Invariants for Bifurcations

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August 3, 2004

Abstract

Bifurcation problems with one parameter are studied here. We develop a method for computing a topological invariant, the number of fold points in a stable one-parameter unfolding for any given bifurcation of finite codimension.

We introduce another topological invariant, the algebraic number of folds. The invariant gives the number of complex solutions to the equations of fold points in a stabilization, an upper bound for the number of fold points in any unfolding. It can be computed by algebraic methods, we show that it is finite for germs of finite codimension. An open question is whether this value is always attained as the maximum number of fold points in a stable unfolding.

We compute these two invariants for simple bifurcations in one dimension, answering the question above in the affirmative. We discuss other invariants in the literature and verify that the algebraic number of folds and the Milnor number form a complete set of invariants for simple bifurcations in one dimension.

1 Introduction

In this paper we study germs of smooth bifurcation problems $f(x, \lambda)$, with n variables $x \in \mathbf{R}^n$ and one parameter $\lambda \in \mathbf{R}$. We consider invariants that arise in the classification of such problems. Our main goal is to bring into the classification of bifurcation problems an approach used in the study of map-germs for the definition of invariants associated to real singularities.

Any invariant associated to a map-germ $f : \mathbf{R}^{n+1}, (0, 0) \longrightarrow \mathbf{R}^n, 0$ is also an invariant of the bifurcation problem $f(x, \lambda)$. For instance, although the expressions $x^2+\lambda$ and $x+\lambda^2$ represent completely different bifurcations, they will have the same invariants as maps $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$. Thus, in the study of bifurcation problems some invariants have to be introduced, in order to take into account the special role of the parameter λ.

A similar procedure is followed in [18], where a method for counting the number of branches in a bifurcation diagram is obtained using the formula of Eisenbud and Levine [7] for the degree of an isolated singularity. The number of branches for positive (resp. negative) λ provides an invariant for bifurcations. These invariants are discussed in section 3.

An important issue in bifurcation theory is to understand the geometry of stabilizations of a given bifurcation problem. The important information is the number of folds of $(x, \lambda) \mapsto F(x, \lambda, t)$ for fixed $t \neq 0$, small. This number usually depends on the choice of stabilization and on the sign of t: the transition set generally divides the unfolding parameter space into more than one connected component. This is in strong contrast to the complex theory, where the complement of the transition set is connected.

We use the results in [7, 18] to develop in section 3 a method for counting the number of folds appearing in a stable one-parameter perturbation of f. The maximum value (over all possible stabilizations) of this number is the geometric number of folds associated to the bifurcation problem.

We introduce in section 4 a new topological invariant of a bifurcation problem f, the algebraic number of folds, that we denote by $\beta(f)$. The invariant $\beta(f)$ is the codimension of an ideal associated to the bifurcation f and it counts the number of complex solutions to the equations for fold points. We show that f has finite codimension if and only if $\beta(f) < \infty$.

One open question is whether the geometric and algebraic number of folds coincide for bifurcation problems in one variable, i.e. if there is a stable perturbation with exactly $\beta(f)$ folds. We have computed the invariants for all the simple bifurcations in one variable (section 5) and found the answer to be yes, in the case of simple germs, but it remains open in general.

The question of whether the number of complex solutions for stabilizations can be realized for the real case has arisen in several situations and can be quite difficult. In the context of multiplicity of stable map germs of discrete algebra type, this was shown to be true by Damon and Galligo [5]. For plane curves the existence of a maximal deformation was shown by A'Campo [1] and by Gusein-Zade [13]. Arnol'd [2] and Entov [8] have shown the existence of maximal morsifications for singularities of type A_k and D_k.

The failure to realize in the real case the number occurring for the complex case can happen for situations such as multiplicity of stable map germs studied by Iarrobino [14]; for vanishing cycles for images of the stabilization of $\mathcal{A}$-finite germs from $\mathbf{C}^2, 0$ to $\mathbf{C}^3, 0$, studied by Marar and Mond [17]; and for vanishing cycles of bifurcation sets studied by Damon [4].

A similar question is raised in [19] for map-germs from $\mathbf{R}^n$ to $\mathbf{R}^p$, $n \geq p$ under $\mathcal{A}$-equivalence.

2 Preliminary results and definitions

For basic results we refer the reader to Golubitsky and Schaeffer [11] and to Keyfitz [15], whose notation we use. We recall some definitions and notation used in the study germs of smooth *bifurcation problems* $f(x,\lambda)$, $f : \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}^n, 0$. The set of all such germs forms a free module, $\overrightarrow{\mathcal{E}_{x\lambda}}$, of rank n over the ring $\mathcal{E}_{x\lambda}$ of germs of smooth functions $g : \mathbf{R}^n \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}$. The ring $\mathcal{E}_{x\lambda}$ has a unique maximal ideal $\mathcal{M}$ and a subring $\mathcal{E}_\lambda$ of germs of functions depending only on λ.

To each bifurcation germ $f(x,\lambda)$ we associate its *bifurcation diagram*, the germ of the set $\{(\lambda,x) : f(x,\lambda) = 0\}$, the set of equilibria of the differential equation $\dot{x} = f(x,\lambda)$.

Two bifurcation germs $f(x,\lambda)$ and $g(x,\lambda)$ are *bifurcation equivalent* (called here *b-equivalent*) if and only if there are germs of maps $S(x,\lambda)$, $X(x,\lambda)$ and $\Lambda(\lambda)$ such that

$$f(x,\lambda) = S(x,\lambda)g\left(X(x,\lambda),\Lambda(\lambda)\right)$$

where $S(x,\lambda)$ is linear for each (x,λ) and $\det S(0,0) > 0$ and where $d_x X(0,0)$ is invertible with $\Lambda'(0) > 0$.

The usual concepts of *strong* and *unipotent* equivalence can be extended to this context as well as those of codimension and unfolding. For definitions and calculation methods, see [10], [11] and references therein.

Recall that the tangent space at f to the b-strong-equivalence class of f, called the *restricted tangent space* of f, is the $\mathcal{E}_{x\lambda}$ submodule of $\overrightarrow{\mathcal{E}_{x\lambda}}$ given by

$$\mathrm{RT}(f) = \left\langle f_i e_j, x_i \frac{\partial f_k}{\partial x_j}, \lambda \frac{\partial f_k}{\partial x_j} \quad i,j,k = 1,\ldots n \right\rangle_{\mathcal{E}_{x\lambda}} \subset \overrightarrow{\mathcal{E}_{x\lambda}}$$

where e_j denotes the elements of the standard basis of $\mathbf{R}^n$. A bifurcation problem f has finite codimension if and only if $\mathrm{RT}(f)$ has finite codimension in $\overrightarrow{\mathcal{E}_{x\lambda}}$.

For $n = 1$, all bifurcation problems of codimension seven or less have been classified by Keyfitz [15]. The only non trivial bifurcations of codimension zero, called *stable* bifurcations, are the folds $f(x,\lambda) = \pm x^2 \pm \lambda$ and their suspensions in $\mathbf{R}^n$, $(x_1^2 + \lambda, x_2, \ldots, x_n)$, also called folds in this paper.

2.1 The fold ideal

Let $\mathcal{B}(f) \subset \mathcal{E}_{x\lambda}$ be the ideal $\mathcal{B}(f) = \langle f_1(x,\lambda), \ldots, f_n(x,\lambda), J_x f(x,\lambda)\rangle$, where $J_x f$ denotes the determinant of $d_x f$. We call $\mathcal{B}(f)$ the *fold ideal* of f.

Theorem 1 *A bifurcation problem $f : \mathbf{R}^n \times \mathbf{R}, 0 \longrightarrow \mathbf{R}^n, 0$ has finite codimension if and only if $\mathcal{B}(f)$ has finite codimension.*

Proof: Assume f has finite codimension and therefore $\mathrm{RT}(f) \supset \mathcal{M}^k \overrightarrow{\mathcal{E}_{x\lambda}}$ for some k. Following the argument of Gaffney in [9], Lemma 2.12, we will show that $\mathcal{B}(f)$ contains $\mathcal{M}^{kn}$. In fact, let $u \in \mathcal{M}^{kn}$, say $u = \prod_{i=1}^n u_i$ where $u_i \in \mathcal{M}^k$. Since $\mathrm{RT}(f)$ contains the germs $u_i e_i$, then the matrix equation

$$d_x f \cdot A = \begin{pmatrix} u_1 & 0 & \ldots & 0 \\ 0 & u_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & u_n \end{pmatrix}$$

has a solution A with entries in $\mathcal{E}_{x\lambda}$, modulo $\langle f_i e_j \rangle_{i,j=1,\ldots,n}$. Taking the determinant of each side it follows that $u \in \langle J_x f \rangle \pmod{\langle f_1, \ldots, f_n\rangle}$, and thus the condition is necessary.

To see the sufficiency, assume $\mathcal{B}(f) \supset \mathcal{M}^k$ for some k. It is enough to prove that $d_x f(\overrightarrow{\mathcal{M}}) \supset \langle J_x f e_i \quad i = 1, \ldots, n\rangle_{\mathcal{E}_{x\lambda}}$. The proof is again just a parametrized version of Gaffney's argument.

Let $A(i,j)$ be the cofactor of the element $\dfrac{\partial f_i}{\partial x_j}$ in the matrix $d_x f$. Then we can write the vector whose only nonzero entry is $J_x f$ in the l-th position as:

$$\begin{pmatrix} 0 \\ \vdots \\ J_x f \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \ldots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_l}{\partial x_1} & \ldots & \dfrac{\partial f_l}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \ldots & \dfrac{\partial f_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} A(l,1) \\ \vdots \\ A(l,l) \\ \vdots \\ A(l,n) \end{pmatrix}$$

proving the result. □

3 Branches and Folds

Given a finite codimension bifurcation problem f, consider a representative of a one-parameter unfolding $F(x,\lambda,t)$ of f defined in a neighbourhood $U \subset \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}$ of the origin. The representative $F : U \longrightarrow \mathbf{R}^n$, is a *stabilization* of f if it has the property that for $t \neq 0$ the only singularities of $F_t(x,y) = F(x,\lambda,t)$ in $U_t = U \cap \mathbf{R}^n \times \mathbf{R} \times \{t\}$ are folds.

In this section, given a stabilization $F(x,\lambda,t)$ of a bifurcation problem $f(x,\lambda)$, we obtain a formula for the number, $\mathbf{b}_+(F)$, of fold points of $F(x,\lambda,t)$ for small fixed $t > 0$. This number usually depends on the choice of one-parameter unfolding and also on the sign of t. The theory for real bifurcation problems is in strong contrast to the complex case: all stabilizations of a complex bifurcation correspond to equivalent problems.

The formula for $\mathbf{b}_+(F)$ is obtained by applying to the germ of $g(y,t) = ((F(y,t), J_xF(y,t))$, where $y = (x,\lambda)$, the results of Nishimura, Fukuda and Aoki [18] on the number of branches of a bifurcation problem. The folds in the stabilization F are branches in the higher dimensional problem $g(y,t)$.

Given a map germ $g : \mathbf{R}^{n+1}, 0 \longrightarrow \mathbf{R}^n, 0$, let $\mathbf{r}(g)$ denote the number of half-branches in the bifurcation diagram of g, i.e. the number of connected components of $g^{-1}(0) - \{0\}$. Denote by $\mathbf{r}_+(g)$ and $\mathbf{r}_-(g)$, the number of half branches with $\lambda > 0$ (resp. $\lambda < 0$) in the bifurcation diagram of g and $\mathbf{r}_\pm(g) = \mathbf{r}_+(g) - \mathbf{r}_-(g)$. The numbers $\mathbf{r}_+(g)$ and $\mathbf{r}_-(g)$ are invariants for b-equivalence.

Nishimura, Fukuda and Aoki [18], have shown that $\mathbf{r}(g)$ is twice the topological degree of the germ $\Phi_1 : \mathbf{R}^{n+1}, 0 \longrightarrow \mathbf{R}^{n+1}, 0$, given by $\Phi_1(y,\lambda) = (g(y,\lambda), \lambda J_y g(y,\lambda))$, where $J_y g(y,\lambda)$ is the Jacobian of g. Similarly, $\mathbf{r}(g)$ is twice the topological degree of the germ $\Phi_2 : \mathbf{R}^{n+1}, 0 \longrightarrow \mathbf{R}^{n+1}, 0$, given by $\Phi_2(y,\lambda) = (g(y,\lambda), J_y g(y,\lambda))$. Therefore, $\mathbf{r}(g)$, $\mathbf{r}_+(g)$ and $\mathbf{r}_-(g)$ are topological invariants and the degrees of Φ_1 and Φ_2 can be computed using the important result of Eisenbud and Levine [7] that we shall describe briefly.

Let $\langle \Phi \rangle$ be the ideal in $\mathcal{E}_y$ generated by the components of a map $\Phi(y)$, $\Phi : \mathbf{R}^m, 0 \longrightarrow \mathbf{R}^m, 0$. If $\langle \Phi \rangle$ has finite codimension, consider the algebra $Q(\Phi) = \mathcal{E}_y / \langle \Phi \rangle$ and let I be an ideal of $Q(\Phi)$ that is maximal with respect to the property $I^2 = 0$. Then Eisenbud and Levine [7] show that if $Q(\Phi)$ is finite-dimensional then $|\text{degree}(\Phi)| = \dim_{\mathbf{R}} Q(\Phi) - 2\dim_{\mathbf{R}} I$.

The algebra $Q(\Phi)$ has a minimal ideal, called the socle, generated by the class J_0 of $J_y\Phi$. Another way to obtain the degree, from [7], is to consider any linear functional l in $Q(\Phi)$ such that $l(J_0) > 0$. Then the degree of Φ is the signature of the bilinear form $L : Q(\Phi) \times Q(\Phi) \longrightarrow \mathbf{R}$ given by $L(p,q) = l(pq)$.

Given a stabilization $F(x,\lambda,t)$ of a finite codimension bifurcation problem $f(x,\lambda)$, consider the algebras

$$Q_1(F) = \mathcal{E}_{x\lambda t}/\langle F, J_xF, tJ_{x\lambda}(F, J_xF)\rangle$$

and

$$Q_2(F) = \mathcal{E}_{x\lambda t}/\langle F, J_xF, J_{x\lambda}(F, J_xF)\rangle \ .$$

Lemma 1 *If f has finite codimension and F is a stabilization of f then the algebras $Q_1(F)$ and $Q_2(F)$ have finite dimension as real vector spaces.*

Proof: Without loss of generality we may assume F is analytic, since f has finite codimension. Let $F_{\mathbf{C}} : \mathbf{C}^{n+1} \times \mathbf{C}, 0 \longrightarrow \mathbf{C}^n, 0$ be the complexification of F, that we will denote by F in the remainder of this proof. If we show that the varieties of the ideals $\langle F, J_xF, tJ_{x\lambda}(F, J_xF)\rangle$ and $\langle F, J_xF, J_{x\lambda}(F, J_xF)\rangle$ reduce to 0 then the result follows from Hilbert Nullstellensatz.

We start by the second ideal. The equations $F = 0$, $J_xF = 0$ are defining conditions for folds in the unfolding F. Folds satisfying the last equation $J_{x\lambda}(F, J_xF) = 0$ are degenerate: at these points either the gradient $\nabla_x(J_xF)$ is equal to zero or the gradient is orthogonal to the kernel of D_xF. Both possibilities characterize points that are more degenerate than folds and therefore this implies $t = 0$, since F is stable for $t \neq 0$. For $t = 0$, the first two equations reduce to $f(x,\lambda) = 0$, $J_xf(x,\lambda) = 0$ and have a unique solution $(x,\lambda) = (0,0)$ since we have already shown that the fold ideal has finite codimension when $\text{cod}(f) < \infty$ and thus the claim holds.

For the first ideal, there are two possibilities, $t = 0$ and $t \neq 0$, both covered by the arguments above. □

Each one of the algebras $Q_1(F)$ and $Q_2(F)$ has a socle generated, respectively, by the residue classes of

$$s_1 = J_{x\lambda t}\left(F, J_xF, tJ_{x\lambda}(F, J_xF)\right) \qquad \text{and} \qquad s_2 = J_{x\lambda t}\left(F, J_xF, J_{x\lambda}(F, J_xF)\right) \ .$$

In each algebra $Q_i(F)$, let l_i be a linear functional satisfying $l_i(s_i) > 0$ and let L_i be the bilinear form $L_i(p,q) = l_i(pq)$, as in the Eisenbud and Levine [7] result.

Theorem 2 *If F is a stabilization of a bifurcation problem f of finite codimension, then the number $\mathbf{b}_+(F)$ of folds in F for $t > 0$ is $\mathbf{b}_+(F) = signature(L_1) + signature(L_2)$.*

Proof: Consider the map germs Φ_1 and $\Phi_2 : \mathbf{R}^{n+2}, 0 \longrightarrow \mathbf{R}^{n+2}, 0$ given by

$$\Phi_1 = (F, J_x F, t J_{x\lambda}(F, J_x F)) \qquad \text{and} \qquad \Phi_2 = (F, J_x F, J_{x\lambda}(F, J_x F)) \ .$$

Their algebras, $Q_1(F)$ and $Q_2(F)$, are finite dimensional by Lemma1. Therefore, by the results of [7] the degree of each Φ_i is the signature of L_i.

On the other hand, the fold points of F are precisely the nondegenerate zeros of

$$g(x, \lambda, t) = (F(x, \lambda, t), J_x F(x, \lambda, t))$$

i.e., the branches of the bifurcation problem $g(y, t)$ with $y = (x, \lambda)$, and bifurcation parameter t. Therefore, the total number of fold points of F for $t > 0$ and $t < 0$ equals the total number, $\mathbf{r}(g)$, of half-branches of g and the difference between the number of folds with $t > 0$ and $t < 0$ is $\mathbf{r}_\pm(g)$. From [18] it follows that $\mathbf{r}(g) = 2$ degree(Φ_1) and $\mathbf{r}_\pm(g) = 2$ degree(Φ_2). □

From the computation of $\mathbf{r}_\pm$ in the proof above, it follows:

Corollary 1 *If F is a stabilization of a bifurcation problem f of finite codimension, then the number of folds for $t > 0$ is congruent modulo 2 to the number of folds for $t < 0$.*

Another formulation of Theorem 2, less easy to compute, can be obtained using the results of [7]:

Theorem 3 *If F is a stabilization of a bifurcation problem f of finite codimension, then the number $\mathbf{b}_+(F)$ of folds in F for $t > 0$ is $\mathbf{b}_+(F) = dim(Q_1(F)) - dim(Q_2(F)) - 2\dim_{\mathbf{R}}(I_1) + 2\dim_{\mathbf{R}}(I_2)$, where each $I_i \subset Q_i(F)$ is an ideal that is maximal with respect to the property $I_i^2 = 0$.*

The singular set Σ in the unfolding of a bifurcation problem of finite codimension was defined in section 4 in the proof of Theorem 1. It is shown in [11] that generic points in Σ are hysteresis, bifurcation or double limit points. Crossing a double limit point only changes the relative position of folds in a bifurcation diagram. Crossing a hysteresis or a bifurcation point creates (or destroys) a pair of folds. Thus it follows:

Corollary 2 *If F is a stable germ in a versal unfolding of a bifurcation problem f of finite codimension, then the number of folds is constant modulo 2.*

3.1 Weighted homogeneous bifurcations

We say that $g : \mathbf{R}^m \longrightarrow \mathbf{R}^p$ is *weighted homogeneous* if we may assign weights $w(x_i) > 0$, $i = 1, \ldots, m$ so that each of the coordinate functions g_i, $i = 1, \ldots, p$ is weighted homogeneous, of weight $w(g_i)$.

If the germ f has a weighted homogeneous representative, the calculation of $\mathbf{b}_+(F)$ is easier (see [20] and [3] for details). In particular, we get:

Corollary 3 *Let $f : \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$ be a polynomial bifurcation problem that is weighted homogeneous and such that the germ of f at the origin has finite codimension. If $F(x, \lambda, t)$ is a weighted homogeneous stabilization of f, where t has odd weight, then $\mathbf{b}_+(F) = \mathbf{b}_-(F)$.*

Proof: We show that the bilinear form L_2 of Theorem 2 has signature zero. The map $\Phi_2 = (F, J_xF, J_{x\lambda}(F, J_xF))$ is weighted homogeneous and this induces a filtration of $Q_2(F)$ where the maximum weight M is that of the socle, $s_2 = J_y\Phi_2(y)$ with $y = (x, \lambda, t)$. Except in the trivial case $Q_2(F) = \{0\}$, the maximum weight M is given by

$$M = w(s_2) = 4w(J_xF) - 2w(\lambda) - w(t) \ .$$

Consider the subspaces $Q_{1/2}$, $Q_<$ and $Q_>$ of $Q_2(F)$ of elements of weight $= M/2$, $< M/2$ and $> M/2$, respectively. Since the multiplication is additive on weigths, the signature of the bilinear form L_2 is determined by the signature of its restriction to $Q_{1/2}$: the multiplication dually pairs $Q_<$ with $Q_>$, so these subspaces give no contribution to the signature. Since M is odd then $Q_{1/2}$ is empty and the signature is zero. □

A similar construction for the map $\Phi_1 = (F, J_xF, tJ_{x\lambda}(F, J_xF))$ shows that in this case we only have to compute the signature of the bilinear form L_1 in the subspace of $Q_1(F)$ of elements of weight

$$\frac{M}{2} = \frac{1}{2}w(s_1) = 2w(J_xF) - w(\lambda) \ .$$

The hypothesis of Corollary 3 on the stabilization F is not very restrictive: let f be weighted homogeneous with finite codimension and denote by $T(f)$ the tangent space at f to the b-orbit of f. Suppose there is a complementary subspace to $T(f)$ consisting of germs of weighted homogeneous maps of the form $p(x,\lambda) = (p_1(x,\lambda),\ldots,p_n(x,\lambda))$, such that $w(p_i) < w(f_i)$. Then for every stable germ $g(x,\lambda)$ in an unfolding of f there is a weighted homogeneous stabilization $F(x,\lambda,t)$ of f, with $w(t) = 1$, such that for some $t \neq 0$ the germs $g(x,\lambda)$ and $F_t(x,\lambda)$ are b-equivalent and Corollary 3 can be applied.

4 Algebraic Folds

An upper bound for the number of folds appearing in any stabilization of a given bifurcation problem is the number of folds appearing in a complexification of the problem. We define the *algebraic number of folds* $\beta(f)$ of a bifurcation f as the codimension of $\mathcal{B}(f)$. It is an invariant for b-equivalence that counts the number of simple solutions $(x,\lambda) \in \mathbf{C}^{n+1}$ of $F(x,\lambda,\alpha) = 0$, $J_xF(x,\lambda,\alpha) = 0$ for generic fixed α on an unfolding F of f, as we show below, i.e. it counts the number of points where F_α is equivalent to a fold.

Theorem 4 *For a bifurcation problem* $f : \mathbf{R}^n \times \mathbf{R}, 0 \longrightarrow \mathbf{R}^n, 0$ *of finite codimension,* $\beta(f)$ *is an invariant for b-equivalence giving the number of folds appearing in a stable deformation of the complexification of* f*.*

Proof: Recall that, by Theorem 1, $\mathcal{B}(f)$ has finite codimension. Let $\tilde{F} : \mathbf{C}^n \times \mathbf{C} \times \mathbf{C}^u, 0 \longrightarrow \mathbf{C}^n, 0$ be a versal unfolding of the complexification of f, and $F : W \longrightarrow \mathbf{C}^n$ a representative of $\tilde{F}$ defined in a neighbourhood W of the origin in $\mathbf{C}^n \times \mathbf{C} \times \mathbf{C}^u$. We can take $W = U \times T$, where U is a neighbourhood of 0 in $\mathbf{C}^n \times \mathbf{C}$ and T a neighbourhood of 0 in $\mathbf{C}^u$.

Let Σ be the set of $u \in T$ for which there exists a point $(x,\lambda) \in U$, which is a singular point of $F_u = F(.,.,u)$, more degenerate than a fold. By this we mean that either the kernel of d_xF has dimension more than one, or the curve $F_u = 0$ has a contact with the kernel of d_xF more degenerate than a quadratic.

The transition set Σ is a proper analytic set of $\mathbf{C}^u$, and therefore its complement in T is connected. Then for any u and $\tilde{u}$ in the complement of Σ we have that F_u and $F_{\tilde{u}}$ are b-equivalent. Thus, for f complex, the number of folds does not depend on the choice of stabilization.

Let F_α be a representative of a stabilization of the complexification of f defined in a neighbourhood of the origin. Then for each $\alpha \neq 0$

$$\beta(F_\alpha) = \sum_{(x_i,\lambda_i)} \dim_{\mathbf{C}} \frac{\mathcal{E}_{x\lambda}\big|_{(x_i,\,\lambda_i)}}{\left\langle J_x F_\alpha\big|_{(x_i,\,\lambda_i)}, F_\alpha\big|_{(x_i,\,\lambda_i)}\right\rangle}$$

where the summation is taken over all the (x_i, λ_i) that are fold points and $\big|_{(x_i,\,\lambda_i)}$ stands for germs at (x_i, λ_i).

Each fold point contributes 1 to the summation, since

$$\dim \left(\frac{\mathcal{E}_{x\lambda}}{\mathcal{B}(x_1^2 \pm \lambda, x_2, \ldots, x_n)} \right) = 1$$

and thus $\beta(F_\alpha)$ is the number of folds for F_α. Since $\mathcal{B}(F_\alpha)$ defines a family of complete intersection with isolated singularity, it follows [16] that the multiplicity $\beta(F_\alpha)$ is constant and therefore $\beta(F_\alpha) = \beta(f)$ □

The invariant $\beta(f)$ counts the number of complex solutions of $F_\alpha = 0$, $J_x F_\alpha = 0$, with multiplicity, for an unfolding F of f. In the real case, for a stabilization $F_t(x, \lambda)$ of f, the number of real solutions of $F_t = 0$, $J_x F_t = 0$ depends on the choice of stabilization F_t.

The stabilization can be viewed as a path in a versal unfolding of f. In the real case the space of unfolding parameters may be separated by the transition set Σ. Then $\mathbf{b}(F_t)$ is constant in each component of the complement of Σ, but may vary from one component to another. Its maximum $\mathbf{b}_{max}(f)$ over all stabilizations of a given germ f is clearly an invariant of b-equivalence.

A natural question is whether $\mathbf{b}_{max}(f) = \beta(f)$. In other words, we want to know if the geometric and algebraic number of folds coincide. This is the case for all simple bifurcations in one spatial dimension studied in [15] (see section 5).

If in the definition of b-equivalence we drop the requirement that the change of parameter Λ does not depend on x we obtain the usual definition of contact equivalence ($\mathcal{K}$-equivalence) of maps. Every invariant of $\mathcal{K}$-equivalence of germs in $\overrightarrow{\mathcal{E}_{x\lambda}}$ is also invariant under b-equivalence, the second being a specialization of the first. An analogous result holds [5] for $\mathcal{K}$-versal unfoldings of germs of maps in $\mathbf{R}^2$ $\mathcal{K}$-codimension and if $G(y, \alpha)$ is a $\mathcal{K}$-versal unfolding of $g = (g_1, g_2)$ then there is a germ of an open, path connected set A of parameters α with the origin in the closure of A, such that for each $\alpha \in A$ the map $y \mapsto G(y, \alpha)$ has exactly m zeros where m is the codimension of $\langle g_1, g_2 \rangle_{\mathcal{E}_y}$.

Conjecture 1 *Let $f : \mathbf{R} \times \mathbf{R}, (0,0) \longrightarrow \mathbf{R}, 0$ be a finite codimension bifurcation problem and $F(x, \lambda, \alpha)$ a b-versal unfolding of f. Then there is a germ of an open, path connected set A of parameters α, with the origin in the closure of A, such that for each $\alpha \in A$ the bifurcation diagram of $x \mapsto F(x, \lambda, \alpha)$ contains exactly $\beta(f)$ fold points.*

A natural way to prove Conjecture 1 would be to reduce it to the $\mathcal{K}$-equivalence case by considering the germ $g(x, \lambda) = (f(x, \lambda), f_x(x, \lambda))$ and a $\mathcal{K}$-versal unfolding of the form $G(x, \lambda, \alpha) = (F(x, \lambda, \alpha), F_x(x, \lambda, \alpha))$. Then for suitable α, the germ G would have precisely $\beta(f)$ zeros and F would factor through any b-versal unfolding of f. Unfortunately it is often not possible to find a $\mathcal{K}$-versal unfolding of the form (F, F_x). For instance, consider the simple bifurcation problem $f(x, \lambda) = x^3 - x\lambda$. An unfolding $G(x, \lambda, \alpha)$ of $g(x, \lambda) = (x^3 - x\lambda, 3x^2 - \lambda)$ is $\mathcal{K}$-versal if and only if $\langle \frac{\partial G}{\partial \alpha_i} \rangle_{\mathbf{R}}$ forms a complement for $\mathrm{T}_{\mathcal{K}} g$ in the $\mathcal{E}_{x\lambda}$-module of germs of maps from the plane to the plane, where $\mathrm{T}_{\mathcal{K}} g = \{(u, v) u, v \in \mathcal{M}^2 + \langle \lambda \rangle\} + \langle (x, 1) \rangle_{\mathcal{E}_{x\lambda}}$. Clearly, the constant germ $(0, 1)$ cannot be written as $(h, h_x) + \eta$ with $h \in \mathcal{E}_{x\lambda}$, $\eta \in \mathrm{T}_{\mathcal{K}} g$.

5 Examples

In this section we compute invariants for some examples of bifurcation problems. Invariants for b-equivalence are the algebraic number of folds, $\beta(f)$ (section 4) and the number, $\mathbf{r}_+(f)$ (resp. $\mathbf{r}_-(f)$), of half branches with $\lambda > 0$ (resp. $\lambda < 0$) in the bifurcation diagram of f as well as $\mathbf{r}_\pm(f) = \mathbf{r}_+(f) - \mathbf{r}_-(f)$ (section 3). We show that Conjecture 1 holds for simple bifurcations in one spatial dimension, as well as for two modal families (c_m and q_m, see below).

We also compute invariants of $\mathcal{K}$-equivalence since they are also invariant under b-equivalence, and a b-versal unfolding of a bifurcation $f(x, \lambda)$ is also a $\mathcal{K}$-versal unfolding of the germ f. One $\mathcal{K}$-invariant treated here is the number, $\mathbf{r}(f)$, of half-branches in the bifurcation diagram of f, i.e. the number of connected components of $f^{-1}(0) - \{0\}$ (section 3). Another $\mathcal{K}$-invariant is the Milnor number $\mu(f)$ of a bifurcation f. If f has finite codimension with respect to $\mathcal{K}$-equivalence, there is no loss of generality in assuming that f is real analytic. Let $f_{\mathbf{C}}$ be its complexification. Then, the complex hypersurface $f_{\mathbf{C}}^{-1}(0)$ defines a complete intersection with isolated singularity, and $\mu(f) = \mu(f_{\mathbf{C}})$ is defined as the rank of the middle dimensional homology of the Milnor fiber of $f_{\mathbf{C}}$ (see [16]).

For $n = 1$, $\mu(f)$ is the codimension of the ideal $\mathcal{I}_\mu(f) \subset \mathcal{E}_{x\lambda}$ given by

$$\mathcal{I}_\mu(f) = \langle f_x(x,\lambda), f_\lambda(x,\lambda)\rangle.$$

It gives an upper bound for the number of Morse critical points appearing in any germ in the unfolding of f. Note that these critical points do not have to be zeros of f and thus the Milnor number cannot be "read" from the bifurcation diagram.

We start with simple bifurcations in one spatial dimension (see [15]):

Table 1 – invariants for simple bifurcations, $n = 1$

normal form f	k	cod(f)	$\beta(f)$	$\mathbf{b}_{max}(f)$	$\mu(f)$	$\mathbf{r}(f)$	$\mathbf{r}_\pm(f)$
$\pm x^3 \pm \lambda^2$		3	4	4	2	2	0
$\pm x^k \pm \lambda$	k even	$k-2$	$k-1$	$k-1$	0	2	± 2
$k \geq 2$	k odd					2	0
$\pm x^2 \pm \lambda^k$	k even	$k-1$	k	k	$k-1$	4	0
$k \geq 2$	k odd					2	± 2
$\pm x^k \pm x\lambda$	k even	$k-1$	k	k	1	4	0
$k \geq 3$	k odd					4	± 2

For these simple bifurcations it is easy to find a suitable unfolding explicitly and to check that Conjecture 1 holds, i.e., $\mathbf{b}_{max}(f) = \beta(f)$. This is also the case of some modal bifurcations as can be seen in Table 2:

Table 2 – invariants for some modal bifurcations

normal form	cod(f)	$\beta(f)$	$\mathbf{b}_{max}(f)$	$\mu(f)$
$c_m(x,\lambda) = \pm(x^3 - 3mx\lambda^2 \pm 2\lambda^3)$ $m \neq 0, 1$	5	6	6	4
$q_m(x,\lambda) = \pm x^4 + 2mx^2\lambda \pm \lambda^2$ $m \neq 0, \pm 1$	5	6	6	3
$\pm x^k \pm \lambda^2$ $k \geq 4$	$2k-3$	$2k-2$	$2k-2$	$k-1$

No two simple bifurcation problems have the same invariants $\beta(f)$ and $\mu(f)$. Modal bifurcations in one dimension, however, are not classified by the

Milnor number and the algebraic number of folds, as can be seen in Tables 2 and 3:

Table 3 – invariants for some modal bifurcations

normal form		cod(f)	$\beta(f)$	$\mu(f)$
$\pm x^k \pm x\lambda^2$	$k \geq 4$	$2k-1$	$2k$	$k+1$
$\pm x^k \pm x\lambda^2 \pm \lambda^3$	$k \geq 4$	$2k-2$	$2k$	$k+1$
$\pm x^{k+1} \pm x^{k-1}\lambda^2 \pm \lambda^3$	$k \geq 3$	$3k-2$	$2k$	$k+1$
$\pm x^k \pm x^{k-2}\lambda \pm \lambda^2$	$k \geq 5$	$2k-4$	$2k-2$	$k-1$

For most cases studied here, if two bifurcation problems have the same codimension and the same values of β and of μ, then they are b-equivalent. The exceptions are, naturally, the germs inside the modal family c_m, as well as germs in the family $q_m(x,\lambda)$ (together with the germ of $\pm x^4 \pm \lambda^2 = q_0$).

6 Acknowledgements

We would like to thank J. Damon for helpful discussions and comments. This work was carried out when I.S.L. was a visitor at Instituto de Ciências Matemáticas de São Carlos, USP, Brasil, and its hospitality is gratefully acknowledged. We acknowledge the finantial support from Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Brazil, grants # 1995/8761-0 and # 97/10735-3, Conselho Nacional de Desenvolvimento Científico e Tecnológico, (CNPq), Brazil, grant # 300066/88-0. CMUP is supported by Fundação para a Ciência e a Tecnologia, Portugal, through Programa Operacional Ciência, Tecnologia e Inovação (POCTI) and Programa Operacional Sociedade da e Informação (POSI) of Quadro Comunitário de Apoio III (2000–2006) with European Union funding (FEDER) and national funding.

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