The  $\mathbf{H}$  and  $\mathbf{V}$  solenoidality equations, the associated solvability conditions and the associated fluctuation equations at order  $n$ ,  $n \geq 1$  are

$$\partial \cdot \mathbf{H}^{(n)} + \nabla \cdot \mathbf{H}^{(n-1)} = 0, \quad (43)$$

$$\nabla \cdot \langle \mathbf{H}^{(n+1)} \rangle = 0, \quad (44)$$

$$\partial \cdot \{ \mathbf{H}^{(n)} \} + \nabla \cdot \{ \mathbf{H}^{(n-1)} \} = 0, \quad (45)$$

$$\partial \cdot \mathbf{V}^{(n)} + \nabla \cdot \mathbf{V}^{(n-1)} = 0, \quad (46)$$

$$\nabla \cdot \langle \mathbf{V}^{(n+1)} \rangle = 0, \quad (47)$$

$$\partial \cdot \{ \mathbf{V}^{(n)} \} + \nabla \cdot \{ \mathbf{V}^{(n-1)} \} = 0. \quad (48)$$

Although the last three equations are already embedded in the block matrix formalism, it is worth writing them apart, since they will be useful in further simplifications. In deriving (45) and (48) we used (44) and (47).

In following sections we will need to evaluate explicitly terms of the form  $\langle \mathbf{A}^{(\beta)} \mathbf{W}^{(\alpha)} \rangle$  and  $\mathbf{A}^{(\beta)} \langle \mathbf{W}^{(\alpha)} \rangle$  for  $\alpha, \beta = 0, 1, 2, \dots$ . We will also need to evaluate  $\mathbf{A}^{(\beta)}(\zeta \mathbf{Q})$ , where  $\zeta = \zeta(\mathbf{X})$  is a scalar function depending only on the slow variables and  $\mathbf{Q} = \mathbf{Q}(\mathbf{x})$  is a block vector function depending only on the fast variables, for  $\beta = 1, 2, 3, \dots$ . Let us use the following notation for the components of  $\mathbf{Q}$ :

$$\mathbf{Q} = \begin{bmatrix} Q^p \\ \mathbf{Q}^V \\ \mathbf{Q}^H \\ Q^T \end{bmatrix}.$$

With the previously introduced simplifications in mind, it is easy to show the equalities below<sup>1</sup>.

### Order 0

$$\langle \mathbf{A}^{(0)} \mathbf{W}^{(\alpha)} \rangle = \begin{bmatrix} 0 \\ \lambda_0 \langle \mathbf{V}^{(\alpha)} \rangle + \langle \mathbf{V}^{(\alpha)} \cdot \partial \tilde{\mathbf{V}} \rangle + \Omega \times \langle \mathbf{V}^{(\alpha)} \rangle - \langle \mathbf{H}^{(\alpha)} \cdot \partial \tilde{\mathbf{H}} \rangle - \alpha \mathbf{G} \langle T^{(\alpha)} \rangle \\ \lambda_0 \langle \mathbf{H}^{(\alpha)} \rangle \\ \langle \mathbf{V}^{(\alpha)} \cdot \partial \tilde{T} \rangle - \sigma \langle (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \mathbf{H}^{(\alpha)}) \rangle + \lambda_0 \langle T^{(\alpha)} \rangle \end{bmatrix}, \quad (49)$$

$$\begin{aligned} \mathbf{A}^{(0)} \langle \mathbf{W}^{(\alpha)} \rangle &= \lambda_0 \begin{bmatrix} 0 \\ \langle \mathbf{V}^{(\alpha)} \rangle \\ \langle \mathbf{H}^{(\alpha)} \rangle \\ \langle T^{(\alpha)} \rangle \end{bmatrix} + \begin{bmatrix} 0 \\ \Omega \times \langle \mathbf{V}^{(\alpha)} \rangle - \alpha \mathbf{G} \langle T^{(\alpha)} \rangle \\ 0 \\ 0 \end{bmatrix} \\ &+ \sum_{i=1}^3 \begin{bmatrix} 0 \\ \partial_i \tilde{\mathbf{V}} \\ \partial_i \tilde{\mathbf{H}} \\ \partial_i \tilde{T} \end{bmatrix} \langle V_i^{(\alpha)} \rangle + \sum_{i=1}^3 \begin{bmatrix} 0 \\ -\partial_i \tilde{\mathbf{H}} \\ -\partial_i \tilde{\mathbf{V}} \\ 0 \end{bmatrix} \langle H_i^{(\alpha)} \rangle. \end{aligned} \quad (50)$$

### Order 1

$$\langle \mathbf{A}^{(1)} \mathbf{W}^{(\alpha)} \rangle = \begin{bmatrix} \nabla \cdot \langle \mathbf{V}^{(\alpha)} \rangle \\ \nabla \langle p^{(\alpha)} \rangle + \lambda_1 \langle \mathbf{V}^{(\alpha)} \rangle + \langle \tilde{\mathbf{V}} \cdot \nabla \mathbf{V}^{(\alpha)} \rangle - \langle \tilde{\mathbf{H}} \cdot \nabla \mathbf{H}^{(\alpha)} \rangle \\ \langle \tilde{\mathbf{H}} \nabla \cdot \mathbf{V}^{(\alpha)} \rangle - \langle \tilde{\mathbf{H}} \cdot \nabla \mathbf{V}^{(\alpha)} \rangle + \lambda_1 \langle \mathbf{H}^{(\alpha)} \rangle - \langle \tilde{\mathbf{V}} \nabla \cdot \mathbf{H}^{(\alpha)} \rangle + \langle \tilde{\mathbf{V}} \cdot \nabla \mathbf{H}^{(\alpha)} \rangle \\ -\sigma \langle (\partial \times \tilde{\mathbf{H}}) \cdot (\nabla \times \mathbf{H}^{(\alpha)}) \rangle + \lambda_1 \langle T^{(\alpha)} \rangle \end{bmatrix}, \quad (51)$$

$$\mathbf{A}^{(1)} \langle \mathbf{W}^{(\alpha)} \rangle = \lambda_1 \begin{bmatrix} 0 \\ \langle \mathbf{V}^{(\alpha)} \rangle \\ \langle \mathbf{H}^{(\alpha)} \rangle \\ \langle T^{(\alpha)} \rangle \end{bmatrix} + \begin{bmatrix} 0 \\ \nabla \langle p^{(\alpha)} \rangle \\ 0 \\ 0 \end{bmatrix} + \sum_{i,j=1}^3 \begin{bmatrix} 0 \\ \mathbf{e}_i \tilde{V}_j \\ -\mathbf{e}_i \tilde{H}_j \\ 0 \end{bmatrix} \nabla_j \langle V_i^{(\alpha)} \rangle$$

<sup>1</sup>When evaluating  $\langle \mathbf{A}^{(0)} \mathbf{W}^{(i)} \rangle$  we use the equation (10) for  $\mathbf{A}^{(0)}$ , since the averages of the curls vanish due to the periodicity. In the remaining calculations we use (41) for  $\mathbf{A}^{(0)}$  and (42) for  $\mathbf{A}^{(1)}$ .



$$+ \sum_{i,j=1}^3 \begin{bmatrix} 0 \\ -\mathbf{e}_i \tilde{H}_j \\ \mathbf{e}_i \tilde{V}_j \\ -\sigma(\partial_j \tilde{H}_i - \partial_i \tilde{H}_j) \end{bmatrix} \nabla_j \langle H_i^{(\alpha)} \rangle + \sum_{j=1}^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tilde{V}_j \end{bmatrix} \nabla_j \langle T^{(\alpha)} \rangle, \quad (52)$$

$$\mathbf{A}^{(1)}(\zeta \mathbf{Q}) = \zeta \lambda_1 \begin{bmatrix} 0 \\ \mathbf{Q}^{\mathbf{V}} \\ \mathbf{Q}^{\mathbf{H}} \\ Q^T \end{bmatrix} + \begin{bmatrix} \mathbf{Q}_j^{\mathbf{V}} \\ Q^p \mathbf{e}_j - 2\nu \partial_j \mathbf{Q}^{\mathbf{V}} + \tilde{V}_j \mathbf{Q}^{\mathbf{V}} - \tilde{H}_j \mathbf{Q}^{\mathbf{H}} \\ \tilde{\mathbf{H}} Q_j^{\mathbf{V}} - \tilde{H}_j \mathbf{Q}^{\mathbf{V}} - 2\eta \partial_j \mathbf{Q}^{\mathbf{H}} - \tilde{\mathbf{V}} Q_j^{\mathbf{H}} + \tilde{V}_j \mathbf{Q}^{\mathbf{H}} \\ -\sigma(\partial_j \tilde{H}_k - \partial_k \tilde{H}_j) Q_k^{\mathbf{H}} - 2k \partial_j Q^T + \tilde{V}_j Q^T \end{bmatrix} \nabla_j \zeta. \quad (53)$$

**Order 2**

$$\langle \mathbf{A}^{(2)} \mathbf{W}^{(\alpha)} \rangle = \mathbf{A}^{(2)} \langle \mathbf{W}^{(\alpha)} \rangle = \begin{bmatrix} 0 \\ \lambda_2 \langle \mathbf{V}^{(\alpha)} \rangle - \nu \nabla^2 \langle \mathbf{V}^{(\alpha)} \rangle \\ \lambda_2 \langle \mathbf{H}^{(\alpha)} \rangle - \eta \nabla^2 \langle \mathbf{H}^{(\alpha)} \rangle \\ \lambda_2 \langle T^{(\alpha)} \rangle - k \nabla^2 \langle T^{(\alpha)} \rangle \end{bmatrix}, \quad (54)$$

$$\mathbf{A}^{(2)}(\zeta \mathbf{Q}) = \zeta \lambda_2 \begin{bmatrix} 0 \\ \mathbf{Q}^{\mathbf{V}} \\ \mathbf{Q}^{\mathbf{H}} \\ Q^T \end{bmatrix} + \begin{bmatrix} 0 \\ -\nu \mathbf{Q}^{\mathbf{V}} \\ -\eta \mathbf{Q}^{\mathbf{H}} \\ -k Q^T \end{bmatrix} \nabla^2 \zeta. \quad (55)$$

**Order n** ( $n > 2$ )

$$\langle \mathbf{A}^{(n)} \mathbf{W}^{(\alpha)} \rangle = \mathbf{A}^{(n)} \langle \mathbf{W}^{(\alpha)} \rangle = \lambda_n \begin{bmatrix} 0 \\ \langle \mathbf{V}^{(\alpha)} \rangle \\ \langle \mathbf{H}^{(\alpha)} \rangle \\ \langle T^{(\alpha)} \rangle \end{bmatrix}, \quad (56)$$

$$\mathbf{A}^{(n)}(\zeta \mathbf{Q}) = \zeta \lambda_n \begin{bmatrix} 0 \\ \mathbf{Q}^{\mathbf{V}} \\ \mathbf{Q}^{\mathbf{H}} \\ Q^T \end{bmatrix}. \quad (57)$$

## 7 Symmetries of the basic fields

The following symmetries of the basic fields are consistent with the basic equations:

1. Suppose that the fields  $\tilde{\mathbf{V}}$  and  $\tilde{\mathbf{H}}$  are **parity-invariant** ( $\mathbf{f}(-\mathbf{x}) = -\mathbf{f}(\mathbf{x})$ ). The pressure  $\tilde{p}$  must be **parity-antiinvariant** ( $\mathbf{f}(-\mathbf{x}) = \mathbf{f}(\mathbf{x})$ ) and  $\tilde{T}$  must be parity-invariant, in order to satisfy the Navier-Stokes equation. For the temperature equation to be satisfied, we must neglect the conductivity term, making  $\sigma = 0$ .
2. Suppose that the fields  $\tilde{\mathbf{V}}$  and  $\tilde{\mathbf{H}}$  are symmetric about  $x_3$  axis. A **vector field**  $\mathbf{Q}$  is called **symmetric about the  $x_3$  axis** if

$$Q_1(-x_1, -x_2, x_3) = -Q_1(x_1, x_2, x_3)$$

$$Q_2(-x_1, -x_2, x_3) = -Q_2(x_1, x_2, x_3)$$

$$Q_3(-x_1, -x_2, x_3) = Q_3(x_1, x_2, x_3)$$

and **anti-symmetric about the  $x_3$  axis** if

$$Q_1(-x_1, -x_2, x_3) = Q_1(x_1, x_2, x_3)$$

$$Q_2(-x_1, -x_2, x_3) = Q_2(x_1, x_2, x_3)$$

$$Q_3(-x_1, -x_2, x_3) = -Q_3(x_1, x_2, x_3)$$

A **scalar field  $f$**  is said to be **symmetric about the  $x_3$  axis** if  $f(-x_1, -x_2, x_3) = f(x_1, x_2, x_3)$  and **anti-symmetric about the  $x_3$  axis** if  $f(-x_1, -x_2, x_3) = -f(x_1, x_2, x_3)$ . The scalar fields  $\tilde{p}$  and  $\tilde{T}$  must be symmetric scalar fields for the Navier-Stokes equation to hold. This is consistent with the temperature equation with  $\sigma \neq 0$ .

In what follows, a set of fields  $p, \mathbf{V}, \mathbf{H}, T$  is called **symmetric** if it satisfies the symmetry defined in items 1 or 2, and **anti-symmetric** if it satisfies the symmetry opposite to those defined in items 1 or 2. Anti-symmetric basic fields are inconsistent with the basic equations. This can be verified by looking at the behaviour of the first two terms in the Navier-Stokes equation.

It can be shown that  $\mathbf{A}^{(0)}$  preserves symmetry of fields and  $\mathbf{A}^{(1)}$  changes it, if  $\lambda_0 = \lambda_1 = 0$ .

## 8 Solving the equations

We will now analyse the equations at each order and make some order specific simplifications, using symmetries and boundary conditions.

### 8.1 Order 0

#### 8.1.1 Solvability

Expanding (23), we obtain:

$$\begin{cases} \langle \partial \cdot \mathbf{V}^{(0)} \rangle = 0 \\ \langle \lambda_0 \mathbf{V}^{(0)} \rangle + \langle \Omega \times \mathbf{V}^{(0)} \rangle - \langle \alpha \mathbf{G} T^{(0)} \rangle = 0 \\ \langle \lambda_0 \mathbf{H}^{(0)} \rangle = 0 \\ \langle \lambda_0 T^{(0)} \rangle - \langle \sigma (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \mathbf{H}^{(0)}) \rangle = 0 \end{cases} \quad (58)$$

and from (24)

$$\langle \partial \cdot \mathbf{H}^{(0)} \rangle = 0. \quad (59)$$

Equation (59) and the first equation in (58) are trivially satisfied due to the periodicity of the fields. Since we are interested in solutions with  $\langle \mathbf{H}^{(0)} \rangle \neq 0$ , we must have  $\lambda_0 = 0$ . Hence, (58) simplifies to:

$$\Omega \times \langle \mathbf{V}^{(0)} \rangle - \alpha \mathbf{G} \langle T^{(0)} \rangle = 0, \quad (60)$$

$$\lambda_0 = 0, \quad (61)$$

$$\langle (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \mathbf{H}^{(0)}) \rangle = 0. \quad (62)$$

Note also that (62) is satisfied, because  $\mathbf{H}^{(0)}$  turns out to be anti-symmetric. This is a direct consequence of the symmetries of the fluctuating part of  $\mathbf{W}^{(0)}$ , which we discuss below.

### 8.1.2 Fluctuation

From (50) and using (60) we can write

$$\mathbf{A}^{(0)}\langle\mathbf{W}^{(0)}\rangle = \sum_{i=1}^3 \begin{bmatrix} 0 \\ \partial_i \tilde{\mathbf{V}} \\ \partial_i \tilde{\mathbf{H}} \\ \partial_i \tilde{T} \end{bmatrix} \langle V_i^{(0)} \rangle + \sum_{i=1}^3 \begin{bmatrix} 0 \\ -\partial_i \tilde{\mathbf{H}} \\ -\partial_i \tilde{\mathbf{V}} \\ 0 \end{bmatrix} \langle H_i^{(0)} \rangle.$$

Thus, expanding (33), we obtain

$$\mathbf{A}^{(0)}\{\mathbf{W}^{(0)}\} = -\mathbf{A}^{(0)}\langle\mathbf{W}^{(0)}\rangle = \sum_{i=1}^3 \begin{bmatrix} 0 \\ -\partial_i \tilde{\mathbf{V}} \\ -\partial_i \tilde{\mathbf{H}} \\ -\partial_i \tilde{T} \end{bmatrix} \langle V_i^{(0)} \rangle + \sum_{i=1}^3 \begin{bmatrix} 0 \\ \partial_i \tilde{\mathbf{H}} \\ \partial_i \tilde{\mathbf{V}} \\ 0 \end{bmatrix} \langle H_i^{(0)} \rangle.$$

From the last equality, we can express the fluctuation as:

$$\{\mathbf{W}^{(0)}\} = \sum_{i=1}^3 (\tilde{\mathbf{A}}^{(0)})^{-1} \begin{bmatrix} 0 \\ -\partial_i \tilde{\mathbf{V}} \\ -\partial_i \tilde{\mathbf{H}} \\ -\partial_i \tilde{T} \end{bmatrix} \langle V_i^{(0)} \rangle + \sum_{i=1}^3 (\tilde{\mathbf{A}}^{(0)})^{-1} \begin{bmatrix} 0 \\ \partial_i \tilde{\mathbf{H}} \\ \partial_i \tilde{\mathbf{V}} \\ 0 \end{bmatrix} \langle H_i^{(0)} \rangle,$$

i.e.

$$\{\mathbf{W}^{(0)}\} = \sum_{i=1}^3 \mathbf{S}_i^{\mathbf{V}} \langle V_i^{(0)} \rangle + \sum_{i=1}^3 \mathbf{S}_i^{\mathbf{H}} \langle H_i^{(0)} \rangle,$$

where  $\mathbf{S}_i^{\mathbf{V}} = \mathbf{S}_i^{\mathbf{V}}(\mathbf{x})$  and  $\mathbf{S}_i^{\mathbf{H}} = \mathbf{S}_i^{\mathbf{H}}(\mathbf{x})$  are the solutions of the zero order auxiliary problems:

$$A^{(0)}\mathbf{S}_i^{\mathbf{V}} = \begin{bmatrix} 0 \\ -\partial_i \tilde{\mathbf{V}} \\ -\partial_i \tilde{\mathbf{H}} \\ -\partial_i \tilde{T} \end{bmatrix}, \quad A^{(0)}\mathbf{S}_i^{\mathbf{H}} = \begin{bmatrix} 0 \\ \partial_i \tilde{\mathbf{H}} \\ \partial_i \tilde{\mathbf{V}} \\ 0 \end{bmatrix}. \quad (63)$$

Note that  $\mathbf{A}^{(0)}$  preserves the symmetry of fields. Hence,  $\mathbf{S}_i^{\mathbf{V}}$  and  $\mathbf{S}_i^{\mathbf{H}}$  have the same symmetry as the r.h.s. of the respective auxiliary problem (63). This means that their symmetry is opposite to the symmetry of the basic fields. Thus (62) is satisfied.

## 8.2 Notation for the auxiliary problems

We obtain several auxiliary problems when solving the equation for the fluctuation at each order. Also, we need to use some components of these auxiliary problems in further algebraic calculations. Thus it is convenient to introduce the following coherent notation to identify the solutions of the auxiliary problems:

$$\mathbf{X}_I^F$$

where  $\mathbf{X}$  is a name that we chose for the auxiliary problem,  $F$  indicates that the solution must be multiplied by an expression containing an average of  $F$  or one of its components and  $I$  is a set of indices characterising the auxiliary problem.  $\mathbf{X}$ ,  $F$  and  $I$  define a unique auxiliary problem.  $F$  can be either of the fields, i.e.  $p$ ,  $\mathbf{V}$ ,  $\mathbf{H}$ ,  $T$ . The indices in  $i$  range from 1 to 3 and depend on the order of the auxiliary problem and on the field involved in it.

Each auxiliary problem  $\mathbf{X}_I^F$  is a block vector with a structure similar to  $\mathbf{W}$ . To refer to the components

of this block vector we use the notation

$$(\mathbf{X}_I^F)^{CVB}$$

where  $CVB$  can be either of the fields. So we may write,

$$\mathbf{X}_I^F = \begin{bmatrix} (\mathbf{X}_I^F)^p \\ (\mathbf{X}_I^F)^{\mathbf{V}} \\ (\mathbf{X}_I^F)^{\mathbf{H}} \\ (\mathbf{X}_I^F)^T \end{bmatrix}.$$

We use the notation

$$(\mathbf{X}_I^F)_{CVF}^{CVB}$$

where  $CVF$  is an index in the range 1, 2, 3, for individual scalar components of vector fields comprising this block vector.

### 8.3 Order 1

#### 8.3.1 Solvability

From (16) and (17), recalling that  $\langle \mathbf{H}^{(0)} \rangle \neq \mathbf{0}$ , we obtain:

$$\nabla \cdot \langle \mathbf{V}^{(0)} \rangle = 0, \quad (64)$$

$$\nabla \langle p^{(0)} \rangle + \Omega \times \langle \mathbf{V}^{(1)} \rangle - \alpha \mathbf{G} \langle T^{(1)} \rangle = \mathbf{0}, \quad (65)$$

$$\lambda_1 = 0, \quad (66)$$

$$\langle (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \mathbf{H}^{(1)}) \rangle + \langle (\partial \times \tilde{\mathbf{H}}) \cdot (\nabla \times \mathbf{H}^{(0)}) \rangle = \mathbf{0}, \quad (67)$$

$$\nabla \cdot \langle \mathbf{H}^{(0)} \rangle = 0. \quad (68)$$

To obtain the equations above some simplifications were made:

1.  $\langle (\tilde{\mathbf{H}} \cdot \nabla) \mathbf{V}^{(0)} \rangle = \langle (\tilde{\mathbf{V}} \cdot \nabla) \mathbf{H}^{(0)} \rangle = \langle \tilde{\mathbf{H}} \nabla \cdot \mathbf{V}^{(0)} \rangle = \langle \tilde{\mathbf{V}} \nabla \cdot \mathbf{H}^{(0)} \rangle = 0$ , because of symmetry: the fields at order 0 have the symmetry opposite to that of the basic fields.
2.  $\langle \tilde{\mathbf{H}} \partial \cdot \mathbf{V}^{(1)} \rangle = \langle \tilde{\mathbf{V}} \partial \cdot \mathbf{H}^{(1)} \rangle = 0$ , since using the incompressibility of  $\mathbf{V}$  and  $\mathbf{H}$  at order 1 these terms can be reduced to  $\langle \tilde{\mathbf{H}} \nabla \cdot \mathbf{V}^{(0)} \rangle$  and  $\langle \tilde{\mathbf{V}} \nabla \cdot \mathbf{H}^{(0)} \rangle$ , which vanish by the previous argument.
3.  $\langle \mathbf{V}^{(1)} \cdot \partial \tilde{\mathbf{V}} \rangle = \langle \mathbf{V}^{(1)} \cdot \partial \tilde{\mathbf{H}} \rangle = \langle \mathbf{V}^{(1)} \cdot \partial \tilde{T} \rangle = \langle \mathbf{H}^{(1)} \cdot \partial \tilde{\mathbf{V}} \rangle = \langle \mathbf{H}^{(1)} \cdot \partial \tilde{\mathbf{H}} \rangle = 0$ ; integrating by parts and bearing in mind that the averages with derivatives on the left vanish, these terms can be reduced to terms similar to those in item 2; using the incompressibility they are therefore equal to zero.

#### 8.3.2 Fluctuation

We start by rewriting the equation for the first order fluctuation (35):

$$A^{(0)} \{ \mathbf{W}^{(1)} \} = -A^{(0)} \langle \mathbf{W}^{(1)} \rangle - A^{(1)} \langle \mathbf{W}^{(0)} \rangle - A^{(1)} \{ \mathbf{W}^{(0)} \}.$$

From (50), using (66), we can write

$$A^{(0)} \langle \mathbf{W}^{(1)} \rangle = \begin{bmatrix} 0 \\ -\nabla \langle p^{(0)} \rangle \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^3 \begin{bmatrix} 0 \\ \partial_i \tilde{\mathbf{V}} \\ \partial_i \tilde{\mathbf{H}} \\ \partial_i \tilde{T} \end{bmatrix} \langle V_i^{(1)} \rangle + \sum_{i=1}^3 \begin{bmatrix} 0 \\ -\partial_i \tilde{\mathbf{H}} \\ -\partial_i \tilde{\mathbf{V}} \\ 0 \end{bmatrix} \langle H_i^{(1)} \rangle$$

and from (52) we obtain

$$A^{(1)}\langle \mathbf{W}^{(0)} \rangle = \begin{bmatrix} 0 \\ \nabla \langle p^{(0)} \rangle \\ 0 \\ 0 \end{bmatrix} + \sum_{i,j=1}^3 \begin{bmatrix} 0 \\ \mathbf{e}_i \tilde{V}_j \\ -\mathbf{e}_i \tilde{H}_j \\ 0 \end{bmatrix} \nabla_j \langle V_i^{(0)} \rangle \\ + \sum_{i,j=1}^3 \begin{bmatrix} 0 \\ -\mathbf{e}_i \tilde{H}_j \\ \mathbf{e}_i \tilde{V}_j \\ -\sigma(\partial_j \tilde{H}_i - \partial_i \tilde{H}_j) \end{bmatrix} \nabla_j \langle H_i^{(0)} \rangle + \sum_{j=1}^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tilde{V}_j \end{bmatrix} \nabla_j \langle T^{(0)} \rangle.$$

To compute  $A^{(1)}\{\mathbf{W}^{(0)}\}$ ,  $\{\mathbf{W}^{(0)}\}$  is expressed as a function of  $\langle \mathbf{W}^{(0)} \rangle$  and of the zeroth order auxiliary problems. Thus, using (53), we obtain:

$$A^{(1)} \left[ \sum_{i=1}^3 \mathbf{S}_i^{\mathbf{V}} \langle V_i^{(0)} \rangle \right] = \sum_{i,j,k=1}^3 \begin{bmatrix} (S_i^{\mathbf{V}})^{\mathbf{V}}_j \\ \mathbf{e}_j (S_i^{\mathbf{V}})^p - 2\nu \partial_j (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{V}} + (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{V}} \tilde{V}_j - (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{H}} \tilde{H}_j \\ \tilde{\mathbf{H}} (S_i^{\mathbf{V}})^{\mathbf{V}}_j - (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{V}} \tilde{H}_j - 2\eta \partial_j (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{H}} - \tilde{\mathbf{V}} (S_i^{\mathbf{V}})^{\mathbf{H}}_j + (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{H}} \tilde{V}_j \\ -\sigma(\partial_j \tilde{H}_k - \partial_k \tilde{H}_j) (S_i^{\mathbf{V}})^{\mathbf{H}}_k - 2k \partial_j (S_i^{\mathbf{V}})^T + (S_i^{\mathbf{V}})^T \tilde{V}_j \end{bmatrix} \nabla_j \langle V_i^{(0)} \rangle$$

and

$$A^{(1)} \left[ \sum_{i=1}^3 \mathbf{S}_i^{\mathbf{H}} \langle H_i^{(0)} \rangle \right] = \sum_{i,j,k=1}^3 \begin{bmatrix} (S_i^{\mathbf{H}})^{\mathbf{V}}_j \\ \mathbf{e}_j (S_i^{\mathbf{H}})^p - 2\nu \partial_j (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{V}} + (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{V}} \tilde{V}_j - (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{H}} \tilde{H}_j \\ \tilde{\mathbf{H}} (S_i^{\mathbf{H}})^{\mathbf{V}}_j - (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{V}} \tilde{H}_j - 2\eta \partial_j (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{H}} - \tilde{\mathbf{V}} (S_i^{\mathbf{H}})^{\mathbf{H}}_j + (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{H}} \tilde{V}_j \\ -\sigma(\partial_j \tilde{H}_k - \partial_k \tilde{H}_j) (S_i^{\mathbf{H}})^{\mathbf{H}}_k - 2k \partial_j (S_i^{\mathbf{H}})^T + (S_i^{\mathbf{H}})^T \tilde{V}_j \end{bmatrix} \nabla_j \langle H_i^{(0)} \rangle.$$

Summing up all the terms we can express the fluctuation at order 1 as:

$$\{\mathbf{W}^{(1)}\} = \sum_{i=1}^3 \mathbf{S}_i^{\mathbf{V}} \langle V_i^{(1)} \rangle + \sum_{i=1}^3 \mathbf{S}_i^{\mathbf{H}} \langle H_i^{(1)} \rangle + \sum_{i,j=1}^3 \mathbf{\Gamma}_{ij}^{\mathbf{V}} \nabla_j \langle V_i^{(0)} \rangle + \sum_{i,j=1}^3 \mathbf{\Gamma}_{ij}^{\mathbf{H}} \nabla_j \langle H_i^{(0)} \rangle + \sum_{j=1}^3 \mathbf{\Gamma}_j^T \nabla_j \langle T^{(0)} \rangle,$$

where

$$A^{(0)} \mathbf{\Gamma}_{ij}^{\mathbf{V}} = \sum_{k=1}^3 \begin{bmatrix} -(S_i^{\mathbf{V}})^{\mathbf{V}}_j \\ -\mathbf{e}_i \tilde{V}_j - \mathbf{e}_j (S_i^{\mathbf{V}})^p + 2\nu \partial_j (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{V}} - (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{V}} \tilde{V}_j + (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{H}} \tilde{H}_j \\ \mathbf{e}_i \tilde{H}_j - \tilde{\mathbf{H}} (S_i^{\mathbf{V}})^{\mathbf{V}}_j + (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{V}} \tilde{H}_j + 2\eta \partial_j (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{H}} + \tilde{\mathbf{V}} (S_i^{\mathbf{V}})^{\mathbf{H}}_j - (\mathbf{S}_i^{\mathbf{V}})^{\mathbf{H}} \tilde{V}_j \\ \sigma(\partial_j \tilde{H}_k - \partial_k \tilde{H}_j) (S_i^{\mathbf{V}})^{\mathbf{H}}_k + 2k \partial_j (S_i^{\mathbf{V}})^T - (S_i^{\mathbf{V}})^T \tilde{V}_j \end{bmatrix}, \\ A^{(0)} \mathbf{\Gamma}_{ij}^{\mathbf{H}} = \sum_{k=1}^3 \begin{bmatrix} -(S_i^{\mathbf{H}})^{\mathbf{V}}_j \\ \mathbf{e}_i \tilde{H}_j - \mathbf{e}_j (S_i^{\mathbf{H}})^p + 2\nu \partial_j (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{V}} - (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{V}} \tilde{V}_j + (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{H}} \tilde{H}_j \\ -\mathbf{e}_i \tilde{V}_j - \tilde{\mathbf{H}} (S_i^{\mathbf{H}})^{\mathbf{V}}_j + (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{V}} \tilde{H}_j + 2\eta \partial_j (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{H}} + \tilde{\mathbf{V}} (S_i^{\mathbf{H}})^{\mathbf{H}}_j - (\mathbf{S}_i^{\mathbf{H}})^{\mathbf{H}} \tilde{V}_j \\ \sigma(\partial_j \tilde{H}_i - \partial_i \tilde{H}_j) + \sigma(\partial_j \tilde{H}_k - \partial_k \tilde{H}_j) (S_i^{\mathbf{H}})^{\mathbf{H}}_k + 2k \partial_j (S_i^{\mathbf{H}})^T - (S_i^{\mathbf{H}})^T \tilde{V}_j \end{bmatrix}$$

and

$$A^{(0)} \mathbf{\Gamma}_j^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\tilde{V}_j \end{bmatrix}$$

are the so called auxiliary problems of order 1.

## 8.4 Order 2

### 8.4.1 Solvability

The only simplifications that can be made in the second order solvability conditions are in the term  $\langle A^{(0)} \mathbf{W}^{(2)} \rangle$ :

$$\begin{aligned}\langle \tilde{\diamond} \partial \cdot \star^{(2)} \rangle &= -\langle \tilde{\diamond} \nabla \cdot \star^{(1)} \rangle \\ \langle \star^{(2)} \cdot \partial \tilde{\diamond} \rangle &= -\langle \tilde{\diamond} \partial \cdot \star^{(2)} \rangle = \langle \tilde{\diamond} \nabla \cdot \star^{(1)} \rangle\end{aligned}$$

where  $\star$  and  $\diamond$  represent either  $\mathbf{H}$  or  $\mathbf{V}$ . From (18) and (19) we get:

$$\nabla \cdot \langle \mathbf{V}^{(1)} \rangle = 0, \quad (69)$$

$$\begin{aligned}\lambda_2 \langle \mathbf{V}^{(0)} \rangle - \nu \nabla^2 \langle \mathbf{V}^{(0)} \rangle + \nabla \langle p^{(1)} \rangle + \langle \tilde{\mathbf{V}} \nabla \cdot \mathbf{V}^{(1)} \rangle \\ + \langle \tilde{\mathbf{V}} \cdot \nabla \mathbf{V}^{(1)} \rangle - \langle \tilde{\mathbf{H}} \nabla \cdot \mathbf{H}^{(1)} \rangle - \langle \tilde{\mathbf{H}} \cdot \nabla \mathbf{H}^{(1)} \rangle + \Omega \times \langle \mathbf{V}^{(2)} \rangle - \alpha \mathbf{G} \langle T^{(2)} \rangle = 0,\end{aligned} \quad (70)$$

$$\begin{aligned}\lambda_2 \langle \mathbf{H}^{(0)} \rangle - \eta \nabla^2 \langle \mathbf{H}^{(0)} \rangle + \langle \tilde{\mathbf{H}} \nabla \cdot \mathbf{V}^{(1)} \rangle \\ - \langle \tilde{\mathbf{H}} \cdot \nabla \mathbf{V}^{(1)} \rangle - \langle \tilde{\mathbf{V}} \nabla \cdot \mathbf{H}^{(1)} \rangle + \langle \tilde{\mathbf{V}} \cdot \nabla \mathbf{H}^{(1)} \rangle = 0,\end{aligned} \quad (71)$$

$$\begin{aligned}\lambda_2 \langle T^{(0)} \rangle - k \nabla^2 \langle T^{(0)} \rangle + \langle \tilde{T} \nabla \cdot \mathbf{V}^{(1)} \rangle \\ - \sigma \langle (\partial \times \tilde{\mathbf{H}}) \cdot (\nabla \times \mathbf{H}^{(1)}) \rangle - \sigma \langle (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \mathbf{H}^{(2)}) \rangle = 0,\end{aligned} \quad (72)$$

$$\nabla \cdot \langle \mathbf{H}^{(1)} \rangle = 0. \quad (73)$$

Note  $\lambda_2 \neq 0$ .

### 8.4.2 Closed equations for $\langle \mathbf{V}^{(0)} \rangle$ , $\langle \mathbf{H}^{(0)} \rangle$ and $\langle T^{(0)} \rangle$

At order 2, we derive a set of closed equations for  $\langle \mathbf{V}^{(0)} \rangle$ ,  $\langle \mathbf{H}^{(0)} \rangle$  and  $\langle T^{(0)} \rangle$ . Equations (70) and (72) involve some components of  $\mathbf{W}^{(2)}$ . In order to use them, we would have to write the second order auxiliary problems explicitly. We shall see that this is not necessary, because we have enough equations of lower order. The equations involving components of  $\mathbf{W}^{(1)}$  are easily related to a linear combination of the components of  $\langle \mathbf{W}^{(0)} \rangle$  and  $\langle \mathbf{W}^{(1)} \rangle$ , using the first order auxiliary problems. The terms involving  $\langle \mathbf{W}^{(1)} \rangle$  vanish in all cases so we will be able to obtain a closed set of equations for  $\langle \mathbf{W}^{(0)} \rangle$ .

From (60), (67) and (71) we get 7 scalar independent equations involving the following seven unknowns:  $\langle \mathbf{V}^{(0)} \rangle$  (3 components),  $\langle \mathbf{H}^{(0)} \rangle$  (3 components) and  $\langle T^{(0)} \rangle$ . After some algebraic calculations, we obtain

$$\Omega \times \langle \mathbf{V}^{(0)} \rangle - \alpha \mathbf{G} \langle T^{(0)} \rangle = 0, \quad (74)$$

$$\begin{aligned}\sum_{kl} \langle \partial_l \tilde{H}_k - \partial_k \tilde{H}_l \rangle \nabla_l \langle H_k^{(0)} \rangle \\ + \sum_{ijkl} \langle (\partial_l \tilde{H}_k - \partial_k \tilde{H}_l) (\partial_l (\Gamma_{ij}^{\mathbf{V}})_k^{\mathbf{H}}) \rangle \nabla_j \langle V_i^{(0)} \rangle \\ + \sum_{ijkl} \langle (\partial_l \tilde{H}_k - \partial_k \tilde{H}_l) (\partial_l (\Gamma_{ij}^{\mathbf{H}})_k^{\mathbf{H}}) \rangle \nabla_j \langle H_i^{(0)} \rangle \\ + \sum_{jkl} \langle (\partial_l \tilde{H}_k - \partial_k \tilde{H}_l) (\partial_l (\Gamma_j^{\mathbf{T}})_k^{\mathbf{H}}) \rangle \nabla_j \langle T^{(0)} \rangle = 0,\end{aligned} \quad (75)$$

$$\begin{aligned}
& \lambda_2 \langle \mathbf{H}^{(0)} \rangle - \eta \nabla^2 \langle \mathbf{H}^{(0)} \rangle \\
& + \sum_{ijk} \langle \tilde{\mathbf{H}} (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{V}} - \tilde{H}_k (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{V}} - \tilde{\mathbf{V}} (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{H}} + \tilde{V}_k (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{H}} \rangle \nabla_j \nabla_k \langle V_i^{(0)} \rangle \\
& + \sum_{ijk} \langle \tilde{\mathbf{H}} (\Gamma_{ij}^{\mathbf{H}})_{\mathbf{k}}^{\mathbf{V}} - \tilde{H}_k (\Gamma_{ij}^{\mathbf{H}})_{\mathbf{k}}^{\mathbf{V}} - \tilde{\mathbf{V}} (\Gamma_{ij}^{\mathbf{H}})_{\mathbf{k}}^{\mathbf{H}} + \tilde{V}_k (\Gamma_{ij}^{\mathbf{H}})_{\mathbf{k}}^{\mathbf{H}} \rangle \nabla_j \nabla_k \langle H_i^{(0)} \rangle \\
& + \sum_{jk} \langle \tilde{\mathbf{H}} (\Gamma_j^{\mathbf{T}})_{\mathbf{k}}^{\mathbf{V}} - \tilde{H}_k (\Gamma_j^{\mathbf{T}})_{\mathbf{k}}^{\mathbf{V}} - \tilde{\mathbf{V}} (\Gamma_j^{\mathbf{T}})_{\mathbf{k}}^{\mathbf{H}} + \tilde{V}_k (\Gamma_j^{\mathbf{T}})_{\mathbf{k}}^{\mathbf{H}} \rangle \nabla_j \nabla_k \langle T^{(0)} \rangle = 0, \tag{76}
\end{aligned}$$

supplemented by

$$\nabla \cdot \langle \mathbf{V}^{(0)} \rangle = 0, \tag{77}$$

$$\nabla \cdot \langle \mathbf{H}^{(0)} \rangle = 0, \tag{78}$$

This is a closed set of equations for  $\langle \mathbf{V}^{(0)} \rangle$ ,  $\langle \mathbf{H}^{(0)} \rangle$  and  $\langle T^{(0)} \rangle$ .

### 8.4.3 Fluctuation

To have a closed set of equations for  $\langle \mathbf{W}^{(0)} \rangle$ , we must get an equation for  $\langle p^{(0)} \rangle$ . Remark that without the Coriolis and Boussinesq terms, the equation (65) reduces to  $\nabla \langle p^{(0)} \rangle = 0$ , like in Dubrulle and Frisch (1991). With those effects present, equation (65) requires some components of  $\langle \mathbf{W}^{(1)} \rangle$  to determine  $\langle p^{(0)} \rangle$ .

The equation for the first order fluctuation is:

$$\mathbf{A}^{(0)} \{ \mathbf{W}^{(2)} \} = -\mathbf{A}^{(0)} \langle \mathbf{W}^{(2)} \rangle - \mathbf{A}^{(1)} \langle \mathbf{W}^{(1)} \rangle - \mathbf{A}^{(2)} \langle \mathbf{W}^{(0)} \rangle - \mathbf{A}^{(1)} \{ \mathbf{W}^{(1)} \} - \mathbf{A}^{(2)} \{ \mathbf{W}^{(0)} \}. \tag{79}$$

After some manipulations to evaluate the r.h.s. of (79) we obtain:

$$\begin{aligned}
\{ \mathbf{W}^{(2)} \} &= \sum_{i=1}^3 \mathbf{S}_i^{\mathbf{V}} \langle V_i^{(2)} \rangle + \sum_{i=1}^3 \mathbf{S}_i^{\mathbf{H}} \langle H_i^{(2)} \rangle + \sum_{i,j=1}^3 \Gamma_{ij}^{\mathbf{V}} \nabla_j \langle V_i^{(1)} \rangle + \sum_{i,j=1}^3 \Gamma_{ij}^{\mathbf{H}} \nabla_j \langle H_i^{(1)} \rangle + \sum_{j=1}^3 \Gamma_j^{\mathbf{T}} \nabla_j \langle T^{(1)} \rangle \\
&+ \sum_{i,j,k=1}^3 \Lambda_{ijk}^{\mathbf{V}} \nabla_k \nabla_j \langle V_i^{(0)} \rangle + \sum_{i,j,k=1}^3 \Lambda_{ijk}^{\mathbf{H}} \nabla_k \nabla_j \langle H_i^{(0)} \rangle + \sum_{j,k=1}^3 \Lambda_{jk}^{\mathbf{T}} \nabla_k \nabla_j \langle T^{(0)} \rangle \\
&+ \sum_{i=1}^3 [(\lambda_2 - \Xi \bullet \nabla^2) \Theta_i^{\mathbf{V}}] \langle V_i^{(0)} \rangle + \sum_{i=1}^3 [(\lambda_2 - \Xi \bullet \nabla^2) \Theta_i^{\mathbf{H}}] \langle H_i^{(0)} \rangle + (\lambda_2 - k \nabla^2) \Theta^{\mathbf{T}} \nabla^2 \langle T^{(0)} \rangle,
\end{aligned}$$

where

$$\begin{aligned}
A^{(0)} \Lambda_{ijk}^{\mathbf{V}} &= \begin{bmatrix} 0 \\ \langle \tilde{\mathbf{V}} (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{V}} + \tilde{V}_k (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{V}} - \tilde{\mathbf{H}} (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{H}} - \tilde{H}_k (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{H}} \rangle \\ 0 \\ 0 \end{bmatrix} \\
&+ \sum_{l=1}^3 \begin{bmatrix} -(\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{V}} \\ -\mathbf{e}_k (\Gamma_{ij}^{\mathbf{V}})^p + 2\nu \partial_k (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{V}} - (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{V}} \tilde{V}_k + (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{H}} \tilde{H}_k \\ -\tilde{\mathbf{H}} (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{V}} + (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{V}} \tilde{H}_k + 2\eta \partial_k (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{H}} + \tilde{\mathbf{V}} (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{H}} - (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{k}}^{\mathbf{H}} \tilde{V}_k \\ \sigma (\partial_k \tilde{H}_l - \partial_l \tilde{H}_k) (\Gamma_{ij}^{\mathbf{V}})_{\mathbf{l}}^{\mathbf{H}} + 2k \partial_k (\Gamma_{ij}^{\mathbf{V}})^{\mathbf{T}} - (\Gamma_{ij}^{\mathbf{V}})^{\mathbf{T}} \tilde{V}_k \end{bmatrix},
\end{aligned}$$

$$A^{(0)}\mathbf{\Lambda}_{ijk}^{\mathbf{H}} = \begin{bmatrix} 0 \\ \langle \tilde{\mathbf{V}}(\Gamma_{ij}^{\mathbf{H}})^{\mathbf{V}} + \tilde{V}_k(\Gamma_{ij}^{\mathbf{H}})^{\mathbf{V}} - \tilde{\mathbf{H}}(\Gamma_{ij}^{\mathbf{H}})^{\mathbf{H}} - \tilde{H}_k(\Gamma_{ij}^{\mathbf{H}})^{\mathbf{H}} \rangle \\ 0 \\ 0 \end{bmatrix} + \sum_{l=1}^3 \begin{bmatrix} -(\Gamma_{ij}^{\mathbf{H}})_k^{\mathbf{V}} \\ -\mathbf{e}_k(\Gamma_{ij}^{\mathbf{H}})^p + 2\nu\partial_k(\Gamma_{ij}^{\mathbf{H}})^{\mathbf{V}} - (\Gamma_{ij}^{\mathbf{H}})^{\mathbf{V}}\tilde{V}_k + (\Gamma_{ij}^{\mathbf{H}})^{\mathbf{H}}\tilde{H}_k \\ -\tilde{\mathbf{H}}(\Gamma_{ij}^{\mathbf{H}})_k^{\mathbf{V}} + (\Gamma_{ij}^{\mathbf{H}})^{\mathbf{V}}\tilde{H}_k + 2\eta\partial_k(\Gamma_{ij}^{\mathbf{H}})^{\mathbf{H}} + \tilde{\mathbf{V}}(\Gamma_{ij}^{\mathbf{H}})_k^{\mathbf{H}} - (\Gamma_{ij}^{\mathbf{H}})^{\mathbf{H}}\tilde{V}_k \\ \sigma(\partial_k\tilde{H}_l - \partial_l\tilde{H}_k)(\Gamma_{ij}^{\mathbf{H}})_l^{\mathbf{H}} + 2k\partial_k(\Gamma_{ij}^{\mathbf{H}})^T - (\Gamma_{ij}^{\mathbf{H}})^T\tilde{V}_k \end{bmatrix},$$

$$A^{(0)}\mathbf{\Lambda}_{jk}^T = \begin{bmatrix} 0 \\ \langle \tilde{\mathbf{V}}(\Gamma_j^T)_k^{\mathbf{V}} + \tilde{V}_k(\Gamma_j^T)^{\mathbf{V}} - \tilde{\mathbf{H}}(\Gamma_j^T)_k^{\mathbf{H}} - \tilde{H}_k(\Gamma_j^T)^{\mathbf{H}} \rangle \\ 0 \\ 0 \end{bmatrix} + \sum_{l=1}^3 \begin{bmatrix} -(\Gamma_j^T)_k^{\mathbf{V}} \\ -\mathbf{e}_k(\Gamma_j^T)^p + 2\nu\partial_k(\Gamma_j^T)^{\mathbf{V}} - (\Gamma_j^T)^{\mathbf{V}}\tilde{V}_k + (\Gamma_j^T)^{\mathbf{H}}\tilde{H}_k \\ -\tilde{\mathbf{H}}(\Gamma_j^T)_k^{\mathbf{V}} + (\Gamma_j^T)^{\mathbf{V}}\tilde{H}_k + 2\eta\partial_k(\Gamma_j^T)^{\mathbf{H}} + \tilde{\mathbf{V}}(\Gamma_j^T)_k^{\mathbf{H}} - (\Gamma_j^T)^{\mathbf{H}}\tilde{V}_k \\ \sigma(\partial_k\tilde{H}_l - \partial_l\tilde{H}_k)(\Gamma_j^T)_l^{\mathbf{H}} + 2k\partial_k(\Gamma_j^T)^T - (\Gamma_j^T)^T\tilde{V}_k \end{bmatrix},$$

$$A^{(0)}\mathbf{\Theta}_i^{\mathbf{V}} = \begin{bmatrix} 0 \\ -(\mathbf{S}_i^{\mathbf{V}})^{\mathbf{V}} - \mathbf{e}_i \\ -(\mathbf{S}_i^{\mathbf{V}})^{\mathbf{H}} \\ -(\mathbf{S}_i^{\mathbf{V}})^T \end{bmatrix},$$

$$A^{(0)}\mathbf{\Theta}_i^{\mathbf{H}} = \begin{bmatrix} 0 \\ -(\mathbf{S}_i^{\mathbf{H}})^{\mathbf{V}} \\ -(\mathbf{S}_i^{\mathbf{H}})^{\mathbf{H}} - \mathbf{e}_i \\ -(\mathbf{S}_i^{\mathbf{H}})^T \end{bmatrix}$$

and

$$A^{(0)}\mathbf{\Theta}_i^{\mathbf{V}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

are the so called auxiliary problems of order 2. The operator  $\Xi$  is defined as

$$\Xi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & k \end{bmatrix}.$$



## 8.5 Order 3

### 8.5.1 Solvability

Averaging at this order, we get:

$$\left\{ \begin{array}{l} \nabla \cdot \langle \mathbf{V}^{(2)} \rangle = 0 \\ \langle \mathbf{V}^{(3)} \cdot \partial \tilde{\mathbf{V}} \rangle + \Omega \times \langle \mathbf{V}^{(3)} \rangle - \langle \mathbf{H}^{(3)} \cdot \partial \tilde{\mathbf{H}} \rangle - \alpha \mathbf{G} \langle T^{(3)} \rangle + \nabla \langle p^{(2)} \rangle \\ + \langle \mathbf{V}^{(2)} \cdot \nabla \tilde{\mathbf{V}} \rangle - \langle \mathbf{H}^{(1)} \cdot \nabla \tilde{\mathbf{H}} \rangle - \langle \tilde{\mathbf{H}} \cdot \nabla \mathbf{H}^{(1)} \rangle + \lambda_2 \langle \mathbf{V}^{(1)} \rangle - \nu \nabla^2 \langle \mathbf{V}^{(1)} \rangle + \lambda_3 \langle \mathbf{V}^{(0)} \rangle = 0 \\ \langle \tilde{\mathbf{H}} \nabla \cdot \mathbf{V}^{(2)} \rangle - \langle \tilde{\mathbf{H}} \cdot \nabla \mathbf{V}^{(2)} \rangle - \langle \tilde{\mathbf{V}} \nabla \cdot \mathbf{H}^{(2)} \rangle \\ + \langle \tilde{\mathbf{V}} \cdot \nabla \mathbf{H}^{(2)} \rangle + \lambda_2 \langle \mathbf{H}^{(1)} \rangle - \eta \nabla^2 \langle \mathbf{H}^{(1)} \rangle + \lambda_3 \langle \mathbf{H}^{(0)} \rangle = 0 \\ \langle \mathbf{V}^{(3)} \cdot \partial \tilde{\mathbf{V}} \rangle - \sigma \langle (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \mathbf{H}^{(3)}) \rangle \\ - \sigma \langle (\partial \times \tilde{\mathbf{H}}) \cdot (\nabla \times \mathbf{H}^{(2)}) \rangle + \lambda_2 \langle T^{(1)} \rangle - k \nabla^2 \langle T^{(1)} \rangle + \lambda_3 \langle T^{(0)} \rangle = 0 \end{array} \right.$$

and

$$\nabla \cdot \langle \mathbf{H}^{(2)} \rangle = 0.$$

### 8.5.2 Closed equations for $\langle \mathbf{V}^{(1)} \rangle$ , $\langle \mathbf{H}^{(1)} \rangle$ and $\langle T^{(1)} \rangle$

We begin by enumerating the solvability conditions not trivially satisfied and not yet used.

- at order 1: equation (65) above.
- at order 2: equations (69), (73), (70) and (72) above.
- at order 3:

$$\nabla \cdot \langle \mathbf{V}^{(2)} \rangle = 0 \tag{80}$$

$$\begin{aligned} & \langle \mathbf{V}^{(3)} \cdot \partial \tilde{\mathbf{V}} \rangle + \Omega \times \langle \mathbf{V}^{(3)} \rangle - \langle \mathbf{H}^{(3)} \cdot \partial \tilde{\mathbf{H}} \rangle - \alpha \mathbf{G} \langle T^{(3)} \rangle + \nabla \langle p^{(2)} \rangle \\ & + \langle \mathbf{V}^{(2)} \cdot \nabla \tilde{\mathbf{V}} \rangle - \langle \mathbf{H}^{(1)} \cdot \nabla \tilde{\mathbf{H}} \rangle - \langle \tilde{\mathbf{H}} \cdot \nabla \mathbf{H}^{(1)} \rangle + \lambda_2 \langle \mathbf{V}^{(1)} \rangle - \nu \nabla^2 \langle \mathbf{V}^{(1)} \rangle + \lambda_3 \langle \mathbf{V}^{(0)} \rangle = 0 \end{aligned} \tag{81}$$

$$\begin{aligned} & \langle \tilde{\mathbf{H}} \nabla \cdot \mathbf{V}^{(2)} \rangle - \langle \tilde{\mathbf{H}} \cdot \nabla \mathbf{V}^{(2)} \rangle - \langle \tilde{\mathbf{V}} \nabla \cdot \mathbf{H}^{(2)} \rangle \\ & + \langle \tilde{\mathbf{V}} \cdot \nabla \mathbf{H}^{(2)} \rangle + \lambda_2 \langle \mathbf{H}^{(1)} \rangle - \eta \nabla^2 \langle \mathbf{H}^{(1)} \rangle + \lambda_3 \langle \mathbf{H}^{(0)} \rangle = 0 \end{aligned} \tag{82}$$

$$\begin{aligned} & \langle \mathbf{V}^{(3)} \cdot \partial \tilde{\mathbf{V}} \rangle - \sigma \langle (\partial \times \tilde{\mathbf{H}}) \cdot (\partial \times \mathbf{H}^{(3)}) \rangle \\ & - \sigma \langle (\partial \times \tilde{\mathbf{H}}) \cdot (\nabla \times \mathbf{H}^{(2)}) \rangle + \lambda_2 \langle T^{(1)} \rangle - k \nabla^2 \langle T^{(1)} \rangle + \lambda_3 \langle T^{(0)} \rangle = 0 \end{aligned} \tag{83}$$

$$\nabla \cdot \langle \mathbf{H}^{(2)} \rangle = 0 \tag{84}$$

Equations (69) and (73) will be used to restrict the subspace of solutions of  $\langle \mathbf{V}^{(1)} \rangle$ ,  $\langle \mathbf{H}^{(1)} \rangle$  and  $\langle T^{(1)} \rangle$ , in the next order. Equations (81) and (83) cannot be used yet, since they use components of  $\mathbf{W}^{(3)}$ .

Taking the curl of equation (65) we get the following vector equation:

$$\Omega \cdot \nabla \langle \mathbf{V}^{(1)} \rangle - \alpha \mathbf{G} \times \nabla \langle T^{(1)} \rangle = 0. \quad (85)$$

In equation (72) we must express both  $\mathbf{H}^{(2)}$  and  $\mathbf{H}^{(1)}$  as a function of the components of  $\langle \mathbf{W}^{(1)} \rangle$  and  $\langle \mathbf{W}^{(0)} \rangle$ . We obtain after some algebraic manipulations:

$$\begin{aligned} & \lambda_2 \langle T^{(0)} \rangle - k \nabla^2 \langle T^{(0)} \rangle \\ & + \sum_{ijk} \langle \tilde{T} (\Gamma_{ij}^{\mathbf{V}})^{\mathbf{V}} + \tilde{V}_k (\Gamma_{ij}^{\mathbf{V}})^T \rangle \nabla_k \nabla_j \langle V_i^{(0)} \rangle \\ & + \sum_{ijk} \langle \tilde{T} (\Gamma_{ij}^{\mathbf{H}})^{\mathbf{V}} + \tilde{V}_k (\Gamma_{ij}^{\mathbf{H}})^T \rangle \nabla_k \nabla_j \langle H_i^{(0)} \rangle \\ & + \sum_{jk} \langle \tilde{T} (\Gamma_j^T)^{\mathbf{V}} + \tilde{V}_k (\Gamma_j^T)^T \rangle \nabla_k \nabla_j \langle T^{(0)} \rangle \\ & + \sum_{\alpha\beta} \langle \partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha \rangle \nabla_\alpha \langle H_\beta^{(1)} \rangle \\ & + \sum_{\alpha\beta i} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) (S_i^{\mathbf{V}})_{\beta}^{\mathbf{H}} \rangle \nabla_\alpha \langle V_i^{(0)} \rangle \\ & + \sum_{\alpha\beta i} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) (S_i^{\mathbf{H}})_{\beta}^{\mathbf{H}} \rangle \nabla_\alpha \langle H_i^{(0)} \rangle \\ & + \sum_{\alpha\beta ij} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) (\Gamma_{ij}^{\mathbf{V}})_{\beta}^{\mathbf{H}} \rangle \nabla_\alpha \nabla_j \langle V_i^{(0)} \rangle \\ & + \sum_{\alpha\beta ij} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) (\Gamma_{ij}^{\mathbf{H}})_{\beta}^{\mathbf{H}} \rangle \nabla_\alpha \nabla_j \langle H_i^{(0)} \rangle \\ & + \sum_{\alpha\beta i} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) (\Gamma_i^T)_{\beta}^{\mathbf{H}} \rangle \nabla_\alpha \nabla_i \langle T^{(0)} \rangle \\ & + \sum_{\alpha\beta ij} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Gamma_{ij}^{\mathbf{V}})_{\beta}^{\mathbf{H}} \rangle \nabla_j \langle V_i^{(1)} \rangle \\ & + \sum_{\alpha\beta ij} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Gamma_{ij}^{\mathbf{H}})_{\beta}^{\mathbf{H}} \rangle \nabla_j \langle H_i^{(1)} \rangle \\ & + \sum_{\alpha\beta i} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Gamma_i^T)_{\beta}^{\mathbf{H}} \rangle \nabla_i \langle T^{(1)} \rangle \\ & + \sum_{\alpha\beta ijk} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Lambda_{ijk}^{\mathbf{V}})_{\beta}^{\mathbf{H}} \rangle \nabla_k \nabla_j \langle V_i^{(0)} \rangle \\ & + \sum_{\alpha\beta ijk} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Lambda_{ijk}^{\mathbf{H}})_{\beta}^{\mathbf{H}} \rangle \nabla_k \nabla_j \langle H_i^{(0)} \rangle \\ & + \sum_{\alpha\beta ik} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Lambda_{ijk}^T)_{\beta}^{\mathbf{H}} \rangle \nabla_k \nabla_i \langle T^{(0)} \rangle \\ & + \lambda_2 \sum_{\alpha\beta i} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Theta_i^{\mathbf{V}})_{\beta}^{\mathbf{H}} \rangle \langle V_i^{(0)} \rangle \\ & - \sum_{\alpha\beta i} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Xi \Theta_i^{\mathbf{V}})_{\beta}^{\mathbf{H}} \rangle \nabla^2 \langle V_i^{(0)} \rangle \\ & + \lambda_2 \sum_{\alpha\beta i} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Theta_i^{\mathbf{H}})_{\beta}^{\mathbf{H}} \rangle \langle H_i^{(0)} \rangle \\ & - \sum_{\alpha\beta i} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Xi \Theta_i^{\mathbf{H}})_{\beta}^{\mathbf{H}} \rangle \nabla^2 \langle H_i^{(0)} \rangle \end{aligned}$$

$$+ \sum_{\alpha\beta} \langle (\partial_\alpha \tilde{H}_\beta - \partial_\beta \tilde{H}_\alpha) \partial_\alpha (\Theta^T)^\mathbf{H} \rangle (\lambda_2 - k\nabla^2) \langle T^{(0)} \rangle = 0. \quad (86)$$

Finally,

$$\begin{aligned} & \sum_{\alpha ij} \langle \tilde{\mathbf{H}} (\Gamma_{ij}^\mathbf{V})_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Gamma_{ij}^\mathbf{V})^\mathbf{V} - \tilde{\mathbf{V}} (\Gamma_{ij}^\mathbf{V})_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Gamma_{ij}^\mathbf{V})^\mathbf{H} \rangle \nabla_\alpha \nabla_j \langle V_i^{(1)} \rangle \\ & + \sum_{\alpha ij} \langle \tilde{\mathbf{H}} (\Gamma_{ij}^\mathbf{H})_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Gamma_{ij}^\mathbf{H})^\mathbf{V} - \tilde{\mathbf{V}} (\Gamma_{ij}^\mathbf{H})_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Gamma_{ij}^\mathbf{H})^\mathbf{H} \rangle \nabla_\alpha \nabla_j \langle H_i^{(1)} \rangle \end{aligned} \quad (87)$$

$$+ \sum_{\alpha j} \langle \tilde{\mathbf{H}} (\Gamma_j^T)_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Gamma_j^T)^\mathbf{V} - \tilde{\mathbf{V}} (\Gamma_j^T)_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Gamma_j^T)^\mathbf{H} \rangle \nabla_\alpha \nabla_j \langle T^{(1)} \rangle \quad (88)$$

$$\begin{aligned} & + \sum_{\alpha ijk} \langle \tilde{\mathbf{H}} (\Lambda_{ijk}^\mathbf{V})_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Lambda_{ijk}^\mathbf{V})^\mathbf{V} - \tilde{\mathbf{V}} (\Lambda_{ijk}^\mathbf{V})_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Lambda_{ijk}^\mathbf{V})^\mathbf{H} \rangle \nabla_\alpha \nabla_k \nabla_j \langle V_i^{(0)} \rangle \\ & + \sum_{\alpha ijk} \langle \tilde{\mathbf{H}} (\Lambda_{ijk}^\mathbf{H})_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Lambda_{ijk}^\mathbf{H})^\mathbf{V} - \tilde{\mathbf{V}} (\Lambda_{ijk}^\mathbf{H})_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Lambda_{ijk}^\mathbf{H})^\mathbf{H} \rangle \nabla_\alpha \nabla_k \nabla_j \langle H_i^{(0)} \rangle \\ & + \sum_{\alpha jk} \langle \tilde{\mathbf{H}} (\Lambda_{jk}^T)_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Lambda_{jk}^T)^\mathbf{V} - \tilde{\mathbf{V}} (\Lambda_{jk}^T)_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Lambda_{jk}^T)^\mathbf{H} \rangle \nabla_\alpha \nabla_k \nabla_j \langle T^{(0)} \rangle \end{aligned} \quad (89)$$

$$\begin{aligned} & + \lambda_2 \sum_{\alpha i} \langle \tilde{\mathbf{H}} (\Theta_i^\mathbf{V})_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Theta_i^\mathbf{V})^\mathbf{V} - \tilde{\mathbf{V}} (\Theta_i^\mathbf{V})_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Theta_i^\mathbf{V})^\mathbf{H} \rangle \nabla_\alpha \langle V_i^{(0)} \rangle \\ & + \lambda_2 \sum_{\alpha i} \langle \tilde{\mathbf{H}} (\Theta_i^\mathbf{H})_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Theta_i^\mathbf{H})^\mathbf{V} - \tilde{\mathbf{V}} (\Theta_i^\mathbf{H})_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Theta_i^\mathbf{H})^\mathbf{H} \rangle \nabla_\alpha \langle H_i^{(0)} \rangle \\ & - \sum_{\alpha i} \langle \tilde{\mathbf{H}} (\Xi \Theta_i^\mathbf{V})_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Xi \Theta_i^\mathbf{V})^\mathbf{V} - \tilde{\mathbf{V}} (\Xi \Theta_i^\mathbf{V})_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Xi \Theta_i^\mathbf{V})^\mathbf{H} \rangle \nabla_\alpha \nabla^2 \langle V_i^{(0)} \rangle \end{aligned} \quad (90)$$

$$- \sum_{\alpha i} \langle \tilde{\mathbf{H}} (\Xi \Theta_i^\mathbf{H})_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Xi \Theta_i^\mathbf{H})^\mathbf{V} - \tilde{\mathbf{V}} (\Xi \Theta_i^\mathbf{H})_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Xi \Theta_i^\mathbf{H})^\mathbf{H} \rangle \nabla_\alpha \nabla^2 \langle H_i^{(0)} \rangle \quad (91)$$

$$\begin{aligned} & + \sum_{\alpha} \langle \tilde{\mathbf{H}} (\Theta^T)_\alpha^\mathbf{V} - \tilde{H}_\alpha (\Theta^T)^\mathbf{V} - \tilde{\mathbf{V}} (\Theta^T)_\alpha^\mathbf{H} + \tilde{H}_\alpha (\Theta^T)^\mathbf{H} \rangle \nabla_\alpha (\lambda_2 - k\nabla^2) \langle T^{(0)} \rangle \\ & + \lambda_2 \langle \mathbf{H}^{(1)} \rangle - \eta \nabla^2 \langle \mathbf{H}^{(1)} \rangle + \lambda_3 \langle \mathbf{H}^{(0)} \rangle = 0. \end{aligned} \quad (92)$$

Equations (85), (86) and (92), supplemented by equations (69) and (73) are a set of closed equations for  $\langle \mathbf{V}^{(1)} \rangle$ ,  $\langle \mathbf{H}^{(1)} \rangle$  and  $\langle T^{(1)} \rangle$ .

### 8.5.3 Closed equation for $\langle p^{(0)} \rangle$

Taking the divergence of equation (65), we obtain the following Poisson equation for  $\langle p^{(0)} \rangle$ :

$$\nabla^2 \langle p^{(0)} \rangle = \mathbf{\Omega} \cdot (\nabla \times \langle \mathbf{V}^{(1)} \rangle) + \alpha \mathbf{G} \cdot \nabla \langle T^{(1)} \rangle. \quad (93)$$

Note that at this point the r.h.s. of the above equation is completely known.

## 8.6 Particular case $\sigma = 0$

### 8.6.1 Closed equations for $\langle \mathbf{V}^{(0)} \rangle$ , $\langle \mathbf{H}^{(0)} \rangle$ and $\langle T^{(0)} \rangle$

In this case we cannot use (67), since all terms involved vanish due to  $\sigma = 0$ . Fortunately, (72) can be simplified due to the same reason. So we obtain

$$\lambda_2 \langle T^{(0)} \rangle - k \nabla^2 \langle T^{(0)} \rangle + \langle \tilde{T} \nabla \cdot \mathbf{V}^{(1)} \rangle = 0.$$

As above, we may express  $\mathbf{V}^{(1)}$  in terms of the auxiliary problems and averages. Using symmetries to simplify some terms involving averages, we obtain

$$\begin{aligned}
& \lambda_2 \langle T^{(0)} \rangle - k \nabla^2 \langle T^{(0)} \rangle \\
& + \sum_{ijk} \langle \tilde{T} (\Gamma_{ij}^{\mathbf{V}})_k^{\mathbf{V}} + \tilde{V}_k (\Gamma_{ij}^{\mathbf{V}})^T \rangle \nabla_k \nabla_j \langle V_i^{(0)} \rangle \\
& + \sum_{ijk} \langle \tilde{T} (\Gamma_{ij}^{\mathbf{H}})_k^{\mathbf{V}} + \tilde{V}_k (\Gamma_{ij}^{\mathbf{H}})^T \rangle \nabla_k \nabla_j \langle H_i^{(0)} \rangle \\
& + \sum_{jk} \langle \tilde{T} (\Gamma_j^T)_k^{\mathbf{V}} + \tilde{V}_k (\Gamma_j^T)^T \rangle \nabla_k \nabla_j \langle T^{(0)} \rangle = 0.
\end{aligned} \tag{94}$$

So, in this case, (74), (94) and (76) are the closed set of equations for  $\langle \mathbf{V}^{(0)} \rangle$ ,  $\langle \mathbf{H}^{(0)} \rangle$  and  $\langle T^{(0)} \rangle$ .

### 8.6.2 Closed equations for $\langle \mathbf{V}^{(1)} \rangle$ , $\langle \mathbf{H}^{(1)} \rangle$ and $\langle T^{(1)} \rangle$

Here (72) is of no use any more since its expanded form includes only averages of order zero. (83) reduces to

$$\lambda_2 \langle T^{(1)} \rangle - k \nabla^2 \langle T^{(1)} \rangle + \lambda_3 \langle T^{(0)} \rangle + \langle \mathbf{V}^{(3)} \cdot \partial \tilde{\mathbf{V}} \rangle = 0,$$

which can be further simplified to

$$\lambda_2 \langle T^{(1)} \rangle - k \nabla^2 \langle T^{(1)} \rangle + \lambda_3 \langle T^{(0)} \rangle + \langle \tilde{\mathbf{V}} \nabla \cdot \mathbf{V}^{(2)} \rangle = 0.$$

Using the expression for the fluctuation at second order, we may write

$$\begin{aligned}
& \sum_{ijk} \langle \tilde{\mathbf{V}} (\Gamma_{ij}^{\mathbf{V}})_k^{\mathbf{V}} \rangle \nabla_k \nabla_j \langle V_i^{(1)} \rangle \\
& + \sum_{ijk} \langle \tilde{\mathbf{V}} (\Gamma_{ij}^{\mathbf{H}})_k^{\mathbf{V}} \rangle \nabla_k \nabla_j \langle H_i^{(1)} \rangle \\
& + \sum_{jk} \langle \tilde{\mathbf{V}} (\Gamma_j^T)_k^{\mathbf{V}} \rangle \nabla_k \nabla_j \langle T^{(1)} \rangle \\
& + \sum_{ijkl} \langle \tilde{\mathbf{V}} (\Lambda_{ijk}^{\mathbf{V}})_l^{\mathbf{V}} \rangle \nabla_k \nabla_j \langle V_i^{(0)} \rangle \\
& + \sum_{ijkl} \langle \tilde{\mathbf{V}} (\Lambda_{ijk}^{\mathbf{H}})_{kl}^{\mathbf{V}} \rangle \nabla_k \nabla_j \langle H_i^{(0)} \rangle \\
& + \sum_{jkl} \langle \tilde{\mathbf{V}} (\Lambda_{jk}^T)_l^{\mathbf{V}} \rangle \nabla_l \nabla_k \nabla_j \langle T^{(0)} \rangle \\
& + \lambda_2 \sum_{ij} \langle \tilde{\mathbf{V}} (\Theta_i^{\mathbf{V}})_j^{\mathbf{V}} \rangle \nabla_j \langle V_i^{(0)} \rangle \\
& + \lambda_2 \sum_{ij} \langle \tilde{\mathbf{V}} (\Theta_i^{\mathbf{H}})_j^{\mathbf{V}} \rangle \nabla_j \langle H_i^{(0)} \rangle \\
& + \lambda_2 \sum_j \langle \tilde{\mathbf{V}} (\Theta^T)_j^{\mathbf{V}} \rangle \nabla_j \langle T^{(0)} \rangle \\
& - \sum_{ij} \langle \tilde{\mathbf{V}} (\Xi \Theta_i^{\mathbf{V}})_j^{\mathbf{V}} \rangle \nabla_j \nabla^2 \langle V_i^{(0)} \rangle \\
& - \sum_{ij} \langle \tilde{\mathbf{V}} (\Xi \Theta_i^{\mathbf{H}})_j^{\mathbf{V}} \rangle \nabla_j \nabla^2 \langle H_i^{(0)} \rangle \\
& - \sum_j \langle \tilde{\mathbf{V}} (\Xi \Theta^T)_j^{\mathbf{V}} \rangle \nabla_j \nabla^2 \langle T^{(0)} \rangle = 0.
\end{aligned}$$

## 9 Conclusion

We have derived a closed set of equations for the dominant slow scale mode. However these equations require the knowledge of the solution of the auxiliary problems. The equations obtained by averaging are valid, but they do not guarantee the solvability of the auxiliary problems. The solvability conditions amount as usual to the condition that the right hand side of the fluctuation equations at each order is orthogonal to the kernel of the adjoint of the operator  $\mathbf{A}^{(0)}$ . In the kinematic dynamo problem, this is equivalent to averaging the equations, since the kernel mentioned above comprises only constants. The symmetry analysis yields two possibilities, as mentioned before (7).

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