Linear Equivalence and ODE-equivalence for Coupled Cell Networks

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Abstract

Coupled cell systems are systems of ODEs, defined by 'admissible' vector fields, associated with a network whose nodes represent variables and whose edges specify couplings between nodes. It is known that non-isomorphic networks can correspond to the same space of admissible vector fields. Such networks are said to be 'ODE-equivalent'. We prove that two networks are ODE-equivalent if and only if they determine the same space of *linear* vector fields; moreover, the variable associated with each node may be assumed 1-dimensional for that purpose. We briefly discuss the combinatorics of the resulting linear algebra problem.

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1 Introduction

Networks of nonlinear dynamical systems have become the topic of considerable attention recently, mainly because a wide variety of physical and biological systems can naturally be modelled by such networks, see Wang [12], Stewart [10]. The theoretical understanding of such systems is also under intensive development. Of course, every (finite) network of dynamical systems can be considered as a single dynamical system, and every dynamical system is trivially a network with only one node and no edges, so it might seem that networks offer no gain in generality. However, networks possess additional structure, namely,

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canonical observables—the dynamical behaviour of the individual nodes [5]. These observables can be compared, revealing such features as synchrony or phase-relations, and it is precisely these features that are important in many applications. Any theoretical treatment of network dynamics must therefore take this additional structure into account, so conventional dynamical systems theory must be modified to preserve that structure. The topology (or 'architecture') of the network imposes constraints on the dynamics, with the result that many new phenomena become 'generic' for a given architecture, see for example Golubitsky *et al.* [4].

A network (or graph) is a schematic representation of a set of dynamical systems (that is, ordinary differential equations or ODEs) that are coupled together. The nodes of the graph ('cells' of the network) represent the individual dynamical sytems, and the directed edges ('arrows') represent couplings. One formulation of this idea is known as 'coupled cell systems', and it provides a convenient formal framework for the theory. In this formulation, introduced by Stewart *et al.* [11] and extended into a technically more convenient form by Golubitsky *et al.* [7], both arrows and cells are labelled to indicate various 'types' of dynamical behaviour. To each cell c is associated a choice of 'cell phase space' P_c, which we will assume is a finite-dimensional vector space $\mathbf{R}^k$ over $\mathbf{R}$, where k may depend on c. (More generally, it could be a finite-dimensional smooth manifold, but we do not consider this generalization here.) The overall phase space of the coupled cell system is P, the direct product of the P_c over all cells c.

Associated with each network G is a class of differential equations on P, which correspond to 'admissible' vector fields on P. These are the ODEs that are compatible with the network topology and the choice of cell phase spaces. The admissible vector fields can be characterised in terms of an algebraic structure known as the 'symmetry groupoid' of the network. A groupoid is similar to a group, except that product of two elements may not always be defined. The symmetry groupoid $\mathcal{B}_G$ consists of all 'input isomorphisms' between pairs of cells c, d—that is, type-preserving bijections between the set of arrows entering cell c and the corresponding set for cell d. The admissible vector fields then turn out to be precisely those that are equivariant under a natural action of the groupoid $\mathcal{B}_G$ on P, in a sense that generalizes the usual notion of equivariance under the action of a group [5, 6].

It was observed in [7] that topologically distinct coupled cell networks can give rise to the same space of admissible vector fields (for a suitable choice of cell phase spaces), a phenomenon known as 'ODE-equivalence'. The aim of this paper is to investigate the conditions under which two networks are ODE-equivalent. Here we prove two main theorems. The first (Theorem 7.1 below) reduces the problem of ODE-equivalence to 'linear equivalence', where two networks (with suitably identified phase spaces) are linearly equivalent if they determine the same space of *linear* admissible vector fields. (The definition we use is actually more technical.) The second (a simple but useful corollary) is that when deciding linear equivalence, it can without loss of generality be assumed that each cell phase space is 1-dimensional (Corollary 7.7).

We also discuss the characterization of linearly equivalent networks, reducing this question to a combinatorial condition in linear algebra. In a sense, this condition completely solves the problem of linear equivalence, hence of ODE-equivalence. However, the relation between network topology and the linear algebra condition is deceptively simple; in particular, there seems to be no straightforward combinatorial condition on the two networks that determines linear equivalence, other than a suitably 'encoded' form of the linear algebra condition. This topic will be the subject of future work by Aguiar and Dias [1].

Sections 2, 3, 4 of the paper provide formal definitions for, and basic properties of, coupled cell networks, the associated symmetry groupoid, and admissible vector fields. Section 5 defines ODE-equivalence. Section 6 discusses linear equivalence, including a typical example that shows how the network topology encodes a linear algebra condition. Section 7 proves the main theorem that ODE-equivalence is the same as linear equivalence, and deduces as a corollary that linear equivalence does not depend on the choice of cell phase spaces (provided their dimensions are at least 1), so that when deciding linear (hence ODE) equivalence, all cells may be assumed to have 1-dimensional phase spaces. Finally Section 8 provides a brief discussion of the combinatorial issues associated with linear equivalence.

2 Coupled Cell Networks

A coupled cell network can be represented schematically by a directed graph (see for example Figures 1, 2, 3 below) whose nodes correspond to cells and whose edges represent couplings. We employ the following definition, introduced by Golubitsky*et al.* [7], which permits multiple arrows and self-coupling. This formulation has several technical advantages over the more restricted version described in [11].

Definition 2.1 [7] In the *multiarrow formalism*, a *coupled cell network* G consists of:

(a) A finite set $\mathcal{C} = \{1, \ldots, n\}$ of *nodes* (or *cells*).

(b) An equivalence relation $\sim_C$ on the nodes in $\mathcal{C}$.
The *type* or *cell label* of cell c is the $\sim_C$-equivalence class $[c]_C$ of c.

(c) Associated with each node c is a finite set of *input terminals* $I(c)$. Each input terminal $i \in I(c)$ is the receptacle for one *arrow* or *edge* that begins in *tail cell* $\tau(i)$ and ends in terminal i. That arrow is denoted by $e = (\tau(i), i)$, and it has *head cell* c and *head terminal* i. Let $\mathcal{E}$ denote the set of all arrows.

(d) An equivalence relation $\sim_E$ on the edges in $\mathcal{E}$.
The *type* or *coupling label* of edge e is the $\sim_E$-equivalence class $[e]_E$ of e.

(e) Equivalent edges have equivalent tails and heads. That is, if $(\tau(i), i) \sim_E (\tau(j), j)$ where $i \in I(c)$ and $j \in I(d)$, then $\tau(i) \sim_C \tau(j)$ and $c \sim_C d$.

We write $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$. ◇

Observe that in this definition of coupled cell network, self-coupling is permitted since $\tau(i) = c$ for a terminal i in cell c is permitted. Also multiarrows are permitted since we can have $\tau(i) = \tau(j)$ for two distinct terminals i, j in the same cell c.

Remark 2.2 It is possible to avoid explicit use of terminals since they are in one-to-one correspondence with arrows (via the map $(\tau(i), i) \mapsto i$). We therefore follow [7] and omit explicit terminals from all network diagrams. Implicitly, a terminal is determined by the head end of the corresponding arrow. ◇

3 Symmetry Groupoid of a Coupled Cell Network

Given a graph $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ as in Definition 2.1, we can define the 'symmetry groupoid' $\mathcal{B}_G$ of G. This definition centres upon the notion of 'input set'.

Definition 3.1 Following [**7**], let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network. The relation $\sim_I$ of *input-equivalence* on $\mathcal{C}$ is defined by $c \sim_I d$ if and only if there exists a bijection

$$\beta : \ I(c) \to I(d)$$

that preserves edge type. That is, for every input terminal $i \in I(c)$

$$(\tau(i), i) \sim_E (\tau(\beta(i)), \beta(i))$$

Any such β is called an *input isomorphism* from cell c to cell d. We denote the set of all input isomorphisms from cell c to cell d by $B(c, d)$, and define

$$\mathcal{B}_G = \dot{\bigcup_{c,d \,\in\, \mathcal{C}}} B(c, d)$$

where $\dot{\cup}$ indicates disjoint union. A natural product operation can be defined on $\mathcal{B}_G$ as follows: elements $\beta_2 \in B(c, d)$ and $\beta_1 \in B(a, b)$ can be multiplied only when $b = c$, and in this case $\beta_2\beta_1 \in B(a, d)$ is the usual composition of functions. Now $\mathcal{B}_G$ is a groupoid whose objects are the nodes of G, and the $\mathcal{B}_G$-morphisms are the elements of the sets $B(c, d)$, with the product operation between the morphisms as defined above. Some references on groupoids are Brandt [2], Brown [3] and Higgins [8]. Following [7, 11] we call $\mathcal{B}_G$ the *symmetry groupoid of the network* G. For any $c \in \mathcal{C}$, the subset $B(c, c)$ is a group, the *vertex group* corresponding to c. $\diamond$

Structure of $B(c, d)$

Let $B(c, d) \subseteq \mathcal{B}_G$. We can specify the structure of the set $B(c, d)$ in terms of the structure of G. We distinguish three cases:

1. If $c \not\sim_I d$ then $B(c, d) = \emptyset$.

2. If $c = d$ we can define an equivalence relation $\equiv_c$ on $I(c)$ by

$$j_1 \equiv_c j_2 \iff (\tau(j_1), j_1) \sim_E (\tau(j_2), j_2) \tag{3.1}$$

   where $j_1, j_2 \in I(c)$. If $K_1, K_2, \ldots, K_{r(c)}$ are the $\equiv_c$-equivalence classes (on $I(c)$), then

$$B(c, c) = \mathbf{S}_{K_1} \times \cdots \times \mathbf{S}_{K_{r(c)}} \subseteq \mathbf{S}_n \tag{3.2}$$

   where each $\mathbf{S}_{K_i}$ comprises all permutations of the set K_i, extended by the identity on $I(c) \setminus K_i$, and $n = |\mathcal{C}|$.

3. If $c \neq d$ and $c \sim_I d$ (so that $B(c, d) \neq \emptyset$), then for *any* $\beta \in B(c, d)$ we have

$$B(c, d) = \beta B(c, c) = B(d, d)\beta$$

For proofs of the above facts see [11], end of Section 3.

4 Admissible Vector Fields

We make now precise the connection between coupled cell systems and coupled cell networks. Essentially, the network is a schematic diagram (graph), whereas the system is a set of ODEs whose couplings correspond to the edges of the network. To obtain these ODEs we must associate variables x_c with cells c, that is, we must choose a phase space for each cell.

By a *coupled cell system* we mean a network of dynamical systems coupled together, where we use a labelled directed graph G (that is, a coupled cell network in the sense of Definition 2.1), whose nodes correspond to *cells*, and whose edges represent *couplings*. The term 'coupling' here is used in the sense that the output of certain cells affects the time-evolution of other cells.

Again, we follow the treatment of Stewart *et al.* [11] and Golubitsky *et al.* [7]. Consider a coupled cell network $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$, with symmetry groupoid $\mathcal{B}_G$. We wish to define a class $\mathcal{F}_G^P$ of 'admissible' vector fields corresponding to G. This class consists of all vector fields that are 'compatible' with the labelled graph structure, and it depends on a choice of 'total phase space' P.

To each cell $c \in \mathcal{C}$ we associate a *cell phase space* P_c, which for simplicity we assume is a nonzero finite-dimensional real vector space.

If c, d are in the same $\sim_C$-equivalence class, then we insist that $P_c = P_d$, and we identify these spaces canonically. The *total phase space* is

$$P = \prod_{c \in \mathcal{C}} P_c$$

with coordinate system

$$x = (x_c)_{c \in \mathcal{C}}$$

on P. If $\mathcal{D} = (d_1, \ldots, d_s)$ is any finite ordered subset of s cells in $\mathcal{C}$ we define

$$P_{\mathcal{D}} = P_{d_1} \times \cdots \times P_{d_s}$$

and we write

$$x_{\mathcal{D}} = (x_{d_1}, \ldots, x_{d_s})$$

where $x_{d_i} \in P_{d_i}$. Note that the same cell can appear more than once in $\mathcal{D}$. (This condition must be permitted because of the multiarrow formalism.)

Suppose that $c \sim_I d$ and consider the ordered sets $\mathcal{D}_1 = \tau(I(c)), \mathcal{D}_2 = \tau(I(d))$ of $\mathcal{C}$. Let $\beta \in B(c, d)$. Then β is a bijection between $I(c)$ and $I(d)$. Moreover for all $i \in I(c)$ we have $(\tau(i), i) \sim_E (\tau(\beta(i)), \beta(i))$, and so $\tau(i) \sim_C \tau(\beta(i))$. We can define the *pullback map*

$$\beta^* : \ P_{\mathcal{D}_2} \to P_{\mathcal{D}_1}$$

by

$$\big(\beta^*(z)\big)_{\tau(j)} = z_{\tau(\beta(j))}$$

for all $\tau(j) \in \mathcal{D}_1$ and $z \in P_{\mathcal{D}_2}$. If $\tau\left(I(c)\right) = (\tau(i_1), \ldots, \tau(i_s))$ then $x_{\tau(I(c))} = \left(x_{\tau(i_1)}, \ldots, x_{\tau(i_s)}\right)$ and $\beta^*\left(x_{\tau(I(d))}\right) = \left(x_{\tau(\beta(i_1))}, \ldots, x_{\tau(\beta(i_s))}\right)$.

We use pullback maps to relate different components of a vector field associated with a given coupled cell network.

For a given cell c the *internal phase space* is P_c and the *coupling phase space* is

$$P_{\tau(I(c))} = P_{\tau(i_1)} \times \cdots \times P_{\tau(i_s)}$$

where as before $\tau(I(c))$ denotes the ordered set of cells $(\tau(i_1), \ldots, \tau(i_s))$.

Definition 4.1 [7] Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network with symmetry groupoid $\mathcal{B}_G$. For a given choice of the P_c, a (smooth) vector field $f: P \to P$ is *$\mathcal{B}_G$-equivariant* or *G-admissible* if:

(a) *Domain Condition*: For any $c \in \mathcal{C}$ the component $f_c(x)$ depends only on the internal phase space variable x_c and the coupling phase space variables $x_{\tau(I(c))}$. That is, there exists a (smooth) function $\hat{f}_c: P_c \times P_{\tau(I(c))} \to P_c$ such that

$$f_c(x) = \hat{f}_c\big(x_c, x_{\tau(I(c))}\big)$$

(b) *Pullback Condition*: For all $c, d \in \mathcal{C}$ and $\beta \in B(c, d)$

$$\hat{f}_d\big(x_d, x_{\tau(I(d))}\big) = \hat{f}_c\big(x_d, \beta^*\left(x_{\tau(I(d))}\right)\big)$$

for all $x \in P$.

◇

Theorem 4.2 *Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network and $\mathcal{B}_G$ the corresponding symmetry groupoid. A vector field $f: P \to P$ for a given choice of the P_c is $\mathcal{B}_G$-equivariant if and only if for each connected component $\mathcal{Q}$ of $\mathcal{B}_G$ (that is, each $\sim_I$-equivalence class)*

(a) $\hat{f}_c$ is $B(c, c)$-invariant for some $c \in \mathcal{Q}$.

(b) For $d \in \mathcal{Q}$ such that $d \neq c$, given (any) $\beta \in B(c, d)$, we have

$$f_d\big(x_d, x_{\tau(I(d))}\big) = \hat{f}_c\big(x_d, \beta^*\left(x_{\tau(I(d))}\right)\big)$$

Proof This is a generalization of Lemma [11] 4.5 and is proved the same way. □

Now we introduce notation for the space of G-admissible vector fields on P:

Definition 4.3 Let G be a coupled cell network. For a given choice of the P_c, define $\mathcal{F}_G^P$ to consist of all smooth G-admissible vector fields on P. Clearly $\mathcal{F}_G^P$ is a vector space over $\mathbf{R}$. Like all function spaces, it can be equipped with a variety of topologies, but here only the vector space structure is relevant. Let $\mathcal{P}_G^P$ be the subspace of $\mathcal{F}_G^P$ consisting of the G-admissible polynomial vector fields on P, and let $\mathcal{L}_G^P$ be the subspace of $\mathcal{P}_G^P$ consisting of the G-admissible linear vector fields on P. ◇

The space of $\mathcal{B}_G$-equivariant maps has a natural decomposition according to the 'connected components' of the groupoid $\mathcal{B}_G$, and this decomposition is inherited by the polynomial and linear vector fields:

Definition 4.4 Let $\mathcal{Q} \subseteq \mathcal{C}$ be an $\sim_I$-equivalence class. Define

$$\begin{aligned} \mathcal{F}_G^P(\mathcal{Q}) &= \{f \in \mathcal{F}_G^P : \ f_s(x) = 0, \ \forall s \notin \mathcal{Q}\} \\ \mathcal{P}_G^P(\mathcal{Q}) &= \{f \in \mathcal{P}_G^P : \ f_s(x) = 0, \ \forall s \notin \mathcal{Q}\} \\ \mathcal{L}_G^P(\mathcal{Q}) &= \{f \in \mathcal{L}_G^P : \ f_s(x) = 0, \ \forall s \notin \mathcal{Q}\} \end{aligned}$$

We say that vector fields in $\mathcal{F}_G^P(\mathcal{Q})$, $\mathcal{P}_G^P(\mathcal{Q})$, and $\mathcal{L}_G^P(\mathcal{Q})$ are *supported on* $\mathcal{Q}$. $\diamond$

Remark 4.5 From the above theorem there are direct sum decompositions

$$\mathcal{F}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{F}_G^P(\mathcal{Q}) \qquad \mathcal{P}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{P}_G^P(\mathcal{Q}) \qquad \mathcal{L}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{L}_G^P(\mathcal{Q})$$

where $\mathcal{Q}$ runs over the $\sim_I$-equivalence classes of G. $\diamond$

For detailed proofs see [11], end of Section 4, especially Proposition 4.6.

5 ODE-equivalence

As pointed by Golubitsky *et al.* [7], in the class of coupled cell networks that permits self-coupling and multiarrows, it is possible for two different coupled cell networks G_1 and G_2 to generate the same space of admissible vector fields. Figure 1 shows a simple example, taken from Golubitsky *et al.* [7]. In G_1 both cells have the same cell type, and similarly for G_2. Suppose that the phase space for all four cells is $\mathbf{R}^k$ and identify these spaces canonically. Then the total phase space for both G_1 and G_2 is $\mathbf{R}^k \times \mathbf{R}^k$.

The admissible vector fields for G_1 have the form

$$H(x_1, x_2) = (h(x_1, x_1, x_2), h(x_2, x_2, x_1))$$

where $h : \ \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$ is any smooth function, and the admissible vector fields for G_2 have the form

$$F(x_1, x_2) = (f(x_1, x_2), f(x_2, x_1))$$

where $f : \ \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$ is any smooth function. It is now easy to see that the set $\{H\}$ of all H is the same as the set $\{F\}$ of all F. Namely, given f we can set $h(x, y, z) = f(x, z)$, so that $\{H\} \subseteq \{F\}$. Given h we can set $f(a, b) = h(a, a, b)$ so that $\{F\} \subseteq \{H\}$. Therefore the spaces $\mathcal{F}_{G_1}^{P_1}$ and $\mathcal{F}_{G_2}^{P_2}$ are the same.

For a less trivial example, see Figure 2 of Section 6. Note that the above comparison of admissible vector fields involves identifying cells in the two networks, a step that we formalise in general in terms of a bijection between the two sets of cells.

In the next definition, given a coupled cell network G_i and a choice of total phase space P_i for G_i, we denote by $P_{i,c}$ the cell phase space corresponding to cell c of $\mathcal{C}_i$.

Figure 1: Two coupled cell networks G_1 (on the left) and G_2 (on the right) that generate the same space of admissible vector fields.

Definition 5.1 Two coupled cell networks G_1 and G_2 are γ-*ODE-equivalent* if:

1. There is a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence, such that:

2. If we choose cell phase spaces $P_c \neq 0$ for G_1, and define the corresponding choice of cell phase spaces for G_2 by
$$P_{2,\gamma(c)} = P_{1,c}$$
so that the corresponding total phase spaces are
$$P_1 = \prod_{c\in\mathcal{C}_1} P_{1,c} \qquad P_2 = \prod_{c\in\mathcal{C}_1} P_{1,\gamma(c)}$$
then:

3. The condition
$$\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2} \tag{5.3}$$
is satisfied.

Two coupled cell networks G_1 and G_2 are *ODE-equivalent* if they are γ-ODE-equivalent for some bijection γ. $\diamond$

Remarks 5.2 (a) The cells of G_2 can be renumbered so that $\gamma = \text{id}$. In this case, we omit explicit reference to γ.

(b) It is shown in Section 7 below that if (5.3) holds for some choice of nonzero cell phase spaces P_c, then it holds for all such choices. We postpone proving this fact until we have looked at a typical example, which makes the result obvious.

It follows that ODE-equivalence of two networks depends only on their architecture, and not on the particular choice of cell phase spaces. Note, however, the appearance of the bijection γ that associates cells in the two networks (and must preserve cell-equivalence), and the requirement that $P_{\gamma(c)} = P_c$.

$\diamond$

Isomorphic networks (in the usual graph-theoretic sense) are always ODE-equivalent. As pointed out by Golubitsky *et al.* [7], ODE-equivalent networks are not necessarily isomorphic (see for instance Figure 1). The aim of this paper is to describe necessary and sufficient conditions for two coupled cell networks to be ODE-equivalent. In the next section we define the notion of 'linear equivalence' between two networks. We show in Section 7 that two coupled cell networks are ODE-equivalent if and only if they are linearly equivalent.

6 Linear Equivalence

In this section we define the notion of 'linear equivalence' (Definition 6.4 below). We start with an example to illustrate the ideas involved, and, in particular, the effect of multiple arrows.

Example 6.1 Consider the coupled cell networks G_1 and G_2 of Figure 2. Here all cells are cell-equivalent in each graph, and the $\sim_I$-equivalence classes of both graphs are:

$$\mathcal{Q}_1 = \{1,2,3\},\ \mathcal{Q}_2 = \{4\}$$

The identity function on $\{1,2,3,4\} = \mathcal{C}_1 = \mathcal{C}_2$ preserves cell-equivalence and input-equivalence.

Figure 2: Coupled cell networks G_1 (left) and G_2 (right). The number k attached to the right of each edge symbolizes k edges of that type.

First, choose all cell phase spaces to be $P_c = \mathbf{R}$. We now describe the linear admissible vector fields for both graphs, that is, the spaces $\mathcal{L}^P_{G_1}$ and $\mathcal{L}^P_{G_2}$ of linear groupoid-equivariant maps. Let Y_c denote coordinates on the phase space of cell c, for $c = 1, \ldots, 4$, in both graphs. Any linear G_1-admissible vector field $F = (f_1, f_2, f_3, f_4) :\ \mathbf{R}^4 \to \mathbf{R}^4$ has the form:

$$\begin{aligned}
f_1(Y_1) &= aY_1 \\
f_2(Y_2) &= aY_2 \\
f_3(Y_3) &= aY_3 \\
f_4(Y_4, Y_1, Y_2, Y_3) &= bY_4 + c(5Y_1 + Y_3) + d(2Y_1 + Y_2 + Y_3)
\end{aligned}$$

where a, b, c, d are real constants, and any linear G_2-admissible vector field $G = (g_1, g_2, g_3, g_4) :$ $\mathbf{R}^4 \to \mathbf{R}^4$ has the form:

$$\begin{aligned}
g_1(Y_1) &= eY_1 \\
g_2(Y_2) &= eY_2 \\
g_3(Y_3) &= eY_3 \\
g_4(Y_4, Y_1, Y_2, Y_3) &= hY_4 + j(5Y_1 + Y_3) + l(5Y_2 + 3Y_3)
\end{aligned}$$

where e, h, j, l are real constants. Now recall Definition 4.1, and use the notation $\mathbf{R}\{z_1, \ldots, z_m\}$ for the real vector space spanned by $z_1, \ldots, z_m$. It is clear that

$$\mathbf{R}\{Y_4,\ 5Y_1 + Y_3,\ 2Y_1 + Y_2 + Y_3\} = \mathbf{R}\{Y_4,\ 5Y_1 + Y_3,\ 5Y_2 + 3Y_3\} \tag{6.4}$$

since $5Y_2+3Y_3 = 5(2Y_1+Y_2+Y_3)-2(5Y_1+Y_3)$ and $2Y_1+Y_2+Y_3 = \frac{2}{5}(5Y_1+Y_3)+\frac{1}{5}(5Y_2+3Y_3)$. Therefore the space $\mathcal{L}^P_{G_1}$ of linear G_1-admissible vector fields on $\mathbf{R}^4$ equals the space $\mathcal{L}^P_{G_2}$ of linear G_2-admissible vector fields on $\mathbf{R}^4$. We prove in Theorem 7.1 that this is a necessary and sufficient condition for the graphs G_1 and G_2 to be ODE-equivalent. $\diamond$

If we let $P_c = \mathbf{R}^k$ for $k > 1$ the identical calculation can be carried over, with the only change being that the Y_j now represent arbitrary vectors in $\mathbf{R}^k$. However, condition (6.4) can be interpreted as the condition that the rows of the 3×4 matrices

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 5 & 0 & 3 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 5 & 0 & 1 & 0 \\ 0 & 5 & 3 & 0 \end{bmatrix}$$

should span the same subspaces of $\mathbf{R}^4$. The entries in these matrices are determined by the corresponding network topology, so this condition does not depend on the size of k. This fact generalises, see Corollary 7.7 below. (It is also easy to give an independent proof, along the lines of the above example.)

Definition of Linear Equivalence

We introduce some notation before we define linear equivalence between coupled cell networks. Consider two coupled cell networks $G_i = (\mathcal{C}_i, \mathcal{E}_i, \sim_{C_i}, \sim_{E_i})$ for $i = 1, 2$ such that there is a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ preserving cell-equivalence and input-equivalence. Given a connected component $\mathcal{Q}$ of $\mathcal{B}_{G_1}$ and $c \in \mathcal{Q}$, consider

$$I_1(c) = K_1 \,\dot{\cup} \cdots \dot{\cup}\, K_{r_1(c)}$$

where $K_1, \ldots, K_{r_1(c)}$ are the $\equiv_c$-equivalence classes (on $I_1(c)$) and $n_1(c)$ is the cardinality of $I_1(c)$. (See (3.1) for the definition of the relation $\equiv_c$.) Consider

$$I_2(\gamma(c)) = L_1 \,\dot{\cup} \cdots \dot{\cup}\, L_{r_2(\gamma(c))}$$

where $L_1, \ldots, L_{r_2(\gamma(c))}$ are the $\equiv_{\gamma(c)}$-equivalence classes on $I_2(\gamma(c))$, and $n_2(\gamma(c))$ is the cardinality of $I_2(\gamma(c))$.

We use the notation $\mathbf{R}[z_1, \ldots, z_m]$ for the polynomial ring in indeterminates $z_1, \ldots, z_m$ over $\mathbf{R}$, and $\mathbf{R}\{z_1, \ldots, z_m\}$ for the real vector space spanned by $z_1, \ldots, z_m$. Let

$$R_1 = \mathbf{R}[Y_{\tau_1(1)}, \ldots, Y_{\tau_1(n_1(c))}]$$

be the real vector space of polynomials in the indeterminates $Y_{\tau_1(1)}, \ldots, Y_{\tau_1(n_1(c))}$, where $\tau_1(1), \ldots, \tau_1(n_1(c)) \in \mathcal{C}_1$.

Remark 6.2 We avoid notational complications here if we permit repetition of the indeterminates (that is, we allow $z_i = z_j$ when $i \neq j$), and interpret the resulting ring of polynomials to be the same as the ring obtained when any repeated indeterminates are replaced by the corresponding single indeterminate. Again, this convention arises from the multiarrow formalism. It amounts to performing calculations in the polynomial ring $\mathbf{R}[z_1, \ldots, z_m]$ where the z_j are independent indeterminates, and then applying a ring homomorphism to identify various z_i. $\diamond$

Let

$$R_2 = \mathbf{R}[Y_{\gamma^{-1}(\tau_2(1))}, \ldots, Y_{\gamma^{-1}(\tau_2(n_2(\gamma(c))))}]$$

be the real vector space of polynomials in the indeterminates $Y_{\gamma^{-1}(\tau_2(1))}, \ldots, Y_{\gamma^{-1}(\tau_2(n_2(\gamma(c))))}$, where $\tau_2(1), \ldots, \tau_2(n_2(\gamma(c))) \in \mathcal{C}_2$.

Consider the subspace S_1^c of R_1 defined by

$$S_1^c = \mathbf{R}\left\{ Y_c, \sum_{i \in K_1} Y_{\tau_1(i)},\ \ldots,\ \sum_{i \in K_{r_1(c)}} Y_{\tau_1(i)} \right\} \tag{6.5}$$

Thus S_1^c contains the linear polynomials of R_1 that are $B_1(c, c)$-invariant. Similarly, let

$$S_2^{\gamma(c)} = \mathbf{R}\left\{ Y_c, \sum_{i \in L_1} Y_{\gamma^{-1}(\tau_2(i))},\ \ldots,\ \sum_{i \in L_{r_2(\gamma(c))}} Y_{\gamma^{-1}(\tau_2(i))} \right\} \subseteq R_2 \tag{6.6}$$

be formed by the linear polynomials of R_2 that are $B_2(\gamma(c), \gamma(c))$-invariant.

Example 6.3 Recall the coupled cell networks G_1 and G_2 of Figure 2. For both networks, all cells are cell-equivalent and the $\sim_I$-equivalence classes are $\mathcal{Q}_1 = \{1, 2, 3\}$ and $\mathcal{Q}_2 = \{4\}$. Thus the identity function γ on $\{1, 2, 3, 4\} = \mathcal{C}_1 = \mathcal{C}_2$ preserves cell-equivalence and input-equivalence. Consider $\mathcal{Q}_2$ and recall (6.5) and (6.6) where now $c = 4 = \gamma(4)$. Then

$$S_1^4 = \mathbf{R}\left\{Y_4,\ 5Y_1 + Y_3,\ 2Y_1 + Y_2 + Y_3\right\}$$

and

$$S_2^{\gamma(4)} = \mathbf{R}\left\{Y_4,\ 5Y_1 + Y_3,\ 5Y_2 + 3Y_3\right\}$$

As noted earlier, $S_1^4 = S_2^{\gamma(4)}$. Moreover,

$$S_1^i = S_2^{\gamma(i)} = \mathbf{R}\left\{Y_i\right\}$$

for $i = 1, 2, 3$.

Definition 6.4 Two coupled cell networks G_1 and G_2 are γ-*linearly equivalent* if:

1. There is a bijection $\gamma :\ \mathcal{C}_1 \to \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence, such that:

2. For each connected component $\mathcal{Q}$ of the network G_1 and for each $c \in \mathcal{Q}$ we have

$$S_1^c = S_2^{\gamma(c)}$$

where $S_1^c, S_2^{\gamma(c)}$ are as defined in (6.5) and (6.6).

Two coupled cell networks G_1 and G_2 are *linearly equivalent* if they are γ-linearly equivalent for some γ. $\diamond$

Note that this definition is independent of the dimensions of the P_c. Again, we may renumber to make γ the identity.

Example 6.5 We return to Example 6.3. Recall the coupled cell networks G_1 and G_2 of Figure 2. Let γ denote the identity on the set $\{1,2,3,4\}$. Then G_1 and G_2 are γ-linearly equivalent since $S_1^c = S_2^{\gamma(c)}$ for all $c \in \{1,2,3,4\}$. We show in Theorem 7.1 that this is necessary and sufficient for G_1 and G_2 to be ODE-equivalent. As a corollary, $\mathcal{L}_{G_1}^P = \mathcal{L}_{G_2}^P$ for *any* choice of P compatible with cell-equivalence.

7 Linear Equivalence and ODE-equivalence

We now come to the main theorem of this paper, which reduces ODE-equivalence to linear equivalence, and its corollary, that the cell phase spaces may be assumed 1-dimensional in that context. Recall Definition 5.1 of ODE-equivalence and Definition 6.4 of linear equivalence of coupled cell networks. Our main result is:

Theorem 7.1 *Let $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ be a bijection that preserves cell-equivalence and input-equivalence. Then two coupled cell networks G_1 and G_2 are γ-ODE-equivalent if and only if they are γ-linearly equivalent.*

Proof The proof is divided in two steps. We prove in Proposition 7.2 below that given two coupled cell networks G_1 and G_2 and a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ preserving cell-equivalence and input-equivalence, together with a choice of total phase space P_1 for G_1 and P_2 for G_2 according to Definition 5.1, then $\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2}$ if and only if $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$. The rest of the proof consists in proving in Proposition 7.3 below that $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$ if and only if G_1 and G_2 are γ-linearly equivalent for some bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ preserving cell-equivalence and input-equivalence. As a corollary, we deduce that $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$ if and only if $\mathcal{L}_{G_1}^{P_1} = \mathcal{L}_{G_2}^{P_2}$, and that in this context we may without loss of generality assume that all cell phase spaces are 1-dimensional. $\square$

In the rest of the section we state and prove Propositions 7.2 and 7.3.

Proposition 7.2 *Let G_1 and G_2 be two coupled cell networks such that there is bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence. Consider a choice of total phase space $P_1 = \prod_{c\in\mathcal{C}_1} P_{1,c}$ for G_1, and let $P_2 = \prod_{c\in\mathcal{C}_1} P_{1,\gamma(c)}$ be the corresponding phase space for G_2. Then $\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2}$ if and only if $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$.*

Proof Trivially, if $\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2}$ then $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$. Suppose now that $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$. By Theorem 4.2, every smooth equivariant vector field $f \in \mathcal{F}_{G_i}^{P_i}$ is determined uniquely by its components f_c where c runs through a set of representatives for the connected components (that is, the $\sim_I$-equivalence classes) of the groupoid $\mathcal{B}_{G_i}$. Note that since $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ is a bijection that preserves input-equivalence, if $\mathcal{Q}$ is a connected component of the groupoid $\mathcal{B}_{G_1}$ then $\gamma(\mathcal{Q})$ is a connected component of $\mathcal{B}_{G_2}$, and if $\mathcal{R}$ is a set of representatives for the connected components of $\mathcal{B}_{G_1}$ then $\gamma(\mathcal{R})$ is a set of representatives for the connected components of $\mathcal{B}_{G_2}$. The only constraints on f_c are that it depends only on $x_c, x_{\tau(I(c))}$ and is invariant under the vertex group $B(c,c)$. Thus every smooth equivariant vector field f is determined uniquely by a finite set of $B(c,c)$-invariant functions, where c runs through a set of representatives for the connected components of the groupoid. Moreover, if $d \sim_I c$ then

f_d is related to f_c by a pullback map β^* for $\beta \in B(c, d)$. Pullbacks permute variables, hence preserve smoothness (and also map polynomials to polynomials).

Schwarz [9] proves that in general for any compact Lie group Γ with an orthogonal action on $\mathbf{R}^n$, if the algebra of Γ-invariant polynomials is generated by $\rho_1, \ldots, \rho_k$ (and by Hilbert's basis theorem such a finite basis always exist), then any Γ-invariant C^∞-function of n variables is a C^∞-function of the generators $\rho_1, \ldots, \rho_k$. Since $\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$, the vector space of polynomial $B_1(c, c)$-invariants (where $B_1(c, c) \subseteq \mathcal{B}_{G_1}$) coincides with the vector space of polynomial $B_2(\gamma(c), \gamma(c))$-invariants (where $B_2(\gamma(c), \gamma(c)) \subseteq \mathcal{B}_{G_2}$). In particular, the two spaces share a set of invariant polynomial generators. Thus, given an $\sim_I$-equivalence class $\mathcal{Q} \subseteq \mathcal{C}_1$, the equality

$$\mathcal{P}_{G_1}^{P_1}(\mathcal{Q}) = \mathcal{P}_{G_2}^{P_2}(\gamma(\mathcal{Q}))$$

implies that

$$\mathcal{F}_{G_1}^{P_1}(\mathcal{Q}) = \mathcal{F}_{G_2}^{P_2}(\gamma(\mathcal{Q}))$$

Now Theorem 4.2 implies that $\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2}$. □

Proposition 7.3 *Assume the conditions of Proposition 7.2. Then*

$$\mathcal{P}_{G_1}^{P_1} = \mathcal{P}_{G_2}^{P_2}$$

if and only if G_1 and G_2 are γ-linearly equivalent.

Before we prove Proposition 7.3, we state and prove two lemmas that explore the structure of the symmetry groupoids of coupled cell networks.

Lemma 7.4 *Consider $V_1^{d_1}, \ldots, V_s^{d_s}$ where each V_i is a nonzero finite-dimensional vector space of dimension k_i, and denote coordinates on $V_i^{d_i}$ by $x_i = (x_{i,1}, \ldots, x_{i,d_i})$. Let*

$$\Gamma = \mathbf{S}_{d_1} \times \cdots \times \mathbf{S}_{d_s}$$

and

$$V = V_1^{d_1} \oplus \cdots \oplus V_s^{d_s}$$

Define a Γ-action on V by: if $\sigma \in \mathbf{S}_{d_i}$, then

$$\sigma \cdot x = (x_1, \ldots, x_{i-1}, \sigma \cdot x_i, x_{i+1}, \ldots, x_s)$$

where

$$\sigma \cdot x_i = (x_{i,\sigma^{-1}(1)}, \ldots, x_{i,\sigma^{-1}(d_i)})$$

Then any real Γ-invariant polynomial is a sum of polynomials of the form

$$q_1(x_1) q_2(x_2) \cdots q_s(x_s)$$

where for $j = 1, \ldots, s$, each $q_j(x_j)$ is $\mathbf{S}_{d_j}$-invariant.

Proof The idea of the proof is simple but the notation is complicated. Essentially, we use the fact that any invariant can be obtained as a linear combination of symmetrized monomials, so the proof reduces to computations with monomials.

In detail, recall that $p: V \to \mathbf{R}$ is Γ-invariant if and only if

$$p(\sigma \cdot x) = p(x) \quad \forall \sigma \in \Gamma,\ x \in V$$

This condition holds if and only if $p: V \to \mathbf{R}$ is $\mathbf{S}_{d_i}$-invariant, where $\mathbf{S}_{d_i}$ acts nontrivially only on $V_i^{d_i}$.

Denote by $\mathbf{Z}_0^+$ the set of nonnegative integers. Monomials in x_1 have the form

$$x_{1,1}^{I_1} \cdots x_{1,d_1}^{I_{d_1}}$$

where $I_1, \ldots, I_{d_1} \in (\mathbf{Z}_0^+)^{k_1}$, and each $x_{1,j}^{I_j}$ is a monomial in the k_1 components of $x_{1,j}$.

Let $p: V \to \mathbf{R}$ be a Γ-invariant polynomial, and write it as linear combination of monomials in x_1 with coefficients in $\mathbf{R}[x_2, \ldots, x_s]$. Suppose that $p(x)$ contains a term that is a scalar multiple of

$$x_{1,1}^{I_1} \cdots x_{1,d_1}^{I_{d_1}}\, q(x_2, \ldots, x_s)$$

Since p is $\mathbf{S}_{d_1}$-invariant and $\mathbf{S}_{d_1}$ acts trivially on $x_2, \ldots, x_s$, then $p(x)$ must also contain

$$x_{1,\sigma(1)}^{I_1} \cdots x_{1,\sigma(d_1)}^{I_{d_1}}\, q(x_2, \ldots, x_s)$$

for all $\sigma \in \mathbf{S}_{d_1}$. It follows that $p(x)$ contains a scalar multiple of

$$\left(\sum_{\sigma \in \mathbf{S}_{d_1}} x_{1,\sigma(1)}^{I_1} \cdots x_{1,\sigma(d_1)}^{I_{d_1}} \right) q(x_2, \ldots, x_s) = q_1(x_1) \cdot q(x_2, \ldots, x_s)$$

where $q_1(x_1) = \sum_{\sigma \in \mathbf{S}_{d_1}} x_{1,\sigma(1)}^{I_1} \cdots x_{1,\sigma(d_1)}^{I_{d_1}}$. Now we repeat the same argument for $q(x_2, \ldots, x_s)$ inductively. $\square$

Remark 7.5 This proof can be presented in a more abstract way: inductively, consider the polynomial $\mathbf{S}_{d_{j+1}}$-invariants over the ring of polynomial invariants for the subgroup $\mathbf{S}_{d_1} \times \cdots \times \mathbf{S}_{d_j} \times \mathbf{1} \times \cdots \times \mathbf{1}$. $\diamond$

Lemma 7.6 *Let V be a nonzero finite-dimensional real vector space of dimension d, and denote coordinates on V^t by $y = (y_1, \ldots, y_t)$. Let $\Gamma = \mathbf{S}_t$ and consider the action of Γ on V^t defined by:*

$$\sigma \cdot y = \left(y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(t)}\right) \quad (\sigma \in \mathbf{S}_t,\ y \in V)$$

Then the ring of the Γ-invariant polynomials from V^t to $\mathbf{R}$ is generated by the set of all Γ-invariant polynomials of the form

$$y_1^I + \cdots + y_t^I$$

where $I \in (\mathbf{Z}_0^+)^d$ and each y_i^I is a monomial in the d components of y_i.

Proof Choose coordinates $(y_1, \ldots, y_t)$ on V^t, where $y_i = (y_{i,1}, \ldots, y_{i,d})$. Thus, if $\sigma \in \mathbf{S}_t$,

$$\sigma \cdot (y_1, \ldots, y_t) = \big(y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(t)}\big)$$

where

$$y_{\sigma^{-1}(i)} = \big(y_{\sigma^{-1}(i),1}, \ldots, y_{\sigma^{-1}(i),d}\big)$$

A real polynomial $\mathbf{S}_t$-invariant on V^t is a linear combination of $\mathbf{S}_t$-invariants of the form

$$\sum_{\sigma \in \mathbf{S}_t} y_{\sigma(1)}^{I_1} \cdots y_{\sigma(t)}^{I_t} \tag{7.7}$$

where $I_i \in (\mathbf{Z}_0^+)^d$.

To continue the proof we need some terminology. Say that a polynomial (7.7) is of *type* m, where $1 \le m \le t$, if only m sets of indices, without loss of generality, $I_1, \ldots, I_m$, are non-zero. That is, $I_{m+1} = \cdots = I_t = (0, \ldots, 0)$, and $I_j \neq (0, \ldots, 0)$ for $j = 1, \ldots, m$.

Now observe that if $m = 1$, then given *any* $I_1 \in (\mathbf{Z}_0^+)^d$, an expression (7.7) of type 1 has the form

$$p_{I_1}(y) = \sum_{\sigma \in \mathbf{S}_t} y_{\sigma(1)}^{I_1} = y_1^{I_1} + \cdots + y_t^{I_1}$$

The proof of the lemma is carried out by induction on the type m of the Γ-invariant polynomial. Suppose that any polynomial of the form (7.7) of type less than or equal to m is a polynomial in polynomials of type 1. We prove that the same holds for polynomials of type $m+1$. Consider

$$p_{I_1,\ldots,I_{m+1}}(y) = \sum_{\sigma \in \mathbf{S}_t} y_{\sigma(1)}^{I_1} y_{\sigma(2)}^{I_2} \cdots y_{\sigma(m+1)}^{I_{m+1}}$$

Take the $\mathbf{S}_t$-invariant polynomial

$$p(y) = p_{I_1}(y) \cdots p_{I_{m+1}}(y)$$

where

$$p_{I_i}(y) = \sum_{\sigma \in \mathbf{S}_t} y_{\sigma(1)}^{I_i}$$

Note that $p_{I_j}(y)$ for all j is a Γ-invariant polynomial of type 1. Moreover

$$p(y) = p_{I_1,\ldots,I_{m+1}}(y) + \sum_i \beta_i r_i(y)$$

where each $\beta_i \in \mathbf{R}$ and each $r_i(y)$ is an $\mathbf{S}_t$-invariant of the form (7.7) of type less than or equal to m. By hypothesis $r_i(y)$ is Γ-invariant and can be written as a polynomial in Γ-invariant polynomials of type 1. Thus

$$p_{I_1,\ldots,I_{m+1}}(y) = p(y) - \sum_i \beta_i r_i(y)$$

and so $p_{I_1,\ldots,I_{m+1}}(y)$ is a Γ-invariant polynomial that can be written as a polynomial of Γ-invariant polynomials of type 1. $\square$

Again, this proof can be presented more abstractly, along the lines of Remark 7.5.

Proof of Proposition 7.3 Let $G_1,\ G_2$ be two coupled cell networks and let $\gamma:\ \mathcal{C}_1 \to \mathcal{C}_2$ be a bijection that preserves cell-equivalence and input-equivalence. Renumber the cells in $\mathcal{C}_2$ so that the bijection γ is the identity. Thus $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ and $P_1 = P_2 = P$.

Trivially if $\mathcal{P}^P_{G_1} = \mathcal{P}^P_{G_2}$ then for each connected component $\mathcal{Q}$ of G_1 (and G_2) and for each $c \in \mathcal{Q}$ we have $S^c_1 = S^c_2$. We now prove the converse. Suppose that for each connected component $\mathcal{Q}$ of G_1 (and G_2) and for each $c \in \mathcal{Q}$ we have

$$S^c_1 = S^c_2$$

Let f be any admissible polynomial vector field in $\mathcal{P}^P_{G_1}$. By Theorem 4.2, for any component $\mathcal{Q}$ of $\mathcal{B}_{G_1}$ and $c \in \mathcal{Q}$, the property of $\mathcal{B}_{G_1}$-equivariance of f on the component $\widehat{f_c}:\ P_c \times P_{\tau(I_1(c))} \to P_c$ is equivalent to $B_1(c,c)$-invariance of $\widehat{f_c}$. Moreover, $B_1(c,c)$-invariance of $\widehat{f_c}$ is equivalent to $B_1(c,c)$-invariance of each real component of $\widehat{f_c}$. We choose the same coordinate system for P_1 and P_2, and then prove that if $S^c_1 = S^c_2$ then the ring of the real polynomial $B_1(c,c)$-invariants is the same as the ring of the real polynomial $B_2(c,c)$-invariants. By Lemmas 7.4 and 7.6 the ring of real polynomial $B_1(c,c)$-invariants is generated as a ring by the real polynomial $B_1(c,c)$-invariants of type 1. Similarly, the ring of real polynomial $B_2(c,c)$-invariants is generated as a ring by the real polynomial $B_2(c,c)$-invariants of type 1. Now it is enough to prove that if $S^c_1 = S^c_2$, then any type 1 real polynomial $B_1(c,c)$-invariant is a real polynomial $B_2(c,c)$-invariant. (Or equivalently, that any type 1 real polynomial $B_2(c,c)$-invariant is a real polynomial $B_1(c,c)$-invariant.)

As before, $I_1(c) = K_1 \dot\cup \cdots \dot\cup K_{r_1(c)}$ where $K_1, \ldots, K_{r_1(c)}$ are the $\equiv_c$-equivalence classes on $I_1(c)$. Thus

$$B_1(c,c) = \mathbf{S}_{K_1} \times \cdots \times \mathbf{S}_{K_{r_1(c)}}$$

Similarly, $I_2(c) = L_1 \dot\cup \cdots \dot\cup L_{r_2(c)}$ where $L_1, \ldots, L_{r_2(c)}$ are the $\equiv_c$-equivalence classes on $I_2(c)$, and

$$B_2(c,c) = \mathbf{S}_{L_1} \times \cdots \times \mathbf{S}_{L_{r_2(c)}}$$

By Lemma 7.4 any real $B_1(c,c)$-invariant polynomial is a product of real $\mathbf{S}_{K_i}$-invariant polynomials. Set $K_i = \{1, \ldots, t\}$, so that $\mathbf{S}_t = \mathbf{S}_{K_i}$ and let $V^t = P_{\tau(K_i)}$ where $V = P_{\tau(l)}$ (for any $l \in K_i$) is a non-zero finite-dimensional real vector space. Suppose that V has dimension d, and denote coordinates on V^t by $y = (y_{\tau(1)}, \ldots, y_{\tau(t)})$, where each $y_{\tau(i)} = (y_{\tau(i),1}, \ldots, y_{\tau(i),d})$. Thus, if $\sigma \in \mathbf{S}_t$ then

$$\sigma \cdot (y_{\tau(1)}, \ldots, y_{\tau(t)}) = (y_{\tau(\sigma^{-1}(1))}, \ldots, y_{\tau(\sigma^{-1}(t))})$$

where

$$y_{\tau(\sigma^{-1}(i))} = (y_{\tau(\sigma^{-1}(i)),1}, \ldots, y_{\tau(\sigma^{-1}(i)),d})$$

Given $I \in (\mathbf{Z}^+_0)^d$, the polynomial

$$p_I(y) = y^I_{\tau(1)} + \cdots + y^I_{\tau(t)}$$

is $\mathbf{S}_t$-invariant (and so $B_1(c,c)$-invariant) and

$$Y_{\tau(1)} + \cdots + Y_{\tau(t)} \in S^c_1$$

where S_1^c is as defined in (6.5). By hypothesis, $S_1^c = S_2^c$, where S_2^c is defined in (6.6). Thus there exist real coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots,$ such that

$$Y_{\tau(1)} + \cdots + Y_{\tau(t)} = \alpha_0 p_0(Y) + \alpha_{j_1} p_{j_1}(Y) + \alpha_{j_2} p_{j_2}(Y) + \cdots \tag{7.8}$$

where $p_0(Y) = Y_c$ and $p_{j_p}(Y) = \sum_{i \in L_{j_p}} Y_{\tau(i)}$ for $p \geq 1$. Here $L_{j_1}, L_{j_2}, \ldots$ denote $\equiv_c$-equivalence classes on $I_2(c)$.

We claim that the $p_{j_p}(Y)$ that appear in (7.8) can be chosen to depend only on $Y_{\tau(k)}$, where the cell phase space $P_{\tau(k)} = P_{\tau(1)}$. To see this note that for all $m \in \{1, \ldots, t\} = K_i \subseteq I_1(c)$ we know that since $(\tau(1), 1) \sim_{E_1} (\tau(m), m)$ then $\tau(1) \sim_{C_1} \tau(m)$ and so $P_{\tau(1)} = P_{\tau(m)}$. Also, all the cells in the same $\equiv_c$-equivalence class are cell-equivalent. Thus if some $p_{j_p}(Y)$ (with $\alpha_{j_p} \neq 0$) in (7.8) depends on $Y_{\tau(l)}, Y_{\tau(k)}$ such that $P_{\tau(l)} = P_{\tau(1)}$, then as $\tau(k) \sim_{C_2} \tau(l)$ since $(\tau(l), l) \sim_{E_2} (\tau(k), k)$ where $l, k \in L_{j_p} \subseteq I_2(c)$, we have that $P_{\tau(k)} = P_{\tau(l)} = P_{\tau(1)}$. This proves the claim.

Set $Y_{\tau(j)} = y^I_{\tau(j)}$ for all j. Substituting all of this into equation (7.8) we get

$$p_I(y) = Y_{\tau(1)} + \cdots + Y_{\tau(t)} = q_I(y)$$

where

$$q_I(y) = \alpha_0 y^I_c + \alpha_{j_1} \left(\sum_{i \in L_{j_1}} y^I_{\tau(i)} \right) + \alpha_{j_2} \left(\sum_{i \in L_{j_2}} y^I_{\tau(i)} \right) + \cdots$$

is a $B_2(c,c)$-invariant. □

Corollary 7.7 *The following conditions on two networks G_1, G_2 are equivalent:*

(a) *G_1 and G_2 are γ-linearly equivalent.*

(b) *With the identification $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$, the spaces $\mathcal{L}^P_{G_1}$ and $\mathcal{L}^P_{G_2}$ are equal for all P.*

(c) *With the identification $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$, the spaces $\mathcal{L}^P_{G_1}$ and $\mathcal{L}^P_{G_2}$ are equal when all cell phase spaces are taken to be $\mathbf{R}$.*

Proof Condition (a) implies γ-ODE-equivalence for any choice of P, and γ-ODE-equivalence implies (b) by restricting to linear vector fields. Then (c) is special case of (b). Finally, (c) clearly implies (a). □

8 Combinatorics of Linear Equivalence

Given two coupled cell networks, G_1 and G_2, Theorem 7.1 implies that in order to verify their ODE-equivalence, we need only check their linear equivalence. That is, we must decide whether there exists some bijection between the corresponding sets of cells $\mathcal{C}_i$, preserving cell-equivalence and input-equivalence, such that for each $c \in \mathcal{C}_1$ the vector spaces S_1^c and $S_2^{\gamma(c)}$ are equal. A problem concerning the topology of networks as directed graphs has now become a linear algebra problem of finding when two vector spaces of linear polynomials are equal. Standard methods of linear algebra can solve this problem efficiently in any given case.

It might seem likely that linear equivalence is simpler to work with than ODE-equivalence, in the sense that linear equivalence can be read off easily from the topologies two networks concerned, modulo elementary linear algebra. But it is not clear whether the above linear algebra problem can be simplified significantly by exploiting the network topology (say by defining some kind of 'normal form' for the network, with a topological procedure that reduces any given network to normal form) in a way that is not a trivial encoding of the corresponding linear algebra computation. To illustrate the combinatorial complexities that may arise when determining linear equivalence, we generalize Example 6.1.

Recall Example 6.3, corresponding to the coupled cell networks of Figure 2. The two networks are ODE-equivalent since they are linearly equivalent, taking γ to be the identity function on $\mathcal{C}_1 = \mathcal{C}_2 = \{1,2,3,4\}$. Other examples of coupled cell networks can easily be constructed that are also linearly equivalent to G_1 and G_2, in the following way. Note that taking

$$V = \{\lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3 + \lambda_4 Y_4 : \ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbf{R}\}$$

then

$$S_1^4 = S_2^{\gamma(4)} = \{\lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3 + \lambda_4 Y_4 \in V : \ \lambda_1 + 3\lambda_2 = 5\lambda_3\}$$

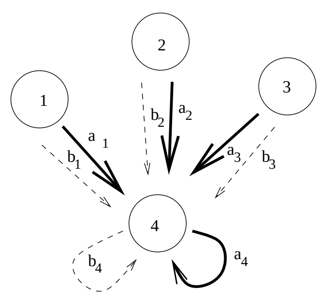

Figure 3: A coupled cell network with four identical cells, two edge-equivalence classes. The symbols a_i and b_i attached to the right of each edge symbolizes the number of edges of that type. Thus a_i, b_i denote nonnegative integers.

Now consider Figure 3. Any coupled cell network with four identical cells and two edge-equivalence classes, as in Figure 3, is γ-linearly equivalent to G_1 of Figure 2 provided that

$$\mathbf{R}\{Y_4,\ a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4,\ b_1Y_1 + b_2Y_2 + b_3Y_3 + b_4Y_4\} =$$

$$\{\lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3 + \lambda_4 Y_4 \in V : \ \lambda_1 + 3\lambda_2 = 5\lambda_3\}$$

Here a_i, b_i are nonnegative integers and γ is the identity on $\{1,2,3,4\}$.

Other questions that we can pose include the following. Given an ODE-equivalence class of coupled cell networks, is there a canonical 'normal form' — perhaps a graph, or a set of graphs, such that the number of edges is minimal among all the graphs of that ODE-class? Moreover, given a graph G, when can we find a subgraph that is ODE-equivalent to G? These questions are addressed by Aguiar and Dias [1].

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