RANDOM PERTURBATIONS OF DIFFEOMORPHISMS WITH DOMINATED SPLITTING

JOSÉ F. ALVES, VÍTOR ARAÚJO, AND CARLOS H. VÁSQUEZ

Abstract. We prove that the statistical properties of random perturbations of a nonuniformly hyperbolic diffeomorphism are described by a finite number of stationary measures. We also give necessary and sufficient conditions for the stochastic stability of such dynamical systems. We show that a certain C^2-open class of nonuniformly hyperbolic diffeomorphisms introduced in [Alves, J; Bonatti, C. and Viana, V., *SRB measures for partially hyperbolic systems with mostly expanding central direction*, Invent. Math., 140 (2000), 351-398] are stochastically stable. Our setting encompasses that of partially hyperbolic diffeomorphisms as well as uniformly hyperbolic diffeomorphisms. Moreover, the techniques used enable us to obtain SRB measures in this setting through zero-noise limit measures.

1. Introduction

Let $f : M \to M$ be a smooth map defined on a compact Riemannian manifold M. We write $\|\cdot\|$ for the induced norm on TM and fix some normalized Riemannian volume form m on M that we call *Lebesgue measure*. We say that an f-invariant Borel probability measure μ on M is an *SRB measure* if for a positive Lebesgue measure set of points $x \in M$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f^n x\big) = \int \varphi \, d\mu, \tag{1}$$

for every continuous map $\varphi : M \to \mathbb{R}$. We define the *basin* $B(\mu)$ of μ as the set of those points x in M for which (1) holds for all continuous φ. These measures, introduced by Sinai, Ruelle and Bowen in a series of results [20, 9, 8, 19], describe the statistical behavior of *many* orbits (positive Lebesgue measure subset) when the number of iterates goes to infinity, and their existence, construction and properties have become a major focus of attention in the Ergodic Theory of smooth transformations.

Date: April 7, 2004.

2000 *Mathematics Subject Classification.* Primary: 37D25. Secondary: 37C40, 37H15.

Key words and phrases. dominated splitting, non-uniform hyperbolicity, SRB measures, random perturbations, stochastic stability.

Trabalho parcialmente financiado por CMUP. O CMUP é financiado por FCT, no âmbito de POCTI-POSI do Quadro Comunitário de Apoio III (2000-2006), com fundos comunitários (FEDER) e nacionais. J.F.A. and V.A. were also partially supported by grant FCT/SAPIENS/36581/99. Part of this work was done while V.A. enjoyed a leave from CMUP at IMPA and C.V enjoyed a leave at CMUP.

Here we study the stochastic stability of SRB measures for a class of diffeomorphisms with a dominated splitting of the tangent bundle and nonuniform expansion and contraction along the directions of the splitting, obtaining a necessary and sufficient condition for stochastic stability, satisfied by open classes of C^2 diffeomorphisms. Moreover, our setting encompasses that of uniformly hyperbolic diffeomorphisms and of partially hyperbolic ones as well. Hence our methods are sufficiently general to provide stochastic stability also for maps in these classes. Our results are consistent with the program in order to obtain a global view of the dynamical systems given by Palis a few years ago [14].

In the following subsections we explain our setting, state the main results and derive some easy generalizations as corollaries.

1.1. **Nonuniformly hyperbolic diffeomorphisms.** Let $f : M \to M$ be a diffeomorphism for which there is a strictly forward f-invariant open set $U \subset M$, that is $f(\overline{U}) \subset U$, and there is a continuous Df-invariant splitting $E^{cs} \oplus E^{cu}$ of $T_U M$, the tangent bundle over U. The bundles E^{cs} and E^{cu} will be called *center-stable* and *center-unstable* and have dimensions u and s, respectively, with $u, s \geq 1$ and $s + u = \dim(M)$. We will assume several conditions on the splitting of $T_U M$:

(a) *Dominated decomposition:* there exists a constant $0 < \lambda < 1$ for which
$$\|Df|E^{cs}_x\| \cdot \|Df^{-1}|E^{cu}_{fx}\| \leq \lambda \quad \text{for all } x \in U.$$

(b) *Nonuniform expansion along the central-unstable direction:* there is $c_u > 0$ such that for Lebesgue almost all $x \in U$
$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j x}\| \leq -c_u.$$

(c) *Nonuniform contraction along the central-stable direction:* there is $c_s > 0$ such that for Lebesgue almost all $x \in U$
$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df|E^{cs}_{f^j x}\| \leq -c_s.$$

We will refer to a diffeomorphism f satisfying (a)-(c) above simply as a *nonuniformly hyperbolic diffeomorphism*.

Theorem 6.3 of [2] shows that a nonuniformly hyperbolic diffeomorphism f has some ergodic *Gibbs cu-state* μ supported in $\Lambda = \bigcap_{j=0}^{\infty} f^j(U)$, that is, μ is an invariant probability measure whose $\dim E^{cu}$ larger Lyapunov exponents are positive and whose conditional measures along the corresponding local unstable manifolds are almost everywhere absolutely continuous with respect to Lebesgue measure on such manifolds. Moreover, if the derivative of f has uniform behavior over at least one the subbundles E^{cs} or E^{cu}, then μ is an SRB measure. This is a well known consequence of the absolute continuity of the conditional measures of μ and absolute continuity of the stable lamination [15]: the union of the stable manifolds through the points whose time averages are given by μ is a positive Lebesgue measure subset of the basin of μ.

As shown in [2] there are some cases where non-uniform behavior on the directions of E^{cs} and E^{cu} are enough for ensuring the existence of SRB measures for f. A sufficient condition is that there exist a positive Lebesgue measure set of points $x \in U$ with many (positive density at infinity) *simultaneous α-hyperbolic times* n with respect to the two subbundles: there is some $0 < \alpha < 1$ such that

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \le \alpha^k \quad \text{and} \quad \prod_{j=n-k}^{n-1} \|Df|E^{cs}_{f^j(x)}\| \le \alpha^k \tag{2}$$

for every $0 \le k \le n$. Under this assumption [2, Proposition 6.4] gives that f has a finite number of ergodic Gibbs cu-states which are SRB measures, and whose basins cover a full Lebesgue measure subset of U.

Finally, we refer to a simple condition presented in [2] implying the existence of many simultaneous hyperbolic times for a diffeomorphism f with a dominated splitting for which Df has not necessarily uniform behavior over the subbundles E^{cs} and E^{cs}: let

$$A^u = \sup_{f(U)} -\log \|Df^{-1}|E^{cu}\| \quad \text{and} \quad A^s = \sup_{U} -\log \|Df|E^{cs}\|.$$

If the constants c_u and c_s verify

$$\frac{c_u}{A^u} + \frac{c_s}{A^s} > 1, \tag{3}$$

then [2, Proposition 6.5] gives that there exists $\alpha < 1$ such that the simultaneous α-hyperbolic times of Lebesgue almost every $x \in U$ have positive density at infinity.

1.2. **Statement of results.** In this work we are interested in studying random perturbations of a nonuniformly hyperbolic diffeomorphism $f\colon M \to M$. For that, we take a continuous map

$$\begin{array}{rccl} \Phi: & T & \longrightarrow & \mathrm{Diff}^2(M,M) \\ & t & \longmapsto & f_t \end{array}$$

from a metric space T into the space of C^2 diffeomorphisms from M to M, with $f = f_{t^*}$ for some fixed $t^* \in T$. Given $x \in M$, a *random orbit* of x will be a sequence $(f^n_{\underline{t}} x)_{n\ge 1}$ where $\underline{t}$ denotes an element $(t_1, t_2, t_3, \dots)$ in the product space $T^{\mathbb{N}}$ and

$$f^n_{\underline{t}} = f_{t_n} \circ \cdots \circ f_{t_1} \quad \text{for} \quad n \ge 1.$$

We also take a family $(\theta_\varepsilon)_{\varepsilon>0}$ of probability measures on T such that their supports $\mathrm{supp}\,(\theta_\varepsilon)$ form a nested family of connected compact sets and $\mathrm{supp}\,(\theta_\varepsilon) \to \{t^*\}$ when $\varepsilon \to 0$. We assume some quite general non-degeneracy conditions on the map Φ and the measures θ_ε (see the beginning of Section 3) and refer to $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ as a *random perturbation* of f.

In the setting of random perturbations of a map, a Borel probability measure μ^ε on M is said to be a *physical measure* if for a positive Lebesgue measure set of points $x \in M$ one has

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j_{\underline{t}} x) = \int \varphi \, d\mu^\varepsilon, \tag{4}$$

for all continuous $\varphi\colon M \to \mathbb{R}$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. We denote the set of points $x \in M$ for which (4) holds for every φ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$ by $B(\mu^\varepsilon)$ and call it the *basin of* μ^ε.

The map $f\colon M \to M$ is said to be *stochastically stable* if the weak* accumulation points (when $\varepsilon > 0$ goes to zero) of the physical measures of the random perturbation are convex linear combinations of the SRB measures of f (for this notion of stochastic stability, see [4]).

Theorem A. *Let $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ be a random perturbation of a C^2 nonuniformly hyperbolic diffeomorphism f with positive density of simultaneous hyperbolic times at infinity. There is $l \in \mathbb{N}$ such that for small enough $\varepsilon > 0$ there exist physical measures $\mu_1^\varepsilon, \dots, \mu_l^\varepsilon$ with pairwise disjoint supports with* supp $(\mu_i^\varepsilon) \subset B(\mu_i^\varepsilon)$ *for $i = 1, \dots, l$, satisfying*

1. *for each $x \in M$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$ there is $i = i(x, \underline{t}) \in \{1, \dots, l\}$ such that*
$$\mu_i^\varepsilon = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j x};$$
2. *for $1 \le i \le l$ and any probability measure η with support contained in $B(\mu_i^\varepsilon)$ we have*
$$\mu_i^\varepsilon = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} (f_{\underline{t}}^j)_* \eta \qquad \text{for } \theta_\varepsilon^{\mathbb{N}} \text{ almost every } \underline{t} \in T^{\mathbb{N}};$$
3. *the number of physical measures is at most the number of SRB measures.*

The first item above means that almost all random orbits of every point of M have time averages given by a physical measure out of $\mu_1^\varepsilon, \dots, \mu_l^\varepsilon$, a property akin to the one obtained in [2] for points in the topological basin of partially hyperbolic sets.

Now we present a notion that will play an important role in the study of the stochastic stability of a diffeomorphism with a dominated splitting. We say that f is *non-uniformly expanding along the center-unstable direction for random orbits* if there is $c > 0$ such that
$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1} | E^{cu}_{f_{\underline{t}}^j x}\| \le -c$$
for Lebesgue almost every $x \in U$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$, at least for small $\varepsilon > 0$.

The main result below gives a characterization of the stochastically stable nonuniformly hyperbolic diffeomorphisms. To the best of our knowledge this is the only result pointing in the direction of general stochastic stability for diffeomorphisms with a dominated splitting.

Theorem B. *Let f be a nonuniformly hyperbolic diffeomorphism with positive density at infinity of simultaneous hyperbolic times. Then f is stochastically stable if, and only if, f is non-uniformly expanding along the center-unstable direction for random orbits.*

We exhibit examples of nonuniformly hyperbolic diffeomorphisms which are stochastically stable. The ones we have in mind were presented in [2, Appendix A]. These form an open class $\mathcal{D}$ of diffeomorphisms defined on the d-dimensional torus $M = \mathbb{T}^d$, $d \ge 2$, having the whole M as a nonuniformly hyperbolic region, $TM = E^{cs} \oplus E^{cu}$, and exhibiting nonuniform expansion (resp. contraction) along the E^{cs} (resp. E^{cu}) direction with positive frequency of simultaneous hyperbolic times; see Section 7 for a complete description. Here we prove their stochastic stability by showing that they satisfy the condition in Theorem B above.

Theorem C. *Every $f \in \mathcal{D}$ is non-uniformly expanding for random orbits along the center-unstable direction.*

In particular, $\mathcal{D}$ is an open class of stochastically stable nonuniformly hyperbolic diffeomorphisms and of strongly nonuniformly hyperbolic maps. Further developments in [21] show that these diffeomorphisms have a unique SRB measure, and in [22] it was already proved that these maps are also statistically stable.

Moreover, the generality of the conditions needed to get both Theorems A and B enables us to obtain as corollaries of our method the stochastic stability of some families of diffeomorphisms with dominated splittings.

Obviously uniform expansion along the center-unstable bundle and nonuniform contraction along the center-stable bundle fit in our conditions. This setting was studied in [16, 7], where it was shown that these conditions are enough to obtain SRB measures for f. Since uniform expansion along the center-unstable direction and dominated splitting are robust in a whole C^1-neighborhood of f, we get nonuniform expansion for random orbits along the center-unstable direction for free.

Corollary D. *Let $f : M \to M$ be a C^2 diffeomorphism admitting a strictly forward invariant open set U with a dominated splitting $T_U M = E^{cs} \oplus E^u$, where the center-unstable bundle is uniformly expanding. If f is nonuniformly contracting along the center-stable direction, then f is stochastically stable.*

Another easy remark is that uniform contraction along the center-stable bundle with nonuniform expansion along the center-unstable direction, together with nonuniform expansion for random orbits along this direction, also fit in our setting.

Corollary E. *Let $f : M \to M$ be a C^2 diffeomorphism admitting a strictly forward invariant open set U with a dominated splitting $T_U M = E^s \oplus E^{cu}$, where the center-stable bundle is uniformly contracting. If f is nonuniformly expanding along the center-unstable direction and nonuniform expanding on random orbits along the same direction, then f is stochastically stable.*

In each of these corollaries the nonuniform bundle may admit a further dominated splitting into a uniformly behaved bundle and a central one, where the nonuniform expanding or contracting conditions will apply. This is the setting of *partially hyperbolic diffeomorphisms* (see [6] for definitions and main features), where the tangent bundle splits into three subbundles

$$T_U M = E^s \oplus E^c \oplus E^u$$

such that both $(E^s \oplus E^c) \oplus E^u$ and $E^s \oplus (E^c \oplus E^u)$ are dominated splittings, E^s is uniformly contracted and E^u is uniformly expanding. Finally, the uniformly hyperbolic case (when $E^c = \{0\}$) is also clearly in our setting (this result was first proved in [24]).

Corollary F. *Let $f : M \to M$ be a C^2 diffeomorphism admitting a strictly forward invariant open set U with a (uniformly) hyperbolic splitting $T_U M = E^s \oplus E^u$. Then f is stochastically stable.*

We also use random perturbations to obtain SRB measures as zero-noise limits of physical measures, obtaining a condition on Lebesgue almost points on a C^2 neighborhood of maps for the existence of SRB measures in the setting of attractors with dominated decomposition.

We say that f is *strongly nonuniformly hyperbolic* if there exists a neighborhood $\mathcal{V} \subset \mathrm{Diff}^2(M)$ and a constant $c > 0$ such that for any given sequence $\underline{g} = (g_1, g_2, g_3, \dots)$ of maps in $\mathcal{V}$ the sets

$$H_n^{cu}(\underline{g}) = \left\{ x \in M : \sum_{j=0}^{n-1} \log \|Df^{-1} \mid E^{cu}_{\underline{g}^j(x)}\| < -cn \right\}$$

and

$$H_n^{cs}(\underline{g}) = \left\{ x \in M : \sum_{j=0}^{n-1} \log \|Df \mid E^{cs}_{\underline{g}^j(x)}\| < -cn \right\}$$

satisfy $m\big(M \setminus (H_n^{cs}(\underline{g}) \cup H_n^{cu}(\underline{g}))\big) \to 0$ when $n \to \infty$, where $\underline{g}^j = g_j \circ \cdots \circ g_1$ for $j \geq 1$ and $\underline{g}^0 = Id$.

Theorem G. *Let f be a strongly nonuniformly hyperbolic diffeomorphism on a C^2-neighborhood $\mathcal{V}$. Then every $g \in \mathcal{V}$ has an at most finite number of ergodic Gibbs cu-states which are SRB measures.*

This last result promps some questions that we feel are extremely interesting to develop future research in this setting: whether the basins of the SRB measures cover Lebesgue almost every point in U; the variation of the number of SRB measures with $g \in \mathcal{V}$ and the variation of the Lyapunov exponents, are some examples. We remark that the robust class of nonuniformly hyperbolic diffeomorphisms which are stochastically stable presented above is also an example of an class of strongly nonuniformly hyperbolic maps.

We observe finally that every statement is trivially true if the region U coincides with the entire manifold, as the examples in Section 7 and in [7, 2]. In addition, the C^2 smoothness is not strictly needed for our arguments: $C^{1+\alpha}$ for some $\alpha > 0$ is all that is really needed in our arguments and in our references.

The paper is organized as follows. We start by stating some preliminary results in Section 2 and by studying random perturbations in Section 3, where the proof of Theorem A is explained. Then we prove the necessary condition of Theorem B in Section 4, where we also outline the main steps of the proof of sufficiency. The proof of Theorem B is completed in Section 5. In Section 6 we present the arguments leading to Thereom G. Finally in Section 7 we describe the construction of an open class of strongly nonuniformly hyperbolic diffeomorphisms.

2. Preliminary results

Here we outline some local geometrical and dynamical consequences of the hypothesis assumed in the previous subsection, referring mainly to previous works [2] and [1] for proofs.

2.1. **Curvature of center-unstable disks.** It was shown in [2] that f satisfies a bounded curvature property over disks having the tangent space at each point contained in a cone field around the center-unstable direction. Here we present the "random version" of the main results in [2, Section 2].

Given $0 < a < 1$ we define the *center-unstable cone field* $C_a^{cu} = (C_a^{cu}(x))_{x\in U}$ *of width* a by

$$C_a^{cu}(x) = \{v_1 + v_2 \in E_x^{cs} \oplus E_x^{cu} \colon \|v_1\| \le a\|v_2\|\} \tag{5}$$

and the *center-stable cone field* $C_a^{cs} = (C_a^{cs}(x))_{x\in U}$ *of width* a in the same manner but reversing the roles of the bundles in (5).

Up to increasing λ slightly we may fix a and ε small enough so that condition (a) of Subsection 1.1 (dominated decomposition) extends to vectors in the cone fields for all maps nearby f, i.e.

$$\|Df_t(x)v^{cs}\| \cdot \|Df_t^{-1}(f_t x)v^{cu}\| \le \lambda\|v^{cs}\| \cdot \|v^{cu}\| \tag{6}$$

for all $v^{cs} \in C_a^{cs}(x)$, $v^{cu} \in C_a^{cu}(f_t x)$, $x \in U$ and $t \in \operatorname{supp}(\theta_\varepsilon)$. Moreover, the domination property above together with the continuity of Φ and the closeness of t to t^* imply that $Df_t C_a^{cu}(x)$ is contained in a cone of width λa centered around $Df_t E_x^{cu}$, defined as above with respect to the splitting $Df_t E_x^{cs} \oplus Df_t E_x^{cu}$. Since the subspaces $Df_t E_x^{cs}, Df_t E_x^{cu}$ are close to $E_{f_t x}^{cs}, E_{f_t x}^{cu}$ respectively, then $Df_t C_a^{cu}(x) \subset C_a^{cu}(f_t x)$ if $\varepsilon > 0$ is small enough.

Given an embedded submanifold $S \subset U$ we say that S is *tangent to the center-unstable cone field* if $T_x S \subset C_a^{cu}(x)$ for all $x \in S$. Hence $f_t(S)$ is also tangent to the center-unstable cone field. The curvature of these submanifolds and their iterates will be approximated in local coordinates by the notion of Hölder variation of the tangent bundle as follows.

Let us take δ_0 sufficiently small so that if we take $V_x = B(x, \delta_0)$, then the exponential map $\exp_x : V_x \to T_x M$ is a diffeomorphism onto its image for all $x \in M$. We are going to identify V_x through the local chart $\exp_x^{-1}$ with the neighborhood $U_x = \exp_x V_x$ of the origin in $T_x M$. Identifying x with the origin in $T_x M$ we get that E_x^{cu} is contained in $C_a^{cu}(y)$ for all $y \in U_x$, reducing δ_0 if needed. Then the intersection of E_x^{cs} with $C_a^{cu}(y)$ is the zero vector. So if $x \in S$ then $T_y S$ is the graph of a linear map $A_x(y) : E_x^{cu} \to E_x^{cs}$ for $y \in U_x \cap S$.

For $C > 0$ and $\zeta \in]0,1[$ we say that the *tangent bundle of* S *is* (C,ζ)*-Hölder* if

$$\|A_x(y)\| \le C \operatorname{dist}_S(x,y)^\zeta \quad \text{for all} \quad y \in U_x \cap S \quad \text{and} \quad x \in U, \tag{7}$$

where $\operatorname{dist}_S(x,y)$ is *the distance along* S defined by the length of the shortest smooth curve from x to y inside S.

Up to choosing smaller $a > 0$ and $\varepsilon > 0$ we may assume that there are $\lambda < \lambda_1 < 1$ and $0 < \zeta < 1$ such that for all norm one vectors $v^{cs} \in C_a^{cs}(x), v^{cu} \in C_a^{cu}(x)$, $x \in U$ it holds

$$\|Df_t(x)v^{cs}\| \cdot \|Df_t^{-1}(f_t x)v^{cu}\|^{1+\zeta} \le \lambda_1.$$

For these values of λ_1 and ζ, given a C^1 submanifold $S \subset U$ tangent to the center-unstable cone field we define

$$\kappa(S) = \inf\{C > 0 : TS \text{ is } (C, \zeta)\text{-Hölder}\}. \tag{8}$$

The proofs of the results that we present below may be obtained by mimicking the proofs of the corresponding ones in [2], and we leave it as an exercise to the reader. The basic ingredients in those proofs are the cone invariance and dominated decomposition properties that we have already extended for nearby perturbations f_t of the diffeomorphism f.

Proposition 2.1. *There is $C_1 > 0$ such that for every C^1 submanifold $S \subset U$ tangent to the center-unstable cone field and every $\underline{t} \in T^{\mathbb{N}}$*

1. *there exists n_1 such that $\kappa(f_{\underline{t}}^n S) \le C_1$ for all $n \ge n_1$;*
2. *if $\kappa(S) \le C_1$ then $\kappa(f_{\underline{t}}^n S) \le C_1$ for all $n \ge 1$;*
3. *in particular, if S is as in the previous item, then*
$$J_n : f_{\underline{t}}^n S \ni x \mapsto \log|\det(Df|T_x f_{\underline{t}}^n S)|$$
is (L_1, ζ)-Hölder continuous with $L_1 > 0$ depending only on C_1 and f, for every $n \ge 1$.

Proof. See [2, Proposition 2.2] and [2, Corollary 2.4]. □

The bounds provided by Proposition 2.1 may be seen as bounds on the curvature of embedded disks tangent to the center-unstable cone field.

2.2. **Hyperbolic times.** From the condition of nonuniform expansion along the center-unstable direction we will be able to deduce some uniform expansion at certain times which are precisely defined through the following notion.

Definition 2.2. Given $0 < \alpha < 1$ we say that $n \ge 1$ is a α-hyperbolic time for $(\underline{t}, x) \in T^{\mathbb{N}} \times U$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f_{\underline{t}}^j x}\| \le \alpha^k \quad \text{for all} \quad k = 1, \dots, n.$$

The condition of non-uniform expansion for random orbits along the center-unstable direction is enough to ensure that almost all points have infinitely many hyperbolic times according to the following result.

Proposition 2.3. *There exists $\gamma > 0$ depending only on f and α such that for $\theta^{\mathbb{N}} \times \mathrm{m}$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times U$ and a sufficiently big integer $N \ge 1$, there exist $1 \le n_1 < \cdots < n_k \le N$, with $k \ge \gamma N$, which are hyperbolic times for $(\underline{t}, x)$.*

Proof. See [2, Corollary 3.2]. □

Let n be a α-hyperbolic time for $(\underline{t}, x) \in T^{\mathbb{N}} \times U$. This implies that $Df^{-k}|E^{cu}_{f_{\underline{t}}^n x}$ is a contraction for all $k = 1, \dots, n$. In addition, if $a > 0$ and $\varepsilon > 0$ are taken small enough in the definition of the cone fields and the random perturbations, then taking $\delta_1 > 0$ also small, we have by continuity

$$\|Df_t^{-1}|E^{cu}_{f_t y}\| \le \alpha^{-1/2}\|Df^{-1}|E^{cu}_{fx}\| \tag{9}$$

for all $t \in \operatorname{supp}(\theta_\varepsilon^{\mathbb{N}})$, $x \in \overline{fU}$ and $y \in U$ with $\operatorname{dist}(x,y) < \delta_1$. As a consequence of this we obtain the following result:

Lemma 2.4. *Given any C^1 disk $\Delta \subset U$ tangent to center-unstable cone field, $x \in \Delta$ and $n \geq 1$ a α-hyperbolic time for $(\underline{t}, x)$, we have*

$$\operatorname{dist}_{f_{\underline{t}}^{n-k}\Delta}(f_{\underline{t}}^{n-k}(y), f_{\underline{t}}^{n-k}(x)) \leq \alpha^{k/2} \operatorname{dist}_{f_{\underline{t}}^{n}\Delta}(f_{\underline{t}}^{n}y, f_{\underline{t}}^{n}x), \quad k = 1, \ldots, n,$$

for every point $y \in \Delta$ such that $\operatorname{dist}_{f_{\underline{t}}^n(\Delta)}(f_{\underline{t}}^n(y), f_{\underline{t}}^n(x)) \leq \delta_1$.

Proof. See [2, Lemma 2.7]. □

Using the previous lemma and the Hölder continuity property given by Proposition 2.1 it is not difficult to deduce the following bounded distortion result:

Proposition 2.5. *There exists $C_2 > 1$ such that, given any C^1 disk Δ tangent to the center-unstable cone field with $\kappa(\Delta) \leq C_1$, and given any $x \in \Delta$ and $n \geq 1$ a α-hyperbolic time for $(\underline{t}, x)$, then*

$$\frac{1}{C_2} \leq \frac{|\det Df_{\underline{t}}^n | T_y\Delta|}{|\det Df_{\underline{t}}^n | T_x\Delta|} \leq C_2$$

for every $y \in \Delta$ such that $\operatorname{dist}_{f_{\underline{t}}^n(\Delta)}(f_{\underline{t}}^n(y), f_{\underline{t}}^n(x)) \leq \delta_1$.

Proof. See [2, Proposition 2.8]. □

3. Random perturbations

To begin our study of random perturbations $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ of a diffeomorphism f admitting a partially hyperbolic set U, we observe that if we take $\varepsilon > 0$ small enough, then all random orbits $(f_{\underline{t}}^n x)_{n\geq 1}$ for every $x \in U$ are contained in a compact subset K of U when $\underline{t} \in \operatorname{supp}(\theta_\varepsilon^{\mathbb{N}})$, just by continuity of Φ and because U is strictly forward invariant.

Moreover if we introduce the skew-product map $F : T^{\mathbb{N}} \times M \to T^{\mathbb{N}} \times M$ given by $(\underline{t}, z) \mapsto (\sigma(\underline{t}), f_{t_1}(z))$, where σ is the left shift on sequences $\underline{t} = (t_1, t_2, \ldots) \in T^{\mathbb{N}}$, then we have that

$$\hat{\Lambda}_\varepsilon = \bigcap_{n\geq 0} F^n(\operatorname{supp}(\theta_\varepsilon^{\mathbb{N}}) \times U) \quad \text{with} \quad \Lambda_\varepsilon = \pi(\hat{\Lambda}_\varepsilon) \subset K,$$

where $\pi : T^{\mathbb{N}} \times M \to M$ is the natural projection. $\hat{\Lambda}_\varepsilon$ is a forward F invariant set, the *ε-random attractor*.

As mentioned before, we will assume that the random perturbations of the partially hyperbolic map f satisfy some *non-degeneracy conditions*: there is $0 < \varepsilon_0 < 1$ such that for every $0 < \varepsilon < \varepsilon_0$ we may find $n_0 = n_0(\varepsilon) \in \mathbb{N}$ satisfying the following conditions for all $x \in M$ and all $n \geq n_0$:

1. there is $\xi = \xi(\varepsilon) > 0$ such that $\left\{f_{\underline{t}}^n x \colon \underline{t} \in (\operatorname{supp} \theta_\varepsilon)^{\mathbb{N}}\right\} \supset B(f^n x, \xi)$;
2. defining $f_{\odot}^n x : T^{\mathbb{N}} \to M$ as $f_{\odot}^n x(\underline{t}) = f_{\underline{t}}^n(x)$, then $(f_{\odot}^n x)_* \theta_\varepsilon^{\mathbb{N}} \ll \mathrm{m}$.

Condition 1 says that the set of perturbed iterates of any given point covers a full neighborhood of the unperturbed iterates after a threshold for all sufficiently small noise levels. Condition 2 means that each set of perturbation vectors having positive $\theta_\varepsilon^{\mathbb{N}}$ measure must send each point $x \in M$ onto positive Lebesgue measure subsets of M, after a certain number of iterates.

Examples 1 and 2 in [3] show that every smooth map $f : M \to M$ of a compact manifold always has a random perturbation satisfying the non-degeneracy conditions 1 and 2, as long as we take $T = \mathbb{R}^p$, $t^* = 0$ and also θ_ε equal to Lebesgue measure restricted to the ball of radius ε around 0 (normalized to become a probability measure), for a sufficiently big number $p \in \mathbb{N}$ of parameters. For manifolds whose tangent bundle is trivial (parallelizable manifolds) the random perturbations which consist in adding at each step a random noise to the unperturbed dynamics clearly satisfy non-degeneracy conditions 1 and 2 for $n_0 = 1$.

The attractor $\cap_{n\geq 0} f^n(U)$ for f will be denoted by Λ. The conditions above imply that for small $\varepsilon > 0$ every f_t is C^2-close to $f \equiv f_{t^*}$. Then for every neighborhood V of Λ we have $\Lambda_\varepsilon \subset V$ for all small enough ε. Thus the compact set K containing Λ_ε may be taken as a neighborhood of Λ. We assume this in the rest of the paper (this is important in Subsection 4.2).

3.1. **The number of physical measures.** In the setting of random perturbations of a map, we say that a set $A \subset M$ is *forward invariant* if for some given small $\varepsilon > 0$ and for all $t \in \operatorname{supp}(\theta_\varepsilon)$ we have $f_t A \subset A$. A probability measure μ is said to be *stationary*, if for every continuous $\varphi : M \to \mathbb{R}$ it holds the following relation, similar to the non-random setting of invariant measures:

$$\int \varphi \, d\mu = \int \int \varphi\big(f_t x\big) \, d\mu(x) \, d\theta_\varepsilon(t). \tag{10}$$

Remark 3.1. If $(\mu^\varepsilon)_{\varepsilon>0}$ is a family of stationary measures having μ_0 as a weak* accumulation point when ε goes to 0, then it follows from (10) and the convergence of $\operatorname{supp}(\theta_\varepsilon)$ to $\{t^*\}$ that μ_0 must be invariant by $f = f_{t^*}$.

Condition (10) is equivalent to saying that $F_*(\theta_\varepsilon^{\mathbb{N}} \times \mu) = \theta_\varepsilon^{\mathbb{N}} \times \mu$ and it is easy to see that a stationary measure μ satisfies

$$x \in \operatorname{supp}(\mu) \quad \Rightarrow \quad f_t x \in \operatorname{supp}(\mu) \text{ for all } t \in \operatorname{supp}(\theta_\varepsilon),$$

just by continuity of Φ. This means that if μ is a stationary measure, then $\operatorname{supp}(\mu)$ is a forward invariant set. Then the interior of $\operatorname{supp}(\mu)$ is nonempty, by non-degeneracy condition 1, and forward invariant by the continuity of the maps f_t. Obviously a physical measure is stationary. Moreover if μ is a physical measure then $\theta_\varepsilon^{\mathbb{N}} \times \mu$ if F ergodic.

The following is a general result from [3] (see also [5, Section 2] for an alternative approach) for random perturbations of a diffeomorphism satisfying the non-degeneracy conditions 1 and 2 above, which implies the first item of Theorem A.

Theorem 3.2. *Assume that $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ is a random perturbation of a C^1 diffeomorphism f satisfying non-degeneracy conditions* 1 *and* 2*. Then for each $\varepsilon > 0$ there are physical* *measures* $\mu_1^\varepsilon, \dots \mu_{l(\varepsilon)}^\varepsilon$, *and for each* $x \in M$ *there is a* $\theta_\varepsilon^{\mathbb{N}}$ *mod* 0 *partition* $T_1(x), \dots, T_{l(\varepsilon)}(x)$ *of* $T^{\mathbb{N}}$ *such that for* $1 \leq i \leq l(\varepsilon)$

$$\mu_i^\varepsilon = \lim_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j x} \quad \textit{for} \quad \underline{t} \in T_i(x).$$

Moreover the supports of the physical measures are pairwise disjoint, have nonempty interior and are contained in their basins: $\operatorname{supp}(\mu_i^\varepsilon) \subset B(\mu_i^\varepsilon), i = 1, \dots, l(\varepsilon)$.

Now we prove the second item of Theorem A. First we observe that since the basin of each physical measure contains its support, then it also has nonempty interior. Let $\mu^\varepsilon = \mu_i^\varepsilon$ be a physical measure and take any probability measure η with support contained in $B(\mu_i^\varepsilon)$. Given any continuous function $\varphi\colon M \to \mathbb{R}$, we have for each $x \in \operatorname{supp}(\eta)$ and θ_ε almost every $\underline{t} \in T^{\mathbb{N}}$

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_{\underline{t}}^j x) = \int \varphi \, d\mu^\varepsilon$$

by definition of $B(\mu^\varepsilon)$. Taking integrals over $\operatorname{supp}(\eta)$ with respect to η on both sides of the equality, Dominated Convergence Theorem gives

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int \varphi(f_{\underline{t}}^j x) \, d\eta(x) = \int \varphi \, d\mu^\varepsilon.$$

(Recall that we are taking integrals over the support of the probability measure η). Since

$$\int \varphi \circ f_{\underline{t}}^j \, d\eta = \int \varphi \, d(f_{\underline{t}}^j)_* \eta$$

for every integer $j \geq 0$, we have proved item 2 of Theorem A.

To conclude the proof of Theorem A all we need is to show that $l(\varepsilon)$ is constant for every sufficiently small $\varepsilon > 0$ and that l is at most the number of SRB measures of f. We start by recalling that, by assumption, $\{\operatorname{supp}(\theta_\varepsilon)\}_{\varepsilon > 0}$ is a nested family of sets. This implies that if μ^ε is a physical measure, then $\operatorname{supp}(\mu^\varepsilon)$ is forward invariant with respect to f_t for all $t \in \operatorname{supp}(\theta_{\varepsilon'})$, whenever $0 < \varepsilon' < \varepsilon$. Since non-degeneracy conditions are satisfied in $\operatorname{supp}(\mu^\epsilon)$ by the probability measure $\theta_{\varepsilon'}$, then Theorem 3.2 ensures that there exists at least one physical measure $\mu^{\varepsilon'}$ with $\operatorname{supp}(\mu^{\varepsilon'}) \subset \operatorname{supp}(\mu^\varepsilon)$ for $0 < \varepsilon' < \varepsilon$. This shows that the number of physical measures is a nondecreasing integer function of ε when $\varepsilon \to 0$. Hence if we show that l is bounded from above, we conclude that l is constant for all small enough $\varepsilon > 0$.

We observe that $\operatorname{supp}(\mu^\varepsilon)$ is forward invariant under $f = f_{t^*}$ and, moreover, conditions (a)-(c) of the definition of a nonuniformly hyperbolic diffeomorphism together with simultaneous hyperbolic times hold in $\operatorname{supp}(\mu^\varepsilon)$ because they hold Lebesgue almost everywhere in U (by assumption) and $\operatorname{supp}(\mu^\varepsilon)$ has nonempty interior. Thus [2, Proposition 6.4] guarantees the existence of at least one SRB measure μ with $\operatorname{supp}(\mu) \subset \operatorname{supp}(\mu^\varepsilon)$.

We have seen that each support of a physical measure μ^ε must contain at least the support of one SRB measure for the unperturbed map f. Since the number of SRB measures is finite we have $l \le p$, where p is the number of those measures. This concludes the proof of Theorem A.

4. Stochastic stability

In this section we start the proof of Theorem B. We first prove that nonuniform expansion along the center-unstable direction for random orbits is a necessary condition for stochastic stability.

4.1. **Stochastic stability implies nonuniform hyperbolicity.** Let f be a stochastically stable nonuniformly hyperbolic diffeomorphism. Taking $\varepsilon > 0$ small we know from Theorem A that there is a finite number of physical measures $\mu_1^\varepsilon, \dots \mu_l^\varepsilon$ and, for each $x \in U$, there is a $\theta_\varepsilon^{\mathbb{N}}$ mod 0 partition $T_1(x), \dots, T_l(x)$ of $T^{\mathbb{N}}$ for which

$$\mu_i^\varepsilon = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j(x)} \quad \text{for each } \underline{t} \in T_i(x).$$

Furthermore, since $\log \|Df^{-1}|E_x^{cu}\|$ is a continuous map, then we have for every $x \in U$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f_{\underline{t}}^j x}\| = \int \log \|Df^{-1}|E_x^{cu}\| d\mu_i^\varepsilon,$$

for some physical measure μ_i^ε with $1 \le i \le l$. Hence, for proving that f is nonuniformly expanding along the center-unstable direction for random orbits it suffices to show that there is $c_0 > 0$ such that if $1 \le i \le l$ then, for small $\varepsilon > 0$,

$$\int \log \|Df^{-1}|E_x^{cu}\| d\mu_i^\varepsilon < c_0.$$

Since we are assuming the stochastic stability of f, then there must be some $\varepsilon_0 > 0$ such that if $\varepsilon \le \varepsilon_0$, then

$$\left| \int \log \|Df^{-1}|E_x^{cu}\| d\mu_i^\varepsilon - \int \log \|Df^{-1}|E_x^{cu}\| d\mu_\varepsilon \right| < c/2 \tag{11}$$

for some convex linear combination μ_ε of the SRB measures of f; see [1, Lemma 5.1]. This means that there are real numbers $w_1(\varepsilon), \dots, w_p(\varepsilon) \ge 0$ with $w_1(\varepsilon) + \dots + w_p(\varepsilon) = 1$ for which $\mu_\varepsilon = w_1(\varepsilon)\mu_1 + \dots + w_p(\varepsilon)\mu_p$. Since μ_i is an SRB measure for $1 \le i \le p$, we have for Lebesgue almost every $x \in B(\mu_i)$

$$\int \log \|Df^{-1}|E_x^{cu}\| d\mu_i = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| \le -c < 0.$$

This implies that

$$\int \log \|Df^{-1}|E_x^{cu}\| d\mu_\varepsilon \le -c,$$

and so, by (11),

$$\int \log \|Df^{-1}|E^{cu}_x\| d\mu^\varepsilon_i \leq -c/2.$$

This completes the proof of necessity in Theorem B.

4.2. **Nonuniform hyperbolicity implies stochastic stability.** Let us explain why nonuniform expansion along the center-unstable direction for random orbits is a sufficient condition for stochastic stability.

In order to prove that $f = f_{t^*}$ is stochastically stable, it suffices to show that every weak* accumulation point μ of any family $(\mu^\varepsilon)_{\varepsilon>0}$, where each μ^ε is a physical measure of level ε, are absolutely continuous with respect to the Lebesgue measure along local center-unstable disks, i.e., that every such μ is a Gibbs cu-state for f. This follows from the following combined results.

Theorem 4.1. *Let f be a nonuniformly hyperbolic diffeomorphism with positive density at infinity of simultaneous hyperbolic times. Then every ergodic Gibbs cu-state is an SRB measure for f. Moreover there are finitely many SRB measures whose basins cover a full Lebesgue measure subset of the topological basin U of the attractor.*

Proof. See [2, Section 6 and Proposition 6.3]. □

Theorem 4.2. *Let f be a C^2 diffeomorphism having a topological attractor with dominated splitting and let μ be a Gibbs cu-state for f. Then every ergodic component of μ is a Gibbs cu-state.*

Assuming further that f is nonuniformly expanding along the center-unstable direction, then every ergodic SRB measure is a Gibbs cu-state.

Proof. See [22, Theorem A and Corollary D]. □

These results ensure that every Gibbs cu-state μ for f has finitely many ergodic components which are SRB measures for f and also that μ can be written as a linear convex combination of these SRB measures. Hence it is enough to prove the following.

Theorem 4.3. *If f is non-uniformly expanding along the central direction for random orbits and $(\Phi, (\theta_\varepsilon)_{\varepsilon>0})$ is a random perturbation of f, then every weak* accumulation point μ^0 of $(\mu^\varepsilon)_{\varepsilon>0}$, when $\varepsilon \to 0$, is a* Gibbs cu-state *for f.*

If this holds, then according to the notions given at the Introduction f is stochastically stable.

In the following section we prove Theorem 4.3. The strategy is to adapt notions from [2] and [22] to deduce that in our setting stationary ergodic measures μ^ε admit *cylinders with mass uniformly bounded away from zero* having leaves which are *uniformly contracted backward* by a special sequence of perturbations $\underline{t} \in \operatorname{supp}(\theta_\varepsilon^{\mathbb{N}})$. This is Proposition 5.1 in the next section.

Having this it is not difficult to show that these cylinders accumulate, when $\varepsilon \to 0$, to cylinders having the same properties with respect to any weak* accumulation point of $(\mu^\varepsilon)_{\varepsilon>0}$. This is the content of Proposition 5.5 of the following section.

Now we just have to show that every f-invariant measure admitting a cylinder as above must be a Gibbs cu-state, which is the statement of Proposition 5.8 also in the next section.

Combining these steps we prove Theorem 4.3 and conclude the proof of Theorem B.

5. Center-unstable cylinders

Now we show that μ^ε admits a cylinder with very specific properties in the setting of non-uniformly expanding maps along the center direction for random orbits.

Let μ^ε be a physical measure of level ε for some small $\varepsilon > 0$ and take m_D the normalized Lebesgue measure on some C^1 disk D tangent to the center-unstable cone field such that m_D-almost every point of D is in $B(\mu^\varepsilon)$. It is possible to choose such a disk, because $B(\mu^\varepsilon)$ has nonempty interior. Now define for each $n \geq 1$

$$\mu_n^{\underline{t}} = \frac{1}{n}\sum_{j=0}^{n-1}(f_{\underline{t}}^j)_* \, \mathrm{m}_D \, .$$

We know from Theorem A that each μ^ε is the weak* limit of the sequence $(\mu_n^{\underline{t}})_n$ for a $\theta_\varepsilon^{\mathbb{N}}$ generic $\underline{t}$ by item 2 of Theorem A. We fix a $\theta_\varepsilon^{\mathbb{N}}$ generic $\underline{t}$ in everything that follows within this subsection.

A cylinder $\mathcal{C} \subset M$ is the image of a C^1 diffeomorphism $\phi : B^u \times B^s \hookrightarrow M$ where B^k is the k-dimensional unit ball of $\mathbb{R}^k, k = s, u$. We will say that a C^1 disk D *crosses* $\mathcal{C}$ if $D \cap \mathcal{C}$ is a graph over B^u: there exists $g : B^u \to B^s$ such that $D \cap \mathcal{C} = \{\phi(w, g(w)) : w \in B^u\}$.

The following is the main result of this subsection.

Proposition 5.1. *Let μ^ε be a stationary probability measure for $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ where f is a nonuniformly hyperbolic C^2 diffeomorphism. Then there are $\beta = \beta(f, c_u) > 0, \rho = \rho(f, c_u) > 0, d = d(f, c_u) > 0$, a cylinder $\mathcal{C} = \phi(\mathcal{B}^u \times B^s)$ and a family $\mathcal{K}$ of disks tangent to the center-unstable cone field which cross $\mathcal{C}$ and whose union is the set K such that*

1. *$\mu^\varepsilon(K \cap \mathcal{C}) \geq \beta$ and both $\phi(B^u \times 0)$ and $\phi(u \times B^s)$ are disks containing a subdisk with radius $\geq \rho$ for all $u \in B^u$;*
2. *for every disk $\gamma \in \mathcal{K}$ there exists a sequence $\underline{s} \in \operatorname{supp} \theta_\varepsilon^{\mathbb{N}}$ such that $(f_{\underline{s}}^n)^{-1} \mid \gamma$ is a $\alpha^{n/2}$-contraction: for $w, z \in \gamma$*
$$\operatorname{dist}_{(f_{\underline{s}}^n)^{-1}\gamma}((f_{\underline{s}}^n)^{-1}(w), (f_{\underline{s}}^n)^{-1}(z)) \leq \alpha^{n/2} \operatorname{dist}_\gamma(w, z)$$
where $\operatorname{dist}_\gamma$ is the induced distance on γ by the Riemannian metric on M;
3. *the disintegration $\{\mu_\gamma^\varepsilon\}_\gamma$ of $\mu^\varepsilon \mid \mathcal{C}$ along the disks $\gamma \in \mathcal{K}$ has densities with respect to the Lebesgue induced measure m_γ on γ uniformly bounded from above and below: $d^{-1} \leq (d\mu_\gamma^\varepsilon / dm_\gamma) \leq d$, μ_γ^ε almost everywhere and for almost every $\gamma \in \mathcal{K}$.*

The proof is essentially the same as in [2]: to consider a component of the average $\mu_n^{\underline{t}}$ calculated at hyperbolic times. We present the proof for completeness and because item (2) of the above statement is not in [2, 22].

Proof. To control the densities of the push-forwards at hyperbolic times we set

$$A = \{x \in D : \operatorname{dist}_D(x, \partial D) \geq \delta_1\}$$

where dist_D is the distance along D, and take δ_1 small enough so that $m_D(A) > 0$. Then we define for each $n \geq 1$ (we recall that $\underline{t}$ is $\theta_\varepsilon^{\mathbb{N}}$ generic fixed from the beginning)

$$H_n = \{x \in A : n \text{ is a hyperbolic time for } (\underline{t}, x)\}.$$

We note that Lemma 2.4 ensures that $\mathrm{dist}_{f_{\underline{t}}^n(D)}(f_{\underline{t}}^n(x), \partial f_{\underline{t}}^n(D)) \geq \delta_1$ for every $x \in H_n$.

Let $D_n(x, \delta_1)$ be the δ_1-neighborhood of $f_{\underline{t}}^n(x)$ inside $f_{\underline{t}}^n(D)$. Then Proposition 2.5 ensures that the density of $((f_{\underline{t}}^n)_* m_D) \mid D_n(x, \delta_1)$ with respect to $m_{D_n(x,\delta_1)}$ is uniformly bounded from above and from below if we normalize both measures.

To extend this control of the density to a significant portion of D we use the following

Lemma 5.2. *There is $\omega > 0$ (depending only on M, the curvature of center-unstable disks and on the dimension u of the center-unstable bundle) such that for all $n \geq 1$ we can find a finite subset $\hat{H}_n$ of H_n satisfying*

1. *$\hat{B}_n = \{B(x, \delta_1/4), x \in \hat{H}_n\}$ is a pairwise disjoint collection;*
2. *the union $B_n = \cup \hat{B}_n$ is such that $((f_{\underline{t}}^n)_* m_D)(B_n) \geq \omega \cdot m_D(H_n)$.*

Proof. See [2, Proposition 3.3 and Lemma 3.4]. □

Now we define a component of the average $\mu_n^{\underline{t}}$

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} ((f_{\underline{t}}^j)_* m_D) \mid B_j \tag{12}$$

and check that the mass carried by ν_n does not vanish.

Lemma 5.3. *There is $\beta_0 > 0$ such that $\nu_n(\cup_{j=0}^{n-1} f_{\underline{t}}^j(D)) \geq \beta_0$ for all big enough $n \geq 1$.*

Proof. We note that

$$\frac{1}{n} \sum_{j=0}^{n-1} m_D(H_j) = \int \int \chi_{H_j}(x)\, dm_D(x)\, d\#_n(j) = \int \left(\int \chi_{H_j}(x)\, d\#_n(j) \right) dm_D(x)$$

where $\#_n$ is the uniform distribution on $\{0, \ldots, n-1\}$. By Proposition 2.3 for big n we must have that the inner integral is bounded from below by $\gamma > 0$. By Lemma 5.2 the mass of ν_n is bounded from below by $\omega \cdot n^{-1} \sum_{j=0}^{n-1} m_D(H_j) \geq \omega\gamma m_D(D)$ for big enough n. We just have to take $\beta_0 = \omega\gamma m_D(D)$ since $\mathrm{supp}\,(\nu_n) \subset \cup_{j=0}^{n-1} f_{\underline{t}}^j(D)$. □

With these settings the support of ν_n is a finite union $\cup_{j=0}^{n-1} B_j$ of disks having size bounded from above and below. Let ν be an accumulation point of $(\nu_n)_{n \geq 1}$ in the weak* topology: $\nu = \lim_k \nu_{n_k}$. Then the support of ν is contained in $B_\infty = \cap_{n \geq 1} \overline{\cup_{j>n} B_j}$.

This construction shows that for $y \in B_\infty$ there are sequences $k_j \to \infty$ of integers and disks $D_j = D_{k_j}(x_{k_j}, \delta_1/4)$ and points $y_j \in D_j$ such that $y_j \to y$ when $j \to \infty$. We know from subsections 2.1 and 2.2 that D_j are C^1 center-unstable disks containing a inner δ_1-ball. Moreover the sequence $(D_j)_j$ is relatively compact by the Ascoli-Arzela Theorem, hence up to taking subsequences we have $x_{k_j} \to x$ and $D_j \to D_x$ in the C^1 topology when $j \to \infty$ for some $x \in B_\infty$ and a disk D_x centered at x with radius $\delta_1/4$. Thus $y \in \overline{D_x} \subset B_\infty$.

Let $(j(n))_{n\geq 1}$ be the subsequence of indexes such that $D_n = D_{k_{j(n)}} \to D_x$ as above when $n\to\infty$. Then $(t_{k_{j(n)}})_n$ admits a convergent subsequence to some $s_1 \in \operatorname{supp}\theta_\varepsilon$.

To avoid too many subscripts we let that subsequence be indexed by k_n^0 with $n\geq 1$. This is a subsequence of $(k_j)_j$. By definition of hyperbolic times we know that $(f_{t_{k_n^0}})^{-1}$ is a $\alpha^{1/2}$-contraction on $D_{k_n^0}$ for all $n\geq 1$. Hence by the C^1 convergence of the disks and the C^2 continuity of the family Φ, we must have that $(f_{s_1})^{-1}$ is a $\alpha^{1/2}$-contraction on D_x.

We also have that $(t_{k_n^0-1})_n$ admits a subsequence tending to some $s_2\in\operatorname{supp}\theta_\varepsilon$ indexed by $(k_n^1)_n$, which is a subsequence of $(k_n^0)_n$. In general we have that $t_{k_n^{\ell-1}-\ell}\to s_\ell$ when $n\to\infty$ where $(k_n^\ell)_n$ is a subsequence of $(k_n^{\ell-1})_n$ for every $\ell\geq 0$. The same continuity arguments as above ensure that $(f_{s_\ell}\circ\cdots\circ f_{s_1})^{-1}$ is a $\alpha^{\ell/2}$-contraction on D_x.

This shows that for every accumulation disk $D_x\in B_\infty$ as above there exists a subsequence $\underline{s}\in\operatorname{supp}\theta_\varepsilon^{\mathbb{N}}$ such that $(f_{\underline{s}}^j)^{-1}\mid D_x$ is a α^j-contraction for every $j\geq 1$. Hence D_x does not depend on the choices made during the construction of convergent subsequences of disks, since it is a Pesin's unstable manifold through $(\underline{s},x)$ for $F: T^{\mathbb{N}}\times M\to T^{\mathbb{N}}\times M$ — see [5, 10, 11, 23] for references on invariant manifolds for random maps.

In what follows, let $\mathcal{B}$ be the family of center-unstable disks in B_∞ obtained through this limit process.

5.1. **Constructing the cylinder.** Now we start the construction of the cylinder. Given any disk $D\in\mathcal{B}$, the compactness of B_∞ and the uniformity of δ_0 (the radius of invertibility of the exponential map of M defined in Subsection 2.1) enables us to construct a (open) cylinder $\mathcal{C}$ over any subdisk D_0 of D with radius $\rho\in(0,\delta_0)$ by considering the images under the exponential map of vectors in T_zM orthogonal to T_zD_0 and with norm less than ρ. We assume that the connected components υ of every center-unstable disk γ that crosses $\mathcal{C}$ have diameter smaller than 2ρ inside γ. We call $\mathcal{C}$ a ρ-cylinder.

We assume that $\rho<\delta_1/100$. If we consider $\hat{B}_j(\rho)$ the sets obtained from $\hat{B}_j$ removing the ρ-neighborhood of the boundary of every disk in $\hat{B}_j$ and let $B_j(\rho)$ be the union of the points in $\hat{B}_j(\delta)$, then setting

$$\nu_{n,\rho}=\nu_n\mid\cup_{j=0}^{n-1}B_j(\delta)$$

we see that $((f_{\underline{t}}^j)_*m_D)\mid B_j(\rho)\geq(1-\delta)\cdot((f_{\underline{t}}^j)_*m_D)\mid B_j$ for some $\delta=\delta(\rho)>0$. The value of $\delta>0$ may be taken independently of j because by the bounded distortion property at hyperbolic times (Proposition 2.5) the relative mass removed from the disks is comparable for all iterates.

Hence for a sufficiently small $\rho>0$ as above we may assume that $\nu_{n,\rho}(M)\geq\beta_0/2$ for all n big enough. We fix this value of $\rho>0$ from now on. Letting ν_ρ be an accumulation point of $\nu_{n,\rho}$ for a subsequence of $(\nu_{n_k})_k$ we have $\nu_\rho\leq\nu$.

The ρ-cylinders $\mathcal{C}$ constructed as above have uniform size (depending on ρ only). Thus a finite collection of them is enough to cover B_∞ and we can take an open cover $\mathcal{C}_1,\dots,\mathcal{C}_k$ with the least possible number of elements.

We claim that this number does not depend on $\varepsilon\geq 0$. Indeed, the ρ-cylinders contain balls in M with radius $r>0$ dependent only on ρ. The attractor Λ is compact and so there is a minimum N for the number of r-balls needed to cover Λ. The union of the balls of one such cover is a neighborhood of Λ in M and so also a neighborhood of Λ_ε for all sufficiently small $\varepsilon > 0$ (recall the comments on the beginning of Section 3). Since $B_\infty \subset \Lambda_\varepsilon$ the minimum number of ρ-cylinders covering B_∞ obtained above is at most N.

This shows that for some cylinder $\mathcal{C} \in \{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ we must have

$$\nu(\mathcal{C}) \geq \nu_\rho(\mathcal{C}) \geq \frac{\nu_\rho(B_\infty)}{N} = \frac{\nu_\rho(M)}{N} \geq \frac{\beta_0}{2N}.$$

According to the construction of the ρ-cylinders, for every disk $D(\rho) \in \hat{B}_j(\rho)$ such that $D(\rho) \cap \mathcal{C} \neq \emptyset$, then the components of $D \cap \mathcal{C}$ cross $\mathcal{C}$, where D is the corresponding disk in $\hat{B}_j$ whose truncation gives $D(\rho)$, $j \geq 1$. Moreover by an arbitrarily small change in ρ we may assume that $\nu(\partial\mathcal{C}) = 0$.

Let us denote by $\mathcal{K}_n$ the components of the intersection $D \cap \mathcal{C}$ that cross $\mathcal{C}$, for all $D \in \hat{B}_n$ and $n \geq 1$, and let $K_n = \cup \mathcal{K}_n$ be the union of the points in $\mathcal{K}_n$. In addition let $\mathcal{K}$ be the set of disks from $\mathcal{B}$ that cross $\mathcal{C}$ and K the set of all points in $\mathcal{K}$. Then for all $n \geq 1$

$$\nu_{n,\rho}(\mathcal{C}) = \nu_{n,\rho}(\mathcal{C} \cap \cup_{j=0}^{n-1} K_j) \leq \nu_n(\mathcal{C} \cap \cup_{j=0}^{n-1} K_j).$$

Hence taking limits of subsequences (recall that $\nu(\partial\mathcal{C}) = 0$ and $\nu_\rho \leq \nu$) we arrive at

$$\frac{\beta_0}{2N} \leq \nu_\rho(\mathcal{C}) \leq \limsup_{n\to\infty} \nu_n(\mathcal{C} \cap \cup_{j=0}^{n-1} K_j).$$

But since K contains the set of accumulation points of $(\cup_{j=0}^{n-1} K_j)_{n\geq 1}$ and ν_n is defined by the average (12), we have that

$$\limsup_{n\to\infty} \nu_n(\mathcal{C} \cap \cup_{j=0}^{n-1} K_j) \leq \nu(\mathcal{C} \cap K)$$

and so $\nu(\mathcal{C} \cap K) \geq \beta_0/(2N)$.

However $\mu^\varepsilon \geq \nu$ by construction, hence $\mu^\varepsilon(\mathcal{C} \cap K) \geq \beta_0/(2N)$ also. We stress that either β_0, N or ρ do not depend on the choice of ε nor of $\underline{t}$. We have proved items (1) and (2) of the statement of the proposition.

Lemma 5.4. *There exists $d > 0$ and a family of conditional measures $(\nu_\delta)_\delta$ of $\nu \mid K$ along the disks $\delta \in \mathcal{K}$ such that $d^{-1} \cdot m_\delta(E) \leq \nu_\delta(E) \leq d \cdot m_\delta(E)$ for every Borel set $E \subset \delta$.*

Proof. This is the same as Lemma 4.4 of [2] with straightforward adaptations to our setting. □

This completes the proof of Proposition 5.5. □

5.2. **Accumulation cylinders.** In what follows we fix a decreasing sequence $\varepsilon_k \to 0$ when $k \to \infty$ and a sequence $\mu_k = \mu^{\varepsilon_k}$ of ergodic stationary measures.

We observe that since $(\mathrm{supp}\ \theta_\varepsilon)_{\varepsilon>0}$ is a nested family of connected compact subsets shrinking to $\{t^*\}$ when $\varepsilon \to 0$, and for each $\varepsilon > 0$ and any stationary measure μ^ε the set $\mathrm{supp}\ \mu^\varepsilon$ is f_t-invariant for all $t \in \mathrm{supp}\ \theta_\varepsilon$, we may choose the sequence μ_k so that $(\mathrm{supp}\ \mu_k)_k$ is a nested family of f-invariant compact subsets.

Proposition 5.5. *Let μ be a weak* accumulation point of $(\mu_k)_k$. Then there exists a cylinder $\mathcal{C}$ and a family $\mathcal{K}$ of disks tangent to the center-unstable cone field which cross $\mathcal{C}$, whose union is the set K, such that*

1. $\mu(K \cap \mathcal{C}) \geq \beta$;
2. *the disintegration $\{\mu_\gamma\}_\gamma$ of $\mu \mid \mathcal{C}$ along the disks $\gamma \in \mathcal{K}$ has densities with respect to the Lebesgue induced measure m_γ on γ uniformly bounded from above and below: $d^{-1} \leq (d\mu_\gamma/dm_\gamma) \leq d$; moreover the ergodic decomposition $\{\mu_x\}_x$ of μ is such that $\mu_x = \mu_\gamma$, μ_γ almost everywhere and for almost every $\gamma \in \mathcal{K}$;*
3. $(f^n)^{-1} \mid \gamma$ *is a $\alpha^{n/2}$-contraction on every disk $\gamma \in \mathcal{K}$, where $f = f_{t^*}$.*
4. *letting $\tau(x) = \min\{k \geq 1 : f^{-k}(x) \in K\}$ and setting $R_k = K \cap \tau^{-1}(k)$ for all $k \geq 1$ we have that there exist constants $C_3 > 0$ and $\lambda_0 \in (0,1)$ such that $\mu(R_k) \leq C_3 \lambda_0^k$ for every $k \geq 1$.*

The values of β and $d > 0$ above are the same from Proposition 5.1. The proof follows [22] closely with minor adaptations to the present setting.

Proof. Let μ_k be as stated in the beginning of the subsection and let $\mathcal{C}_k, \mathcal{K}_n$ and K_n be the corresponding cylinders, families of disks and sets from Proposition 5.1. We assume that $\mu_k \to \mu$ in the weak* topology when $k \to \infty$. Then μ is an f-invariant probability measure (Remark 3.1).

The compactness of M ensures that for some subsequence k_n the cylinder $\mathcal{C}_{k_n}$ tends to a cylinder $\mathcal{C}$. In fact, each C_k is a diffeomorphic image of $\phi_k : B^u \times B^s \hookrightarrow M$, with B^ℓ the ℓ-dimensional unit ball of $\mathbb{R}^\ell, \ell = s, u$. By the Ascoli-Arzela Theorem there is a subsequence $(k_n)_{n\geq 1}$ such that $\phi_{k_n}(B^u \times 0)$ converges in the C^1-topology to a disk $D_0 = \phi(B^u \times 0)$ in M. Since the diameters of $\phi_{k_n}(B^u \times 0)$ and $\phi_{k_n}(0 \times B^s)$ are uniformly bounded from below by $\rho > 0$ (by Proposition 5.1) and by the construction of $\mathcal{C}_k$, defining $\mathcal{C}$ as the set of images under the exponential map of vectors in $T_z M$ orthogonal to $T_z D_0$ and with norm less than ρ, then $\overline{\mathcal{C}}_{k_n}$ tends to $\overline{\mathcal{C}}$ in the Hausdorff topology.

Let $\mathcal{K}$ be the family of disks D in $\mathcal{C}$ which are accumulated by sequences of disks D_n in $\mathcal{K}_{k_n}$ for $n \geq 1$. Since every disk D_n is tangent to the center-unstable cone field of f, the continuity of the cone field on U assures that every disk $D \in \mathcal{K}$ is also a center-unstable disk.

For any fixed $D \in \mathcal{K}$ let $x, y \in D$ and take $(x_n)_n, (y_n)_n$ sequences in $D_n \in \mathcal{K}_{k_n}$ such that $x_n \to x$ and $y_n \to y$ when $n \to \infty$. From item (3) of Proposition 5.1 we know that there are sequences of parameters $(\underline{s}(n))_{n\geq 1}$ such that $\underline{s}(n) \in \operatorname{supp} \theta^{\mathbb{N}}_{\varepsilon_{k_n}}$ and

$$\operatorname{dist}_{(f^j_{\underline{s}(n)})^{-1}(D_n)}((f^j_{\underline{s}(n)})^{-1}(x_n), (f^j_{\underline{s}(n)})^{-1}(y_n)) \leq \alpha^{j/2} \operatorname{dist}_{D_n}(x_n, y_n)$$

for every $j \geq 1$ and for every given $n \geq 1$. Fixing $j \geq 1$ we get

$$(s_1(n), \ldots, s_j(n)) \to (t^*, \ldots, t^*) \quad \text{when} \quad n \to \infty,$$

because $\operatorname{supp}(\theta^{\mathbb{N}}_{\varepsilon_{k_n}}) \to \{t^*\}$. The continuity of $f_t(x)$ with respect to $(t, x) \in T \times M$ implies that

$$\operatorname{dist}_{f^{-j}(D)}(f^{-j}(x), f^{-j}(y)) \leq \alpha^{j/2} \operatorname{dist}_D(x, y) \tag{13}$$

for every given $j \geq 1$. Hence f^{-j} is an $\alpha^{j/2}$-contraction on every $D \in \mathcal{K}$ and D is a center-unstable disk in U. This means that E^{cu}_x is uniformly expanded by Df for every $x \in D$. The domination property for the splitting $E^{cu} \oplus E^{cs}$ guarantees that any eventual expansion along the complementary direction is weaker than this. Thus D is contained in the unique local strong-unstable manifold $W^u_{loc}(x)$ tangent to E^{cu}_x, see [15].

Now since $\mu_k(\mathcal{C}_k \cap \mathcal{K}_k) \geq \beta > 0$ for all $k \geq 1$ from Proposition 5.1, if we fix $\delta > 0$ then for all n big enough we get $K_{k_n} \subset B(K, \delta)$. Letting $\delta > 0$ be such that $\mu(\partial B(K, \delta)) = 0$ (this holds except for an at most countable set of values of δ) then

$$\mu(B(K,\delta)) = \lim_{n\to\infty} \mu_{k_n}(B(K,\delta)) \geq \beta > 0.$$

Moreover $K = \cap_{\delta>0} B(K,\delta)$ thus $\mu(K) = \inf_{\delta>0} \mu(B(K,\delta)) \geq \beta$. We have proved items (1) and (3) of the statement of Proposition 5.5.

Lemma 5.6. *There is a family $\{\mu_\gamma\}_\gamma$ of conditional measures of $\mu \mid K$ along the unstable disks $\gamma \in \mathcal{K}$ such that for every Borel set $E \subset D$*

$$\frac{1}{d} \cdot m_\gamma(E) \leq \mu_\gamma(E) \leq d \cdot m_\gamma(E).$$

In addition, μ_γ coincides with the ergodic component μ_x for m_γ-almost every $x \in \gamma$ and for $\hat\mu$-almost every $\gamma \in \mathcal{K}$

Proof. This is [22, Lemma 3.3 and 3.4] essentially verbatim. □

This lemma gives item (2) of the statement of Proposition 5.5.

Finally we use the uniform contraction property (13) of the disks in $\mathcal{K}$ and the uniform bound on the density along the disks of $\mathcal{K}$ to deduce item (4).

Let τ and R_k be as in item (4) of the statement for $k \geq 1$. Then the f-invariance of μ implies $f^k_*(\mu_\gamma) = \mu_{f^{-k}(\gamma)}$ on R_k ($f^{-k}(R_k) \subset K$ by definition of R_k) for $\hat\mu$-almost every $\gamma \in \mathcal{K}$ and

$$\begin{aligned} \mu(R_k) = \mu(f^{-k}(R_k)) &= \int_K \mu_\gamma(f^{-k}(R_k))\, d\hat\mu(\gamma) = \int_K \mu_{f^{-k}(\gamma)}(R_k)\, d\hat\mu(\gamma) \\ &\leq d \int_K m_{f^{-k}(\gamma)}(R_k)\, d\hat\mu(\gamma). \end{aligned}$$

Now since each $\gamma \in \mathcal{K}$ is a submanifold of M and $f^k \mid f^{-k}(\gamma) : f^{-k}(\gamma) \to \gamma$ is a diffeomorphism

$$m_{f^{-k}(\gamma)}(R_k) = m_{f^{-k}(\gamma)}\big(f^{-k}(f^k(R_k))\big) = \int_{f^k(R_k)} \mathrm{Jac}_x(f^{-k} \mid \gamma)\, dm_\gamma(x).$$

The uniform bound C_1 on the curvature for all center-unstable disks from Proposition 2.1 provides a uniform constant $C > 0$ (dependent on C_1 only) that together with the uniform contraction (13) and the dimension u of every $\gamma \in \mathcal{K}$ gives $\mathrm{Jac}_x(f^{-k} \mid \gamma) \leq C(\alpha^{k/2})^u$. Hence

$$\mu(R_k) \leq d \cdot C\alpha^{uk/2} \int_K m_\gamma(f^k(R_k))\, d\hat\mu(\gamma) \leq C_3 \lambda_0^k$$

where $\lambda_0 = \alpha^{u/2}$ and $C_3 = d \cdot Cm(K) \ge d \cdot C \int_K m_\gamma(f^k(R_k))\, d\hat\mu(\gamma)$. This concludes the proof of Proposition 5.5. $\square$

5.3. **Absolute continuity of accumulation measure.** Here we fix $\varepsilon_k \to 0$, $\mu_k = \mu^{\varepsilon_k}$ and $\mu = \lim_{k\to\infty} \mu_k$ in the weak* topology as in the discussion of the previous subsection. We denote by $\mathcal{C}$ the cylinder and by K the compact subset in the statement of Proposition 5.5 with respect to μ. We also denote by $\mathcal{C}^k$ the cylinder and by K^k the compact subset from the statement of Proposition 5.1 with respect to each $\mu_k, k \ge 1$.

For each $n \ge 1$ we define $K_n = \{x \in M : \tau(x) \le n\}$, recall the definition of τ from Proposition 5.5. For any $k \ge 1$ we set

$$A_n^k = \{x \in K^k : (f_{\underline{t}}^j)^{-1}(x) \in M \setminus K^k \quad \text{for all} \quad j = 1, \dots, n \quad \text{and} \quad \underline{t} \in \operatorname{supp}(\theta_{\varepsilon_k}^{\mathbb{N}})\}$$

and define $K_n^k = M \setminus A_n^k$ and $\mu_k^n = \mu_k \mid K_n^k$.

The following result is a consequence of the assumption that f_t is is C^2-close to $f \equiv f_{t^*}$ when t is close to t^* together with item 4 of Proposition 5.5.

Lemma 5.7. *There is $C_0 > 0$ and for any given $n \in \mathbb{N}$ there is $\ell \ge 1$ such that (where $\lambda_0 \in (0,1)$ is given by Proposition 5.5)*

$$\mu_k(A_n^k) < C_0 \lambda_0^{n+1} \quad \text{for every} \quad k > \ell.$$

Moreover for every fixed $n \ge 1$ we have $\mu_k^n \to \mu \mid K_n$ when $k \to \infty$ in the weak* topology.

Proof. From item 4 of Proposition 5.5 we know that for any given n we have that

$$A_n = \{x \in K : f^{-j}(x) \in M \setminus K, \quad j = 1, \dots, n\}$$

satisfies $\mu(A_n) \le C_3 \sum_{j>n} \lambda_0^j = C' \lambda_0^{n+1}$, where $C' = C_3/(1-\lambda_0)$.

Since K is closed and μ is a Borel regular measure, fixing the number n of iterates involved we have that for small enough $\zeta_1, \zeta_2 > 0$ the set

$$A_n(\zeta_1,\zeta_2) = \{x \in B(K,\zeta_1) : f^{-j}(x) \in M \setminus B(K,\zeta_2), \quad j = 1, \dots, n\}$$

also satisfies $\mu(A_n(\zeta_1,\zeta_2)) < 2C'\lambda_0^{n+1}$, where $B(K,\zeta) = \cup_{x\in K} B(x,\zeta)$ is the ζ-neighborhood of K in M for any $\zeta > 0$. Moreover for a fixed $\zeta_1 > 0$ the set $A_n(\zeta_1,\zeta_2)$ is an open neighborhood of A_n and does not depend on $\zeta_2 > 0$ for all small enough values of $\zeta_2 > 0$. So through an arbitrarily small change in ζ_1 we may assume that $\mu(\partial A_n(\zeta_1,\zeta_2)) = 0$ in what follows.

We note that for fixed n and ζ_1 as above, there exists $\ell_1 \in \mathbb{N}$ such that for small enough $\zeta_2 < \zeta_1$ we have $K^k \subset B(K,\zeta_2/2) \subset B(K,\zeta_1)$ for all $k > \ell_1$. Since f_t depends continuously on t and the number n of iterates is fixed, we may take ℓ_1 big enough and $\zeta_2 > 0$ small enough such that every $x \in A_n^k$ satisfies $\operatorname{dist}(f_{\underline{t}}^j(x), K^k) \ge \zeta_2/2$ for all $k \ge \ell_1, j = 1, \dots, n$ and $\underline{t} \in \operatorname{supp} \theta_{\varepsilon_k}^{\mathbb{N}}$.

This shows that for these values of ℓ_1, ζ_2 and ζ_1 it holds that $A_n^k \subset A_n(\zeta_1,\zeta_2)$ for all $k \ge \ell_1$. But $\mu_k \to \mu$ in the weak* topology so by the assumption on the boundary of $A_n(\zeta_1,\zeta_2)$ we arrive at $\mu_k(A_n^k) \le \mu_k(A_n(\zeta_1,\zeta_2)) \le 4C'\lambda_0^{n+1}$ for all big enough k. The first statement of the lemma is obtained resetting ℓ to a bigger value (if needed) and letting $C_0 = 4C'$.

For small enough $\xi_1, \xi_2 > 0$ we now define

$$A_n^k(\xi_1, \xi_2) = \{x \in B(K^k, \xi_1) : (f_{\underline{t}}^j)^{-1}(x) \in M \setminus B(K^k, \xi_2),\ j = 1, \dots, n,\ \underline{t} \in \operatorname{supp}(\theta_{\varepsilon_k}^{\mathbb{N}})\}$$

which is an open neighborhood of A_n^k that also does not depend on the value of $\xi_2 > 0$. Again for fixed n and $\xi_1 > 0$ there exists $\ell_2 \in \mathbb{N}$ such that for small enough $\xi_2 < \xi_1$ we have $K \subset B(K^k, \xi_2/2) \subset B(K^k, \xi_1)$ for all $k > \ell_2$. Since $t^* \in \operatorname{supp}(\theta_{\varepsilon_k}^{\mathbb{N}})$ we easily check that $A_n \subset A_n^k(\xi_1, \xi_2)$ for all $k \geq \ell_2$.

Choosing small values of $\zeta_1, \zeta_2 > 0$ and taking ℓ big enough the above arguments ensure that for all $k \geq \ell$ it holds

$$A_n \subset A_n(\zeta_1, \zeta_2) \subset A_n^k(2\zeta_1, \zeta_2/2) \subset A_n(3\zeta_1, \zeta_2/3) \tag{14}$$

and we can simultaneously assume that we have

$$\mu(\partial A_n(\zeta_1, \zeta_2)) = 0 = \mu(\partial A_n(3\zeta_1, \zeta_2/3)). \tag{15}$$

Let us take an open set B such that $\mu(\partial B) = 0$ (the collection of all such sets generates the Borel σ-algebra μ mod 0). Using (14) we get for all $k \geq \ell$

$$\mu_k(B \cap [M \setminus A_n(3\zeta_1, \zeta_2/3)]) \leq \mu_k(B \cap [M \setminus A_n^k(2\zeta_1, \zeta_2/2)]) \leq \mu_k(B \cap [M \setminus A_n(\zeta_1, \zeta_2)])$$

and letting $k \to \infty$ and using (15) we arrive at

$$\begin{aligned} \mu(B \cap [M \setminus A_n(3\zeta_1, \zeta_2/3)]) &\leq \liminf_{k\to\infty} \mu_k(B \cap [M \setminus A_n^k(2\zeta_1, \zeta_2/2)]) \\ &\leq \limsup_{k\to\infty} \mu_k(B \cap [M \setminus A_n^k(2\zeta_1, \zeta_2/2)]) \\ &\leq \mu(B \cap [M \setminus A_n(\zeta_1, \zeta_2)]). \end{aligned}$$

Finally letting $\zeta_2 \to 0$ first and then $\zeta_1 \to 0$ also the compact sets $M \setminus A_n(3\zeta_1, \zeta_2/3)$ and $M \setminus A_n(\zeta_1, \zeta_2)$ both grow to $M \setminus A_n$ which clearly equals K_n. In the same way $M \setminus A_n^k(2\zeta_1, \zeta_2/2)$ grows to $M \setminus A_n^k = K_n^k$.

This together with the last sequence of inequalities shows that $\mu_k^n(B) \to (\mu \mid K_n)(B)$ when $n \to \infty$, finishing the proof. □

The uniform bound from this lemma is essential to prove

Proposition 5.8. *The probability measure μ is absolutely continuous along the center-unstable direction, that is, μ is a Gibbs cu-state.*

Proof. We know that μ almost every $x \in K$ is such that the corresponding μ_x from the ergodic decomposition of μ is a Gibbs *cu*-state — this follows from item (2) of Proposition 5.5 and the definition of Gibbs *cu*-state. In what follows we define $\eta = \int_K \mu_x \, d\mu(x)$ and prove that $\mu = \eta$.

We note first that η is clearly f-invariant since every μ_x is f-invariant by construction. Moreover each μ_x is a Gibbs *cu*-state by definition so η is a Gibbs *cu*-state. We also remark that $\eta \mid K_n = \mu \mid K_n$ for all $n \geq 1$ as a consequence of the invariance of η and because $\eta \mid K = \mu \mid K$.

For each $n \geq 1$ let us define $\eta_n = \eta \mid K_n$. Then in the weak* topology $\eta_n \to \eta$ when $n \to \infty$ because of the following simple fact.

Lemma 5.9. *For η-almost every z there exists $j \le 0$ such that $f^j(z) \in K$.*

Proof. Given B a Borel subset we have that $\mu_x(B) = \lim_{n\to+\infty} n^{-1}\sum_{j=0}^{n-1}\chi_B(f^j(x))$ for μ-almost every x — recall that $\mu(K) > 0$. Thus $\eta(B) > 0$ ensures that for some $x \in K$ there exists $j \ge 0$ such that $f^j(x) \in B$. □

Moreover Lemma 5.7 implies that $\mu_k^n \to \mu_k$ when $n \to \infty$ in the weak* topology in a uniform way, since $\mu_k(A_n^k) \to 0$ when $n \to \infty$ uniformly in k. In addition, the same lemma ensures that (see also the remark above) for any fixed $n \in \mathbb{N}$ it holds that $\mu_k^n \to \eta_n$ when $k \to \infty$ in the weak* topology.

Let $\zeta > 0$ and a continuous $\varphi : M \to \mathbb{R}$ be given. Then we may find a big enough $n \ge 1$ such that

$$|\eta(\varphi) - \eta_n(\varphi)| \le \zeta \quad \text{and} \quad |\mu_k^n(\varphi) - \mu_k(\varphi)| \le \zeta$$

for every sufficiently big $k \ge 1$ — we stress that for the second inequality we need the uniform bound on $\mu_k(A_n^k)$ provided by Lemma 5.7.

Having fixed n we may now take k big enough keeping the above inequalities and satisfying also

$$|\mu_k(\varphi) - \mu(\varphi)| \le \zeta \quad \text{and} \quad |\eta_n(\varphi) - \mu_k^n(\varphi)| \le \zeta.$$

Finally putting this all together we arrive at

$$\begin{aligned} |\eta(\varphi) - \mu(\varphi)| \;\; \le \;\; & |\eta(\varphi) - \eta_n(\varphi)| + |\eta_n(\varphi) - \mu_k^n(\varphi)| + \\ & |\mu_k^n(\varphi) - \mu_k(\varphi)| + |\mu_k(\varphi) - \mu(\varphi)| \le 4\zeta. \end{aligned}$$

□

As explained in the beginning of Subsection 4.2 this is precisely what is needed to conclude stochastic stability for f. Theorem B is proved.

6. Nonuniform hyperbolicity and existence of SRB measures

Here we prove Theorem G. Let $\mathcal{V}$ be a C^2-neighborhood were f in strongly nonuniformly hyperbolic with $c > 0$ as defined in Subsection 1.2. Then it is straightforward to see that f is nonuniformly expanding along the center-unstable direction for random orbits just by taking any random perturbation $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ of f and letting ε be sufficiently small — all we need is that the image of $\Phi \mid \operatorname{supp}(\theta_\varepsilon)$ be in $\mathcal{V}$. On the other hand, we also have that for every small enough $\varepsilon > 0$

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df|E^{cs}_{f_{\underline{t}}^j x}\| \le -c \tag{16}$$

for Lebesgue almost every $x \in U$ and *every* $\underline{t} \in T^{\mathbb{N}}$. That is, f is nonuniformly contracting along the center-stable direction for random orbits.

6.1. **Existence of SRB measures.** We are in the setting of Theorem 4.3. We know that μ^ε desintegrates along the leaves of certain cylinders with uniformly bounded density and that every weak* accumulation point μ of $(\mu^\varepsilon)_{\varepsilon>0}$, when $\varepsilon \to 0$, is a Gibbs cu-state for f.

The nonuniform contraction on random orbits implies that $\mu^\varepsilon(\log \|Df \mid E^{cs}\|) \le -c < 0$ for every physical measure and for all small enough $\varepsilon > 0$. Since $\log \|Df \mid E^{cs}\|$ is a continuous function on M, we conclude that $\mu(\log \|Df \mid E^{cs}\|) \le -c < 0$ also. Hence there exists an ergodic component η of μ such that

- η is a Gibbs cu-state (by Theorem 4.2);
- $\eta(\log \|Df \mid E^{cs}\|) \le -c/2 < 0$ (by the Ergodic Decomposition Theorem).

This means that every Lyapunov exponent of η is less than $-c/2$ along the center-stable direction. By standard arguments using the absolute continuity of the stable lamination and the absolute continuity of η along the unstable lamination, we conclude that the ergodic basin of η has positive Lebesgue measure in M, see e.g. [18, 7, 2]. We have shown that every weak* accumulation point of physical measures admits an ergodic component which is a SRB measure for f.

6.2. **The number of SRB measures.** Now we prove that the number of ergodic SRB measures is finite. Indeed, according to Theorem 4.2, every SRB measure μ is a Gibbs cu-state. Moreover, the nonuniform expansion along the center-unstable direction provides a cylinder $\mathcal{C}$ with the same properties listed in Proposition 5.5. This follows by the same arguments presented in Subsection 5.1 in the random setting, and can be found in [22]. In addition, since f is nonuniformly contracting along the center-stable direction and $B(\mu)$ has positive volume, Lebesgue almost every point $x \in B(\mu)$ will satisfy (16), so we obtain $\mu(\log \|Df \mid E^{cs}\|) \le -c < 0$. Thus the Lyapunov exponents of μ are negative along the center-stable direction.

Now if there are infinitely many such SRB measures, we may take a sequence μ_n of pairwise distinct ergodic SRB measures and a sequence $\mathcal{C}_n$ of cylinders as above which, by relative compactness, must accumulate on an invariant measure μ and a cylinder $\mathcal{C}$, respectively. Then $\mathcal{C}$ will satisfy each item of Proposition 5.5 with respect to μ. Thus μ will be a Gibbs cu-state — the details of the arguments are the same as the ones used in the random setting and can be seen in [22].

The negative Lyapunov exponents along the center-unstable direction for each μ_n ensure that through Lebesgue almost every point of the disks of the cylinders $\mathcal{C}_n$ there passes a local stable manifold, which will be entirely contained in the basin $B(\mu_n)$.

The accumulation of $\mathcal{C}_n$ on $\mathcal{C}$ ensures that the disks of $\mathcal{C}_n$ will be uniformly close to the disks in $\mathcal{C}$. Hence for big n the local stable manifolds through disks of $\mathcal{C}_n$ will intersect the disks of $\mathcal{C}$.

The absolute continuity of the stable lamination guarantees that the disks in $\mathcal{C}$ will intersect these leaves in a set of positive μ measure. But each disk in $\mathcal{C}$ is a local unstable manifold, so that backward time averages are the same on every point of the disks. The previous discussion shows that the backward time averages on a positive μ measure subset equal the time averages of μ_n, for every big n. Thus, by ergodicity, for some big n we have that μ_m equals μ_n for all $m > n$.

We conclude that there are finitely many $\mu_1, \dots, \mu_p$ ergodic SRB measures for f in U.

We note that if we exchange f by some other $g \in \mathcal{V}$ in the definitions of $H_N^*(\mathcal{V}), * =$cs or cu, the functions $\|Dg^\pm \mid E^*\|$ change by at most a constant factor. Thus taking the neighborhood $\mathcal{V}$ small enough ensures that we may apply the previous reazonings to every $g \in \mathcal{V}$, with just small changes in the value of c. This concludes the proof of Theorem G.

7. A stochastically stable class

In this section we present a robust class of partially hyperbolic diffeomorphisms whose center-unstable direction is nonuniformly expanding. Here we take U equal to M. This presentation follows closely [2] and we just sketch the main points. The C^1 open classes of transitive non-Anosov diffeomorphisms presented in [7, Section 6], as well as other robust examples from [12], are constructed in a similar way.

We start with a linear Anosov diffeomorphism $\hat{f}$ on the d-dimensional torus $M = \mathbb{T}^d$, $d \geq 2$. We write $TM = E^u \oplus E^s$ the corresponding hyperbolic decomposition of the tangent fiber bundle. Let V be a small closed domain in M for which there exist unit open cubes K^0 and K^1 in $\mathbb{R}^d$ such that $V \subset \pi(K^0)$ and $\hat{f}(V) \subset \pi(K^1)$, where $\pi : \mathbb{R}^d \to \mathbb{T}^d$ is the canonical projection. Now, let f be a diffeomorphism on $\mathbb{T}^d$ such that

(A) f admits invariant cone fields C^{cu} and C^{cs}, with small width $a > 0$ and containing, respectively, the unstable bundle E^u and the stable bundle E^s of the Anosov diffeomorphism $\hat{f}$;

(B) f is *volume hyperbolic*: there is $\sigma_1 > 1$ so that
$$|\det(Df|T_x\Delta^{cu})| > \sigma_1 \quad \text{and} \quad |\det(Df|T_x\Delta^{cs})| < \sigma_1^{-1}$$
for any $x \in M$ and any disks Δ^{cu}, Δ^{cs} tangent to C^{cu}, C^{cs}, respectively.

(C) f is C^1-close to $\hat{f}$ in the complement of V, so that there exists $\sigma_2 < 1$ satisfying
$$\|(Df|T_x\Delta^{cu})^{-1}\| < \sigma_2 \quad \text{and} \quad \|Df|T_x\Delta^{cs}\| < \sigma_2$$
for any $x \in (M \setminus V)$ and any disks Δ^{cu}, Δ^{cs} tangent to C^{cu}, C^{cs}, respectively.

(D) there exist some small $\delta_0 > 0$ satisfying
$$\|(Df|T_x\Delta^{cu})^{-1}\| < 1 + \delta_0 \quad \text{and} \quad \|Df|T_x\Delta^{cs}\| < 1 + \delta_0$$
for any $x \in V$ and any disks Δ^{cu} and Δ^{cs} tangent to C^{cu} and C^{cs}, respectively.

Closeness in (C) should be enough to ensure that $f(V)$ is also contained in the projection of a unit open cube. If $\tilde{f}$ is a torus diffeomorphism satisfying (A), (B), (D), and coinciding with $\hat{f}$ outside V, then any map f in a C^1 neighborhood of $\tilde{f}$ satisfies all the previous conditions. Results in [2, Appendix] show in particular that for any f satisfying (A)–(D) there exist $c_s, c_u > 0$ such that f is nonuniformly expanding along its center-unstable direction and nonuniformly contracting along its center-stable direction as in Subsection 1.1. Moreover [2, Remark A.6] also shows that by taking δ_0 small enough, then $A^u = \sup_{f(U)} -\log\|Df^{-1}|E^{cu}\|$ and $A^s = \sup_U -\log\|Df|E^{cs}\|$ can be made arbitrarily close to 0, so that condition (3) holds in a whole C^1-neighborhood of f. Hence, since these maps can be taken to be of class C^2, there exists a C^2 open set of maps having positive density at infinity of simultaneous hyperbolic times.

7.1. **Behavior over random orbits.** Here we take f as described above and T a small neighborhood of f in the C^2 topology in such a way that conditions (A)–(D) hold for every $g \in T$. We define $\Phi(t) = f_t = t$ and take $(\theta_\varepsilon)_{\varepsilon>0}$ a family of measures on T as before. We are going to show that any such f is nonuniformly expanding along the center-unstable direction on random orbits and, in the process, we will also see that this is a neighborhood of strongly nonuniformly hyperbolic maps. This will be done by showing that there is $c > 0$ such that for any disk Δ^{cu} tangent to the center-unstable cone field and $\theta_\varepsilon^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times \Delta^{cu}$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|\big(Df|T_{f_{\underline{t}}^j x}\Delta_j^{cu}(\underline{t})\big)^{-1}\| \le -c, \tag{17}$$

where $\Delta_j^{cu}(\underline{t}) = f_{\underline{t}}^j \Delta^{cu}$. To explain this, let $B_1, \ldots, B_p, B_{p+1} = V$ be any partition of $\mathbb{T}^d$ into small domains, in the same sense as before: there exist open unit cubes K_i^0 and K_i^1 in $\mathbb{R}^d$ such that

$$B_i \subset \pi(K_i^0) \quad \text{and} \quad f(B_i) \subset \pi(K_i^1). \tag{18}$$

Let us fix Δ^{cu} any disk tangent to the center-unstable cone field and define m to be Lebesgue measure in Δ^{cu} normalized so that $m(\Delta^{cu}) = 1$.

Lemma 7.1. *There is $\zeta > 0$ such that for $\theta_\varepsilon^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times \Delta^{cu}$ and large enough $n \ge 1$ we have*

$$\#\{0 \le j < n : f_{\underline{t}}^j(x) \in B_1 \cup \ldots \cup B_p\} \ge \zeta n. \tag{19}$$

Moreover, there is $0 < \tau < 1$ for which the set I_n of points $(\underline{t}, x) \in T^{\mathbb{N}} \times \Delta^{cu}$ whose orbits do not spend a fraction ζ of the time in $B_1 \cup \ldots \cup B_p$ up to iterate n is such that $(\theta_\varepsilon^{\mathbb{N}} \times m)(I_n) \le \tau^n$ for large $n \ge 1$.

Proof. Let us fix $n \ge 1$ and $\underline{t} \in T^{\mathbb{N}}$. For a sequence $\underline{i} = (i_0, \ldots, i_{n-1}) \in \{1, \ldots, p+1\}^n$ we write

$$[\underline{i}] = B_{i_0} \cap (f_{\underline{t}}^1)^{-1}(B_{i_1}) \cap \cdots \cap (f_{\underline{t}}^{n-1})^{-1}(B_{i_{n-1}})$$

and define $g(\underline{i}) = \#\{0 \le j < n : i_j \le p\}$. We start by observing that for $\zeta > 0$ the number of sequences $\underline{i}$ such that $g(\underline{i}) < \zeta n$ is bounded by

$$\sum_{k<\zeta n} \binom{n}{k} p^k \le \sum_{k\le\zeta n} \binom{n}{k} p^{\zeta n}.$$

Using Stirling's formula (cf. [7, Section 6.3]) the expression on the right hand side is bounded by $(e^\gamma p^\zeta)^n$, where $\gamma > 0$ depends only on ζ and $\gamma(\zeta) \to 0$ when $\zeta \to 0$.

Assumption (B) ensures that $m([\underline{i}]) \le \sigma^{-(1-\zeta)n}$ (recall that $m(M) = 1$). Hence the measure of the union $I_n(\underline{t})$ of all the sets $[\underline{i}]$ with $g(\underline{i}) < \zeta n$ is bounded by

$$\sigma^{-(1-\zeta)n}(e^\gamma p^\zeta)^n.$$

Since $\sigma > 1$ we may choose ζ so small that $e^{\gamma} p^{\zeta} < \sigma^{(1-\zeta)}$. Then $m(I_n(\underline{t})) \le \tau^n$ with $\tau = e^{\gamma+\zeta-1} p^{\zeta} < 1$ for big enough $n \ge N$. Note that τ and N do not depend on $\underline{t}$.

Remark 7.2. If $x \in M \setminus I_n(\underline{t})$, then

$$\#\{0 \le j < n : f_{\underline{t}}^j(x) \in B_{p+1}\} \le (1-\zeta) \cdot n$$

by definition of $I_n(\underline{t})$.

Setting

$$I_n = \bigcup_{\underline{t} \in T^{\mathbb{N}}} \big(\{\underline{t}\} \times I_n(\underline{t})\big)$$

we have $(\theta_\varepsilon^{\mathbb{N}} \times m)(I_n) \le \tau^n$ for big $n \ge N$, by Fubini's Theorem. Since $\sum_n (\theta_\varepsilon^{\mathbb{N}} \times m)(I_n) < \infty$ then Borel-Cantelli's Lemma implies

$$(\theta_\varepsilon^{\mathbb{N}} \times m)\left(\bigcap_{n \ge 1} \bigcup_{k \ge n} I_k\right) = 0$$

and this means that $\theta_\varepsilon^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ satisfies (19). □

Lemma 7.3. *For $0 < \lambda < 1$ there are $\eta > 0$ and $c_0 > 0$ such that, if f_t also satisfies conditions (C) and (D) for all $t \in T$, then we have*

1. $m\big(\{x \in M : \sum_{j=0}^{n-1} \log \|\big(Df | T_{f_{\underline{t}}^j x} \Delta_j^{cu}(\underline{t})\big)^{-1}\| \le -cn\}\big) \ge 1 - \tau^n$ *for all $\underline{t} \in T^{\mathbb{N}}$ and for every large n;*
2. $\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|\big(Df | T_{f_{\underline{t}}^j x} \Delta_j^{cu}(\underline{t})\big)^{-1}\| \le -c$ *for $\theta_\varepsilon^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times \Delta^{cu}$.*

Proof. Let $\{B_1, \ldots, B_p, B_{p+1}, \ldots, B_{p+1}\}$ be a measurable cover of M as before and $\zeta > 0$ be the constant provided by Lemma 7.1. We fix $\eta > 0$ sufficiently small so that $\lambda^\zeta (1+\eta) \le e^{-c}$ holds for some $c > 0$ and take $x \in M \setminus I_n(\underline{t})$ for some $n \ge 1$ and $\underline{t} \in T^{\mathbb{N}}$. Conditions (C) and (D) now imply

$$\prod_{j=0}^{n-1} \|\big(Df | T_{f_{\underline{t}}^j x} \Delta_j^{cu}(\underline{t})\big)^{-1}\| \le \lambda^{\zeta n} (1+\eta)^{(1-\zeta)n} \le e^{-cn}. \tag{20}$$

by Remark 7.2. Hence the set in item 1 is contained in $M \setminus I_n(\underline{t})$, proving the statement of this item by the second part of the statement of Lemma 7.1.

This also means that the second item of the statement holds for $\theta^{\mathbb{N}} \times m$ almost every $(\underline{t}, x) \in T^{\mathbb{N}} \times M$ by the statement of Lemma 7.1.

□

Since Δ^{cu} was an arbitrary disk tangent to the center-unstable cone, we conclude from Lemma 7.3 that f is nonuniformly expanding along the center-unstable direction for random orbits. Moreover the same arguments apply verbatim to $\|Df \mid T_{f_{\underline{t}}^j x} \Delta^{cs}(\underline{t})\|$ and any disk Δ^{cs} tangent to the center-stable cone, so that the first item of Lemma 7.3 holds for this cone also.

This means that for a given $\underline{t} \in T^{\mathbb{N}}$ both

$$m(H_n^{cu}(\underline{t})) \ge 1 - \tau^n \quad \text{and} \quad m(H_n^{cs}(\underline{t})) \ge 1 - \tau^n$$

for every big enough $n \geq 1$, showing that T is a C^2 open class of strongly nonunifomrly hyperbolic diffeomorphisms.

References

[1] J. F. Alves, V. Araújo, *Random perturbations of non-uniformly expanding maps*, Astérisque, 286 (2003), 25-62.

[2] J. F. Alves, C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems with mostly expanding central direction*, Invent. Math., 140 (2000), 351-398.

[3] V. Araújo, *Attractors and time averages for random maps*, Ann. de l'Inst. H. Poincaré - Anal. non-Lin. 17 (2000), 307-369.

[4] V. Araújo, *Infinitely Many Stochastically Stable Attractors*, Nonlinearity, 14 (2001), 583-596.

[5] M. Brin and Yu. Kifer, *Dynamics of Markov chains and stable manifolds for random diffeomorphisms*, Erg. Th. & Dyn. Sys., 7 (1987), 351-374.

[6] C. Bonatti, L. Díaz and M. Viana, *Dynamics beyond uniform hyperbolicity: a global geometric and probabilistic approach*, Preprint IMPA, Serie A (2003/221), 2003.

[7] C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems with mostly contracting central direction*, Isr. J. Math. 115 (2000), 157-193.

[8] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, *Lect. Notes in Math.*, 470, Springer-Verlag, New-York, 1975.

[9] R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows* , Invent. Math., 29 (1975), 181-202.

[10] P.-D. Liu and M. Qian, Smooth ergodic theory of random dynamical systems, *Lecture Notes in Math.* 1606, Springer-Verlag, Berlin, 1995.

[11] P.-D. Liu, *Dynamics of random transformations: smooth ergodic theory*, Erg. Th. and Dyn. Syst., 21 (2001), 1279-1319.

[12] R. Mañé, *Contributions to the stability conjecture*, Topology, 17 (1978), 383-396.

[13] R. Mañé, Ergodic theory and differentiable dynamics, Springer-Verlag, New-York, 1987.

[14] ,J. Palis, *A global view of Dynamics and a conjecture on the denseness of finitude of attractors*, Astérisque, "Géométrie complexe et systèmes dynamiques (Orsay, 1995)", 261, 2000, 335-347.

[15] Ya. Pesin. *Families of invariant manifolds corresponding to non-zero characteristic exponents.* Math. USSR. Izv., 10 (1976), 1261–1302.

[16] Ya. Pesin and Ya. Sinai *Gibbs measures for partially hyperbolic attractors.* Ergod. Th. & Dynam. Sys., 2 (1982), 417–438.

[17] V. Pliss, *On a conjecture due to Smale*, Diff. Uravnenija, 8 (1972), 262-268.

[18] Pugh, C and M. Shub, *Ergodic Attractors*, Trans. Amer. Math. Soc., 312 (1989), 1-54.

[19] D. Ruelle, *A measure associated with Axiom A attractors*, Amer. J. Math., 98 (1976), 619-654.

[20] Ya. Sinai, *Gibbs measures in ergodic theory* , Russian Math. Surveys, 27 (1972), 21-69.

[21] A. Tahzibi, *Stably ergodic systems that are not partially hyperbolic*, Israel Jr. of Math., to appear. [`www.preprint.impa.br/Shadows/SERIE_C/2002/9.html`]

[22] C. Vásquez, *Statistical stability for diffeomorphisms with dominated splitting*, PhD. Thesis, IMPA, 2003. [`www.preprint.impa.br/Shadows/SERIE_A/2003/200.html`]

[23] M. Viana, Stochastic Dynamics of Deterministic Systems, *Lect. Notes XXI Braz. Math Colloq.*, IMPA, Rio de Janeiro, 1997.

[24] L.-S. Young, *Stochastic stability of hyperbolic attractors*, Erg. Th. & Dyn. Sys. 6 (1986), 311-319.

Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal
E-mail address: `jfalves@fc.up.pt`
URL: `http://www.fc.up.pt/cmup/home/jfalves`

Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal
E-mail address: `vdaraujo@fc.up.pt`
URL: `http://www.fc.up.pt/cmup/home/vdaraujo`

Departamento de Matemática, Universidad Catolica del Norte, Av. Angamos 0610, Antofagasta-Chile
E-mail address: `cavasquez@ucn.cl`