On the existence in a problem of the calculus of variations without convexity assumptions

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Abstract

We obtain new sufficient conditions for the existence in a problem of the calculus of variations without convexity assumptions.

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1 Introduction

Consider the following problem of the calculus of variations

$$\int_a^b L(x(t), \dot{x}(t))dt \to \inf, \qquad (1)$$
$$x(a) = A, \quad x(b) = B,$$

where $L : R \times R \to R$ is a sufficiently smooth function satisfying the usual growth condition. We obtain new sufficient conditions for the existence of solution to problem (1) without convexity assumptions. The approach we use is based on the equivalence between problem (1) and the time optimal control problem

$$\Theta \to \inf,$$
$$\frac{d(t,y)(\theta)}{d\theta} = \frac{(1, u(\theta))}{L(y(\theta), u(\theta))},$$
$$(t,y)(0) = (a, A), \quad (t,y)(\Theta) = (b, B),$$

established by Gamkrelidze [6]. The corresponding convexified time optimal control problem always has a solution $(\check{t}(\cdot), \check{y}(\cdot))$ satisfying

$$\frac{d(\check{t},\check{y})(\theta)}{d\theta} = \lambda_1(\theta)\frac{(1, u_1(\theta))}{L(\check{y}(\theta), u_1(\theta))} + \lambda_2(\theta)\frac{(1, u_2(\theta))}{L(\check{y}(\theta), u_2(\theta))},$$

where $u_k(\cdot)$ and $\lambda_k(\cdot)$, $k = 1, 2$, are measurable functions such that $\lambda_k(\theta) \in [0, 1]$, $\lambda_1(\theta)+\lambda_2(\theta) = 1$. Under some local monotonicity or concavity conditions on the function $L(\cdot, u)$ we show that if θ_0 is a regular point of the functions $\lambda_k(\cdot)$, $k = 1, 2$, and $0 < \lambda_k(\theta) < 1$, then exist $0 < \tau_1 < \tau_2 < \tau_3$ such that the solution $(\hat{t}(\cdot), \hat{y}(\cdot))$ to the Cauchy problem

$$\frac{d(\hat{t},\hat{y})(\theta)}{d\theta} = \frac{(1, u_1(\theta))}{L(\hat{y}(\theta), u_1(\theta))}, \quad \theta \in [\theta_0, \theta_0 + \tau_1],$$
$$\frac{d(\hat{t},\hat{y})(\theta)}{d\theta} = \frac{(1, u_2(\theta))}{L(\hat{y}(\theta), u_2(\theta))}, \quad \theta \in [\theta_0 + \tau_1, \theta_0 + \tau_2],$$
$$(\hat{t},\hat{y})(\theta_0) = (\check{t},\check{y})(\theta_0),$$

satisfies $(\hat{t},\hat{y})(\theta_0+\tau_2) = (\check{t},\check{y})(\theta_0+\tau_3)$. This implies that the optimal solution to the convexified problem must be a trajectory of the original control system, that is, that problem (1) has a solution.

The existence for a problem of calculus of variations without convexity assumptions was considered by many authors, if the Lagrangian has the form $L(t, x, u) = f(t, x) + g(t, u)$, where the function $f(t, \cdot)$ satisfies some special conditions, like linearity, monotonicity or concavity (see, for example, [1, 2, 3, 7]).

We consider only the one-dimensional case. One of the results presented here (Theorem 1) gives sufficient conditions without assumptions on the structure of L. Theorem 2 concerns the

Lagrangian of the form $L(x,u) = f(x) + g(u)$ and is proved under local concavity assumptions ($f''(x) < 0$ whenever $f'(y) = 0$). This theorem is not contained in the Cellina and Colombo result [2] and does not generalized it even in the one-dimensional case. Sufficient conditions of existence in the case $L(x,u) = f(x)g(u)$ are contained in Theorem 3. The known results on the bang-bang property of optimal solutions of optimal control problems, [4, 8, 9], for example, cannot be applied to study the time optimal problem that we face here. For this reason we developed new techniques adequate to the problem.

We use the following notations. If $C \subset R^m$ is a subset, then we denote by co$C$ and by cl$C$ its convex hull and closure, respectively. The components of a vector G are denoted by $G^{(j)}$. Let $G : R^m \to R^m$ be a twice differentiable function. Its second derivative at $z \in R^m$ along the vectors $h_1, h_2 \in R^m$ we denote by $\nabla^2 G(z)[h_1, h_2] \in R^m$. Let $G : R \to R^m$ be a function, and let $\{s_n\}_{n=1}^{\infty}$ be a sequence that tends to zero. The limit $\lim_{n\to\infty}(G(s_n) - G(0))/s_n$ is denoted by $D_{\{s_n\}}G(0)$, if it exists. Assume that the limit $D_{\{s_n\}}G(0)$ exists. Then we denote by $D^2_{\{s_n\}}G(0)$ the limit $\lim_{n\to\infty} 2(G(s_n) - G(0) - D_{\{s_n\}}G(0)s_n)/s_n^2$, if it exists.

2 Main results and examples

Put $H(y,u,p,q) = (p+qu)/L(y,u)$. We will consider problems satisfying the following condition (C):

1. the function $L : R \times R \to R$ is twice continuously differentiable;
2. there exist constants $c > 0$ and $\epsilon > 0$ such that $L(y,u) \geq c(1 + |u|)^{1+\epsilon}$, $(y,u) \in R \times R$;
3. for any $(y,p,q) \in R \times R \times R$, $p^2 + q^2 > 0$, there exist at most two points u_1 and u_2 satisfying
$$H(y,u_k,p,q) \geq H(y,u,p,q), \quad k = 1,2, \quad \forall\, u \in R; \tag{2}$$
4. if u_1 and u_2 satisfy (2), then
$$\frac{\partial^2 H(y,u_k,p,q)}{\partial u^2} < 0, \quad k = 1,2. \tag{3}$$

Theorem 1 *Assume that*

1. *condition (C) is satisfied;*
2. *u_1 and u_2 satisfying (2) also satisfy*
$$\frac{\partial L(y,u_1)}{\partial y} u_2 \neq \frac{\partial L(y,u_2)}{\partial y} u_1. \tag{4}$$

Then problem (1) has a solution.

If condition (4) is not satisfied, then the techniques we use can be applied only to problems with a special structure of the function L.

Theorem 2 *Assume that*

1. $L(y,u) = f(y) + g(u)$, *where* $f : R \to R$ *and* $g : R \to R$ *are functions;*

2. *condition (C) is satisfied;*

3. $f''(y) < 0$ *whenever* $f'(y) = 0$.

Then problem (1) has a solution.

Theorem 3 *Assume that*

1. $L(y,u) = f(y)g(u)$, *where* $f : R \to R$ *and* $g : R \to R$ *are positive functions;*

2. *condition (C) is satisfied;*

3. $f''(y) < 0$ *whenever* $f'(y) = 0$*;*

4. u_1 *and* u_2 *satisfying (2) also satisfy*

$$g(u_1)u_2 \neq g(u_2)u_1. \tag{5}$$

Then problem (1) has a solution.

Consider a few examples. Let

$$L(x, \dot{x}) = (1 + e^x)\left(1 + (1 - \dot{x}^2)^2\right).$$

This function satisfies all conditions of Theorem 1 and problem (1) has a solution. The function

$$L(x, \dot{x}) = \left(1 + \frac{1}{1 + x^2}\right) + \left(1 + (1 - \dot{x}^2)^2\right)$$

does not satisfy (4), but Theorem 2 guarantees the existence of solution to problem (1). If

$$L(x, \dot{x}) = \left(1 + \frac{1}{1 + x^2}\right)\left(1 + (1 - \dot{x}^2)^2\right),$$

then condition (4) is not satisfied. However the existence of solution to problem (1) follows from Theorem 3.

3 Reduction to a time optimal control problem

Recall the following result [6].

Lemma 1 *Assume that the function L is continuously differentiable and $L(x,u) \geq c > 0$, for all (x,u). Let $(\check{t}(\cdot), \check{y}(\cdot), \check{u}(\cdot), \check{\Theta})$ be an optimal process in the time-optimal control problem*

$$\begin{aligned}
&\Theta \to \inf, \\
&\frac{dt(\theta)}{d\theta} = \frac{1}{L(y(\theta), u(\theta))}, \quad \theta \in [0, \Theta], \\
&\frac{dy(\theta)}{d\theta} = \frac{u(\theta)}{L(y(\theta), u(\theta))}, \quad \theta \in [0, \Theta], \\
&u(\theta) \in R, \\
&t(0) = a, \quad t(\Theta) = b, \quad y(0) = A, \quad y(\Theta) = B.
\end{aligned}$$

Then there exists an absolutely continuous inverse function $\check{\theta}(t) = \check{t}^{-1}(t)$, and the function $\check{x}(t) = \check{y}(\check{\theta}(t))$, $t \in [a,b]$, is a solution to problem (1).

Set

$$z = (t, y) \in R \times R, \quad f(z, u) = (1, u)/L(y, u), \quad z_* = (a, A), \quad z^* = (b, B).$$

By Lemma 1 it suffice to prove that the time optimal control problem

$$\begin{aligned}
&\Theta \to \inf, \\
&\frac{dz}{d\theta} = f(z, u), \quad u \in R, \\
&z(0) = z_*, \quad z(\Theta) = z^*,
\end{aligned} \tag{6}$$

has a solution. Consider the convexification of problem (6)

$$\begin{aligned}
&\Theta \to \inf, \\
&\frac{dz}{d\theta} \in \mathrm{cl}\,\mathrm{co} f(z, R), \\
&z(0) = z_*, \quad z(\Theta) = z^*.
\end{aligned} \tag{7}$$

This problem always has a solution $\check{z}(\theta) = (\check{t}, \check{y})(\theta)$, $\theta \in [0, \check{\Theta}]$. Applying the necessary conditions of optimality for differential inclusions [5, Theorem 3.6.1], after simple calculations we see that there exist a constant p and an absolutely continuous function $q(\cdot)$ satisfying

$$\begin{aligned}
&\frac{d(\check{t}, \check{y}, q)(\theta)}{d\theta} \in \mathrm{co}\left\{ \left(f(\check{y}(\theta), u), \frac{\partial H(\check{y}(\theta), u, p, q(\theta))}{\partial y} \right) \middle| \, u \in U(\check{y}(\theta), p, q(\theta)) \right\} \\
&\max_w H(\check{y}(\theta), w, p, q(\theta)) \equiv (\text{const}) \geq 0,
\end{aligned}$$

where

$$U(y,p,q) = \{u \mid H(y,u,p,q) = \max_w H(y,w,p,q).$$

The set-valued map $U(y,p,q)$ is upper semi-continuous. The set $U(\check{y}(\theta),p,q(\theta))$ contains at most two points $\tilde{u}_1$ and $\tilde{u}_2$ (condition (C)). By the Implicit function theorem (3) imply that there exist continuously differentiable functions $u_k(y,p,q)$, $k=1,2$, satisfying $\tilde{u}_k = u_k(\check{y}(\theta),p,q(\theta))$, $k=1,2$, and such that

$$U(\check{y}(s),p,q(s)) \subset \{u_k(\check{y}(s),p,q(s)) \mid k=1,2\}, \quad s \in]\theta-\delta,\theta+\delta[\cap[0,\check{\Theta}], \ \ \delta>0.$$

Thus, applying the Filippov lemma we see that there exist measurable functions $\lambda_k(\cdot)$, $k=1,2$, such that $\lambda_k(\theta)\in[0,1]$, $\lambda_1(\theta)+\lambda_2(\theta)=1$, $\theta\in[0,\check{\Theta}]$, and

$$\frac{d(\check{t},\check{y})(\theta)}{d\theta} = \lambda_1(\theta)f(\check{y}(\theta),u_1(\theta)) + \lambda_2(\theta)f(\check{y}(\theta),u_2(\theta)), \tag{8}$$

$$\frac{dq(\theta)}{d\theta} = -\left(\lambda_1(\theta)\frac{\partial H(\check{y}(\theta),u_1(\theta),p,q(\theta))}{\partial y} + \lambda_2(\theta)\frac{\partial H(\check{y}(\theta),u_2(\theta),p,q(\theta))}{\partial y}\right), \tag{9}$$

$$H(\check{y}(\theta),u_k(\theta),p,q(\theta)) \geq H(\check{y}(\theta),u,p,q(\theta)), \quad k=1,2, \quad \forall\, u\in R, \tag{10}$$

$$H(\check{y}(\theta),u_k(\theta),p,q(\theta)) \equiv (\text{const}) \geq 0, \tag{11}$$

where the functions $u_k(\cdot)$, $k=1,2$, are locally absolutely continuous. It suffice to prove that $\lambda_1(\theta)\in\{0,1\}$ for almost all points $\theta\in[0,\check{\Theta}]$.

4 Proof of Theorem 1

Suppose that $\theta=0$ is a regular point of the functions $\lambda_k(\cdot)$ and $du_k(\cdot)/d\theta$, $k=1,2$, and $u_1(0)\neq u_2(0)$. We show that $\lambda_1(0)\in\{0,1\}$. Set

$$F_k(z,\theta) = f(y,u_k(\theta)), \quad k=1,2, \quad \text{and} \quad F(z,\theta) = \lambda_1(\theta)F_1(z,\theta)+\lambda_2(\theta)F_2(z,\theta).$$

Lemma 2 *Let $z(\cdot)$ be a solution to the Cauchy problem $dz/d\theta = F(z,\theta)$, $z(\theta_0)=z_*$. Then*

$$z(\theta) = z_* + \int_{\theta_0}^{\theta} F(z_*,s)ds + \frac{(\theta-\theta_0)^2}{2}\nabla_z F(z_*,\theta_0)F(z_*,\theta_0) + o((\theta-\theta_0)^2).$$

Proof. Indeed, we have

$$z(\theta) = z_* + \int_{\theta_0}^{\theta} F(z(s),s)ds = z_* + \int_{\theta_0}^{\theta} F(z_* + \int_{\theta_0}^{s} F(z(r),r)dr, s)ds$$

$$= z_* + \int_{\theta_0}^{\theta}\left(F(z_*,s) + \nabla_z F(z_*,s)\int_{\theta_0}^{s} F(z(r),r)dr + o((s-\theta_0))\right)ds$$

$$= z_* + \int_{\theta_0}^{\theta}\left(F(z_*,s) + s\nabla_z F(z_*,s)F(z_*,\theta_0) + o((s-\theta_0))\right)ds$$

$$= z_* + \int_{\theta_0}^{\theta} (F(z_*, s) + (s - \theta_0)\nabla_z F(z_*, \theta_0)F(z_*, \theta_0)$$

$$+(s - \theta_0)(\nabla_z F(z_*, s) - \nabla_z F(z_*, \theta_0))F(z_*, \theta_0) + o((s - \theta_0)))ds$$

$$= z_* + \int_{\theta_0}^{\theta} F(z_*, s)ds + \frac{(\theta - \theta_0)^2}{2}\nabla_z F(z_*, \theta_0)F(z_*, \theta_0) + o((\theta - \theta_0)^2). \quad \square$$

Lemma 3 *Let $\hat{z}(\cdot)$ be a solution to the following Cauchy problem*

$$\frac{d\hat{z}(\theta)}{d\theta} = F_1(\hat{z}(\theta), \theta), \quad \theta \in [0, \tau],$$
$$\frac{d\hat{z}(\theta)}{d\theta} = F_2(\hat{z}(\theta), \theta), \quad \theta \in [\tau, \hat{\theta}],$$
$$\hat{z}(0) = z_*,$$

where $0 < \tau < \hat{\theta}$. Then

$$\hat{z}(\hat{\theta}) = z_* + \int_0^{\tau} F_1(z_*, s)ds + \int_{\tau}^{\hat{\theta}} F_2(z_*, s)ds$$

$$+\frac{\tau^2}{2}\nabla_z F_1(z_*, 0)F_1(z_*, 0) + (\hat{\theta} - \tau)\tau\nabla_z F_2(z_*, 0)F_1(z_*, 0)$$

$$+\frac{(\hat{\theta} - \tau)^2}{2}\nabla_z F_2(z_*, 0)F_2(z_*, 0) + o(\hat{\theta}^2).$$

Proof. As in the proof of Lemma 2 we have

$$\hat{z}(\tau) = z_* + \int_0^{\tau} F_1(z_*, s)ds + \frac{\tau^2}{2}\nabla_z F_1(z_*, 0)F_1(z_*, 0) + o(\tau^2)$$

and

$$\hat{z}(\hat{\theta}) = \hat{z}(\tau) + \int_{\tau}^{\hat{\theta}} F_2(\hat{z}(\tau), s)ds + \frac{(\hat{\theta} - \tau)^2}{2}\nabla_z F_2(\hat{z}(\tau), \tau)F_2(\hat{z}(\tau), \tau) + o((\hat{\theta} - \tau)^2).$$

Combining these two equalities, we get

$$\hat{z}(\hat{\theta}) = z_* + \int_0^{\tau} F_1(z_*, s)ds + \frac{\tau^2}{2}\nabla_z F_1(z_*, 0)F_1(z_*, 0)$$

$$+\int_{\tau}^{\hat{\theta}} F_2\left(z_* + \int_0^{\tau} F_1(z_*, r)dr, s\right)ds + \frac{(\hat{\theta} - \tau)^2}{2}\nabla_z F_2(z_*, 0)F_2(z_*, 0) + o(\hat{\theta}^2)$$

$$= z_* + \int_0^{\tau} F_1(z_*, s)ds + \int_{\tau}^{\hat{\theta}} F_2(z_*, s)ds$$

$$+\frac{\tau^2}{2}\nabla_z F_1(z_*,0)F_1(z_*,0)+(\hat\theta-\tau)\tau\nabla_z F_2(z_*,0)F_1(z_*,0)$$
$$+\frac{(\hat\theta-\tau)^2}{2}\nabla_z F_2(z_*,0)F_2(z_*,0)+o(\hat\theta^2). \quad \square$$

Set $\omega=(\check\theta,\hat\theta,\tau)$ and define the functions $\Phi_m : R^3 \to R^3$, $m=1,2$, as follows:

$$\Phi_1^{(1)}=\int_0^{\check\theta} F^{(1)}(z_*,s)ds,$$
$$\Phi_1^{(2)}=\int_0^{\tau} F_1^{(1)}(z_*,s)ds+\int_\tau^{\check\theta} F_2^{(1)}(z_*,s)ds,$$
$$\Phi_1^{(3)}=\int_0^{\check\theta} F^{(2)}(z_*,s)ds-\left(\int_0^{\tau} F_1^{(2)}(z_*,s)ds+\int_\tau^{\check\theta} F_2^{(2)}(z_*,s)ds\right),$$
$$\Phi_2^{(1)}=\frac{\check\theta^2}{2}\left(\nabla_z F(z_*,0)F(z_*,0)\right)^{(1)},$$
$$\Phi_2^{(2)}=\frac{\tau^2}{2}\left(\nabla_z F_1(z_*,0)F_1(z_*,0)\right)^{(1)}+(\hat\theta-\tau)\tau\left(\nabla_z F_2(z_*,0)F_1(z_*,0)\right)^{(1)}$$
$$+\frac{(\hat\theta-\tau)^2}{2}\left(\nabla_z F_2(z_*,0)F_2(z_*,0)\right)^{(1)},$$
$$\Phi_2^{(3)}=\frac{\check\theta^2}{2}\left(\nabla_z F(z_*,0)F(z_*,0)\right)^{(2)}-\left(\frac{\tau^2}{2}\left(\nabla_z F_1(z_*,0)F_1(z_*,0)\right)^{(2)}\right.$$
$$\left.+(\hat\theta-\tau)\tau\left(\nabla_z F_2(z_*,0)F_1(z_*,0)\right)^{(2)}+\frac{(\hat\theta-\tau)^2}{2}\left(\nabla_z F_2(z_*,0)F_2(z_*,0)\right)^{(2)}\right).$$

Set also

$$\Phi_0(t)=(-t,-t,0),$$
$$\Phi(\omega,t)=\Phi_0(t)+\Phi_1(\omega)+\Phi_2(\omega),$$
$$\Psi(\omega,t)=(\check z^{(1)}(\check\theta)-z_*^{(1)}-t,\hat z^{(1)}(\hat\theta)-z_*^{(1)}-t,\check z^{(2)}(\check\theta)-\hat z^{(2)}(\hat\theta)).$$

Then from Lemma 2 and Lemma 3 we have

$$\Psi(\omega,t)=\Phi(\omega,t)+R(\omega), \tag{12}$$

where $|R(\omega)|=o(|\omega|^2)$. The condition

$$\Psi(\omega,t)=0 \tag{13}$$

implies $\check z^{(1)}(\check\theta)=\hat z^{(1)}(\hat\theta)=z_*^{(1)}+t$ and $\check z^{(2)}(\check\theta)=\hat z^{(2)}(\hat\theta)$, that is, $\check t(\check\theta)=\hat t(\hat\theta)=a+t$ and $\check y(\check\theta)=\hat y(\hat\theta)$.

Lemma 4 *Equation (13) defines a differentiable function $\omega = \omega(t)$. Its derivative at $t = 0$ is given by*

$$\frac{d\omega(0)}{dt} = \begin{pmatrix} \frac{d\check{\theta}(0)}{dt} \\ \frac{d\hat{\theta}(0)}{dt} \\ \frac{d\tau(0)}{dt} \end{pmatrix} = \frac{L(A, u_1(0))L(A, u_2(0))}{\lambda_1(0)L(A, u_2(0)) + \lambda_2(0)L(A, u_1(0))} \begin{pmatrix} 1 \\ 1 \\ \lambda_1(0) \end{pmatrix}. \tag{14}$$

Proof. By definition of Φ and from (12) we have

$$\frac{\partial \Psi(0,0)}{\partial \omega} = \frac{\partial \Phi_1(0)}{\partial \omega} = \begin{pmatrix} F^{(1)}(z_*, 0) & 0 & 0 \\ 0 & F_2^{(1)}(z_*, 0) & F_1^{(1)}(z_*, 0) - F_2^{(1)}(z_*, 0) \\ F^{(2)}(z_*, 0) & -F_2^{(2)}(z_*, 0) & -F_1^{(2)}(z_*, 0) + F_2^{(2)}(z_*, 0) \end{pmatrix}.$$

Since

$$\begin{aligned} \det \frac{\partial \Phi_1(0)}{\partial \omega} &= F^{(1)}(z_*, 0) \left(F_1^{(1)}(z_*, 0) F_2^{(2)}(z_*, 0) - F_1^{(2)}(z_*, 0) F_2^{(1)}(z_*, 0) \right) \\ &= \left(\frac{\lambda_1(0)}{L(A, u_1(0))} + \frac{\lambda_2(0)}{L(A, u_2(0))} \right) \frac{(u_2(0) - u_1(0))}{L(A, u_1(0))L(A, u_2(0))} \neq 0, \end{aligned}$$

by the Implicit function theorem equation (13) defines a differentiable function $\omega = \omega(t)$. Solving the equation

$$\frac{\partial \Psi(0,0)}{\partial \omega} \frac{d\omega(0)}{dt} + \frac{\partial \Psi(0,0)}{\partial t} = 0,$$

we obtain (14). □

Consider a sequence $\{t_n\}$, $t_n \downarrow 0$, and suppose that the limits $D_{\{t_n\}}\lambda_k(0)$, $k = 1, 2$, $D_{\{t_n\}} d\omega(0)/dt$, and $D_{\{t_n\}} d\Psi(\omega(0), 0)/dt$ exist. Then the equality $D_{\{t_n\}} d\Psi(\omega(0), 0)/dt = 0$ is equivalent to

$$\frac{\partial \Phi_1(0)}{\partial \omega} D_{\{t_n\}} \frac{d\omega(0)}{dt} = v, \tag{15}$$

where $v = v_1 + v_2$, and

$$v_1 = -\left(\frac{L(A, u_1(0))L(A, u_2(0))}{\lambda_1(0)L(A, u_2(0)) + \lambda_2(0)L(A, u_1(0))} \right)^2 (V_1 + V_2 + V_3),$$

$$V_1 = -\left(\frac{\lambda_1(0)}{L^2(A, u_1(0))} \frac{\partial L(A, u_1(0))}{\partial u} \frac{du_1(0)}{d\theta} + \frac{\lambda_2(0)}{L^2(A, u_2(0))} \frac{\partial L(A, u_2(0))}{\partial u} \frac{du_2(0)}{d\theta} \right) \begin{pmatrix} 1 \\ 1 \\ \lambda_1(0) \end{pmatrix},$$

$$V_2 = -\lambda_1(0)\lambda_2(0)$$

$$\times \begin{pmatrix} 0 \\ -\frac{1}{L^2(A, u_1(0))} \frac{\partial L(A, u_1(0))}{\partial u} \frac{du_1(0)}{d\theta} + \frac{1}{L^2(A, u_2(0))} \frac{\partial L(A, u_2(0))}{\partial u} \frac{du_2(0)}{d\theta} \\ \left(-\frac{u_1(0)}{L^2(A, u_1(0))} \frac{\partial L(A, u_1(0))}{\partial u} + \frac{1}{L(A, u_1(0))} \right) \frac{du_1(0)}{d\theta} + \left(\frac{u_2(0)}{L^2(A, u_2(0))} \frac{\partial L(A, u_2(0))}{\partial u} - \frac{1}{L(A, u_2(0))} \right) \frac{du_2(0)}{d\theta} \end{pmatrix},$$

$$V_3 = D_{\{t_n\}}\lambda_1(0)\begin{pmatrix} \frac{1}{L(A,u_1(0))} - \frac{1}{L(A,u_2(0))} \\ 0 \\ \frac{u_1(0)}{L(A,u_1(0))} - \frac{u_2(0)}{L(A,u_2(0))} \end{pmatrix},$$

$$v_2 = -\left(\frac{L(A,u_1(0))L(A,u_2(0))}{\lambda_1(0)L(A,u_2(0)) + \lambda_2(0)L(A,u_1(0))}\right)^2 \begin{pmatrix} v_2^{(1)} \\ v_2^{(2)} \\ v_2^{(3)} \end{pmatrix},$$

$$v_2^{(1)} = -\frac{(\partial L(A,u_2(0))/\partial y)u_2(0)}{L(A,u_2(0))^3}$$

$$+\lambda_1(0)\left(-\frac{(\partial L(A,u_2(0))/\partial y)u_1(0)}{L(A,u_1(0))L(A,u_2(0))^2} - \frac{(\partial L(A,u_1(0))/\partial y)u_2(0)}{L(A,u_1(0))^2 L(A,u_2(0))} + 2\frac{(\partial L(A,u_2(0))/\partial y)u_2(0)}{L(A,u_2(0))^3}\right)$$

$$+\lambda_1^2(0)\left(-\frac{(\partial L(A,u_1(0))/\partial y)u_1(0)}{L(A,u_1(0))^3} - \frac{(\partial L(A,u_2(0))/\partial y)u_2(0)}{L(A,u_2(0))^3}\right.$$

$$\left.+\frac{(\partial L(A,u_1(0))/\partial y)u_2(0)}{L(A,u_1(0))^2 L(A,u_2(0))} + \frac{(\partial L(A,u_2(0))/\partial y)u_1(0)}{L(A,u_1(0))L(A,u_2(0))^2}\right),$$

$$v_2^{(2)} = -\frac{(\partial L(A,u_2(0))/\partial y)u_2(0)}{L(A,u_2(0))^3}$$

$$+2\lambda_1(0)\left(\frac{(\partial L(A,u_2(0))/\partial y)u_2(0)}{L(A,u_2(0))^3} - \frac{(\partial L(A,u_2(0))/\partial y)u_1(0)}{L(A,u_1(0))L(A,u_2(0))^2}\right)$$

$$+\lambda_1^2(0)\left(-\frac{(\partial L(A,u_1(0))/\partial y)u_1(0)}{L(A,u_1(0))^3} - \frac{(\partial L(A,u_2(0))/\partial y)u_2(0)}{L(A,u_2(0))^3} + 2\frac{(\partial L(A,u_2(0))/\partial y)u_1(0)}{L(A,u_1(0))L(A,u_2(0))^2}\right),$$

$$v_2^{(3)} = \lambda_1(0)\left(-\frac{(\partial L(A,u_1(0))/\partial y)u_1(0)u_2(0)}{L(A,u_1(0))^2 L(A,u_2(0))} + \frac{(\partial L(A,u_2(0))/\partial y)u_1(0)u_2(0)}{L(A,u_1(0))L(A,u_2(0))^2}\right)$$

$$+\lambda_1^2(0)\left(\frac{(\partial L(A,u_1(0))/\partial y)u_1(0)u_2(0)}{L(A,u_1(0))^2 L(A,u_2(0))} - \frac{(\partial L(A,u_2(0))/\partial y)u_1(0)u_2(0)}{L(A,u_1(0))L(A,u_2(0))^2}\right).$$

To calculate the derivatives $\partial L(A,u_k(0))/\partial u$, $k=1,2$, we need the following lemma.

Lemma 5 *The following equality holds*

$$\frac{\partial L(A,u_k(0))}{\partial u} = \frac{L(A,u_2(0)) - L(A,u_1(0))}{u_2(0) - u_1(0)}, \quad k = 1,2.$$

Proof. Without loss of generality $L(A,u_2(0)) \neq L(A,u_1(0))$. By the Pontriagin maximum principle we have

$$\frac{\partial H(A,u_k(0),p,q(0))}{\partial u} = 0, \quad k = 1,2,$$

or, equivalently,

$$\frac{\partial L(A, u_k(0))}{\partial u}(p + q(0)u_k(0)) = q(0)L(A, u_k(0)), \quad k = 1, 2. \tag{16}$$

Since $H(A, u_2(0), p, q(0)) = H(A, u_1(0), p, q(0))$, we have

$$p = q(0)\frac{u_2(0)L(A, u_1(0)) - u_1(0)L(A, u_2(0))}{L(A, u_2(0)) - L(A, u_1(0))}. \tag{17}$$

From this we see that $q(0) \neq 0$. Substituting (17) for p in (16), we get the result. □

From this lemma we obtain

$$V_1 = -\frac{L(A, u_2(0)) - L(A, u_1(0))}{u_2(0) - u_1(0)} \left(\frac{\lambda_1(0)}{L^2(A, u_1(0))} \frac{du_1(0)}{d\theta} + \frac{\lambda_2(0)}{L^2(A, u_2(0))} \frac{du_2(0)}{d\theta} \right) \begin{pmatrix} 1 \\ 1 \\ \lambda_1(0) \end{pmatrix}$$

and

$$V_2 = -\lambda_1(0)\lambda_2(0)$$

$$\times \begin{pmatrix} 0 \\ \frac{L(A,u_2(0))-L(A,u_1(0))}{u_2(0)-u_1(0)} \left(-\frac{1}{L^2(A,u_1(0))} \frac{du_1(0)}{d\theta} + \frac{1}{L^2(A,u_2(0))} \frac{du_2(0)}{d\theta} \right) \\ \frac{L(A,u_2(0))-L(A,u_1(0))}{u_2(0)-u_1(0)} \left(\frac{u_2(0)}{L^2(A,u_2(0))} \frac{du_2(0)}{d\theta} - \frac{u_1(0)}{L^2(A,u_1(0))} \frac{du_1(0)}{d\theta} \right) + \frac{1}{L(A,u_1(0))} \frac{du_1(0)}{d\theta} - \frac{1}{L(A,u_2(0))} \frac{du_2(0)}{d\theta} \end{pmatrix}.$$

Lemma 6 *The following equality holds*

$$D_{\{t_n\}} d\omega^{(1)}(0)/dt - D_{\{t_n\}} d\omega^{(2)}(0)/dt$$

$$= \frac{\lambda_1(0)\lambda_2(0)L(A, u_1(0))L(A, u_2(0))}{(\lambda_1(0)L(A, u_2(0)) + \lambda_2(0)L(A, u_1(0)))^2} \left(\frac{\partial L(y, u_1(0))}{\partial y} u_2(0) - \frac{\partial L(y, u_2(0))}{\partial y} u_1(0) \right). \tag{18}$$

Proof. From (15) we have

$$D_{\{t_n\}} d\omega^{(1)}(0)/dt - D_{\{t_n\}} d\omega^{(2)}(0)/dt$$

$$= \frac{(v^{(1)} - v^{(2)})(u_2(0)L(A, u_1(0)) - u_1(0)L(A, u_2(0))) + v^{(3)}(L(A, u_2(0)) - L(A, u_1(0)))}{u_2(0) - u_1(0)}.$$

Substituting the obtained values $v^{(k)}$, $k = 1, 2, 3$, we get (18). □

Note that the right-hand side of (18) does not depend on the sequence $\{t_n\}$. Since

$$\frac{\partial L(y, u_1(0))}{\partial y} u_2(0) \neq \frac{\partial L(y, u_2(0))}{\partial y} u_1(0),$$

without loss of generality

$$\frac{\partial L(y,u_1(0))}{\partial y}u_2(0)-\frac{\partial L(y,u_2(0))}{\partial y}u_1(0)>0.$$

(If this condition is not satisfied, we change the indices.) Thus we have

$$\check{\theta}(t)-\hat{\theta}(t)=\frac{\lambda_1(0)\lambda_2(0)L(A,u_1(0))L(A,u_2(0))}{(\lambda_1(0)L(A,u_2(0))+\lambda_2(0)L(A,u_1(0)))^2}$$

$$\times\left(\frac{\partial L(y,u_1(0))}{\partial y}u_2(0)-\frac{\partial L(y,u_2(0))}{\partial y}u_1(0)\right)t^2+o(t^2)>0$$

whenever $t>0$ is sufficiently small and $\lambda_1(0)\in]0,1[$. Therefore the trajectory $\check{z}(\cdot)$ cannot be optimal, if $\lambda_1(0)\in]0,1[$. This contradiction proves Theorem 1.

5 Proof of Theorems 2 and 3

Suppose that $\theta=0$ is a regular point of the functions $\lambda_k(\cdot)$ and $du_k(\cdot)/d\theta$, $k=1,2$, and $u_1(0)\neq u_2(0)$. We show that $\lambda_1(0)\in\{0,1\}$. If $(\partial L(A,u_1(0))/\partial y)u_2\neq(\partial L(A,u_2(0))/\partial y)u_1$, then arguing as in the proof of Theorem 1 we have the results. Suppose that $(\partial L(A,u_1(0))/\partial y)u_2=(\partial L(A,u_2(0))/\partial y)u_1$.

As in the proof of Theorem 1 we show that if $z(\cdot)$ is a solution to the Cauchy problem $dz/d\theta=F(z,\theta)$, $z(\theta_0)=z_*$, then we have

$$z(\theta)=z_*+\int_{\theta_0}^{\theta}F(z_*,s)ds+\int_{\theta_0}^{\theta}\nabla_zF(z_*,s)\int_0^sF(z_*,r)drds$$

$$+\frac{(\theta-\theta_0)^3}{6}\Big((\nabla_zF(z_*,\theta_0))^2F(z_*,\theta_0)+\nabla_z^2F(z_*,\theta_0)[F(z_*,\theta_0),F(z_*,\theta_0)]\Big)+o((\theta-\theta_0)^3),\quad (19)$$

and if $\hat{z}(\cdot)$ is a solution to the Cauchy problem

$$\begin{aligned}&\frac{d\hat{z}(\theta)}{d\theta}=F_1(\hat{z}(\theta),\theta),\quad \theta\in[0,\tau],\\&\frac{d\hat{z}(\theta)}{d\theta}=F_2(\hat{z}(\theta),\theta),\quad \theta\in[\tau,\hat{\theta}],\\&\hat{z}(0)=z_*,\end{aligned}$$

where $0<\tau<\hat{\theta}$, then

$$\hat{z}(\hat{\theta})=z_*+\int_0^{\tau}F_1(z_*,s)ds+\int_{\tau}^{\hat{\theta}}F_2(z_*,s)ds+\int_0^{\tau}\nabla_zF_1(z_*,s)\int_0^sF_1(z_*,r)drds$$

$$+\int_{\tau}^{\hat{\theta}}\nabla_zF_2(z_*,s)ds\int_0^{\tau}F_1(z_*,r)dr+\int_{\tau}^{\hat{\theta}}\nabla_zF_2(z_*,s)\int_{\tau}^{s}F_2(z_*,r)drds$$

$$+\frac{\tau^3}{6}\Big((\nabla_z F_1(z_*,0))^2 F_1(z_*,0)+\nabla_z^2 F_1(z_*,\theta_0)[F_1(z_*,\theta_0),F_1(z_*,\theta_0)]\Big)$$

$$+\frac{(\hat\theta-\tau)\tau^2}{2}\Big(\nabla_z F_2(z_*,0)\nabla_z F_1(z_*,0)F_1(z_*,0)+\nabla_z^2 F_2(z_*,\theta_0)[F_1(z_*,\theta_0),F_1(z_*,\theta_0)]\Big)$$

$$+\frac{(\hat\theta-\tau)^2\tau}{2}\Big((\nabla_z F_2(z_*,0))^2 F_1(z_*,0)+\nabla_z^2 F_2(z_*,\theta_0)[F_1(z_*,\theta_0),F_2(z_*,\theta_0)]\Big)$$

$$+\frac{(\hat\theta-\tau)^3}{6}\Big((\nabla_z F_2(z_*,0))^2 F_2(z_*,0)+\nabla_z^2 F_2(z_*,\theta_0)[F_2(z_*,\theta_0),F_2(z_*,\theta_0)]\Big)+o(\hat\theta^3). \tag{20}$$

From (19) and (20) we see that the function

$$\Psi(\omega,t)=(\check z^{(1)}(\check\theta)-z_*^{(1)}-t,\hat z^{(1)}(\hat\theta)-z_*^{(1)}-t,\check z^{(2)}(\check\theta)-\hat z^{(2)}(\hat\theta))$$

can be represented as

$$\Psi(\omega,t)=\Phi(\omega,t)+R(\omega), \tag{21}$$

where $|R(\omega)|=o(|\omega|^3)$, and

$$\Phi(\omega,t)=\Phi_0(t)+\Phi_1(\omega)+\Phi_2(\omega)+\Phi_3(\omega),$$

with $\Phi_0(t)=(-t,-t,0)$ and

$$\Phi_1^{(1)}=\int_0^{\check\theta}F^{(1)}(z_*,s)ds,$$

$$\Phi_1^{(2)}=\int_0^{\tau}F_1^{(1)}(z_*,s)ds+\int_\tau^{\check\theta}F_2^{(1)}(z_*,s)ds,$$

$$\Phi_1^{(3)}=\int_0^{\check\theta}F^{(2)}(z_*,s)ds-\left(\int_0^{\tau}F_1^{(2)}(z_*,s)ds+\int_\tau^{\check\theta}F_2^{(2)}(z_*,s)ds\right),$$

$$\Phi_2^{(1)}=\int_{\theta_0}^{\check\theta}\int_0^s(\nabla_z F(z_*,s)F(z_*,r))^{(1)}drds,$$

$$\Phi_2^{(2)}=\int_0^{\tau}\int_0^s(\nabla_z F_1(z_*,s)F_1(z_*,r))^{(1)}drds+\int_\tau^{\hat\theta}\int_0^{\tau}(\nabla_z F_2(z_*,s)F_1(z_*,r))^{(1)}drds$$
$$+\int_\tau^{\hat\theta}\int_\tau^{s}(\nabla_z F_2(z_*,s)F_2(z_*,r))^{(1)}drds,$$

$$\Phi_2^{(3)}=\int_{\theta_0}^{\check\theta}\int_0^s(\nabla_z F(z_*,s)F(z_*,r))^{(2)}drds-\Big(\int_0^{\tau}\int_0^s(\nabla_z F_1(z_*,s)F_1(z_*,r))^{(2)}drds$$
$$+\int_\tau^{\hat\theta}\int_0^{\tau}(\nabla_z F_2(z_*,s)F_1(z_*,r))^{(2)}drds+\int_\tau^{\hat\theta}\int_\tau^{s}(\nabla_z F_2(z_*,s)F_2(z_*,r))^{(2)}drds\Big),$$

$$\Phi_3^{(1)}=+\frac{\check\theta^3}{6}\Big((\nabla_z F(z_*,\theta_0))^2 F(z_*,\theta_0)+\nabla_z^2 F(z_*,\theta_0)[F(z_*,\theta_0),F(z_*,\theta_0)]\Big)^{(1)},$$

$$\Phi_3^{(2)}=+\frac{\tau^3}{6}\Big((\nabla_z F_1(z_*,0))^2 F_1(z_*,0)+\nabla_z^2 F_1(z_*,\theta_0)[F_1(z_*,\theta_0),F_1(z_*,\theta_0)]\Big)^{(1)}$$

$$+\frac{(\hat\theta-\tau)\tau^2}{2}\left(\nabla_z F_2(z_*,0)\nabla_z F_1(z_*,0)F_1(z_*,0)+\nabla_z^2F_2(z_*,\theta_0)[F_1(z_*,\theta_0),F_1(z_*,\theta_0)]\right)^{(1)}$$
$$+\frac{(\hat\theta-\tau)^2\tau}{2}\left((\nabla_z F_2(z_*,0))^2F_1(z_*,0)+\nabla_z^2F_2(z_*,\theta_0)[F_1(z_*,\theta_0),F_2(z_*,\theta_0)]\right)^{(1)}$$
$$+\frac{(\hat\theta-\tau)^3}{6}\left((\nabla_z F_2(z_*,0))^2F_2(z_*,0)+\nabla_z^2F_2(z_*,\theta_0)[F_2(z_*,\theta_0),F_2(z_*,\theta_0)]\right)^{(1)}$$
$$\Phi_3^{(3)}=+\frac{\check\theta^3}{6}\left((\nabla_z F(z_*,\theta_0))^2F(z_*,\theta_0)+\nabla_z^2F(z_*,\theta_0)[F(z_*,\theta_0),F(z_*,\theta_0)]\right)^{(2)}$$
$$-\left(\frac{\tau^3}{6}\left((\nabla_z F_1(z_*,0))^2F_1(z_*,0)+\nabla_z^2F_1(z_*,\theta_0)[F_1(z_*,\theta_0),F_1(z_*,\theta_0)]\right)^{(2)}\right.$$
$$+\frac{(\hat\theta-\tau)\tau^2}{2}\left(\nabla_z F_2(z_*,0)\nabla_z F_1(z_*,0)F_1(z_*,0)+\nabla_z^2F_2(z_*,\theta_0)[F_1(z_*,\theta_0),F_1(z_*,\theta_0)]\right)^{(2)}$$
$$+\frac{(\hat\theta-\tau)^2\tau}{2}\left((\nabla_z F_2(z_*,0))^2F_1(z_*,0)+\nabla_z^2F_2(z_*,\theta_0)[F_1(z_*,\theta_0),F_2(z_*,\theta_0)]\right)^{(2)}$$
$$\left.+\frac{(\hat\theta-\tau)^3}{6}\left((\nabla_z F_2(z_*,0))^2F_2(z_*,0)+\nabla_z^2F_2(z_*,\theta_0)[F_2(z_*,\theta_0),F_2(z_*,\theta_0)]\right)^{(2)}\right).$$

Consider a sequence $\{t_n\}$, $t_n\downarrow 0$, and suppose that the limits $D^2_{\{t_n\}}\lambda_k(0)$, $D^2_{\{t_n\}}u_k(0)$, $k=1,2$, $D^2_{\{t_n\}}d\Psi(\omega(0),0)/dt$, and $D^2_{\{t_n\}}d\omega(0)/dt$ exist. Then the equality $D^2_{\{t_n\}}d\Psi(\omega(0),0)/dt=0$ is equivalent to the system of linear equations

$$\frac{\partial\Phi_1(0)}{\partial\omega}D^2_{\{t_n\}}\frac{d\omega(0)}{dt}=v, \tag{22}$$

where

$$v=-2\left(D_{\{t_n\}}\frac{\partial(\Phi_1+\Phi_2)(0)}{\partial\omega}\right)D_{\{t_n\}}\frac{d\omega(0)}{dt}-\left(D^2_{\{t_n\}}\frac{\partial(\Phi_1+\Phi_2+\Phi_3)(0)}{\partial\omega}\right)\frac{d\omega(0)}{dt}.$$

The components of the vector v are rational functions depending on the function L and its first and second derivatives calculated at the points $(A,u_k(0))$, $k=1,2$, on the values $\lambda_k(0)$, $u_k(0)$, $du_k(0)/d\theta$, $k=1,2$, and on the limits $D_{\{t_n\}}d\omega(0)/dt$, $D_{\{t_n\}}\lambda_k(0)$, $D^2_{\{t_n\}}\lambda_k(0)$, and $D_{\{t_n\}}du_k(0)/d\theta$, $k=1,2$. The limit $D_{\{t_n\}}d\omega(0)/dt$ can be found solving (15). To find $du_k(0)/d\theta$, $k=1,2$, one should differentiate the equalities $\partial H(y(\theta),u_k(\theta),p,q(\theta))/\partial u=0$, $k=1,2$, and use (8) and (9). The limits $D_{\{t_n\}}du_k(0)/d\theta$, $k=1,2$, can be found from the equalities $D_{\{t_n\}}d(\partial H(y(0),u_k(0),p,q(0))/\partial u)/d\theta=0$, $k=1,2$ (see (11)).

Under the condition $(\partial L(A,u_1(0))/\partial y)u_2=(\partial L(A,u_2(0))/\partial y)u_1$ we solved system (22) using the computer algebra system Maple 6 and obtained the following result:

$$D^2_{\{t_n\}}d\omega^{(1)}(0)/dt-D^2_{\{t_n\}}d\omega^{(2)}(0)/dt=\mathcal{A}+\mathcal{B}D_{\{t_n\}}\lambda_1(0), \tag{23}$$

where

$$\mathcal{B}=6\frac{u_2^2(0)}{u_1(0)}\frac{\partial L(A,u_1(0))}{\partial y}\frac{(L(A,u_1(0))-L(A,u_2(0)))L^3(A,u_1(0))}{(\lambda_1(0)L(A,u_2(0))+\lambda_2(0)L(A,u_1(0)))^4}$$

$$\times \left(\frac{u_2(0)L(A,u_1(0)) - u_1(0)L(A,u_2(0))}{u_2(0)-u_1(0)} \right),$$

and $\mathcal{A}$ is a rational function depending on the function L and its first and second derivatives calculated at the points $(A, u_k(0))$, $k = 1, 2$, and on the values $\lambda_k(0)$ and $u_k(0)$, $k = 1, 2$. Since (23) contains the limit $D_{\{t_n\}}\lambda_1(0)$, we see that without additional assumptions on the structure of the function L the method we use does not allow to get any information concerning the difference $\check{\theta} - \hat{\theta}$, if $(\partial L(A, u_1(0))/\partial y)u_2 = (\partial L(A, u_2(0))/\partial y)u_1$. However, if either $L(y, u) = f(y) + g(u)$ or $L(y, u) = f(y)g(u)$, then we have $\mathcal{B} = 0$ and from (23) we get

$$\check{\theta}(t) - \hat{\theta}(t) = \mathcal{A}t^3 + o(t^3). \tag{24}$$

To study the sign of $\mathcal{A}$ we need the following lemma.

Lemma 7 *Let u_k, $\eta_k > 0$, $\xi_k > 0$, $k = 1, 2$, and $\lambda \in]0, 1[$ be real numbers. Assume that*

$$\xi_1 u_2 - \xi_2 u_1 \neq 0 \tag{25}$$

and

$$(\xi_1 u_2 - \xi_2 u_1)((1 - \lambda)u_2\eta_1 + \lambda u_1\eta_2) \leq 0. \tag{26}$$

Then $(\xi_1 u_2 - \xi_2 u_1)u_1 < 0$.

Proof. Suppose that $(\xi_1 u_2 - \xi_2 u_1)u_1 \geq 0$. If $(\xi_1 u_2 - \xi_2 u_1)u_1 = 0$, then (25) implies $u_1 = 0$. From (26) we obtain $\xi_1\eta_1(1 - \lambda)u_2^2 \leq 0$. Therefore $u_2 = 0$. This contradicts (25).

If $(\xi_1 u_2 - \xi_2 u_1)u_1 > 0$, then there are two possibilities:

$$\xi_1 u_2 - \xi_2 u_1 > 0, \quad \text{and} \quad u_1 > 0, \tag{27}$$

and

$$\xi_1 u_2 - \xi_2 u_1 < 0, \quad \text{and} \quad u_1 < 0. \tag{28}$$

If (27) is satisfied, then $u_2 > \xi_2 u_1/\xi_1$ and from (26) we have

$$0 \geq (1 - \lambda)u_2\eta_1 + \lambda u_1\eta_2 > u_1((1 - \lambda)\eta_1\xi_2/\xi_1 + \lambda\eta_2) > 0,$$

a contradiction. If (28) is satisfied, then $u_2 < \xi_2 u_1/\xi_1$ and from (26) we have

$$0 \leq (1 - \lambda)u_2\eta_1 + \lambda u_1\eta_2 < u_1((1 - \lambda)\eta_1\xi_2/\xi_1 + \lambda\eta_2) < 0,$$

a contradiction. Thus $(\xi_1 u_2 - \xi_2 u_1)u_1 < 0$. □

Proof of Theorem 2. Since $L(y, u) = f(y) + g(u)$, the condition $(\partial L(A, u_1(0))/\partial y)u_2 = (\partial L(A, u_2(0))/\partial y)u_1$ is equivalent to $f'(A) = 0$. Therefore $f''(A) < 0$. The function $\mathcal{A}$ has the form

$$\mathcal{A} = \frac{\lambda_1(0)\lambda_2(0)L(A, u_1(0))L(A, u_2(0))f''(A)}{(\lambda_1(0)L(A, u_2(0)) + \lambda_2(0)L(A, u_1(0)))^3}$$

$$\times (u_2(0) - u_1(0))(\lambda_2(0)u_2(0)L(A, u_1(0)) + 2\lambda_1(0)u_1(0)L(A, u_2(0))).$$

Without loss of generality

$$(u_2(0) - u_1(0))(\lambda_2(0)u_2(0)L(A, u_1(0)) + \lambda_1(0)u_1(0)L(A, u_2(0))) \leq 0.$$

(If this condition is not satisfied, we change the indices.) Since $f''(A) < 0$, by Lemma 7 we have $\mathcal{A} > 0$. Hence $\check{\theta}(t) - \hat{\theta}(t) > 0$ whenever $t > 0$ is sufficiently small and $\lambda_1(0) \in]0, 1[$. Therefore the trajectory $\check{z}(\cdot)$ cannot be optimal, if $\lambda_1(0) \in]0, 1[$. We achieve a contradiction and the proof of Theorem 2. □

Proof of Theorem 3. Since $L(y, u) = f(y)g(u)$ and condition (5) is satisfied, the equality $(\partial L(A, u_1(0))/\partial y)u_2 = (\partial L(A, u_2(0))/\partial y)u_1$ is equivalent to $f'(A) = 0$. The function $\mathcal{A}$ has the form

$$\mathcal{A} = \frac{\lambda_1(0)\lambda_2(0)g(u_1(0))g(u_2(0))f''(A)}{(\lambda_1(0)g(u_2(0)) + \lambda_2(0)g(u_1(0)))^3}$$

$$\times (g(u_1(0))u_2(0) - g(u_2(0))u_1(0))(\lambda_2(0)u_2(0)g(u_1(0)) + 2\lambda_1(0)u_1(0)g(u_2(0))).$$

Without loss of generality

$$(g(u_1(0))u_2(0) - g(u_2(0))u_1(0))(\lambda_2(0)u_2(0)g(u_1(0)) + \lambda_1(0)u_1(0)g(u_2(0))) \leq 0.$$

Since $f''(A) < 0$, by Lemma 7 we have $\mathcal{A} > 0$. Hence $\check{\theta}(t) - \hat{\theta}(t) > 0$ whenever $t > 0$ is sufficiently small and $\lambda_1(0) \in]0, 1[$. Therefore the trajectory $\check{z}(\cdot)$ cannot be optimal, if $\lambda_1(0) \in]0, 1[$. This proves Theorem 3. □

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