A UNIFORM COOLING OF ALLOYS

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Abstract

A phase separation process model formed by the Cahn-Hilliard equation and the heat equation is considered. The mobility coefficient in the Cahn-Hilliard equation is assumed to be an increasing function of the temperature. This property is of importance at the latest stage of the cooling process. The existence and uniqueness of a weak solution is proved, and a uniform cooling problem is studied. Obtained results can be used to control the cooling process and to create alloys with a uniform structure.

1 Introduction

In 1958 Cahn and Hilliard introduced the equation

$$u_t = \left(-\kappa^2 u_{xx} - u + u^3\right)_{xx}, \tag{1}$$

describing the dynamics of phase separation in binary systems like alloys, glasses, and polymer mixtures [2]. Here $u(t,x)$ is a perturbation of the concentration of one of the phases. The Cahn-Hilliard equation was largely studied, see [3], for example. In [6] Penrose and Fife derived a thermodynamically consistent model of phase separation process. Later Alt and Pawlow, see [1], proposed another mathematical model for the non isothermal phase separation. These models describe the beginning of phase separation. In this paper we consider a model formed by the Cahn-Hilliard equation with the mobility coefficient κ being an increasing function of temperature, and a heat equation. Although the model may not be thermodynamically consistent it has the most relevant properties of other models. Moreover the mobility coefficient in this model decrease when the temperature drops. This phenomenon is observed at the latest stage of the cooling process. The model under consideration can be used to control the cooling process and to create alloys with a given structure.

The necessity of a controlled cooling process is obvious from the following numerical simulation. A typical behavior of the concentration perturbation u, when the temperature governed by a heat equation drops at the very beginning of the process is presented in Fig. 1-3. At an early stage of cooling all concentration fluctuations are amplified (Fig. 2). This phenomenon is known as granulation process. During the process development some of the concentration peaks disappear. This effect is known as aggregation of grains. Its speed depends on the temperature. Since the temperature inside the alloy is higher than at the neighborhood of the boundary, the speed of aggregation inside is higher than the speed near the boundary (Fig. 3). Controlling the temperature at the boundary of the alloy we can control the cooling process and create materials with desired concentration distribution, for example, a uniform one. How should we control the temperature in order to get a uniform concentration distribution? This is the main problem addressed in the paper.

The paper is organized as follows. In the next section we consider the Cahn-Hilliard equation

$$u_t = \left(-\left(\kappa u_x\right)_x + F_u\left(u\right)\right)_{xx}, \quad (t,x) \in Q_T =]0,T[\times]0,L[, \tag{2}$$

where $\kappa = \kappa\left(t,x\right)$ is a sufficiently smooth function and $F\left(u\right) = 2u^4 - u^2 + 1/8$. The solution satisfies the following initial and boundary conditions

$$u\left(0,x\right) = u_0\left(x\right), \quad x \in [0,L], \tag{3}$$

$$u_x = 0, \quad x \in \left\{0,L\right\}, \tag{4}$$

$$\left(\kappa u_x\right)_{xx} = 0, \quad x \in \left\{0,L\right\}. \tag{5}$$

To prove a mathematical correctness of the model under consideration we establish the existence and uniqueness of a weak solution to this problem.

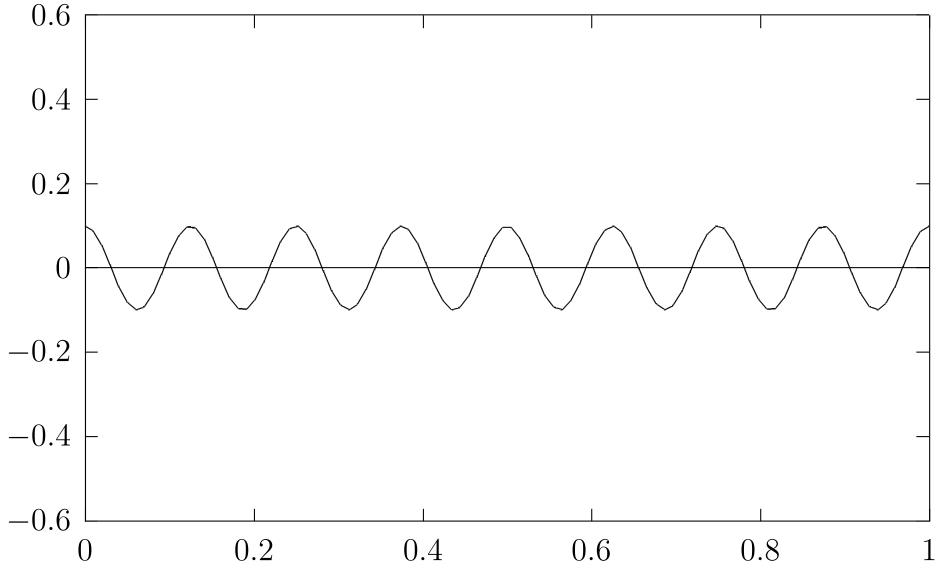

Figure 1: Concentration distribution at $t = 0$.

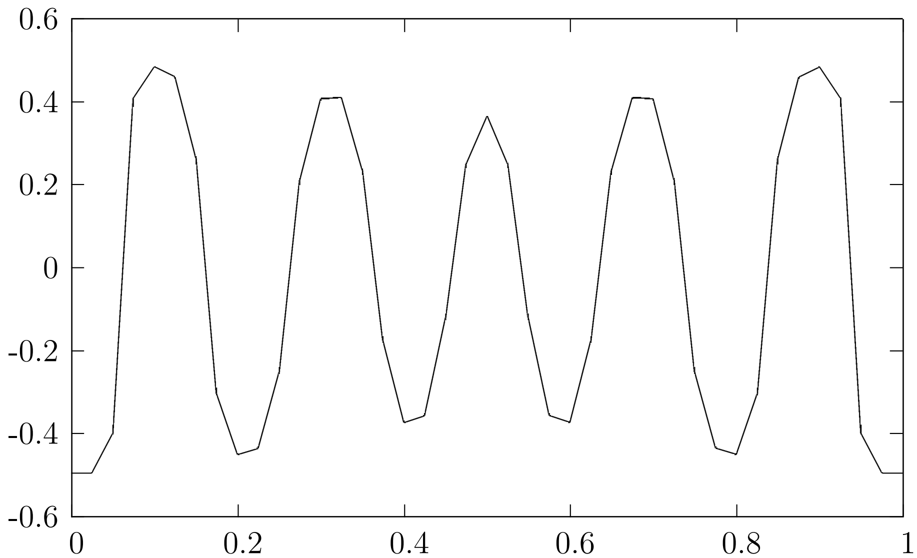

Figure 2: Granulation.

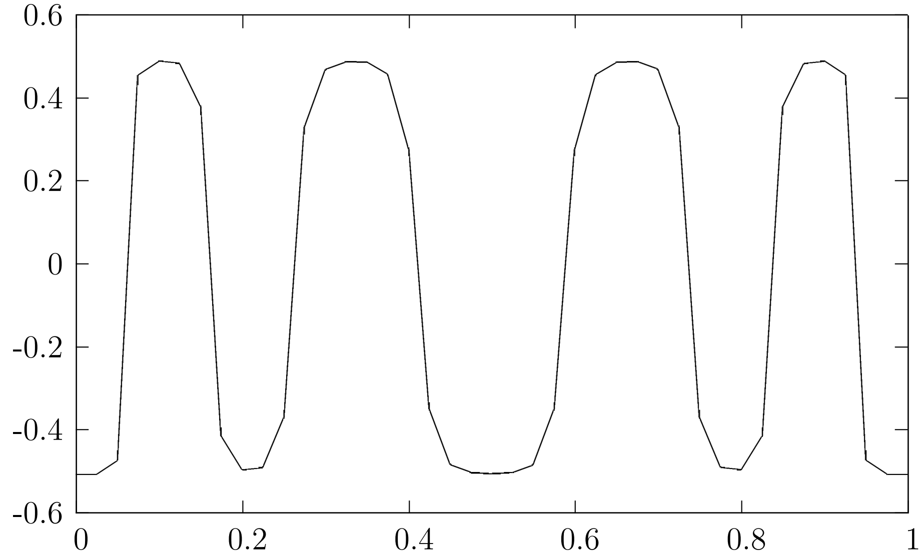

Figure 3: Aggregation of grains

In the last section we consider a uniform cooling problem. The aim is to drive the temperature $\theta\,(t,x)$ satisfying $\theta(0,x) \equiv \Theta_0$ and governed by the heat equation, to a given value $\Theta_T < \Theta_0$ and to guarantee a uniform cooling, that is, to ensure the condition

$$\theta\,(t,x) \approx \frac{1}{L}\int_0^L \theta\,(t,y)\,dy, \quad (t,x) \in Q_T.$$

The temperature satisfies non-homogeneous Robin boundary conditions of the form

$$\begin{cases} \xi\theta_x\,(t,0) & = & \theta\,(t,0) - q\,(t)\,, \\ -\xi\theta_x\,(t,L) & = & \theta\,(t,L) - q\,(t)\,, \end{cases}$$

where the function $q\,(t)$ is a control we have in our disposal. The problem is formalized as an optimal control problem. In order to get an analytical result we consider a finite dimensional approximation to the heat equation [5] and show that the control law ensuring a uniform cooling has the form

$$q\,(t) \approx c_1 + c_2 t.$$

From the practical point of view this means that the uniform cooling problem can be reduced to a simple two-dimensional minimization problem.

2 Existence and uniqueness

We say that $u \in \mathcal{L}^2\left(0, T; H^3\left([0, L]\right)\right)$ is a weak solution to problem (2)-(5) if for any $\upsilon \in H^1\left([0, L]\right)$ the relation

$$\frac{d}{dt}\left(u, \upsilon\right) = \left(\left(\kappa u_x\right)_{xx}, \upsilon_x\right) - \left(u_x F_{uu}\left(u\right), \upsilon_x\right). \tag{6}$$

holds. The main result of this section is

Theorem 1 *Assume that $F\left(u\right) = 2u^4 - u^2 + 1/8$, $\kappa \in \mathcal{C}^3(Q_T)$, and there exist positive constants C_0', C_0, C_1, C_2, and C_3 such that*

$$C_0 \leq \kappa \leq C_0', \quad |\kappa_x| \leq C_1, \quad |k_{xx}| \leq C_2. \tag{7}$$

Then for every $u_0 \in H^3\left([0, L]\right)$ problem (2)-(5) has a unique weak solution.

To prove the theorem we need the following a priopi estimates. Let u be a solution to (2)-(5) .

Lemma 2 *The inequalities*

$$\int_0^L u^2 dx \leq (\text{const}) \int_0^L u_0^2 dx \tag{8}$$

and

$$\int_0^T \int_0^L u_{xx}^2 dx dt \leq (\text{const}) \int_0^L u_0^2 dx \tag{9}$$

hold.

Proof. Multiplying (2) by u and integrating with respect to x we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2 dx = -\int_0^L \kappa_x u_x u_{xx} dx - \int_0^L \kappa u_{xx}^2 dx - \int_0^L F_{uu}\left(u\right) u_x^2 dx.$$

Since

$$\int_0^L \kappa_x u_x u_{xx} dx = \frac{1}{2}\int_0^L \kappa_x \left(u_x^2\right)_x dx,$$

we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2 dx = -\frac{1}{2}\int_0^L \kappa_x \left(u_x^2\right)_x dx - \int_0^L \kappa u_{xx}^2 dx$$
$$= -\int_0^L k u_{xx}^2 dx + \int_0^L \left(\frac{k_{xx}}{2} - F_{uu}(u)\right) u_x^2 dx.$$

From (7) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2 dx \leq -\int_0^L \kappa u_{xx}^2 dx + (\mathrm{const})\int_0^L u_x^2 dx.$$

Since (4) is satisfied, by Young's inequality we have

$$\int_0^L u_x^2 dx = -\int_0^L u u_{xx} dx \leq \frac{\varepsilon^2}{2}\int_0^L u_{xx}^2 dx + \frac{1}{2\varepsilon^2}\int_0^L u^2 dx, \quad \varepsilon > 0.$$

Setting $\epsilon = \sqrt{C_0'}$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2 dx \leq -\frac{C_0'}{2}\int_0^L u_{xx}^2 dx + (\mathrm{const})\int_0^L u^2 dx.$$

Applying the Gronwall inequality we obtain (8) and (9). □

Set

$$\mathcal{J}(u)(t) = \int_0^L e^{\int_0^t \min\left\{0, -\max_{x\in[0,L]}(\kappa_t/\kappa)\right\} d\tau}\left(\frac{\kappa}{2}u_x^2 + F(u)\right) dx,$$

Lemma 3 *The inequality*

$$\mathcal{J}(u)(t) \leq \mathcal{J}(u)(0), \quad t \in [0,T]. \tag{10}$$

holds

Proof. Indeed, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(u)(t) = \frac{1}{2}\int_0^L e^{\int_0^t \min\left\{0, -\max_{x\in[0,L]}(\kappa_t/\kappa)\right\} d\tau}\left[\min\left\{0, -\max_{x\in[0,L]}(\kappa_t/\kappa)\right\}\kappa + \kappa_t\right] u_x^2 dx$$

$$+\int_0^L e^{\int_0^t \min\left\{0, -\max_{x\in[0,L]}(\kappa_t/\kappa)\right\} d\tau}\min\left\{0, -\max_{x\in[0,L]}(\kappa_t/\kappa)\right\} F(u)\, dx$$

$$+\int_0^L e^{\int_0^t \min\left\{0, -\max_{x\in[0,L]}(\kappa_t/\kappa)\right\} d\tau}\left(\kappa u_x u_{xt} + F_u(u)\, u_t\right) dx.$$

Since $F(u) \geq 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}(u)(t) \leq (\mathrm{const})\int_0^L \left(\kappa u_x u_{xt} + F_u(u)\, u_t\right) dx.$$

Integrating by parts we have

$$\int_0^L \left(\kappa u_x u_{xt} + F_u(u)\, u_t\right) dx = \int_0^L \left(-\left(\kappa u_x\right)_x + F_u(u)\right) u_t dx.$$

Using (2) and integrating by parts we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{J}\left(u\right)\left(t\right)\leq-(\text{const})\int_0^L\left[\left(-\left(\kappa u_x\right)_x+F_u\left(u\right)\right)_x\right]^2dx\leq 0.$$

Thus we have $d\mathcal{J}/dt\left(u\right)\left(t\right)\leq 0$. This implies (10). □

From Lemma 3 we have

$$\int_0^L\left(\frac{\kappa}{2}u_x^2+2u^4-u^2+\frac{1}{8}\right)dx\leq(\text{const}).$$

Combining this with Lemma 2 we obtain

$$\int_0^L u_x^2dx+\int_0^L u^4dx\leq(\text{const}). \tag{11}$$

Therefore we have

$$\sup_{x\in[0,L]}|u(t,x)|\leq(\text{const}) \tag{12}$$

whenever $t\in[0,T]$.

Lemma 4 *The inequality*

$$\int_0^T\int_0^L u_{xxx}^2dxdt\leq(\text{const}) \tag{13}$$

holds.

Proof. Multiplying (2) by u_{xx} and integrating with respect to x we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2dx=-\int_0^L u_{xx}((\kappa u_x)_x)_{xx}dx+\int_0^L(F_u(u))_{xx}u_{xx}dx.$$

From this we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2dx=-\int_0^L u_{xxx}(\kappa u_x)_{xx}dx+\int_0^L(F_u(u))_x u_{xxx}dx$$

$$=-\int_0^L\kappa u_{xxx}^2dx-\int_0^L(2\kappa_k u_{xx}+k_{xx}u_x)u_{xxx}dx+\int_0^L(F_u(u))_x u_{xxx}dx.$$

Applying the Young inequality and using (7) we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2dx\leq -C_0\int_0^L u_{xxx}^2dx+\frac{\epsilon^2}{2}\int_0^L u_{xxx}^2dx+(\text{const})\int_0^L(u_{xx}^2+u_x^2)dx$$

$$+\frac{\epsilon^2}{2}\int_0^L u_{xxx}^2dx+(\text{const})\int_0^L F_{uu}^2(u)u_x^2dx$$

Taking $\epsilon = \sqrt{C_0/2}$ and invoking (12) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L u_x^2 dx \le -\frac{C_0}{2}\int_0^L u_{xxx}^2 dx + (\text{const})\int_0^L (u_{xx}^2 + u_x^2) dx$$

From this and estimates (9) and (11) we get (13). □

Proof of Theorem 1. Using the Galerkin method and the a priori estimates from Lemmas 2-4 and following the proof of Theorem 4.2 from [7, Chapter III] we obtain the result. □

3 Uniform Cooling

In this section we study the problem of uniform cooling for the heat equation

$$\theta = \sigma^2 \theta_{xx}, \quad (t, x) \in Q_T, \tag{14}$$

with non-homogeneous Robin boundary conditions

$$\begin{cases} \xi\theta_x(0,t) & = & \theta(t,0) - q(t), \\ & & \\ -\xi\theta_x(t,L) & = & \theta(t,L) - q(t), \end{cases} \qquad t \in [0,T],$$

where $q(t)$ is a control we have in our disposal. At the initial moment of time $t = 0$ the temperature is constant, $\theta(0,x) = \Theta_0$, $x \in [0,L]$. Our aim is to drive the temperature to a given value $\Theta_T < \Theta_0$ (at least approximately) and to guarantee a uniform cooling, that is, to ensure the condition $\theta(t,x) \approx \frac{1}{L}\int_0^L \theta(t,x)\,dx$, $(t,x) \in Q_T$.

The problem can be formalized in the following way: one has to minimize the functional

$$\kappa_1 \int_0^L (\theta(T,x) - \Theta_T)^2\,dx + \kappa_2 \int_0^T \int_0^L \left(\theta(t,x) - \bar{\theta}(t)\right)^2 dxdt + \kappa_3 \int_0^T (\dot{q}(t))^2\,dt,$$

where $\bar{\theta}(t) = L^{-1}\int_0^L \theta(t,y)\,dy$ and $\kappa_j > 0$, $j = 1,2,3$. The last term in the functional is needed to ensure the existence and uniqueness of the optimal control, see [4]. Set $\eta = \theta - q$ and $\alpha = \dot{q}$. Then we have the following optimization problem:

$$\kappa_1 \int_0^L (\theta(T,x) - \Theta_T)^2\,dx + \kappa_2 \int_0^T \int_0^L \left(\theta(t,x) - \bar{\theta}(t)\right)^2 dxdt$$

$$+\kappa_3 \int_0^T (\alpha(t))^2\,dt \to \inf,$$

$$\begin{aligned} &\eta_t(t,x) = \sigma^2\eta_{xx}(t,x) - \alpha(t), \\ &\xi\eta_x(0,t) = \eta(t,0), \\ &\xi\eta_x(t,L) = -\eta(t,L), \\ &\dot{q}(t) = \alpha(t), \ t \in [0,T], \\ &\eta_0(x) = 0, \\ &q(0) = \Theta_0. \end{aligned}$$

Here $\alpha(t)$ is a control. The solution to the initial boundary value problem can be written in the form

$$\theta(t,x) = \sum_{k=1}^{\infty} \eta_k(t)\, \upsilon_k,$$

where $\upsilon_k(x) = \beta_k \cos(\beta_k x) + \sin(\beta_k x)$, β_k are positive solutions of the equation

$$-\tan(L\beta) = \frac{2\beta\xi}{1-\xi^2\beta^2}$$

and the functions $\eta_k(\cdot)$ are solutions to the equations

$$\dot{\eta}_k(t) = -\lambda_k \eta_k(t) - h_k \alpha(t), \quad k = 1,2,\ldots,$$

with

$$\lambda_k = (\sigma\beta_k)^2, \quad h_k = \frac{\gamma_k}{\|\upsilon_k\|^2} \equiv \frac{\int_0^L \upsilon_k dx}{\|\upsilon_k\|^2}.$$

Here $\|\upsilon_k\|$ stands for the L_2-norm of the function υ_k. Thus, the optimization problem can be written in the form

$$\psi(T,\eta(T),q(T)) + \int_0^T \phi(t,\eta(t),\alpha(t))\, dt \to \inf,$$

$$\begin{aligned}
&\dot{\eta}_k(t) = -\lambda_k \eta_k(t) - h_k \alpha(t), \quad k = 1,2,\ldots, \\
&\dot{q}(t) = \alpha(t) \\
&\eta_k(0) = 0, \\
&q(0) = \Theta_0.
\end{aligned}$$

where

$$\psi(T,\eta(T),q(T)) = \kappa_1 L (q(T) - \Theta_T)^2 + \kappa_1 \sum_{k=1}^{\infty} \|\upsilon_k\|^2 \eta_k^2(T)$$

$$+2\kappa_1 (q(T) - \Theta_T) \sum_{k=1}^{\infty} \gamma_k \eta_k(T),$$

and

$$\phi(\eta_k(t),\alpha(t)) = \kappa_2 \left(\sum_{k=1}^{\infty} \|\upsilon_k\|^2 \eta_k^2(t) - \frac{1}{L} \left(\sum_{k=1}^{\infty} \gamma_k \eta_k(t) \right) \right)^2 + \kappa_3 (\alpha(t))^2.$$

Consider a finite dimensional approximation to this problem

$$\psi^N(T,\eta(T),q(T)) + \int_0^T \phi^N(t,\eta(t),\alpha(t))\, dt \to \min,$$

$$
\begin{aligned}
&\dot{\eta}_k(t) = -\lambda_k \eta_k(t) - h_k \alpha(t), \quad k = \overline{1,N}, \\
&\dot{q}(t) = \alpha(t), \\
&\eta_k(0) = 0, \quad k = \overline{1,N}, \\
&q(0) = \Theta_0,
\end{aligned}
$$

where

$$
\psi^N(T, \eta(T), q(T)) = \kappa_1 L (q(T) - \Theta_T)^2 + \kappa_1 \sum_{k=1}^{N} \|\upsilon_k\|^2 \eta_k^2(T)
$$

$$
+2\kappa_1 (q(T) - \Theta_T) \sum_{k=1}^{N} \gamma_k \eta_k(T),
$$

and

$$
\phi^N(\eta_k(t), \alpha(t)) = \kappa_2 \left(\sum_{k=1}^{N} \|\upsilon_k\|^2 \eta_k^2(t) - \frac{1}{L} \left(\sum_{k=1}^{N} \gamma_k \eta_k(t) \right)^2 \right) + \kappa_3 (\alpha(t))^2 .
$$

The optimal control $\hat{\alpha}(t)$ obviously exists. Applying the Pontriagin Maximum Principle we see that there exist functions $\mathsf{p}_k(t)$, $k = \overline{1,N}$, and $\mathsf{p}_q(t)$ satisfying the following conditions:

$$
\begin{aligned}
&\dot{\mathsf{p}}_k(t) = \lambda_k \mathsf{p}_k(t) + \kappa_2 \left(2 \|\upsilon_k\|^2 \eta_k(t) - \frac{2\gamma_k}{L} \sum_{j=1}^{N} \gamma_j \eta_j(t) \right), \\
&\dot{\mathsf{p}}_q(t) = 0, \\
&\mathsf{p}_k(T) = -\kappa_1 \left(2 \|\upsilon_k\|^2 \eta_k(T) + 2\gamma_k (q(T) - \Theta_T) \right), \quad k = \overline{1,N}, \\
&\mathsf{p}_q(T) = -\kappa_1 \left(2L (q(T) - \Theta_T) + 2 \sum_{k=1}^{N} \gamma_k \eta_k(T) \right). \\
&\max_{\alpha \in \mathbb{R}} \left(\left(- \sum_{k=1}^{N} h_k \mathsf{p}_k(t) + \mathsf{p}_q(t) \right) \alpha - \phi(\eta_k(t), \alpha(t)) \right) = \\
&- \sum_{k=1}^{N} h_k \mathsf{p}_k(t) + \mathsf{p}_q(t) - 2\kappa_3 \hat{\alpha}(t).
\end{aligned}
$$

From the maximum condition we find

$$
\hat{\alpha}(t) = \frac{1}{2\kappa_3} \left(- \sum_{j=1}^{N} h_k \mathsf{p}_k(t) + \mathsf{p}_q(t) \right).
$$

Therefore to solve the optimal control problem it suffices to solve the boundary value problem

$$\dot{\eta}_k(t) = -\lambda_k \eta_k(t) - \frac{h_k}{2\kappa_3}\mathsf{p}_q(t) + \frac{h_k}{2\kappa_3}\sum_{j=1}^{N} h_j \mathsf{p}_j(t), \quad k = \overline{1,N}, \tag{15}$$

$$\dot{q}(t) = \frac{1}{2\kappa_3}\left(\mathsf{p}_q(t) - \sum_{j=1}^{N} h_j \mathsf{p}_j(t)\right), \tag{16}$$

$$\dot{\mathsf{p}}_k(t) = \lambda_k \mathsf{p}_k(t) + \kappa_2\left(2\|\upsilon_k\|^2 \eta_k(t) - \frac{2\gamma_k}{L}\sum_{j=1}^{N}\gamma_j\eta_j(t)\right), \quad k = \overline{1,N}, \tag{17}$$

$$\dot{\mathsf{p}}_q(t) = 0, \tag{18}$$

$$\eta_k(0) = 0, \quad k = \overline{1,N}, \tag{19}$$

$$q(0) = \Theta_0, \tag{20}$$

$$\mathsf{p}_k(T) = -\kappa_1\left(2\|\upsilon_k\|^2 \eta_k(T) + 2\gamma_k\left(q(T) - \Theta_T\right)\right), \quad k = \overline{1,N}, \tag{21}$$

$$\mathsf{p}_q(T) = -\kappa_1\left(2L\left(q(T) - \Theta_T\right) + 2\sum_{j=1}^{N}\gamma_j\eta_j(T)\right). \tag{22}$$

Consider the matrix of system (15)-(18)

$$M = \begin{pmatrix} -\Lambda & 0 & \frac{1}{2\kappa_3}HH^* & -\frac{1}{2\kappa_3}H \\ 0 & 0 & -\frac{1}{2\kappa_3}H^* & \frac{1}{2\kappa_3} \\ 2\kappa_2\left(V - \frac{1}{L}\Gamma\Gamma^*\right) & 0 & \Lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where Λ is $N \times N$ diagonal matrix with the elements λ_k, $k = \overline{1,N}$, V is $N \times N$ diagonal matrix with the elements $\|\upsilon_k\|^2$, Γ is a column matrix with the elements γ_k, $k = \overline{1,N}$, and H is a column vector with the elements h_k, $k = \overline{1,N}$. To find the matrix M eigenvalues and eigenvectors consider the system

$$\mu\eta = -\Lambda\eta + \frac{1}{2\kappa_3}HH^*\mathsf{p} - \frac{1}{2\kappa_3}H\mathsf{p}_q, \tag{23}$$

$$\mu q = -\frac{1}{2\kappa_3}H^*\mathsf{p} + \frac{1}{2\kappa_3}\mathsf{p}_q, \tag{24}$$

$$\mu\mathsf{p} = 2\kappa_2\left(V - \frac{1}{L}\Gamma\Gamma^*\right)\eta + \Lambda\mathsf{p}, \tag{25}$$

$$\mu\mathsf{p}_q = 0. \tag{26}$$

We are looking for nontrivial solutions to (23)-(26).

Lemma 5 *If system (23)-(26) with $\mu \neq 0$ has a nontrivial solution, then μ satisfies*

$$\mathcal{F}_{\xi,N}(\mu) = 1 - \frac{\kappa_2}{2\kappa_3} H^* (\mu I_N - \Lambda)^{-1} V (\mu I_N - \Lambda)^{-1} H + \frac{\kappa_2}{L\kappa_3} H^* (\mu I_N - \Lambda)^{-1} \Gamma\Gamma^* (\mu I_N + \Lambda)^{-1} H = 0. \tag{27}$$

Proof. If $\mu \neq 0$ then from (26), we have $\mathsf{p}_q = 0$, and system (23)-(25) takes the form

$$\mu\eta = -\Lambda\eta + \frac{1}{2\kappa_3} H H^* \mathsf{p}, \tag{28}$$

$$\mu q = -\frac{1}{2\kappa_3} H^* \mathsf{p}, \tag{29}$$

$$\mu \mathsf{p} = 2\kappa_2 \left(V - \frac{1}{L}\Gamma\Gamma^* \right) \eta + \Lambda \mathsf{p}. \tag{30}$$

Invoking (29) and (28) we get

$$\eta = -\mu q (\mu I_N + \Lambda)^{-1} H. \tag{31}$$

From (30) and (31) we obtain

$$\mathsf{p} = -2\kappa_2 \mu q (\mu I_N - \Lambda)^{-1} \left(V - \frac{1}{L}\Gamma\Gamma^* \right) (\mu I_N + \Lambda)^{-1} H. \tag{32}$$

If $q = 0$, then $\eta = 0$, $\mathsf{p} = 0$, and $\mathsf{p}_q = 0$. Therefore, $q \neq 0$. Now from (32) and (29) we have (27). □

Lemma 6 *If $\xi = 0$ and $N = \infty$, then*

$$\mathcal{F}_{0,\infty}(i\omega) = 1 + \frac{\kappa_2 L^5}{\kappa_3 \sigma^4 \pi^6} \frac{2\pi\sqrt{\frac{\omega}{2}} \left(\sinh \pi\sqrt{\frac{\omega}{2}} + \sin \pi\sqrt{\frac{\omega}{2}} \right)}{\omega^3 \left(\cosh \pi\sqrt{\frac{\omega}{2}} + \cos \pi\sqrt{\frac{\omega}{2}} \right)}$$

$$- \frac{4\kappa_2 L^5}{\kappa_3 \sigma^4 \pi^6} \frac{\cosh \pi\sqrt{\frac{\omega}{2}} - \cos \pi\sqrt{\frac{\omega}{2}}}{\omega^3 \left(\cosh \pi\sqrt{\frac{\omega}{2}} + \cos \pi\sqrt{\frac{\omega}{2}} \right)} > 1, \quad \omega > 0.$$

Proof. After simple calculations we get

$$\mathcal{F}_{\xi,N}(\mu) = 1 - \frac{\kappa_2}{\kappa_3} \left(\sum_{k=1}^{N} \frac{\gamma_k h_k}{\mu^2 - \lambda_k^2} - \frac{1}{L} \sum_{j=1}^{N} \frac{\gamma_j h_j}{\mu - \lambda_j} \sum_{k=1}^{N} \frac{\gamma_k h_k}{\mu + \lambda_k} \right).$$

Set $\nu = \xi/L$, $\delta_k(\nu) = \beta_k(\nu)L$, and

$$r_k(\nu) = \frac{(\nu \sin \delta_k(\nu) - \cos \delta_k(\nu) + 1)^2}{(\nu^2 \delta_k^2(\nu) + 2\nu + 1)}.$$

Since $\tan\delta_k(\nu) = 2\nu\delta_k(\nu)/(\nu^2\delta_k^2(\nu)-1)$, we have

$$\gamma_k h_k = \frac{2}{L}\frac{\left(\xi\beta_k\sin(\beta_k L)-\cos(\beta_k L)+1\right)^2}{\beta_k^2\left(\xi^2\beta_k^2+2\xi/L+1\right)} = \frac{2L}{\delta_k^2(\nu)}r_k(\nu).$$

It is easy to see that $\delta_k(0) = k\pi$ and $r_k(0) = (1-(-1)^k)^2$. Thus we have

$$\mathcal{F}_{0,\infty}(\mu) = 1 - \frac{\kappa_2}{\kappa_3}\left(\sum_{k=1}^{\infty}\frac{2L^5}{\delta_k^2(0)}\frac{r_k(0)}{L^4\mu^2-\sigma^4\delta_k^4(0)}\right.$$

$$\left.-\frac{1}{L}\sum_{k=1}^{\infty}\frac{2L^3}{\delta_k^2(0)}\frac{r_k(0)}{L^2\mu-\sigma^2\delta_k^2(0)}\sum_{k=1}^{\infty}\frac{2L^3}{\delta_k^2(0)}\frac{r_k(0)}{L^2\mu+\sigma^2\delta_k^2(0)}\right). \tag{33}$$

Set $y = \mu(L/(\sigma\pi))^2$. Then we have

$$\mathcal{F}_{0,\infty}(\mu) = 1 - \frac{8\kappa_2 L^5}{\kappa_3\sigma^4\pi^6}\left(\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2\left((2k-1)^4-y^2\right)}\right.$$

$$\left.-\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2\left((2k-1)^2-y\right)}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2\left((2k-1)^2+y\right)}\right).$$

The series can be easily evaluated using residues. We have

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2\left((2k-1)^4-y^2\right)} = \sum_{k=1}^{\infty}\frac{1}{4\left(k-\frac{1}{2}\right)^2\left(16\left(k-\frac{1}{2}\right)^4-y^2\right)}$$

$$= -\frac{\pi}{2}\sum_{k=1}^{K}\mathrm{res}_{z=z_k}\left[f(z)\cot(\pi z)\right],$$

where $f(z) = \left(4(k-1/2)^2\left(16(k-1/2)^4-y^2\right)\right)^{-1}$ and the residues are calculated at singularities of f. From this we get

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2\left((2k-1)^4-y^2\right)} = \frac{\pi^2}{8(iy)^2}\left(1-\frac{\sqrt{2}}{\pi\sqrt{-iy}}\frac{\sin\pi\sqrt{-\frac{iy}{2}}+\sinh\pi\sqrt{-\frac{iy}{2}}}{\cos\pi\sqrt{-\frac{iy}{2}}+\cosh\pi\sqrt{-\frac{iy}{2}}}\right).$$

Analogously we obtain

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2\left((2k-1)^2+y\right)} = \frac{\pi^2}{8y}-\frac{\pi}{4y^{3/2}}\tan\frac{\pi\sqrt{y}}{2},$$

and

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2\left((2k-1)^2-y\right)} = -\frac{\pi^2}{8y}+\frac{\pi}{4y^{3/2}}\tan\frac{\pi\sqrt{y}}{2}.$$

After simple calculations we have

$$\mathcal{F}_{0,\infty}(i\omega) = 1 + \frac{\kappa_2 L^5}{\kappa_3 \sigma^4 \pi^6} \frac{2\pi\sqrt{\frac{\omega}{2}}\left(\sinh \pi\sqrt{\frac{\omega}{2}} + \sin \pi\sqrt{\frac{\omega}{2}}\right)}{\omega^3\left(\cosh \pi\sqrt{\frac{\omega}{2}} + \cos \pi\sqrt{\frac{\omega}{2}}\right)}$$

$$-\frac{4\kappa_2 L^5}{\kappa_3 \sigma^4 \pi^6} \frac{\cosh \pi\sqrt{\frac{\omega}{2}} - \cos \pi\sqrt{\frac{\omega}{2}}}{\omega^3\left(\cosh \pi\sqrt{\frac{\omega}{2}} + \cos \pi\sqrt{\frac{\omega}{2}}\right)}.$$

Show that $\mathcal{F}_{0,\infty}(i\omega) > 1$, $\omega > 0$. Set $x = \pi\sqrt{\omega/2}$. It suffices to prove the inequality

$$g(x) = 2x(\sinh x + \sin x) - 4(\cosh x - \cos x) > 0, \quad x > 0.$$

Obviously $g(0) = 0$, $g'(0) = 0$, and $g''(x) = 2x(\sinh x - \sin x) > 0$ whenever $x > 0$. Therefore we have $g(x) = \int_0^x \int_0^y g''(z)\,dzdy > 0$, $x > 0$. Since

$$\mathcal{F}_{0,\infty}(0) = 1 + \frac{\kappa_2 L^5}{\kappa_3 \sigma^4 6!} > 1,$$

we obtain $\mathcal{F}_{0,\infty}(i\omega) > 1$ whenever $\omega > 0$. □

Obviously $\mathcal{F}_{0,\infty}(i\omega) = \mathcal{F}_{0,\infty}(-i\omega)$. Therefore $\mathcal{F}_{0,\infty}(i\omega) > 1$ for all real ω. It is easy to see that there exists a positive integer N_0 such that $\mathcal{F}_{0,N}(i\omega) > \mathcal{F}_{0,\infty}(i\omega) - 1/2$, $\omega > 0$, whenever $N > N_0$.

Lemma 7 *Let $N > N_0$. Then there exists ξ_0 such that $\mathcal{F}_{\xi,N}(i\omega) > 0$ whenever $\xi \in [0, \xi_0]$ and ω is real.*

Proof. Since $\lim_{\xi\to 0} |\mathcal{F}_{\xi,N}(i\omega) - \mathcal{F}_{0,N}(i\omega)| = 0$, there exists ξ_0 such that $\mathcal{F}_{\xi,N}(i\omega) > \mathcal{F}_{0,N}(i\omega) - 1/2 > \mathcal{F}_{0,\infty}(i\omega) - 1 > 0$ whenever $\xi \in [0, \xi_0]$. □

Let $\mu \neq 0$ be an eigenvalue of M. Using (31) and (32) we obtain the corresponding eigenvector

$$\begin{pmatrix} \eta(\mu) \\ q(\mu) \\ \mathsf{p}(\mu) \\ \mathsf{p}_q(\mu) \end{pmatrix} = \begin{pmatrix} -\mu(\mu I_N + \Lambda)^{-1} H \\ 1 \\ -2\mu\kappa_2(\mu I_N - \Lambda)^{-1}\left(V - \frac{1}{L}\Gamma\Gamma^*\right)(\mu I_N + \Lambda)^{-1} H \\ 0 \end{pmatrix}. \tag{34}$$

The degree of the polynomial $\mathcal{P}(\mu) = \mathcal{F}(\mu)\prod_{k=1}^{N}(\mu^2 - \lambda_k^2)$ equals $2N$. Obviously the equalities $\mathcal{P}(\mu) = 0$ and $\mathcal{F}(\mu) = 0$ are equivalent and the polynomial $\mathcal{P}$ has exactly $2N$ roots. From Lemma 7 we see that the roots do not belong to the imaginary axis. Thus we have proved the following result.

Lemma 8 *If the numbers N and $1/\xi$ are big enough then matrix M has $2N$ non-zero eigenvalues. These eigenvalues are solutions to the equation $\mathcal{F}_{\xi,N}(\mu) = 0$. The corresponding eigenvectors are given by (34).*

From this lemma we see that $\mu = 0$ is an eigenvalue of M and its multiplicity is equal to two. Simple calculations show that the corresponding eigenvector and the principal vector are given by

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \eta^0 \\ q^0 \\ \mathsf{p}^0 \\ \mathsf{p}_q^0 \end{pmatrix} = \begin{pmatrix} -\Lambda^{-1}H \\ 0 \\ 2\kappa_2\Lambda^{-1}\left(V - \frac{1}{L}\Gamma\Gamma^*\right)\Lambda^{-1}H \\ 2\left(\kappa_3 + \kappa_2 H^*\Lambda^{-1}\left(V - \frac{1}{L}\Gamma\Gamma^*\right)\Lambda^{-1}H\right) \end{pmatrix}.$$

Thus we have the following result.

Lemma 9 *If the numbers N and $1/\xi$ are big enough then the solution to system (15)-(18) has the form*

$$\begin{pmatrix} \eta \\ q \\ p \\ p_q \end{pmatrix}(t) = \sum_{k=1}^{N} c_k^+ e^{\mu_k^+ t} \begin{pmatrix} \eta\left(\mu_k^+\right) \\ q\left(\mu_k^+\right) \\ p\left(\mu_k^+\right) \\ p_q\left(\mu_k^+\right) \end{pmatrix} + \sum_{k=1}^{N} c_k^- e^{\mu_k^- t} \begin{pmatrix} \eta\left(\mu_k^-\right) \\ q\left(\mu_k^-\right) \\ p\left(\mu_k^-\right) \\ p_q\left(\mu_k^-\right) \end{pmatrix}$$

$$+ c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \left(t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \eta^0 \\ q^0 \\ p^0 \\ p_q^0 \end{pmatrix} \right)$$

where c_k^+ are c_k^- complex constants, c_1 and c_2 are real constants, the numbers $\mu_k^\pm$ are the non-zero eigenvalues of matrix M satisfying $\operatorname{Re} \mu_k^+ > 0$ *and* $\operatorname{Re} \mu_k^- < 0$*, for $k = \overline{1, N}$.*

Let $\mathbf{A}^+$ be $N \times N$ matrix with the columns $e^{-\mu_k^+ T}\eta\left(\mu_k^+\right)$, $k = \overline{1, N}$, $\alpha_0 = 0$, $\mathbf{A}^-$ be $N \times N$ matrix with the columns $\eta\left(\mu_k^-\right)$, $k = \overline{1, N}$, $\alpha_1 = \eta_0/T$, $\mathbf{B}^+$ be $N \times N$ matrix with the columns $\mathsf{p}\left(\mu_k^+\right) + 2\kappa_1\left(V\eta\left(\mu_k^+\right) + \Gamma q\left(\mu_k^+\right)\right)$, $k = \overline{1, N}$, $\mathbf{B}^-$ be $N \times N$ matrix with the columns $e^{\mu_k^- T}\left(\mathsf{p}\left(\mu_k^-\right) + 2\kappa_1\left(V\eta\left(\mu_k^-\right) + \Gamma q\left(\mu_k^-\right)\right)\right)$, $k = \overline{1, N}$, $\mathbf{D}^+$ be a row with the elements $\mathsf{p}_q\left(\mu_k^+\right) + 2\kappa_1\left(\left\langle \Gamma, \eta\left(\mu_k^+\right)\right\rangle + Lq\left(\mu_k^+\right)\right)$, and $\mathbf{D}^-$ be a row with the elements $e^{\mu_k^- T}\left(\mathsf{p}_q\left(\mu_k^-\right) + 2\kappa_1\left(\left\langle \Gamma, \eta\left(\mu_k^-\right)\right\rangle + Lq\left(\mu_k^-\right)\right)\right)$. Set $\delta_0 = 2\kappa_1 L$, $\beta_0 = 2\kappa_1 L$, $\beta_1 = \left(\mathsf{p}^0 + 2\kappa_1 V\eta^0 + 2\kappa_1\left(T + q^0\right)\Gamma\right)/T$, and $\delta_1 = \left(2\kappa_1 L\left(T + q^0\right) + \mathsf{p}_q^0\right)/T$. Boundary conditions (19), (21), and (22) can be written in the matrix form

$$\begin{pmatrix} \mathbf{A}^+ & \mathbf{A}^- & \alpha_0 & \alpha_1 \\ \mathbf{B}^+ & \mathbf{B}^- & \beta_0 & \beta_1 \\ \mathbf{D}^+ & \mathbf{D}^- & \delta_0 & \delta_1 \end{pmatrix} \begin{pmatrix} \mathbf{C}^+ \\ \mathbf{C}^- \\ \mathbf{C}_1 \\ \mathbf{C}_2 \end{pmatrix} = \frac{1}{\mathcal{X}} \begin{pmatrix} 0 \\ 2\kappa_1\Theta_T\Gamma \\ 2\kappa_1 L\Theta_T \end{pmatrix}, \tag{35}$$

where $\mathcal{X} = \sum_{k=1}^{N}\left(\left|c_k e^{\mu_k^+ T}\right| + \left|c_k^-\right|\right) + |c_1| + |c_2 T|$, $\mathbf{C}^+$ and $\mathbf{C}^-$ are vectors with the components $\mathbf{C}_k^+ = c_k^+ e^{\mu_k^+ T}/\mathcal{X}$ and $\mathbf{C}_k^- = c_k^-/\mathcal{X}$, respectively, $\mathbf{C}_1 = c_1/\mathcal{X}$, and $\mathbf{C}_2 = c_2 T/\mathcal{X}$.

The solution $\mathbf{C}_k^{\pm}$, $k=\overline{1,N}$, $\mathbf{C}_1$, and $\mathbf{C}_2$ to (35) depends on T. Denote by $\mathcal{C}$ the set of all limits $\lim_{j\to\infty}\left(\mathbf{C}^+,\mathbf{C}^-,\mathbf{C}_1,\mathbf{C}_2\right)(T_j)$, where $\lim_{j\to\infty}T_j=\infty$. Obviously

$$\sum_{k=1}^{N}\left(\left|\mathbf{C}_k^+\right|+\left|\mathbf{C}_k^-\right|\right)+\left|\mathbf{C}_1\right|+\left|\mathbf{C}_2\right|=1$$

for all $(\mathbf{C}^+,\mathbf{C}^-,\mathbf{C}_1,\mathbf{C}_2)\in\mathcal{C}$.

Lemma 10 *If the numbers N and $1/\xi$ are big enough, then there exist at most N numbers q_k, $k=\overline{1,N_1}$, $N_1\le N$, and a constant χ such that*

$$\sum_{k=1}^{N}\left(\left|\mathbf{C}_k^+\right|+\left|\mathbf{C}_k^-\right|\right)\le\chi\left(\left|\mathbf{C}_1\right|+\left|\mathbf{C}_2\right|\right)$$

for all $\left(\mathbf{C}^+,\mathbf{C}^-,\mathbf{C}_1,\mathbf{C}_2\right)\in\mathcal{C}$, whenever $\kappa_2/\kappa_1\ne q_k$, $k=\overline{1,N_1}$.

Proof. Suppose that for any $\chi>0$ there exists a vector

$$\left(\mathbf{C}^+,\mathbf{C}^-,\mathbf{C}_1,\mathbf{C}_2\right)\quad\in\mathcal{C},$$

such that $\sum_{k=1}^{N}\left(\left|\mathbf{C}_k^+\right|+\left|\mathbf{C}_k^-\right|\right)>\chi\left(\left|\mathbf{C}_1\right|+\left|\mathbf{C}_2\right|\right)$. Since the set $\mathcal{C}$ is compact, dividing the inequality by χ and passing to the limit as χ goes to infinity, we see that there exists a vector $\left(\hat{\mathbf{C}}^+,\hat{\mathbf{C}}^-,0,0\right)\in\mathcal{C}$. To prove the lemma it suffices to show that there exist at most N numbers q_k, $k=\overline{1,N_1}$, $N_1\le N$, such that $\hat{\mathbf{C}}^{\pm}=0$ whenever $\kappa_2/\kappa_1\ne q_k$, $k=\overline{1,N_1}$. Let $T_j\to\infty$ be a sequence such that

$$\lim_{j\to\infty}\left(\mathbf{C}^+,\mathbf{C}^-,\mathbf{C}_1,\mathbf{C}_2\right)(T_j)=\left(\hat{\mathbf{C}}^+,\hat{\mathbf{C}}^-,0,0\right).$$

Passing to the limit in (35) as j goes to infinity, we have

$$\begin{pmatrix}0&\mathbf{A}^-\\\mathbf{B}^+&0\\\mathbf{D}^+&0\end{pmatrix}\begin{pmatrix}\hat{\mathbf{C}}^+\\\hat{\mathbf{C}}^-\end{pmatrix}=\begin{pmatrix}0\\2\kappa_1\delta\Gamma\\2\kappa_1\delta L\end{pmatrix},\tag{36}$$

where δ is a constant. From this we obtain $\mathbf{A}^-\hat{\mathbf{C}}^-=0$. Applying the well known Cauchy determinant formula, we get

$$\det\mathbf{A}^-=(\mathrm{const})\begin{vmatrix}\frac{1}{\mu_1^++\lambda_1}&\cdots&\frac{1}{\mu_N^++\lambda_1}\\\vdots&\ddots&\vdots\\\frac{1}{\mu_1^++\lambda_N}&\cdots&\frac{1}{\mu_N^++\lambda_N}\end{vmatrix}$$

$$=(\mathrm{const})\frac{\prod_{1\le j<i\le N}\left(\mu_i-\mu_j\right)\cdot\prod_{1\le j<i\le N}\left(\lambda_i-\lambda_j\right)}{\prod_{i,j=1}^{N}\left(\mu_i+\lambda_j\right)}\ne0.$$

Therefore $\hat{\mathbf{C}}^- = 0$. Set $z = \kappa_2/\kappa_1$. The second line of system (36) can be written in the form

$$(zP + \mathcal{V})\,\hat{\mathbf{C}}^+ = \delta\Gamma,$$

where P is a square matrix with the columns

$$\mu_k^+\left(\mu_k^+ I - \Lambda\right)^{-1}\left(V - \frac{1}{L}\Gamma\Gamma^*\right)\left(\mu_k^+ I + \Lambda\right)^{-1} H,$$

and $\mathcal{V}$ is a square matrix with the columns $-\mu_k^+ V\left(\mu_k^+ I + \Lambda\right)^{-1} H + \Gamma$. The third line of (36) can be written as $P_0\hat{\mathbf{C}}^+ = \delta L$, where P_0 is a vector with the coordinates

$$-\left\langle \Gamma, \mu_k^+\left(\mu_k^+ I + \Lambda\right)^{-1} H\right\rangle + L.$$

If $z = 0$, then we have the system

$$\begin{pmatrix} \frac{\mu_1^+|v_1|^2 h_1}{\mu_1^+ + \lambda_1} - \gamma_1 & \cdots & \frac{\mu_N^+|v_1|^2 h_1}{\mu_N^+ + \lambda_1} - \gamma_1 & \gamma_1 \\ \vdots & \ddots & & \vdots \\ \frac{\mu_1^+|v_1|^2 h_N}{\mu_1^+ + \lambda_N} - \gamma_N & \cdots & \frac{\mu_N^+|v_N|^2 h_N}{\mu_N^+ + \lambda_N} - \gamma_N & \gamma_N \\ \rho_1 & \cdots & \rho_N & L \end{pmatrix} \begin{pmatrix} \hat{\mathbf{C}}_1^+ \\ \vdots \\ \hat{\mathbf{C}}_N^+ \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix},$$

where $\rho_k = \mu_k^+\left\langle \Gamma, \left(\mu_k^+ I + \Lambda\right)^{-1} H\right\rangle - L$. Obviously we have

$$\det\begin{pmatrix} \frac{\mu_1^+|v_1|^2 h_1}{\mu_1^+ + \lambda_1} - \gamma_1 & \cdots & \frac{\mu_N^+|v_1|^2 h_1}{\mu_N^+ + \lambda_1} - \gamma_1 & \gamma_1 \\ \vdots & \ddots & & \vdots \\ \frac{\mu_1^+|v_1|^2 h_N}{\mu_1^+ + \lambda_N} - \gamma_N & \cdots & \frac{\mu_N^+|v_N|^2 h_N}{\mu_N^+ + \lambda_N} - \gamma_N & \gamma_N \\ \rho_1 & \cdots & \rho_N & L \end{pmatrix}$$

$$= \det\begin{pmatrix} \frac{\mu_1^+|v_1|^2 h_1}{\mu_1^+ + \lambda_1} & \cdots & \frac{\mu_N^+|v_N|^2 h_1}{\mu_N^+ + \lambda_1} & \gamma_1 \\ \vdots & \ddots & & \vdots \\ \frac{\mu_1^+|v_1|^2 h_N}{\mu_1^+ + \lambda_N} & \cdots & \frac{\mu_N^+|v_N|^2 h_N}{\mu_N^+ + \lambda_N} & \gamma_N \\ \rho_1 + L & \cdots & \rho_N + L & L \end{pmatrix}$$

$$= (\mathrm{const})\det\begin{pmatrix} \frac{1}{\mu_1^+ + \lambda_1} & \cdots & \frac{1}{\mu_N^+ + \lambda_1} & \frac{\gamma_1}{|v_1|^2 h_1} \\ \vdots & \ddots & & \vdots \\ \frac{1}{\mu_1^+ + \lambda_N} & \cdots & \frac{1}{\mu_N^+ + \lambda_N} & \frac{\gamma_N}{|v_N|^2 h_N} \\ \sum_{k=1}^N \frac{\gamma_k h_k}{\mu_1^+ + \lambda_k} & \cdots & \sum_{k=1}^N \frac{\gamma_k h_k}{\mu_N^+ + \lambda_k} & L \end{pmatrix}$$

$$= (\text{const}) \det \begin{pmatrix} \frac{1}{\mu_1^+ + \lambda_1} & \cdots & \frac{1}{\mu_N^+ + \lambda_1} & \frac{\gamma_1}{|\upsilon_1|^2 h_1} \\ \vdots & \ddots & & \vdots \\ \frac{1}{\mu_1^+ + \lambda_N} & \cdots & \frac{1}{\mu_N^+ + \lambda_N} & \frac{\gamma_N}{|\upsilon_N|^2 h_N} \\ 0 & \cdots & 0 & L - \sum_{k=1}^N \frac{\gamma_k^2}{|\upsilon_k|^2} \end{pmatrix}.$$

From this we see that the determinant is equal to zero if and only if

$$L = \sum_{k=1}^{N} \frac{\gamma_k^2}{|\upsilon_k|^2} = \sum_{k=1}^{N} \frac{2L}{\delta_k^2(\nu)} r_k(\nu)$$

$$= \sum_{k=1}^{N} \frac{2L}{\delta_k^2(\nu)} \frac{(\nu \sin \delta_k(\nu) - \cos \delta_k(\nu) + 1)^2}{\nu^2 \delta_k^2(\nu) + 2\nu + 1}.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8},$$

we have

$$\sum_{k=1}^{N} \frac{2L}{\delta_k^2(0)} r_k(0) = \frac{8L}{\pi^2} \sum_{k=1}^{N} \frac{1}{(2k-1)^2} < L.$$

Therefore there exists ξ_1 such that $\sum_{k=1}^{N} a_k(\xi/L) \neq L$ whenever $\xi \in [0, \xi_1]$. Thus, if $\xi \in [0, \xi_1]$, then the determinant of the system

$$\begin{pmatrix} (zP + \mathcal{V}) & -\Gamma \\ \frac{1}{\kappa_1}\mathbf{D}^+ & -L \end{pmatrix} \begin{pmatrix} \hat{\mathbf{C}}^+ \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is a non-trivial Nth degree polynomial of z with roots $z = q_k$, $k = \overline{1, N}$. This ends the proof. □

Using the compactness of $\mathcal{C}$ and lemmas 9, 5, 6, 7, 8, and 10 we obtain the following result.

Theorem 11 *If the numbers N and $1/\xi$ are big enough then there exist at most N numbers q_k, $k = \overline{1, N_1}$, $N_1 \leq N$ and a constant $T_0 > 0$ such that if $\kappa_2/\kappa_1 \neq q_n$ and $T > T_0$ the optimal trajectory $q(\cdot)$ has the form*

$$q(t) = c_1 + c_2 t + \phi(t),$$

where $\phi(t) \leq 3\chi(c_1 + c_2 T)\left(e^{\mu(t-T)} + e^{-\mu t}\right)$, $t \in [0, T]$, and $\mu = \min_{k=\overline{1,N}} \left|\mathrm{Re}\mu_k^{\pm}\right| > 0$.

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