PHYSICAL MEASURES AT THE BOUNDARY OF HYPERBOLIC MAPS

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ABSTRACT. We consider diffeomorphisms of a compact manifold with a dominated splitting which is hyperbolic except for a "small" subset of points (Hausdorff dimension smaller than one, e.g. a denumerable subset) and prove the existence of physical measures and their stochastical stability. The physical measures are obtained as zero-noise limits which are shown to satisfy the Entropy Formula.

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1. Introduction

Let M be a compact and connected Riemannian manifold and $\mathrm{Diff}^{1+\alpha}(M)$ be the space of $C^{1+\alpha}$ diffeomorphisms of M for a fixed $\alpha > 0$. We write m for some fixed measure induced by a normalized volume form on M that we call *Lebesgue measure*, dist for the Riemannian distance on M and $\|\cdot\|$ for the induced Riemannian norm on TM.

We say that an invariant probability measure μ for a transformation $f_0 : M \to M$ on a manifold M is *physical* if the *ergodic basin*

$$B(\mu) = \left\{ x \in M : \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f_0^j(x)) \to \int \varphi\, d\mu \text{ for all continuous } \varphi : M \to \mathbb{R} \right\}$$

has positive Lebesgue measure. These measures describe the asymptotic average behavior of a large subset of points of the ambient space and are the basis of the understanding of dynamics in a statistical sense. It is a challenging problem in the Ergodic Theory of Dynamical Systems to prove the existence of such invariant measures.

Let $T_{\Omega(f_0)} = E^s \oplus E^u$ be a hyperbolic Df_0-invariant decomposition (Whitney sum) of the tangent bundle of the non-wandering set $\Omega(f_0)$ of f_0. The classical construction of physical measures involves f_0-invariant measures which are absolutely continuous with respect to Lebesgue measure along the unstable direction through the points of $\Omega(f_0)$. These uniformly hyperbolic dynamical systems were the first general class of systems where these measures were shown to exist [10, 32, 34].

An invariant probability measure is called *SRB (Sinai-Ruelle-Bowen) measure*, if it admits positive Lyapunov exponents and its conditional measures along the unstable manifolds (in the sense of Pesin theory [31, 16]) are absolutely continuous with respect to Lebesgue measure induced on the unstable manifolds. For a class of dynamical systems which includes uniformly hyperbolic systems the notions of *physical* and *SRB* measures coincide.

The SRB measures as defined above are related to a class of equilibrium states of a certain potential function. Let $\phi : M \to \mathbb{R}$ be a continuous function. Then a f_0-invariant probability measure μ is a *equilibrium state for the potential* ϕ if

$$h_\mu(f_0) + \int \phi\, d\mu = \sup_{\nu \in \mathcal{M}} \left\{ h_\nu(f_0) + \int \phi\, d\nu \right\},$$

where $\mathcal{M}$ is the set of all f_0-invariant probability measures.

For uniformly hyperbolic diffeomorphisms it turns out that physical (or SRB) measures are the equilibrium states for the potential function $\phi(x) = -\log|\det Df|E^u(x)|$. It is a remarkable fact that *for uniformly hyperbolic systems these three classes of measures (physical, SRB and equilibrium states) coincide*.

We will address the problem of the existence of physical measures on the boundary of uniformly hyperbolic diffeomorphisms. The idea is to add small random noise to a deterministic system f_0 in the boundary of uniformly hyperbolic systems and, for a large class of such maps, we prove that as the level of noise converges to zero, the stationary measures of the random system tend to equilibrium states for f_0 which are physical measures. The stationary measures exist in a very general setting, but the “zero noise” limit measures are not necessarily physical measures. The specific choice of random perturbation is important to obtain physical measures as zero noise limits. The same general idea has been used in [14] to obtain *SRB* measures for partially hyperbolic maps under strong asymptotic growth conditions on every point.

Let $(\theta_\varepsilon)_{\varepsilon>0}$ be a family of Borel probability measures on $(\mathrm{Diff}^{1+\alpha}(M), \mathcal{B}(\mathrm{Diff}^{1+\alpha}(M)))$, where we write $\mathcal{B}(X)$ the Borel $\sigma-$algebra of a topological space X. We will consider random dynamical systems generated by independent and identically distributed diffeomorphisms with θ_ε the probability distribution driving the choice of the maps.

We say that a probability measure μ^ε on M is *stationary for the random system* $(\mathrm{Diff}^{1+\alpha}(M), \theta_\varepsilon)$ if

$$\int\int \varphi(f(x))\, d\mu^\varepsilon(x) d\theta_\varepsilon(f) = \int \varphi\, d\mu^\varepsilon \quad \text{for all continuous } \varphi : M \to \mathbb{R}. \tag{1.1}$$

We will assume that $\mathrm{supp}(\theta_\varepsilon) \to f_0$ when $\varepsilon \to 0$ in a suitable topology. A result based on classical Markov Chain Theory (see [25] or [4]) ensures that *every weak* accumulation point of stationary measures* $(\mu^\varepsilon)_{\varepsilon>0}$ *when* $\varepsilon \to 0$ *is a* f_0*-invariant probability measure,* called a *zero-noise limit measure.* It is then natural to study the kind of zero noise limits that can arise and to define stochastic stability when the limit map f_0 admits physical measures.

We say that a map f_0 is *stochastically stable* (under the random perturbation given by $(\theta_\varepsilon)_{\varepsilon>0}$) if *every accumulation point* μ *of the family of stationary measures* $(\mu^\varepsilon)_{\varepsilon>0}$*, when* $\varepsilon \to 0$*, is a linear convex combination of the physical measures of* f_0.

Stochastic stability has been proved for uniformly expanding maps and uniformly hyperbolic systems [24, 25, 37, 40]. For some non-uniformly hyperbolic systems, like quadratic maps, Hénon maps and Viana maps, stochastic stability has been obtained much more recently [3, 7, 8]. The authors have studied random perturbations of *intermittent maps* and have proved stochastic stability for these maps for some parameters and certain types of random perturbations [5]. The techniques used were extended to higher dimensional local diffeomorphisms exhibiting expansion except at finitely many points, enabling us to obtain physical measures directly as zero-noise limits of stationary measures for certain types of random perturbations, proving also the stochastic stability of these measures.

Stochastic stability results for maps of the 2-torus which are essentially Anosov except at finitely many points were obtained in [14], the physical probability measures of which were constructed in a series of papers using different techniques [22, 21, 20]. Similar results for different kinds of bifurcations away from Anosov maps at fixed points were also studied in [13].

Using ideas akin to [5] we prove the existence of physical probability measures and their stochastic stability for diffeomorphisms which are "almost Anosov" under some geometric and dynamical conditions.

1.1. **Statement of the results.** We assume that $f_0 : U_0 \to f_0(U_0)$ is a $C^{1+\alpha}$ diffeomorphism on a relatively compact open subset U_0 of a manifold M which is strictly invariant, that is, $\mathrm{closure}(f(U_0)) \subset U_0$. During the rest of this paper we set $\Lambda = \cap_{n\geq 0} \mathrm{closure}\, f_0^n(U_0)$. Moreover we suppose there exists a continuous dominated splitting $E \oplus F$ of $T_{U_0}M$ which is Df_0-invariant over Λ, i.e., there exists $\lambda_0 \in (0,1)$ such that for all $x \in U_0$

$$\|Df \mid E(x)\| \cdot \|(Df \mid F(x))^{-1}\| \leq \lambda_0. \tag{1.2}$$

We may see $E\oplus F$ on U_0 as a continuous extension of $E\oplus F$ on Λ. This assumption ensures the existence of

stable cones: $\mathbb{E}^a_x = \{(u,v)\in E(x)\oplus F(x) : \|v\|\le a\cdot\|u\|\}$;
unstable cones: $\mathbb{F}^b_x = \{(u,v)\in E(x)\oplus F(x) : \|u\|\le b\cdot\|v\|\}$;

for all $x\in U$ and $a,b\in(0,1)$, which are Df_0-invariant in the following sense

- if $x, f_0^{-1}(x)\in U$, then $Df_0^{-1}(\mathbb{E}^a_x)\subset \mathbb{E}^{\lambda_0 a}_{f_0^{-1}(x)}$;
- if $x, f_0(x)\in U$, then $Df_0(\mathbb{F}^b_x)\subset \mathbb{F}^{\lambda_0 b}_{f_0(x)}$;

Continuity enables us to unambiguously denote $d_E=\dim(E)$ and $d_F=\dim(F)$, so that $d=d_E+d_F=\dim(M)$. Domination guarantees the absence of tangencies between stable and unstable manifolds, since the angles between the E and F directions are bounded from below away from zero at every point. Let us fix the unit balls of dimensions d_E, d_F

$$\mathbb{B}_E = \{w\in\mathbb{R}^{d_E} : \|w\|_2\le 1\} \quad \text{and} \quad \mathbb{B}_F = \{w\in\mathbb{R}^{d_F} : \|w\|_2\le 1\}$$

where $\|\cdot\|_2$ is the standard Euclidean norm on the corresponding Euclidean space. We say that a $C^{1+\alpha}$ embedding $\Delta:\mathbb{B}_E\to M$ (respectively $\Delta:\mathbb{B}_F\to M$) is a E-disk (resp. F-disk) if the image of $D\Delta(w)$ is contained in $\mathbb{E}^a_{\Delta(w)}$ for all $w\in\mathbb{B}_E$ (resp. $D\Delta(w)(\mathbb{R}^{d_F})\subset\mathbb{F}^b_{\Delta(w)}$ for every $w\in\mathbb{B}_F$).

In what follows we denote by $\mathcal{H}(A)$ the *Hausdorff dimension* of a subset $A\subset M$. We first state the results without mentioning random perturbations.

Theorem A. *Let $f_0: M\to M$ be a $C^{1+\alpha}$ diffeomorphism admitting a strictly forward invariant open set U_0 endowed with a dominated splitting $E\oplus F$ such that*

1. $\|Df_0\mid E(x)\|\le 1$ *and* $\|(Df_0\mid F(x))^{-1}\|\le 1$ *for all* $x\in U_0$*;*
2. $F_1=\{x\in U_0 : \|(Df_0\mid F(x))^{-1}\|=1\}$ *and* $E_1=\{x\in U_0 : \|Df_0\mid E(x)\|=1\}$ *satisfy*
$$\mathcal{H}(\Delta\cap E_1)<1 \quad \textit{and} \quad \mathcal{H}(\hat\Delta\cap F_1)<1,$$
where Δ is any E-disk and $\hat\Delta$ is any F-disk contained in U_0;
3. $|\det(Df_0\mid F(x))|>1$ *for every* $x\in F_1$.

If in addition $f_0\mid\Lambda$ is transitive, then there exists a unique physical measure supported in Λ, with d_F positive Lyapunov exponents along the F-direction and whose basin has full Lebesgue measure in U_0.

We note that if $E_1\cup F_1$ is not invariant and *finite*, then f_0 or some power $f_0^k, k\in\mathbb{Z}$, is an Anosov map, in which case f_0 is also Anosov and the conclusions of Theorem A are known. Moreover from the dominated decomposition assumption (1.2) we easily see that $E_1\cap F_1=\emptyset$. We remark also that *the conditions on E_1 and F_1 in the statement of Theorem A are automatically satisfied whenever E_1 and F_1 is denumerable*.

We clearly may specialize this result for a transitive $C^{1+\alpha}$-diffeomorphism admitting a dominated splitting on the entire manifold and satisfying items (1)-(3) of Theorem A, up to replacing U_0 and Λ by M.

We can adapt the statement of Theorem A to the setting where U_0 has a *partially hyperbolic splitting*, that is, the strictly forward f_0-invariant open subset U_0 admits a continuous splitting $T_{U_0}M = E^s\oplus E^c\oplus E^u$ such that

- both $(E^s \oplus E^c) \oplus E^u$ and $E^s \oplus (E^c \oplus E^u)$ are dominated decompositions;
- E^s is uniformly contracting and E^u is uniformly expanding: there exists $\sigma > 1$ satisfying $\|Df \mid E^s(x)\| \le \sigma^{-1}$ and $\|(Df \mid E^u(x))^{-1}\| \le \sigma^{-1}$ for all $x \in U$;
- the restriction of the splitting to Λ is Df_0-invariant.

Theorem B. *Let $f_0 : M \to M$ be a $C^{1+\alpha}$ diffeomorphism with strictly forward invariant open set U_0 having a partially hyperbolic splitting. We assume that either*

1. $\|Df_0 \mid E^c(x)\| \le 1$ *for all $x \in U_0$, and $K = \{x \in U_0 : \|Df_0 \mid E^c(x)\| = 1\}$ is such that $\mathcal{H}(\Delta \cap K) < 1$ for every $E^s \oplus E^c$-disk Δ contained in U_0;*

or

(2) $\|(Df_0 \mid E^c(x))^{-1}\| \le 1$ *for all $x \in U_0$; $K = \{x \in U_0 : \|(Df_0 \mid E^c(x))^{-1}\| = 1\}$ satisfies $\mathcal{H}(\Delta \cap K) < 1$ for every $E^c \oplus E^u$-disk Δ contained in U_0; and $|\det(Df_0 \mid E^c(x))| > 1$ for every $x \in K$.*

If Λ is transitive, then there exists a unique absolutely continuous f_0-invariant probability measure μ_0 with $\dim E^u$ (case 1) or $\dim E^c + \dim E^u$ (case 2) positive Lyapunov exponents, whose support is contained in Λ and with an ergodic basin of full Lebesgue measure in U_0.

The statement essentially means that *if an attractor admits a partially hyperbolic splitting which is volume hyperbolic, does not admit mixed behavior along the central direction and the neutral points along the central direction form a small subset, then there exists a physical measure.*

Cowieson-Young [14] have obtained similar results, albeit for the existence of *SRB* measures and not necessarily for *physical* ones (see Vásquez [38] where it is shown that in the setting of Theorem B physical measures will necessarily be *SRB* measures). Moreover Cowieson-Young obtained a strong result of existence of *SRB* measure for partially hyperbolic maps with one-dimensional central direction using the same strategy of proof. Here we get rid of the dimensional restriction assuming a dynamical restriction.

These results will be derived from the following more technical one, but also interesting in itself.

Theorem C. *Let $f_0 : M \to M$ be a $C^{1+\alpha}$ diffeomorphism admitting a strictly forward invariant open set U_0 with a dominated splitting satisfying items (1)-(2) of Theorem A. Then*

1. *for any non-degenerate isometric random perturbation $(\theta_\varepsilon)_{\varepsilon>0}$ of f_0, every weak* accumulation point μ of a sequence $(\mu^\varepsilon)_{\varepsilon>0}$ of stationary measures of level ε, when $\varepsilon \to 0$, is an equilibrium state for the potential $-\log|\det(Df_0 \mid F(x))|$, i.e.*
$$h_\mu(f_0) = \int \log|\det(Df_0 \mid F(x))|\, d\mu(x). \tag{1.3}$$
2. *every equilibrium state μ as above is a convex linear combination of*
   (a) *at most finitely many ergodic equilibrium states having positive entropy with d_F positive Lyapunov exponents, with*
   (b) *probability measures having zero entropy whose support has constant unstable Jacobian equal to one, i.e., measures whose Lyapunov exponents are nonpositive.*
3. *every equilibrium state with positive entropy is a physical measure for f_0.*

4. *if* the attractor Λ is transitive*, then there exists at most one equilibrium state with positive entropy.*

Remark 1. We note that if F_1 is denumerable, then necessarily the measures in item (2b) of Theorem C are Dirac measures concentrated on periodic orbits whose tangent map has only non-positive eigenvalues.

The restriction on the random perturbations means the following. We assume that $U_0 \subset M$ admits an open subset $V \subset \text{closure}(V) \subset U$ and an action $\mathcal{V} \to M$, where $\mathcal{V}$ is a small neighborhood of the identity e of a locally compact Lie group G such that for all $x \in \text{closure}(V)$, setting $g_x : \mathcal{V} \to M$, $v \mapsto v \cdot x$, we have

P1: $g_x(\mathcal{V}) \subset U_0$;
P2: $g_x(W)$ is a neighborhood of x for every open subset $W \subset \mathcal{V}$;
P3: for every fixed $v \in \mathcal{V}$ the map $g_v : V \to U_0, x \mapsto v \cdot x$ is an isometry.

Then we define

$$\hat{f} : \mathcal{V} \times M \to M, \quad (v,x) \mapsto v \cdot f_0(x) \tag{1.4}$$

and take a probability measure θ_ε on $\mathcal{V}$, which translates into a probability measure on the family $(\hat{f}_v)_{v \in \mathcal{V}}$.

For more on non-degenerate random perturbation and for examples of non-degenerate isometric random perturbations, see Section 3. In particular, *Theorems A, B and C apply to a bounded topological attracting set for a diffeomorphism on a domain of any Euclidean space*. In Section 3 we show that *this is enough to obtain Theorem A, B and C in full generality* through a tubular neighborhood construction. In particular, Theorem C shows that *in the setting of Theorems A and B the physical measures obtained are stochastically stable*, as explained in Section 7.

This paper is organized as follows. In Section 2 we present some examples in the setting of the main theorems. We outline some general results concerning random maps in Section 3. In Section 4 we derive the main dynamical consequence of our assumptions and then, in Section 5, we prove that equilibrium states for f_0 must be either physical measures or measures with no expansion. Finally we construct equilibrium states using zero-noise limits in Section 6 and put together the results concluding stochastic stability for f_0 and proving Theorems A and B in Section 7.

2. EXAMPLES OF MAPS IN THE SETTING OF THE MAIN THEOREMS

Example 1. *Let $f_0 : \mathbb{T} \to \mathbb{T}$ be a $C^{1+\alpha}$ diffeomorphism, with $0 < \alpha < 1$ and $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, obtained from an Anosov linear automorphism of the 2-torus by weakening the expanding direction F of the fixed point p in such a way that $Df_0(p) \mid F = Id \mid F$. The stable direction E continues to be uniformly contracting throughout and F is still expanded by Df_0 on $\mathbb{T} \setminus \{p\}$.*

This kind of maps where studied by Hu and Young [20, 21, 22]. *In the $C^{1+\alpha}$ case there exists a unique physical probability measure μ for f_0 with one positive Lyapunov exponent. Theorem C then shows that any weak* accumulation point μ^0 of stationary measures μ^ε from additive perturbations is a convex linear combination of δ_p with μ. When f_0 is of class C^2 the only physical* *probability measure for f_0 is δ_p, whose basin contains Lebesgue almost every point of $\mathbb{T}$. Hence Theorem C shows in particular that δ_p is stochastically stable.*

The construction can be adapted to provide maps with finitely many periodic orbits with neutral behavior along the F direction. We note that $E_1 = \emptyset$.

Example 2. *Let us take the product $f_0 \times E_d$, where f_0 is given by Example 1 and $E_d : \mathbb{S}^1 \to \mathbb{S}^1, x \mapsto d \cdot x \mod \mathbb{Z}$, identifying $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$ and letting $d \in \mathbb{N}, d \geq 2$. Then $E_1 = \emptyset$, $E^s = E \times \{0\}$, $E^c = F \times \{0\}$ and, for big enough $d \geq 2$, $E^u = \{(0,0)\} \times \mathbb{R}$. Moreover $F_1 = \{p\} \times \mathbb{S}^1$ and $W^u_{loc}(p) \times \mathbb{S}^1$ is a $E^c \oplus E^u$-disk that contains F_1, and also $\mathcal{H}(F_1) = 1$.*

In this example $\mu = \delta_p \times \lambda$ is the unique physical measure, has positive entropy and only one positive Lyapunov exponent, where λ is Lebesgue measure on $\mathbb{S}^1$.

We note that Examples 1 and 2 can be seen as "derived from Anosov" (DA) maps [39, 12] at the boundary of the set of Anosov diffeomorphisms.

Example 3. *Let $f_0 : \mathbb{S}^1 \times \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{R}^2, (x, \rho e^{i\theta}) \mapsto (g_\alpha(x), (\rho/10 + 1/2) \cdot e^{i(\theta + g_\alpha(x))})$ where again $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and in $\mathbb{R}^2$ we use polar coordinates. If $g_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ is an expanding map, then we have the standard solenoid map. Here we take the $C^{1+\alpha}$ map*

$$g_\alpha(x) = \begin{cases} x + 2^\alpha x^{1+\alpha}, & x \in [0, \frac{1}{2}) \\ x - 2^\alpha (1-x)^{1+\alpha}, & x \in [\frac{1}{2}, 1] \end{cases}$$

for $0 < \alpha < 1$. It is known [35] *that g_α admits a unique absolutely continuous invariant probability measure μ. If $\pi : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ is the natural projection, then on the attractor $\Lambda = \cap_{n \geq 1} f_0^n(\mathbb{S}^1 \times \mathrm{closure}(B(0,1)))$ there is a unique measure ν such that $\pi_*(\nu) = \mu$, which is physical and whose basin contains Lebesgue almost every point of $U_0 = \mathbb{S}^1 \times \mathrm{closure}(B(0,1))$.*

In this case $F_1 = \{0\} \times \mathbb{R}^2$ but every F-disk Δ intersects F_1 at most finitely many times, since Δ must be locally a graph over S^1. Theorem A holds and Theorem C shows that every equilibrium state is a convex linear combination of $\delta_{(0,5/9)}$ with ν ($(0,5/9)$ is the unique fixed point of f_0).

If we let $\alpha \geq 1$, then g_α is of class C^2 and [36] *δ_0 is the unique physical measure for g_α. Theorem C shows that $\delta_{(0,5/9)}$ is stochastically stable. Since $\pi_*(\delta_{(0,5/9)}) = \delta_0$ it is not difficult to see that $\delta_{(0,5/9)}$ has basin containing U_0 Lebesgue modulo zero, so $\delta_{(0,5/9)}$ is the physical measure for Λ.*

Example 4. *Let $f_0 : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2, (t, x, \rho e^{i\theta}) \mapsto (E_d(t), g(E_d(t), x), (\rho/10 + 1/2) \cdot \exp[i(\theta + g(E_d(t), x))])$, where E_d was defined in Example 2, $d \in \mathbb{N}, d \geq 2$ and $g : \mathbb{T} \to \mathbb{S}^1$ of class $C^{1+\alpha}$ is an extension of g_α from Example 3 to $\mathbb{T}$ given by*

$$g(t,x) = \begin{cases} x(1 + 0.1 \cdot \sin^2(\pi t)) + 2^\alpha (1 - 0.1 \cdot \sin^2(\pi t)) x^{1+\alpha}, & x \in [0, \frac{1}{2}) \\ 1 - (1-x)(1 + 0.1 \cdot \sin^2(\pi t)) - 2^\alpha (1 - 0.1 \cdot \sin^2(\pi t))(1-x)^{1+\alpha}, & x \in [\frac{1}{2}, 1] \end{cases}$$

for some fixed $0 < \alpha < 1$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Then we have $E^s = \{(0,0)\} \times \mathbb{R}^2$, $E^c = \{0\} \times \mathbb{S}^1 \times \{0\}$ and $E^u = \mathbb{S}^1 \times \{0\} \times \{0\}$ for big enough d. The conditions on item 2 of Theorem B hold with $K = E_d^{-1}(\{0\}) \times \{0\} \times \mathbb{R}^2$, because $|g_t'(x)| = |D_2 g(t,x)| \geq 1$ and equals 1 only for $(0,0)$.

The natural projection $\pi : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T}$ conjugates f_0 to $f_1 : \mathbb{T} \to \mathbb{T}, (t,x) \mapsto (E_d(t), g(E_d(t), x))$ over the attractor $\Lambda = \cap_{n \geq 1} f_0^n(\mathbb{T} \times \mathrm{closure}(B(0,1)))$. We note that each g_t is conjugate to E_2

through a homeomorphism h_t which depends continuously on $t \in \mathbb{S}^1$ in the C^0 topology. Hence $H(t,x) = (t, h_t(x))$ is a homeomorphism of $\mathbb{T}$ such that $H \circ f_1 = (E_d \times E_2) \circ H$ and since $E_d \times E_2$ is transitive, then f_1 and also f_0 are transitive.

This shows that we can apply Theorem B obtaining the existence of a unique physical measure for f_0.

Example 5. *Let $\mathcal{K} \subset I = [0,1]$ be the middle third Cantor set and $(a_i^n, b_i^n), i = 1, \cdots, 2^{n-1}$ be an enumeration for the gaps of the n-th generation in the construction of $\mathcal{K}$. We define β on any given gap interval (a,b) as*

$$\beta(x) = \begin{cases} x - a, & \text{if } x \in (a, \frac{a+b}{2}) \\ b - x, & \text{if } x \in (\frac{a+b}{2}, b) \end{cases} .$$

Then the map $\beta : I \setminus \mathcal{K} \to I$ is uniformly continuous and so we can continuously extend it to I setting $\beta \mid \mathcal{K} \equiv 0$. Moreover it is easy to see that $\beta \mid (I \setminus \mathcal{K})$ is Lipschitz (with Lipschitz constant 1) and so is its extension to I.

It addition, with respect to Lebesgue measure on I, we get $\int_0^1 \beta < \infty$ and if $g_0 : I \to \mathbb{R}$ is given by $g_0(x) = x + \frac{\int_0^x \beta}{\int_0^1 \beta}$, then $g_0(0) = 0, g_0(1) = 2$ and g_0 induces a C^1 map of the circle onto itself whose derivative is Lipschitz satisfying $g_0' \mid \mathcal{K} \equiv 1$ and $g_0' \mid (I \setminus \mathcal{K}) > 1$.

The map $g_0 : \mathbb{S}^1 \to \mathbb{S}^1$ is mixing since $\sigma(J) = |g_0(J)|/|J| > 1$ for every arc $J \subset \mathbb{S}^1$, where $|\cdot|$ denotes length. Indeed the continuity of the map σ on arcs together with the compactness of the family $\Gamma(\ell) = \{J \subset \mathbb{S}^1 : J \text{ is an arc and } |J| \geq \ell\}$, for any given bound $\ell > 0$ on the length, show that there exists $\sigma(\ell) > 1$ such that $|g_0(J)| \geq \sigma(\ell) \cdot |J|$ for any given arc $\emptyset \neq J \subset \mathbb{S}^1$. Hence for every nonempty arc J there exists $n = n(J) \in \mathbb{N}$ such that $g_0^n(J) = \mathbb{S}^1$.

Replacing g_α by g_0 in the definition of f_0 within Example 3, we get a C^{1+1} map from the solid torus into itself whose topological attractor satisfies the conditions of Theorem A, where F_1 is Cantor set.

3. NON-DEGENERATE RANDOM ISOMETRIC PERTURBATIONS

Let a parameterized family of maps $\hat{f} : X \to \mathrm{Diff}^{1+\alpha}(M), t \mapsto f_t$ be given, where X is a connected compact metric space. We identify a sequence $f_0, f_1, f_2, \ldots$ from $\mathrm{Diff}^{1+\alpha}(M)$ with a sequence $\omega_0, \omega_1, \omega_2, \ldots$ of parameters in X and the probability measure θ_ε can be assumed to be supported on X. We set $\Omega = X^{\mathbb{N}}$ to be the space of sequences $\omega = (\omega_i)_{i \geq 0}$ with elements in X (here we assume that $0 \in \mathbb{N}$). Then we define in Ω the standard infinite product topology, which makes Ω a compact metrizable space. The standard product probability measure $\theta^\varepsilon = \theta_\varepsilon^{\mathbb{N}}$ makes $(\Omega, \mathcal{B}, \theta^\varepsilon)$ a probability space. We write $\mathcal{B} = \mathcal{B}(\Omega)$ for the σ-algebra generated by cylinder sets: the minimal $\sigma-$algebra containing all sets of the form $\{\omega \in \Omega : \omega_0 \in A_0, \omega_1 \in A_2, \omega_2 \in A_2, \cdots, \omega_l \in A_l\}$ for any sequence of Borel subsets $A_i \subset X, i = 0, \cdots, l$ and $l \geq 1$. We use the following skew-product map

$$F : \Omega \times M \to \Omega \times M, \quad (\omega, x) \mapsto (\sigma(\omega), f_{\omega_0}(x))$$

where σ is the left shift on sequences: $(\sigma(\omega))_n = \omega_{n+1}$ for all $n \geq 0$. It is not difficult to see that μ^ε is a stationary measure for the random system $(\hat{f}, \theta_\varepsilon)$ (i.e. satisfying (1.1)) if, and only if, $\theta^\varepsilon \times \mu^\varepsilon$ on $\Omega \times M$ is F-invariant. We say that μ^ε is ergodic if $\theta^\varepsilon \times \mu^\varepsilon$ is F-ergodic.

If we define $\hat{\Omega} = X^{\mathbb{Z}}$ to be the set of all bi-infinite sequences $(\omega_i)_{i\in\mathbb{Z}}$ of elements of X, then we can define G to be the invertible natural extension of F to this space:

$$G : \hat{\Omega} \times M \to \hat{\Omega} \times M, \quad (\omega, x) \mapsto (\sigma(\omega), f_{\omega_0}(x)).$$

This map is invertible and $G^{-1}(\omega, x) = (\sigma^{-1}(\omega), f^{-1}_{\omega_{-1}}(x))$. On $\hat{\Omega}$ we set the natural product topology and the product σ-algebra $\hat{\mathcal{B}} = \mathcal{B}(\hat{\Omega})$ generated by cylinder sets as above but now with indexes in $\mathbb{Z}$. The product probability measure $\hat{\theta}^\varepsilon = \theta_\varepsilon^{\mathbb{Z}}$ makes $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{\theta}^\varepsilon)$ a probability space. We set the following notation for the natural projections

$$\pi_M : \Omega \times M \to M, \quad \hat{\pi}_M : \hat{\Omega} \times M \to M, \quad \hat{\pi}_\Omega : \hat{\Omega} \times M \to \hat{\Omega}, \quad \text{and} \quad \hat{\pi} : \hat{\Omega} \times M \to \Omega \times M.$$

For $\omega \in \hat{\Omega}$ and for $n \in \mathbb{Z}$ we define for all $x \in M$

$$f^n_\omega = (\hat{\pi}_M \circ G^n)(x) = \begin{cases} (f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0})(x), & n > 0 \\ x, & n = 0 \\ (f^{-1}_{\omega_{-n}} \circ \cdots \circ f^{-1}_{\omega_{-1}})(x), & n < 0 \end{cases}.$$

Given $x \in M$ and $\omega \in \hat{\Omega}$ the sequence $(f^n_\omega(x))_{n\geq 1}$ is a *random orbit* of x. Analogously we set $f^n_\omega = \pi_M \circ F^n$ for $n \geq 0$ and $\omega \in \Omega$.

From now on we assume that the family $(\theta_\varepsilon)_{\varepsilon>0}$ of probability measures on X is such that their supports have non-empty interior and $\mathrm{supp}(\theta_\varepsilon) \to \{t_0\}$ when $\varepsilon \to 0$, where $t_0 \in X$ is such that $f_{t_0} = f_0$.

3.1. **Non-degeneracy conditions.** In what follows we write $f^n_x : \Omega \to M$ for the map $\omega \in \Omega \mapsto f^n_\omega(x)$, for every $n \geq 0$. We say that $(\hat{f}, \theta_\varepsilon)_{\varepsilon>0}$ is a *non-degenerate random perturbation* of $f_0 = f_{t_0}$ if, for every small enough ε, there is $\delta_1 = \delta_1(\varepsilon) > 0$ such that for all $x \in U$

ND1: $\{f_t(x) : t \in \mathrm{supp}(\theta_\varepsilon)\}$ contains a ball of radius δ_1 around $f_{t_0}(x)$;

ND2: $(f_x)_*\theta_\varepsilon$ is absolutely continuous with respect to m.

Remark 2. We note that θ_ε cannot have atoms by condition ND2 above.

The following is a finiteness result for non-degenerate random perturbations.

Theorem 3.1. *Let $(\hat{f}, \theta_\varepsilon)_{\varepsilon>0}$ be a non-degenerate random perturbation of f_0. Then for each $\varepsilon > 0$ there are finitely many absolutely continuous ergodic measures $\mu^\varepsilon_1, \ldots \mu^\varepsilon_{l(\varepsilon)}$, and for each $x \in U$ there is a θ^ε mod 0 partition $\Omega_1(x), \ldots, \Omega_{l(\varepsilon)}(x)$ of Ω such that for $1 \leq i \leq l(\varepsilon)$*

$$\mu^\varepsilon_i = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n} \delta_{f^j_\omega x} \quad \textit{for} \quad \omega \in \Omega_i(x).$$

Moreover the interior of the supports of the physical measures are nonempty and pairwise disjoint.

Proof. See [4], [2] or [11]. □

The continuity of the map F is enough to get the *forward invariance of* $\mathrm{supp}(\mu^\varepsilon)$ *for any stationary measure* μ^ε, i.e. if $x \in \mathrm{supp}(\mu^\varepsilon)$ then $f_t(x) \in \mathrm{supp}(\mu^\varepsilon)$ for all $t \in \mathrm{supp}(\theta_\varepsilon)$, since $\theta^\varepsilon \times \mu^\varepsilon$ is F-invariant. By non-degeneracy condition ND1 $\mathrm{supp}(\mu^\varepsilon)$ contains a ball of radius $\delta_1 = \delta_1(\varepsilon)$. Moreover defining the *ergodic basin* of μ^ε by

$$B(\mu^\varepsilon) = \left\{ x \in M : \frac{1}{n}\sum_{j=1}^{n} \varphi(f^j_\omega(x)) \to \int \varphi\, d\mu \text{ for all } \varphi \in C(M,\mathbb{R}) \text{ and } \theta^\varepsilon\text{-a.e. } \omega \in \Omega \right\},$$

then $m(B(\mu^\varepsilon)) > 0$, since $\mu^\varepsilon(B(\mu^\varepsilon)) = 1$ by the Ergodic Theorem applied to $(F, \theta^\varepsilon \times \mu^\varepsilon)$ and $\mu^\varepsilon \ll m$.

These non-degeneracy conditions are not too restrictive since we can always construct a non-degenerate random perturbation of any differentiable map of a compact manifold of finite dimension, with X the closed ball of radius 1 around the origin of a Euclidean space, see [4] and the following subsection.

3.2. **Isometric random perturbations.** We present below the two main types of families of maps we will be dealing with, satisfying conditions P1-P3 stated in Subsection 1.1.

Example 6 (Global additive perturbations)**.** *Let M be a homogeneous space, i.e., a compact connected finite dimensional Lie Group admitting an invariant Riemannian metric. Fixing a neighborhood $\mathcal{U}$ of the identity $e \in M$ we can define a map $f : \mathcal{U} \times M \to M, (u,x) \mapsto L_u(f_0(x))$, where $L_u(x) = u \cdot x$ is the left translation associated to $u \in M$. The invariance of the metric means that left (an also right) translations are isometries, hence fixing $u \in \mathcal{U}$ and taking any $(x,v) \in TM$ we get*

$$\|Df_u(x)\cdot v\| = \|DL_u(f_0(x))(Df_0(x)\cdot v)\| = \|Df_0(x)\cdot v\|. \tag{3.1}$$

In the particular case of $M = \mathbb{T}^d$, the d-dimensional torus, we have $f_u(x) = f_0(x) + u$ and this simplest case suggests the name additive random perturbations *for random perturbations defined using families of maps of this type. It is easy to see that if the probability measure* θ_ε *is absolutely continuous and supported on a open subset X of $\mathcal{U}$, then conditions P1, P2 and P3 are met.*

Example 7 (Local additive perturbations)**.** *If $M = \mathbb{R}^d$ and U_0 is a bounded open subset of M strictly invariant under the diffeomorphisms f_0, i.e.,* $\mathrm{closure}(f_0(U_0)) \subset U_0$*, then we can define a non-degenerate isometric random perturbation setting*

- $V = f_0(U_0)$ *(so that* $\mathrm{closure}(V) = \mathrm{closure}(f_0(U_0)) \subset U_0$*);*
- $G \simeq \mathbb{R}^d$ *the group of translations of* $\mathbb{R}^d$*;*
- $\mathcal{V}$ *a small enough neighborhood of the origin in G.*

Then for $v \in \mathcal{V}$ and $x \in V$ we have $f_v(x) = v \cdot x = x + v$, with the standard notation for vector addition, and clearly $f_v(x) = x + v$ is an isometry and satisfies both conditions P1 and P2.

Now we show that we can construct non-degenerate isometric random perturbations in the setting of Examples 6 and 7. We define the family of maps $\hat{f}$ as in (1.4). The local compactness of G gives a Haar measure ν on G and the isometry condition ensures that $\dim(G) = d$ and that $(\hat{f}_x)_*(\nu \mid \mathcal{V}) \ll m$. Hence for every probability measure θ_ε given by a probability density with respect to ν we have $(\hat{f}_x)_*\theta_\varepsilon \ll m$, and this gives condition ND2.

Moreover whenever $\mathrm{supp}(\theta_\varepsilon)$ has nonempty interior in $\mathcal{V}$ then condition P2, together with the compactness of $\mathrm{closure}(V)$, ensure that there is $\delta=\delta(\varepsilon)>0$ such that condition ND1 is satisfied. Thus we get conditions ND1 and ND2 choosing θ_ε as a probability density in $\mathcal{V}$ whose support has nonempty interior, and setting $X=\mathcal{V}$ for the definition of $\Omega,\hat\Omega$.

3.2.1. *Isometric perturbations of maps in arbitrary manifolds.* Now we show that for any given map f_0 is the setting of Theorems A, B or C, we may define a random isometric perturbation of a particular extension of f_0 as in Example 7, which is partially hyperbolic.

We may assume without loss that M is a compact submanifold of $\mathbb{R}^N$, $N\ge 2d$. Let W_0 be an open *normal tubular neighborhood* of M in $\mathbb{R}^N$, that is, there exists $\Phi: W\to W_0, (x,u)\mapsto x+u$ a (C^∞) diffeomorphism from a neighborhood W of the zero section of the normal bundle $TM^\perp$ of M to W_0. Let also $\pi: W_0\to M$ be the associated projection: $\pi(w)$ is the closest point of w in M for $w\in W_0$, so that the line through the pair of points $w,\pi(w)$ is normal to M at $\pi(w)$, see e.g. [19] or [17]. Now we define for $\rho_0\in(0,1)$

$$F: W\to W,\quad (x,u)\mapsto (f_0(x),\rho_0\cdot u)\quad\text{and}\quad F_0: W_0\to W_0,\quad w\mapsto(\Phi\circ F_0\circ\Phi^{-1})(w).$$

Then clearly F_0 is a diffeomorphism onto its image, $\mathrm{closure}\,F(W_0)\subset W_0$ and $M=\cap_{n\ge0}F^n(W_0)$. Moreover if f_0 admits a dominated splitting $E\oplus F$ in a strictly forward f_0-invariant set $U_0\subset M$, then F_0 has a dominated splitting $E^s\oplus E\oplus F$ in the strictly forward F_0-invariant set $\hat U_0=\pi^{-1}(U_0)\subset W_0$, where $E^s(w)$ is normal to T_wM at $w\in M$ and uniformly contracted by DF_0, as long as ρ_0 is close enough to zero.

We can now define a random isometric perturbation of F_0 and obtain Theorems A and B as corollaries of Theorem C. For that *it is enough to prove Theorem C for non-degenerate random isometric perturbations on an strictly invariant open subset of the Euclidean space*. Then given f_0 we construct F_0 as explained above and note that any F_0-invariant measure must be concentrated on $M\subset\hat U_0$, thus the results obtained for F_0 are easily translated for f_0.

3.2.2. *The random invariant set.* In this setting, letting U_0 denote the strictly forward f_0-invariant set from the statements in Section 1.1 and $U_k=f_0^k(U_0)$ for a given $k\ge1$, we have that for some $\varepsilon_0>0$ small enough

$$\mathcal{W}=\bigcap_{n\ge0}\mathrm{closure}\,G^n(\hat\Omega\times U_k)\subset\hat\Omega\times U_{k-1}\quad\text{and}\quad\hat\Lambda=\hat\pi_M(\mathcal{W})\subset U_{k-1}.$$

Moreover $\mathcal{W}$ is G-invariant (and $\hat\pi(\mathcal{W})$ is F-invariant), where we set $X=\mathrm{closure}\,B(0,\varepsilon_0)$ for the definition of $\hat\Omega$ (and of Ω).

Indeed we have $\mathrm{closure}\,U_k\subset U_{k-1}$ and $d_k=\mathrm{dist}(\mathrm{closure}\,U_k, M\setminus U_{k-1})>0$. Then we may find $\varepsilon_0>0$ such that $\mathrm{dist}(f_v(x),f_0(x))\le d_k/4$ for all $v\in B(0,\varepsilon_0)$ and $x\in U_k$. Hence

$$f_v(U_k)\subset B\left(\mathrm{closure}(U_k),\frac{d_k}{2}\right)\subset U_{k-1}\quad\text{for all}\quad v\in B(0,\varepsilon_0),$$

where $B(A,\delta)=\cup_{z\in A}B(x,\delta)$ is the δ-neighborhood of a subset A, for $\delta>0$. In addition the G-invariance of $\hat\Lambda$ ensures that

$$\text{if}\quad(\omega,x)\in\mathcal{W}\quad\text{then}\quad f^n_\omega(x)\in\hat\Lambda\quad\text{for all}\quad n\in\mathbb{Z}.\tag{3.2}$$

3.3. **Metric entropy for random perturbations.** We outline some definitions of metric entropy for random dynamical systems which we will use and relate them. Let μ^ε be a stationary measure for the random system given by $(\hat f,\theta_\varepsilon)_{\varepsilon>0}$. Since we are dealing with randomly chosen invertible maps the following results relating F- and G-invariant measures will be needed.

Lemma 3.2. [28, Prop. I.1.2] *Every stationary probability measure μ^ε of the random system given by $(\hat f,\theta_\varepsilon)_{\varepsilon>0}$ admits a unique probability measure $\hat\mu^\varepsilon$ on $\hat\Omega\times M$ which is G-invariant and $\hat\pi_*(\hat\mu^\varepsilon)=\theta^\varepsilon\times\mu^\varepsilon$. Moreover $(\hat\pi_\Omega)_*\hat\mu^\varepsilon=\hat\theta^\varepsilon$, $(\hat\pi_M)_*\hat\mu^\varepsilon=\mu^\varepsilon$ and $G^n_*(\hat\theta^\varepsilon\times\mu^\varepsilon)$ tends to $\hat\mu^\varepsilon$ weakly* when $n\to+\infty$.*

We will need to consider weak* accumulation points of G-invariant measures in the following sections, so we state the following property whose proof follows standard lines.

Lemma 3.3. *Let μ^0 be a weak* limit of μ^{ε_k} for a sequence $\varepsilon_k\to 0^+$ when $k\to\infty$. Let $\hat\mu^0$ be a weak* accumulation point of the sequence $\hat\mu^{\varepsilon_k}$. Then $\hat\mu^0=\delta_{\omega_0}\times\mu^0$, where δ_{ω_0} is the Dirac mass at $\omega_0=(\dots,t_0,t_0,t_0,\dots)\in\hat\Omega$.*

Here is one possibility of the calculation of the metric entropy.

Theorem 3.4. [23, Thm. 1.3] *For any finite measurable partition ξ of M*

$$h_{\mu^\varepsilon}((\hat f,\theta_\varepsilon),\xi)=\inf_{n\ge1}\frac1n\int H_{\mu^\varepsilon}\Big(\bigvee_{i=-n}^{n} f^i_\omega(\xi)\Big)d\theta^\varepsilon(\omega)$$

is finite and is called the entropy of the random dynamical system *with respect to ξ and to μ^ε.*

We define $h_{\mu^\varepsilon}(\hat f,\theta_\varepsilon)=\sup_\xi h_{\mu^\varepsilon}((\hat f,\theta_\varepsilon),\xi)$ as the *metric entropy* of the random dynamical system $(\hat f,\theta_\varepsilon)$, where the supremum is taken over all measurable partitions.

Let $\mathcal{B}\times M$ be the minimal $\sigma-$algebra containing all products of the form $A\times M$ with $A\in\mathcal{B}$. We write $\hat{\mathcal{B}}\times M$ for the analogous σ-algebra with $\hat{\mathcal{B}}$ in the place of $\mathcal{B}$. We denote by $h^{\mathcal{B}\times M}_{\theta^\varepsilon\times\mu^\varepsilon}(F)$ the conditional metric entropy of the transformation F with respect to the σ-algebra $\mathcal{B}\times M$. (See e.g. [28, Chpt. 0] for a definition and properties of conditional entropy.) Again we also denote by $h^{\hat{\mathcal{B}}\times M}_{\hat\mu^\varepsilon}(G)$ the conditional entropy of G with measure $\hat\mu^\varepsilon$ with respect to $\hat{\mathcal{B}}\times M$.

Theorem 3.5. [28, Prop. I.2.1 & Thm. I.2.3] *Let μ^ε be a stationary probability measure for the random system given by $(\hat f,\theta_\varepsilon)$. Then $h_{\mu^\varepsilon}(\hat f,\theta_\varepsilon)=h^{\mathcal{B}\times M}_{\theta^\varepsilon\times\mu^\varepsilon}(F)=h^{\hat{\mathcal{B}}\times M}_{\hat\mu^\varepsilon}(G)$.*

The analogous Kolmogorov-Sinai result about generating partitions is also available in this setting. We let $\mathcal{A}=\mathcal{B}(M)$ be the Borel σ-algebra of M. We say that a finite partition ξ of M is a *random generating partition* for $\mathcal{A}$ if $\vee_{i=-\infty}^{+\infty} f^i_\omega(\xi)=\mathcal{A}$ for $\hat\theta^\varepsilon$ almost all $\omega\in\hat\Omega$.

Theorem 3.6. [23, Cor. 1.2] *Let ξ be a random generating partition for $\mathcal{A}$. Then $h_{\mu^\varepsilon}(\hat f,\theta_\varepsilon)=h^{\hat{\mathcal{B}}\times M}_{\hat\mu^\varepsilon}(G,\hat\Omega\times\xi)$.*

We note that in [23] this result is stated only for one-sided sequences. However we know that the Kolmogorov-Sinai Theorem applied to an invertible transformation like G demands that a partition ζ of $\hat\Omega\times M$ be generating in the sense that $\vee_{i\in\mathbb{Z}}G^i(\zeta)$ equals $\hat{\mathcal{B}}\times M$, $\hat\mu^\varepsilon$ mod 0. Since we are calculating a conditional entropy, it is enough that $(\vee_{i\in\mathbb{Z}}G^i(\zeta))\vee(\hat{\mathcal{B}}\times M)$ be the trivial partition in order that $h_{\hat\mu^\varepsilon}^{\hat{\mathcal{B}}\times M}(G,\zeta)=h_{\hat\mu^\varepsilon}^{\hat{\mathcal{B}}\times M}(G)$. In particular, for $\zeta=\hat\Omega\times\xi$, we have $G^i(\zeta)=\{\{\sigma^k(\omega)\}\times f_\omega^k(\xi):\omega\in\hat\Omega\}$ for $i\in\mathbb{Z}$ so

$$\bigvee_{i=-n}^{n}G^i(\zeta)=\left\{\{\omega\}\times\bigvee_{i=-n}^{n}f_\omega^i(\xi):\omega\in\hat\Omega\right\}.$$

Hence ζ generates $(\hat\Omega\times\mathcal{A},\hat\mu^\varepsilon)$ if, and only if, ξ is generating for $\mathcal{A}$, since $(\hat\pi_M)_*\hat\mu^\varepsilon=\mu^\varepsilon$.

4. Expanding and contracting disks

Here we derive the main local dynamical properties of the maps in the setting of Theorem C. We show that F-disks (respectively E-disks) are expanded (resp. contracted) by the action of f_0, and that the rate of expansion (resp. contraction) is uniform for all isometrically perturbed g in a $C^{1+\alpha}$-neighborhood of f_0, but depend on the size of the disks. We also show that the curvature of such disks remains bounded under iteration. These are consequences of the domination condition (1.2) on the splitting together with non-mixing of expanding/contracting behavior along the E and F directions given by condition (1) in Theorem A.

We note that for g sufficiently C^1-close to f_0 and for a small $\zeta\in(0,\alpha)$ and a slightly bigger $\tilde\lambda_0\in(\lambda_0,1)$ we still have for all $x\in U$

$$\|Dg\mid E(x)\|\cdot\|(Dg\mid F(x))^{-1}\|^{1+\zeta}\le\tilde\lambda_0. \tag{4.1}$$

Moreover since closure$(f_0(U))\subset U$, then for g sufficiently C^0-close to f_0 in $\mathrm{Diff}^{1+\alpha}(M)$ we also have closure$(g(U))\subset U$, see Subsection 3.2.2 for more details. We denote by $\mathcal{U}$ a $C^{1+\alpha}$ neighborhood of f_0 where all of the above is valid for $g\in\mathcal{U}$.

4.1. Domination and bounds on expansion/contraction.

The domination condition (1.2) ensures that the splitting $E(x)\oplus F(x)$ varies continuously in Λ and that there are stable and unstable cone fields $\mathbb{E}^a,\mathbb{F}^b$, already defined in Subsection 1.1 for small $a,b>0$, which are Dg-invariant for every g sufficiently C^1-close to f_0. This is a general property of dominated splittings.

We define a norm on $T_{\mathrm{closure}(U)}M$ more adapted to our purposes using the splitting: for every $x\in U$ and $w\in T_xM$ we write

$$w=(u,v)\in E(x)\oplus F(x)\text{ and set }|w|=\max\{\|u\|,\|v\|\}. \tag{4.2}$$

We observe that at $x\in U$

- if $w\in\mathbb{F}_x^b$, then $|w|=\|v\|\le\|(u,v)\|=\|w\|\le\|u\|+\|v\|\le(1+b)\|v\|=(1+b)|w|$;
- if $w\in\mathbb{E}_x^a$, then $|w|=\|u\|\le\|(u,v)\|=\|w\|\le\|u\|+\|v\|\le(1+a)\|u\|=(1+a)|w|$.

Now for $x\in U$ such that $f_0(x)\in U$, decomposing vectors in the E and F directions $w=(u_0,v_0)\in\mathbb{F}_x^b$ we have that $Df_0(x)\cdot w=(u_1,v_1)\in\mathbb{F}_{f_0(x)}^{\lambda_0 b}$ and also

$$\|v_1\|\ge\|(Df_0\mid F(x))^{-1}\|^{-1}\cdot\|v_0\|\ge\|v_0\|\quad\text{and}\quad\|u_1\|\le\|Df_0\mid E(x)\|\cdot\|u_0\|\le\|u_0\|.$$

Hence

$$\frac{|Df_0(x)\cdot w|}{|w|}=\frac{\|v_1\|}{\|v_0\|}\ge \|(Df_0\mid F(x))^{-1}\|^{-1} \quad\text{and}\quad \frac{\|Df_0(x)\cdot w\|}{\|w\|}\ge\frac{\|(Df_0\mid F(x))^{-1}\|^{-1}}{1+b}.$$

We observe that we can make the last expression as close to $\|(Df_0\mid F(x))^{-1}\|^{-1}$ as we like by choosing b very close to zero. Analogous calculations provide

$$\frac{|Df_0^{-1}(x)\cdot w|}{|w|}\ge \|Df_0\mid E(f_0^{-1}(x))\|^{-1} \quad\text{and}\quad \frac{\|Df_0^{-1}(x)\cdot w\|}{\|w\|}\ge\frac{\|Df_0\mid E(f_0^{-1}(x))\|^{-1}}{1+a}$$

for $x\in U$ such that $f_0^{-1}(x)\in U$ and $w\in\mathbb{E}_x^a$. Since the above calculations give approximately the same bounds if we allow small perturbations in the factors involved, then the same conclusion holds for other constants a',b' perhaps closer to 0 if we replace f_0 by any sufficiently C^1-close map g. We collect this in the following lemma, which depends on the domination assumption on the splitting, on the non-contractiveness along F and non-expansiveness along E, and also on the isometric nature (specifically property (3.1)) of the perturbations we are considering.

Lemma 4.1. *Let f_0 be a diffeomorphism admitting a dominated splitting $E\oplus F$ on a strictly forward invariant subset U and $\Lambda=\cap_{n\ge0}\operatorname{closure} f_0^n(U)$. Let $\hat f:\mathcal{U}\times M\to M$ be a family of isometric perturbations of f_0 as in Subsection 3.2. Then there exist*

- *angle bounds $a,b\in(0,1/2)$ defining stable $(\mathbb{E}_x^a)_{x\in\operatorname{closure}(U)}$ and unstable $(\mathbb{F}_x^b)_{x\in\operatorname{closure}(U)}$ cone fields in $T_{\operatorname{closure}(U)}M$;*
- *a neighborhood $\mathcal{V}$ of 0 in $\mathcal{U}$;*
- *an open neighborhood V of Λ satisfying for every $v\in\mathcal{V}$*

$$\operatorname{closure}(V)\subset U,\quad \operatorname{closure}(f_v(V))\subset V\quad\textit{and}\quad \operatorname{closure}(f_v^{-1}(V))\subset U;$$

such that if $x\in V$ and

- *$w\in\mathbb{E}_x^a$, then $|Df_v^{-1}(x)\cdot w|\ge\|Df_0\mid E(f_v^{-1}(x))\|^{-1}\cdot|w|$;*
- *$w\in\mathbb{F}_x^b$, then $|Df_v(x)\cdot w|\ge\|(Df_0\mid F(x))^{-1}\|^{-1}\cdot|w|$.*

4.2. **Uniform bound on the curvature of E,F-disks.** The "curvature" of the E- and F-disks defined at the Introduction will be determined by the notion of Hölder variation of the tangent bundle as follows. Let us take δ_0 sufficiently small so that the exponential map $\exp_x: B(x,\delta_0)\to T_xM$ is a diffeomorphism onto its image for all $x\in\operatorname{closure}(U_0)$, where the distance in M is induced by the Riemannian norm $\|\cdot\|$. We write $V_x=B(x,\delta_0)$ in what follows. We are going to identify V_x through the local chart $\exp_x^{-1}$ with the neighborhood $U_x=\exp_x(V_x)$ of the origin in T_xM, and we also identify x with the origin in T_xM. In this way we get that $E(x)$ (resp. $F(x)$) is contained in $\mathbb{E}_y^a$ (resp. $\mathbb{F}_y^b$) for all $y\in U_x$, reducing δ_0 if needed, and the intersection of $F(x)$ with $\mathbb{E}_y^a$ (and the intersection of $E(x)$ with $\mathbb{F}_y^b$) is the zero vector.

We write Δ also for the image of the respective embedding for every E- or F-disk. Hence if Δ is a E-disk and $y=\Delta(w)$ for some $w\in\mathbb{B}_E$, then the tangent space of Δ at y is the graph of a linear map $A_x(y):T_x\Delta\to F(x)$ for $w\in\Delta^{-1}(V_x)$ (here $T_x\Delta=D\Delta(x)(\mathbb{R}^{d_E})$). The same happens locally for a F-disk exchanging the roles of the bundles E and F above.

For $\zeta \in (0,1)$ given by (4.1) and some $C > 0$ we say that the *tangent bundle of* Δ *is* (C,ζ)-*Hölder* if

$$\|A_x(y)\| \leq C \operatorname{dist}_\Delta(x,y)^\zeta \quad \text{for all} \quad y \in U_x \cap \Delta \quad \text{and} \quad x \in U, \tag{4.3}$$

where $\operatorname{dist}_\Delta(x,y)$ is *the distance along* Δ defined by the length of the shortest smooth curve from x to y inside Δ calculated with respect to the Riemannian norm $\|\cdot\|$ induced on TM.

For a E- or F-disk $\Delta \subset U$ we define

$$\kappa(\Delta) = \inf\{C > 0 : T\Delta \text{ is } (C,\zeta)\text{-Hölder}\}. \tag{4.4}$$

The proof of the following result can be easily adapted from the arguments in [1, Subsection 2.1] with respect to a single map f_0. The basic ingredients are the cone invariance and dominated decomposition properties for f_0 that we have already extended for nearby diffeomorphisms $g \in \mathcal{U}$ with uniform bounds.

Proposition 4.2. *There is $C_1 > 0$ and a small neighborhood X of t_0 such that for every sequence $\omega \in \Omega = X^{\mathbb{N}}$ with $X = \mathcal{U}$*

1. *given a F-disk $\Delta \subset U$*
   (a) *there exists $n_1 \in \mathbb{N}$ such that $\kappa(f^n_\omega(\Delta)) \leq C_1$ for all $n \geq n_1$;*
   (b) *if $\kappa(\Delta) \leq C_1$ then $\kappa(f^n_\omega(\Delta)) \leq C_1$ for all $n \geq 0$;*
   (c) *in particular, if Δ is as in the previous item, then for every fixed $g = f_\omega$ with $\omega \in \Omega$*

   $$J_n : f^n_\omega(\Delta) \ni x \mapsto \log|\det(Dg \mid T_x(f^n_\omega(\Delta))|$$

   is (L_1,ζ)-Hölder continuous with $L_1 > 0$ depending only on C_1 and f_0, for every $n \geq 1$.
2. *for every $n \geq 1$ and any given E-disk Δ such that $(f^j_\omega)^{-1}(\Delta) \subset U$ for all $j = 0,1,\dots,n$ and $\kappa(\Delta) \leq C_1$, then*
   (a) *$\kappa\big((f^n_\omega)^{-1}(\Delta)\big) \leq C_1$ for all $n \geq 1$;*
   (b) *for every $g = f_\omega$ with $\omega \in \Omega$ we have*

   $$J_n : (f^n_\omega)^{-1}(\Delta) \ni x \mapsto \log|\det(Dg \mid T_x(f^n_\omega)^{-1}(\Delta))|$$

   is (L_1,ζ)-Hölder continuous with $L_1 > 0$ depending only on C_1 and f_0.

Proof. See [1, Proposition 2.2] and [1, Corollary 2.4]. □

The bounds provided by Proposition 4.2 may be interpreted as bounds on the curvature of either E-disks or F-disks, since in the case $f_0 \in \mathcal{U} \subset \operatorname{Diff}^2(M)$ we get C_1 as a bound on the curvature tensor of Δ.

4.3. **Locally invariant submanifolds, expansion and contraction.** The domination assumption on U_0, the compactness and f_0-invariance of Λ together with properties (1)-(2) from Theorem A ensure the existence of families of E-disks ($C^{1+\zeta}$ center-stable manifolds) $W^{cs}_\delta(x)$ tangent to $E(x)$ at x and F-disks ($C^{1+\zeta}$ center-unstable manifolds) $W^{cu}_\delta(x)$ tangent to $F(x)$ at x which are locally invariant, for every $x \in \Lambda$ and a small $\delta > 0$, as follows — see Hirsch-Pugh-Shub [18] for details.

There exist continuous families of embeddings $\phi^{cs}:\Lambda\to\mathrm{Emb}^{1+\zeta}(\mathbb{B}_E,M)$ and $\phi^{cu}:\Lambda\to\mathrm{Emb}^{1+\zeta}(\mathbb{B}_F,M)$, where $\zeta\in(0,\alpha)$ is given by (4.1) and $\mathrm{Emb}^{1+\zeta}(\mathbb{B},M)$ is the space of $C^{1+\zeta}$ embeddings from a ball $\mathbb{B}$ in some Euclidean space to M, such that for all $x\in\Lambda$

1. $\phi^{cs}(x)$ is a E-disk and $T_x\phi^{cs}=E(x)$, $\phi^{cu}(x)$ is a F-disk and $T_x\phi^{cu}=F(x)$;
2. writing $W^{cs}_\delta(x)$ for $B(x,\delta)\cap\phi^{cs}(x)(\mathbb{B}_E)$ and $W^{cu}_\delta(x)$ for $B(x,\delta)\cap\phi^{cu}(x)(\mathbb{B}_F)$ we have the local invariance properties: for every $\eta>0$ there exists $\delta>0$ such that for all $x\in\Lambda$
   (a) $f_0^{-1}(W^{cu}_\delta(x))\subset W^{cu}_\eta(f_0^{-1}(x))$;
   (b) $f_0(W^{cs}_\delta(x))\subset W^{cs}_\eta(f_0(x))$.

4.3.1. *Expansion/contraction of inner radius for E/F-disks.* Up to this point we have used some consequences of the dominated decomposition assumption. Now we use assumptions (1)-(2) of Theorem A to understand the dynamical properties of the locally invariant submanifolds.

Given a smooth curve $\gamma:I\to M$ where $I=[0,1]$, we write $\ell(\gamma)=\int_0^1\|\dot\gamma\|$ and $L(\gamma)=\int_0^1|\dot\gamma|$ for the length of this curve with respect to the norms $\|\cdot\|$ and $|\cdot|$. Let

$$\Gamma_E(\upsilon)=\{\gamma:I\to\mathbb{B}_E:\gamma\text{ is smooth and }0<\|\dot\gamma\|\le\upsilon,\gamma(0)=0,\gamma(1)\in\partial\mathbb{B}_E\}$$

and analogously for $\Gamma_F(\upsilon)$ with $\upsilon>0$. We define the *inner radius* of a F-disk Δ (with respect to $|\cdot|$) to be

$$R(\Delta)=\inf\{L(\Delta\circ\gamma)\mid\gamma\in\Gamma_F(\upsilon),\upsilon>0\},$$

and the *inner diameter* of Δ to be

$$\mathrm{diam}_\Delta(\Delta)=\sup\{L(\Delta\circ\gamma)\mid\gamma\in\Gamma_F(\upsilon),\upsilon>0\},$$

and likewise for E-disks. We note that $R=R(\Delta)\ge C\,\mathrm{dist}(\Delta(0),\Delta(\partial\mathbb{B}_F))>0$ where $C>0$ relates the norms $\|\cdot\|$ and $|\cdot|$ and thus $R(\Delta)$ *is a minimum over* $\Gamma_F(\upsilon)$ *for some* $\upsilon>0$. For fixing $\varepsilon>0$ small we can find $\upsilon>0$ and $\gamma\in\Gamma_F(\upsilon)$ such that $R\le L(\Delta\circ\gamma)<R+\varepsilon$, hence we may reparametrize γ such that $\|(\Delta\circ\gamma)'\|$ is a constant in $(C(R+\varepsilon)^{-1},CR^{-1})$. Since $\Gamma_F(\upsilon)$ is a compact family in the C^1 topology, we have that $R(\Delta)$ is assumed at some smooth curve.

Now we consider the family of E-disks having strictly positive inner radius, bounded curvature and bounded inner diameter:

$$\begin{aligned}\mathcal{D}_E(r,K,\delta,k)\;=\;&\{\Delta\in\mathrm{Emb}^{1+\zeta}(\mathbb{B}_E,M):\quad\Delta\text{ is a }E\text{-disk},\quad\Delta(0)\in\mathrm{closure}\,U_k,\\&R(\Delta)\ge r,\quad\kappa(\Delta)\le K\quad\text{and}\quad\mathrm{diam}_\Delta(\Delta)\le\delta\}\end{aligned}$$

for fixed $r,K,\delta>0$ and $k\in\mathbb{N}$, and analogously for $\mathcal{D}_F(r,K,\delta,k)$.

Lemma 4.3. *Given $r,K,\delta>0$ and $k\in\mathbb{N}$ the families $\mathcal{D}_E(r,K,\delta,k)$ and $\mathcal{D}_F(r,K,\delta,k)$ are compact in the C^1 topology of $\mathrm{Emb}^{1+\zeta}(\mathbb{B}_E,M)$ and $\mathrm{Emb}^{1+\zeta}(\mathbb{B}_F,M)$, respectively.*

Proof. We argue for E-disks only since the arguments for F-disks are the same. We note that $\mathcal{D}_E(r,K,\delta,k)$ defines a subset of bundle maps $D\Delta:T\mathbb{B}_E\to TM,(x,v)\mapsto(\Delta(x),D\Delta(x)v)$. The bound on the "curvature" of the disks bounds the Hölder constant of $D\Delta(x)$ for $x\in\mathbb{B}_E$. This Hölder control together with the bounded diameter condition $\int_0^1|D\Delta(\gamma)\dot\gamma|\le\delta$ ensures that $|D\Delta(x)|$ is equibounded on $\mathcal{D}_E(r,K,\delta,k)$. We also get that $\Delta(\mathbb{B}_E)\subset B_1(\mathrm{closure}\,U_0)$.

Finally, the uniform bound on the Hölder constant of $D\Delta$ ensures that $D\Delta$ is a equicontinuous family for $\Delta \in \mathcal{D}_E(r,K,\delta,k)$. The proof finishes applying Ascoli-Arzela Theorem to $\{D\Delta : \Delta \in \mathcal{D}_E(r,K,\delta,k)\}$ and noting that closure(U_0) is compact, any limit point must share the same inner radius and diameter bounds, and also that the cone families are continuous. $\square$

From now on we fix $K = C_1$ from Proposition 4.2, $k \in \mathbb{N}$ big enough, $\delta > 0$ small enough so that every E- and F-disk in the above families be contained in U_0, and write $\mathcal{D}_E(r)$ and $\mathcal{D}_F(r)$ for the families in Lemma 4.3. Let λ be 1-dimensional Lebesgue measure.

If we take $\Delta \in \mathcal{D}_F(r)$ then $\mathcal{H}(\Delta \cap F_1) < 1$ by assumption (2) from Theorem A. Then $\Delta \cap F_1$ is totally disconnected and *curve free* (see e.g. [15]), i.e. for any regular curve $\gamma : I \to \Delta$ we have $\mathcal{H}^1(\gamma(I) \cap F_1) = 0$, where $\mathcal{H}^1$ is 1-dimensional Hausdorff measure. Thus we must have $\lambda(\gamma^{-1}(F_1)) = 0$. For otherwise $\gamma(I) \cap F_1 = \gamma(\gamma^{-1}(F_1))$ and $\mathcal{H}^1(\gamma(I) \cap F_1) = \ell(\gamma \mid \gamma^{-1}(F_1)) = \int_{\gamma^{-1}(F_1)} \|\dot\gamma\| > 0$, since γ is a regular curve, a contradiction.

Lemma 4.1 and the fact that Δ is a F-disk guarantee that $L(g \circ \gamma) = \int_0^1 |Dg(\gamma(t) \cdot \dot\gamma(t)| > \int_{\gamma^{-1}(\Delta \setminus F_1)} |\dot\gamma| + \int_{\gamma^{-1}(F_1)} |\dot\gamma| = L(\gamma)$ for every $g \in \mathcal{U}$ and smooth regular $\gamma : I \to \Delta$. This is enough to show that $R(g(\Delta)) > R(\Delta)$ for $g \in \mathcal{U}$, since $R(g(\Delta))$ is a minimum. The compactness given by Lemma 4.3 assures that there exists $\sigma_F = \sigma_F(r) > 1$ such that

$$R(g(\Delta)) \geq \sigma_F \cdot R(\Delta) \text{ for all } \Delta \in \mathcal{D}_F(r) \quad \text{and} \quad g \in \text{closure } \mathcal{U}, \tag{4.5}$$

taking a smaller $\mathcal{U}$ around f_0 if needed. Clearly we can also get $\sigma_E = \sigma_E(r) > 1$ such that

$$R(f_0^{-1}(\Delta)) \geq \sigma_E \cdot R(\Delta) \text{ for all } \Delta \in \mathcal{D}_E(r) \quad \text{and} \quad g \in \text{closure } \mathcal{U}, \tag{4.6}$$

using the same arguments replacing g by g^{-1}, F_1 by E_1 and taking $\Delta \in \mathcal{D}_E(r)$ above.

Remark 3. These estimates on the inner radius enable us to improve on the local invariance properties from Subsection 4.3 as follows: for every $x \in \Lambda$ and $\delta > 0$ small enough there exists $k = k(x,\delta) \geq 1$ satisfying

1. $f_0^k(W_\delta^{cs}(x)) \subset W_\delta^{cs}(f_0^k(x))$;
2. $f_0^{-k}(W_\delta^{cu}(x)) \subset W_\delta^{cu}(f_0^{-k}(x))$.

Indeed for any given $\delta > 0$ we may find $\eta > 0$ such that $R\big(f_0(W_\delta^{cs}(x))\big) = \eta$ and so

$$R(W_\delta^{cs}(x)) = R\big(f_0^{-1}(f_0(W_\delta^{cs}(x)))\big) \geq \sigma_E(\eta) \cdot R\big(f_0(W_\delta^{cs}(x))\big),$$

thus $R\big(f_0(W_\delta^{cs}(x))\big) \leq \sigma_E(\eta)^{-1} \cdot R(W_\delta^{cs}(x)) < R(W_\delta^{cs}(x))$. This shows that for any given $\upsilon > 0$ there must be an integer $k \geq 1$ such that $R\big(f_0^k(W_\delta^{cs}(x))\big) < \upsilon$, and analogously for the center-unstable disks.

Remark 4. In particular after Remark 3 we ensure that $W_\delta^{cu}(x) \subset \Lambda$ for every $x \in \Lambda$ and $\delta > 0$ small enough. For there is a constant $C > 0$ (see Subsection 4.1) such that, if $y \in W_\delta^{cu}(x)$, then for every $\eta > 0$ we have $R(W_\delta^{cu}(f_0^{-n}(x))) \leq \eta$ and $\text{dist}(f_0^{-n}(x), f_0^{-n}(y)) \leq C\eta$ for big enough $n \geq 0$. Since Λ is f_0-invariant we obtain $\text{dist}(f_0^{-n}(y),\Lambda) < C\eta$ or $y \in f_0^n(U_0)$ for big $n \geq 0$ if η is small enough. Hence $y \in \Lambda$. Moreover this ensures that $W_\delta^{cu}(x)$ is tangent to F at every point.

4.4. **A Local Product Structure for Λ.** The continuity in the $C^{1+\zeta}$ topology of ϕ^{cs} defined in Subsection 4.3 and the inclusion $W^{cu}_\delta(x) \subset \Lambda$ obtained in Remark 4 guarantee that for an open neighborhood V_0 of 0 in $\mathbb{B}_F$ such that $W^{cu}_\delta(x) = \phi^{cu}(x)(V_0), x \in \Lambda$

$$\psi_x : V_0 \times \mathbb{B}_E \to M, \quad (y,z) \mapsto \phi^{cs}\big(\phi^{cu}(x)(y)\big)(z)$$

is a $C^{1+\zeta}$ map for all $x \in \Lambda$. Moreover

- $D_1\psi_x(y,0) = D\big(\phi^{cu}(x)\big)(y) : \mathbb{R}^{d_F} \to F(y)$ is an isomorphism, since $\psi_x(y,0) = \phi^{cu}(x)(y)$ for all $y \in V_0$ and by definition of ϕ^{cu};
- $D_2\psi_x(y,0) : \mathbb{R}^{d_E} \to E(y)$ is an isomorphism, by definition of ϕ^{cs}.

Hence $|\det D\psi_x(y,0)|$ is bounded away from zero for $y \in \Lambda$, because both the angle between $E(y)$ and $F(y)$ (by domination), and $|\det D_2\psi_x(y,0)|$ are bounded from below away from zero for $y \in \Lambda$ (by compactness). Also we note that ψ_x is just the restriction to $W^{cu}_\delta(x) \times \mathbb{B}_E$ of a map $\psi : \Lambda \times \mathbb{B}_E \to M$. This shows that ψ is a local diffeomorphism from a neighborhood of $\Lambda \times 0$ in $\Lambda \times \mathbb{B}_E$ to a neighborhood of Λ in M. Since $\psi \mid (\Lambda \times 0) \equiv \mathrm{Id} \mid \Lambda$ we may choose a neighborhood V_1 of 0 in $\mathbb{B}_E$ so that $\psi_0 = \psi \mid (\Lambda \times V_1)$ is a diffeomorphism onto its image, which we write W_0.

Remark 5. In addition, following the arguments of Remark 4 we get $\mathrm{dist}(f_0^{-n}(y), f_0^{-n}(x)) \to 0$ and $\mathrm{dist}(f_0^n(z), f_0^n(x)) \to 0$ when $n \to +\infty$ for all $x \in \Lambda, y \in W^{cu}_\delta(x)$ and $z \in W^{cs}_\delta(x)$. In particular this shows that *forward time averages along center-stable disks and backward time averages along center-unstable disks are constant.*

The special neighborhood W_0 of Λ together with Remark 5 shows that the stable set $W^s(\Lambda) = \{z \in M : \lim_{n\to+\infty} \mathrm{dist}(f_0^n(z), \Lambda) = 0\}$ of Λ coincides with the union of the stable sets of each point of Λ: $W^s(\Lambda) = \cup_{x\in\Lambda} W^s(x)$.

Lemma 4.4. *There exist constants $h_0, h_1 > 0$ such that for any f_0-invariant ergodic probability measure μ supported in Λ and every F-disk $\Delta \subset \Lambda$, then $m(B(\mu)) \geq h_0 \cdot m_\Delta(B(\mu))$, where m_Δ is the Lebesgue measure induced by m along the submanifold Δ. In addition, if $\Delta = \Delta \cap B(\mu), m_\Delta -$* mod 0, *then there is a ball of radius $\geq h_1 \cdot R(\Delta)$ contained in $B(\mu)$ Lebesgue modulo zero and intersecting Λ.*

Proof. We have $B(\mu) \supset \cup_{y\in\Delta\cap B(\mu)} \phi^{cs}(y)(\mathbb{B}_E) \supset \cup_{y\in\Delta\cap B(\mu)} W^{cs}_\delta(y)$ by Remark 5 and definition of center-stable manifolds. We note that both the angle between the tangent space to $x \in \Delta$ and $E(x)$, and the inner radius of the center-stable leaves are bounded from below away from zero over Λ by β_0 and r_0 respectively. Hence the Lebesgue measure of $B_0 = \cup_{y\in\Delta\cap B(\mu)} W^{cs}_\delta(y)$ is bounded from below by $h_0 \cdot m_\Delta(B(\mu))$, where $h_0 > 0$ depends only on β_0 and r_0. Thus if $\Delta \subset B(\mu), m_\Delta -$ mod 0, then B_0 contains a ball of radius bounded from below by $h_1 \cdot R(\Delta)$ dependent on β_0, on r_0 and on the curvature of F-disks, all uniform over Λ. Clearly $B_0 \cap \Lambda \neq \emptyset$. □

4.4.1. *Disks as graphs.* We can apply the results from Subsection 4.3 to any sequence of maps in $\mathcal{U}$ using the invariance of $\hat{\Lambda}$ (see Subsection 3.2.2) and the "local product structure" from the previous discussion.

Let $K = C_1$ be as fixed in Subsection 4.3.1. Let $k \in \mathbb{N}$ be big enough, $\delta > 0$ and ε_0 small enough be fixed so that every disk in $\mathcal{D}_E(r), \mathcal{D}_F(r)$ centered at $\hat{\Lambda} \subset \mathrm{closure}(U_k) \subset W_0$ be contained in

W_0, where $\hat{\Lambda}$ was defined in Subsection 3.2.2 for $\mathcal{U}$ small enough (corresponding to $\varepsilon_0 > 0$ very small). We consider the family of E- and F-disks which are local graphs as follows

$$\begin{aligned}\mathcal{G}_E(s) \quad = \quad \{\Delta \in \mathcal{D}_E(r) : r > 0 \text{ and for } x \in \Lambda \text{ such that } \Delta(0) \in W^{cs}_{\delta}(x), \text{ there is } \phi : V \to V_0 \\ \text{with } B(0,s) \subset V \subset V_1 \subset \mathbb{B}_E \quad \text{and} \quad \mathrm{Graph}(\phi) \subset \psi_x^{-1}(\Delta)\},\end{aligned}$$

and likewise for $\mathcal{G}_F(s)$ with $s > 0$, exchanging the roles of E, F, V_1 and V_0. We note that *since cones are complementary* (i.e. any d_E-subspace of $\mathbb{E}_x$ together with any d_F-subspace of $\mathbb{F}_x$ span T_xM, $x \in U_0$) then *every E- or F-disk is a local graph for some $s > 0$*. Let also $\delta_0 = \sup\{\operatorname{diam}\psi_x(V_0 \times V_1) : x \in \Lambda\} > 0$, which is finite by compactness.

Lemma 4.5. *Let $0 < r \ll \min\{\delta, \delta_0\}$, $\Delta \in \mathcal{D}_E(r)$ and $\hat{\Delta} \in \mathcal{D}_F(r)$ be given. If $\Delta(0), \hat{\Delta}(0) \in \hat{\Lambda}$ and $\Delta \in \mathcal{G}_E(s), \hat{\Delta} \in \mathcal{G}_F(s)$ for some $s > 0$, then $(f_\omega)^{-1}(\Delta) \in \mathcal{G}_E(\sigma_E(s) \cdot s), f_\omega(\hat{\Delta}) \in \mathcal{G}_F(\sigma_F(s) \cdot s)$ for all $\omega \in \Omega$.*

Proof. It is obvious that $(f_\omega)^{-1}(\Delta)$ is a E-disk and that $f_\omega(\hat{\Delta})$ is a F-disk after (4.5) and (4.6). Moreover $(f_0)^{-1}(\Delta) \in \mathcal{G}_E(\sigma_E(s) \cdot s), f_0(\hat{\Delta}) \in \mathcal{G}_F(\sigma_F(s) \cdot s)$ by the local expression of f_0 on the "local product coordinates" provided by ψ. The expansion on the inner radius of the domains of the graphs is a consequence of the fact that a ball in $W^{cs}_{\delta}(x)$ is a E-disk and any ball in $W^{cu}_{\delta}(x)$ is a F-disk, $x \in \Lambda$. The conclusion for f_ω and any $\omega \in \Omega$ holds since f_ω is taken C^1-close to f_0. □

5. Equilibrium states and physical measures

Here we characterize the equilibrium states μ for f_0 with respect to the potential $-\varphi(x)$ where $\varphi(x) = \log|\det Df_0 \mid F(x)|$, as in (1.3). We start by observing that, in the setting of Theorem C, given any f_0-invariant measure μ the sum $\chi^+(x)$ of the positive Lyapunov exponents of μ-a.e. point x equals

$$\chi^+(x) = \lim_{n\to+\infty} \frac{1}{n} \log|\det Df_0^n \mid F(x)| \tag{5.1}$$

by the Multiplicative Ergodic Theorem [30]. Indeed by condition (1) of Theorem A every Lyapunov exponent along the E direction is non-positive and every Lyapunov exponent along the F direction is non-negative.

Theorem 5.1. *Let $f_0 : M \to M$ be a $C^{1+\alpha}$ diffeomorphism admitting a strictly forward invariant open set U with a dominated splitting satisfying items (1)-(2) of Theorem A. Then every equilibrium state μ with respect to φ as in* (1.3)*, supported in U, is a convex linear combination of*

1. *finitely many ergodic equilibrium states with positive entropy and which are physical probability measures, with*
2. *ergodic equilibrium states having zero entropy whose support has constant unstable Jacobian equal to one, i.e., measures whose Lyapunov exponents are nonpositive.*

Moreover if $f_0 \mid \Lambda$ is transitive, then there is at most one ergodic equilibrium state with positive entropy whose basin covers U_0 Lebesgue almost everywhere.

Proof. We first show that we may assume μ ergodic.

Lemma 5.2. *Almost every ergodic component of an equilibrium state for* φ *is itself an equilibrium state for the same function.*

Proof. Let μ be an f-invariant measure satisfying (1.3). On the one hand, the Ergodic Decomposition Theorem (see e.g Mañé [29]) ensures that

$$\int \varphi \, d\mu = \int\int \varphi(x) \, d\mu_z(x) \, d\mu(z) \quad \text{and} \quad h_\mu(f) = \int h_{\mu_z}(f) \, d\mu(z). \tag{5.2}$$

On the other hand, Ruelle's inequality guarantees for a μ-generic z that (recall (5.1))

$$h_{\mu_z}(f) \leq \int \varphi \, d\mu_z. \tag{5.3}$$

By (5.2) and (5.3), and because μ is an equilibrium state (1.3), we conclude that we have equality in (5.3) for μ-almost every z. ☐

Now let μ be an ergodic equilibrium state for φ supported in U. Thus $\operatorname{supp}(\mu) \subset \Lambda$. Now we have two possibilities.

$h_\mu(f_0) > 0$**:** According to the characterization of measures satisfying the Entropy Formula [26], μ must be an *SRB measure*, i.e., μ *admits a disintegration into conditional measures along unstable manifolds which are absolutely continuous with respect to the volume measure naturally induced on these submanifolds of* M.

$h_\mu(f_0) = 0$**:** Since $\varphi \geq 0$ on Λ by condition (1) of Theorem A, the equality (1.3) shows that $\varphi = 0$ for μ-a.e. x. Hence $\chi^+ = 0$, μ-a.e. and μ has no expansion.

The Ergodic Decomposition then ensures that every equilibrium state will be a convex linear combination of the two types of measures described above. The latter possibility corresponds to item (2) in the statement of Theorem 5.1. The former case with positive entropy needs more detail.

Remark 6. Up until now we have shown that ergodic equilibrium states for $-\varphi$ are either measures with no expansion or *SRB* measures. This is exactly the same conclusion that Cowieson-Young get [14] in a more general setting.

The Entropy Formula (1.3) and the assumption $h_\mu(f_0) > 0$ ensure that there are positive Lyapunov exponents for μ. Hence there exist *Pesin's smooth* ($C^{1+\alpha}$) *unstable manifolds* $W^u(x)$ through μ-a.e. point x. Moreover, as already mentioned, the disintegration μ_x^u of μ along these unstable manifolds $W^u(x)$ is absolutely continuous with respect to the Lebesgue measure m_x^u induced by the volume form of M restricted to $W^u(x)$, for μ-a.e. x.

We claim that $\mu(F_1) = 0$. For otherwise there would be some component with $\mu_x^u(F_1) > 0$ which implies $m_x^u(F_1) > 0$, and so $\mathcal{H}(F_1) \geq \mathcal{H}(F_1 \cap W^u(x)) \geq \dim(W^u(x)) \geq 1$, a contradiction, since $W^u(x)$ is a F-disk.

This means that $\int \log \|(Df_0 \mid F)^{-1}\| \, d\mu < 0$. Hence *the Lyapunov exponents of* μ *along every direction in* F *are strictly positive*. Thus $\dim W^u(x) = \dim F = d_F$ for μ-generic x, and μ is a *Gibbs state along the center-unstable direction* F. These manifolds are asymptotically backward exponentially contracted by Df_0, see [31], hence $W^u(x) \subset \Lambda$, since Λ is a topological attractor (see the arguments in Remark 4).

Fixing a μ-generic x, since $W^u(x)$ is an F-disk Lemma 4.5 ensures that we may assume $R(W^u(x)) \geq \rho$ for some $\rho > 0$ dependent only on $\mathrm{dist}(\Lambda, M \setminus U_0)$. Vásquez shows [38] that the support of any Gibbs cu-state such as μ contains *entire unstable leaves* $W^u(y)$ for μ-a.e. y, so we may also assume that $W^u(x) \cap B(\mu)$ has full Lebesgue measure in $W^u(x)$. Using Lemma 4.4 we get that $B(\mu)$ contains Lebesgue modulo zero a ball of radius uniformly bounded from below by $h_1 \cdot \rho > 0$.

We have shown that each ergodic equilibrium state μ having positive entropy must be a physical measure. Since the ergodic basins of distinct physical measures are disjoint and have volume uniformly bounded from below away from zero, there are at most finitely many such measures. This concludes the proof of items (1) and (2) of Theorem 5.1.

Let $f_0 \mid \Lambda$ have a dense orbit and let us suppose that there two distinct equilibrium states μ_1, μ_2 with positive entropy. Then by the previous discussion there are two balls B_1, B_2 contained in the ergodic basins $B(\mu_1)$ and $B(\mu_2)$ Lebesgue modulo zero, respectively, and intersecting Λ. Since $f_0 \mid \Lambda$ is a transitive diffeomorphism and a regular map, there exists $k \geq 1$ such that $m(f^k(B_1) \cap B_2) > 0$. Thus $\mu_1 = \mu_2$ and *transitiveness of* Λ *is enough to ensure there is only one equilibrium state with positive entropy*.

Let μ be the unique equilibrium state with positive entropy and let us take an open set $B = \psi_x(W^u(x) \times V_1)$ contained in $B(\mu)$ Lebesgue modulo zero, where $x \in \Lambda \cap B(\mu)$ — see Subsection 4.4 for the definition of ψ_x on a F-disk such as $W^u(x) \subset \Lambda$. As already explained, we may assume that $W^u(x) \cap B(\mu)$ has full Lebesgue measure along $W^u(x)$. We set $\delta = R(W^u(x)) > 0$.

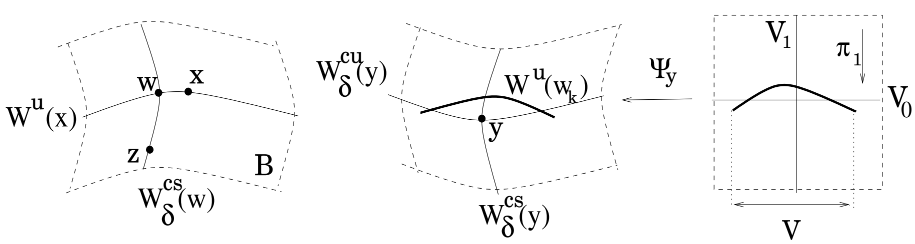

FIGURE 1. The "local product structure" neighborhoods B and $W^{cu}_\delta(y) \times V_1$.

We can take $z \in B \cap \Lambda$ whose forward f_0-orbit is dense in Λ. We may take z as close to x as we like and there are points $(w,u) \in W^u(x) \times V_1$ such that $z = \psi_x(w,u)$. Then $z \in W^{cs}_\delta(w)$ and $W^u(w) = W^u(x)$, see Figure 1. Then by Remark 5 the forward f_0-orbit of w is also dense in Λ. We note that $f_0^n(W^u(w)) \supset W^u(f_0^n(w))$ and also $R(f_0^n(W^u(w)) \geq \delta$ for $n \geq 1$ by Lemma 4.5.

Arguing by contradiction, we suppose that $m(W_0 \setminus B(\mu)) > 0$, where W_0 was defined in Subsection 4.4. Then there exists $y \in \Lambda$ such that both $Z = (\psi_y \mid (W^{cu}_\delta(y) \times V_1))^{-1}(W_0 \setminus B(\mu))$ and $\pi_1(Z)$ have positive Lebesgue measure, where $\pi_1 : W^{cu}_\delta(y) \times V_1 \to W^{cu}_\delta(y)$ is the projection onto the first factor. Moreover we can choose y so that it is a Lebesgue density point of $\pi_1(Z)$.

Let n_k be a sequence such that $w_k = f_0^{n_k}(w) \to y$ when $k \to +\infty$. Since $W^u(w_k)$ is a F-disk, $\psi_y^{-1}(W^u(w_k)) \subset W^{cu}_\delta(y) \times V_1$ is the graph of a map from an open neighborhood V of y in $W^{cu}_\delta(y)$ to V_1, for big enough k. Then $V \cap \pi_1(Z)$ has positive Lebesgue measure and so, after Remark 5, the Lebesgue measure of $W^u(w_k) \cap (W_0 \setminus B(\mu))$ is also positive. But this implies that $W^u(x) \cap (W_0 \setminus B(\mu))$ also has positive Lebesgue measure, contradicting the choice of x.

This shows that $B(\mu)$ has full Lebesgue measure in W_0 and hence in U_0, as in the statement of Theorem 5.1. □

6. ZERO-NOISE LIMITS ARE EQUILIBRIUM MEASURES

Here we prove Theorem C. Let $f_0 : M \to M$, $\hat{f} : X \to C^{1+\alpha}(M,M), t \mapsto f_t$, $f_{t_0} \equiv f$ for fixed $t_0 \in X$, and $(\theta_\varepsilon)_{\varepsilon>0}$ be a family of probability measures on X such that $(\hat{f}, (\theta_\varepsilon)_{\varepsilon>0})$ is a non-degenerate isometric random perturbation of f_0, as in Subsection 3.2.

The main idea is to find a *fixed random generating partition* for the system $(\hat{f}, \theta_\varepsilon)$ for *every small* $\varepsilon > 0$ and use the absolute continuity of the stationary measure μ^ε, together with the conditions on the splitting to obtain a semicontinuity property for entropy on zero-noise limits.

Theorem 6.1. *Let us assume that there exists a finite partition ξ of M (Lebesgue modulo zero) which is generating for random orbits, for every small enough $\varepsilon > 0$.*

Let μ^0 be a weak accumulation point of $(\mu^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \to 0$. If $\mu^{\varepsilon_j} \to \mu^0$ for some $\varepsilon_j \to 0$ when $j \to \infty$, then* $\limsup_{j\to\infty} h_{\mu^{\varepsilon_j}}(\hat{f}, \theta_{\varepsilon_j}) \le h_{\mu^0}(f_0, \xi)$.

Remark 7. Recently Cowieson-Young obtained [14] a similar semicontinuity property without assuming the existence of a uniform generating partition but using either a local entropy condition or that the maps $\hat{f}$ involved be of class C^∞.

The absolute continuity of μ^ε, the conditions on the splitting for f_0 and the isometric perturbations permit us to use a random version of the Entropy Formula

Theorem 6.2. *If an ergodic stationary measure μ^ε for a isometric random perturbation $(\hat{f}, \theta_\varepsilon)$ of f_0, in the setting of Theorem C, is absolutely continuous for any given $\varepsilon > 0$, then*

$$h_{\mu^\varepsilon}(\hat{f}, \theta_\varepsilon) = \int \log|\det Df_0 \mid F(x)|\, d\mu^\varepsilon(x).$$

Moreover if condition (3) of the statement of Theorem A also holds, then in addition to the above there exists $c > 0$ such that $h_{\mu^\varepsilon}(\hat{f}, \theta_\varepsilon) \ge c$ for all $\varepsilon > 0$ small enough.

Putting Theorems 6.1 and 6.2 together shows that $h_{\mu^0}(f_0) \ge \int \log|\det Df_0(x)|\, d\mu^0(x)$, since $\theta_\varepsilon \to \delta_{t_0}$ in the weak* topology when $\varepsilon \to 0$, by the assumptions on the support of θ_ε in Subsection 3. Since the reverse inequality holds in general (that is Ruelle's inequality [33]) we get the first statement of Theorem C. To conclude the proof we just have to recall Theorem 5.1 from Section 5, which provides the second part of the statement of Theorem C.

6.1. **Random Entropy Formula.** Now we explain how to obtain Theorem 6.2.

Let $\varepsilon > 0$ be fixed in what follows. The Lyapunov exponents $\lim_{n\to\infty} n^{-1} \log \|Df^n_\omega(x) \cdot v\|$ exist for $\theta^\varepsilon \times \mu^\varepsilon$-almost every (ω, x) and every $v \in T_xM \setminus \{0\}$, by Oseledets result [30] adapted to this setting, see e.g. [6]. At every given point (ω, x) there are at most $d = \dim(M)$ possible distinct values for the above limit, the *Lyapunov exponents* at (ω, x). We write $\chi^+(\omega, x)$ for the *sum of the positive Lyapunov exponents* at x. Lyapunov exponents are F-invariant by definition, so $\chi^+(\omega, x) = \chi^+(x)$ for $\theta^\varepsilon \times \mu^\varepsilon$-almost every (ω, x) (a consequence of θ^ε being a product measure and σ-ergodic, see [28, Corollary I.1.1]) and χ^+ is constant almost everywhere if μ^ε is ergodic.

The Entropy Formula for random maps is the content of the following result.

Theorem 6.3. *Let a random perturbation* $(\hat{f},\theta_\varepsilon)$ *of a diffeomorphisms* f_0 *be given and assume that the stationary measure* μ^ε *is such that* $\log|\det Df_t(x)| \in L^1(\Omega\times M, \theta_\varepsilon\times\mu^\varepsilon)$. *If* μ^ε *is absolutely continuous with respect to Lebesgue measure on M, then*

$$h_{\mu^\varepsilon}(\hat{f},\theta_\varepsilon) = \int \chi^+ \, d\mu^\varepsilon. \tag{6.1}$$

Proof. See [27] and [28, Chpt. IV]. □

Now since the random perturbations are isometric we have

$$\frac{1}{n}\log\|(Df^n_\omega \mid F(x))^{-1}\| \le \frac{1}{n}\sum_{j=0}^{n-1}\log\|(Df_{\omega_{j+1}} \mid F(f^j_\omega(x)))^{-1}\| \to \int \log\|(Df_0 \mid F(x))^{-1}\| \, d\mu^\varepsilon(x)$$

when $n\to\infty$ for $\theta^\varepsilon\times\mu^\varepsilon$-a.e. (ω,x) by the Ergodic Theorem, if μ^ε is ergodic. By the assumptions on f_0 and $E\oplus F$ this ensures that the Lyapunov exponents in the directions of F are non-negative. In the same way we get for $\theta^\varepsilon\times\mu^\varepsilon$-a.e. (ω,x)

$$\limsup_{n\to\infty}\frac{1}{n}\log\|Df^n_\omega \mid E(x)\| \le \int \log\|Df_0 \mid E(x)\| \, d\mu^\varepsilon(x) \le 0,$$

and so every Lyapunov exponent in the directions of E is non-positive. Since E and F together span $T_{U_0}M$, according to the Multiplicative Ergodic Theorem (Oseledets [30]) the sum χ^+ of the positive Lyapunov exponents (with multiplicities) equals the following limit $\theta^\varepsilon\times\mu^\varepsilon$-almost everywhere

$$\chi^+(x) = \lim_{n\to\infty}\frac{1}{n}\log|\det Df^n_\omega \mid F(x)| = \int \log|\det Df_0 \mid F(x)| \, d\mu^\varepsilon(x) \ge 0.$$

The identity above follows from the Ergodic Theorem, if μ^ε is ergodic, since the value of the limit is F-invariant, thus constant.

Finally since μ^ε is absolutely continuous for random isometric perturbations, the formula in Theorem 6.3 gives the first part of the statement of Theorem 6.2.

Remark 8. The argument above together with conditions (1)-(3) from Theorem A ensure that there exists $c_0>0$ satisfying $h_{\mu^\varepsilon}(\hat{f},\theta_\varepsilon)\ge c_0$ for every small enough $\varepsilon>0$. In fact, condition (3) ensures that $|\det Df_0 \mid F(x)|>1$ for all $x\in\Lambda$. Hence there is $c_0>0$ such that $\log|\det Df_0 \mid F(x)|\ge c_0$ for every x in a neighborhood U_k as in Subsection 3.2.2, for some fixed big $k\ge 1$.

Finally, as shown in Subsection 3.2.2, for any given $k\ge 1$ there is $\varepsilon_0>0$ for which the random invariant set $\hat\Lambda=\hat\Lambda_\varepsilon$ is contained in U_k for all $\varepsilon\in(0,\varepsilon_0)$. Then $\operatorname{supp}\mu^\varepsilon\subset\hat\Lambda_\varepsilon$ will be in the setting of Remark 8 above if condition (3) of Theorem A holds in addition to conditions (1) and (2). This completes the proof of Theorem 6.2.

6.2. **Uniform random generating partition.** Here we construct the uniform random generating partition assumed in Theorem 6.1. In what follows we fix a weak* accumulation point μ^0 of μ^ε when $\varepsilon\to 0$: there exist $\varepsilon_j\to 0$ when $j\to\infty$ such that $\mu=\lim_{j\to+\infty}\mu^{\varepsilon_j}$.

Let us take a finite cover $\{B(x_i,\rho_0/4), i=1,\dots,\ell\}$ of closure(U_k) by $\rho_0/4$-balls, where $\rho_0\in(0,\min\{\delta_0,\operatorname{dist}(M\setminus U_0,U_k)\})$ for some $k\ge 1$ such that $\operatorname{supp}(\mu^{\varepsilon_j})\subset U_k\subset W_0$ for all $j\ge 1$. Recall from Subsection 4.4 that W_0 is a "local product structure" neighborhood of Λ and note that we can choose k as big as we like, if we let j be big enough.

Now since μ^0 is a probability measure, we may assume that $\mu^0(\partial\xi) = 0$, for otherwise we can replace each ball by $B(x_i, \gamma\rho_0/4)$, for some $\gamma \in (1, 3/2)$ and for all $i = 1, \dots, k$. We set ξ to be the finest partition of M obtained through all possible intersections of these balls: $\xi = B(x_1, \gamma\rho_0/4) \vee \cdots \vee B(x_\ell, \gamma\rho_0/4)$. In what follows we let $\rho = \gamma\rho_0/4 \in (0, 3\rho_0/4)$.

Remark 9. The partition ξ is such that all atoms of $\vee_{i=-n}^{n}(f_\omega^i)^{-1}\xi$ have boundary (which is a union of pieces of boundaries of open balls) with zero Lebesgue measure, for all $n \geq 1$ and every $\omega \in \hat{\Omega}$. Moreover since μ^0 is f_0-invariant and $\mu^0(\partial\xi) = 0$, then $\mu^0(\vee_{i=0}^{n-1} f^{-j}\xi) = 0$ for all $n \geq 1$.

Lemma 6.4. *For each $\omega \in \hat{\Omega}$ we have* $\operatorname{diam}\left(\vee_{i=-n}^{n} f_\omega^i(\xi)\right) \to 0$ *when $n \to +\infty$.*

Proof. Let $n \geq 1$, $\omega \in \hat{\Omega}$, $x_0 \in \operatorname{closure}(U_k)$ and $y_0 \in (\vee_{i=-n}^{n} f_\omega^i(\xi))(x_0)$ with $y_0 \neq x_0$. We write $x_k = f_\omega^k(x_0)$ and likewise for y_k, $|k| \leq n$.

Let us suppose that there exists a E-disk $\Delta \in \mathcal{G}_E(s)$ centered at $x_0 = \Delta(0)$ such that $y_0 \in \Delta$ and $0 < s_0 < \operatorname{dist}(y_0, x_0) < \rho$. Then by Lemma 4.5 we see that since $\operatorname{dist}(y_i, x_i) \leq \rho$ for $i = -1, \dots, -n$ we have $\operatorname{dist}(x_0, y_0) \leq \sigma_E(s_0)^{-n} \cdot \rho$. If this holds for arbitrarily big values of $n \geq 1$, then the statement of the lemma is proved.

We now show that the assumption above is always true. Since $x_0, y_0 \in \hat{\Lambda}$ we know that $x_n, y_n \in \hat{\Lambda} \subset W_0$. By definition we have $\operatorname{dist}(x_0, y_0), \operatorname{dist}(y_n, x_n) < \rho < \delta_0$. Hence there exist $w_0, w_n \in \Lambda$ such that both $x_0, y_0 \in \psi_{w_0}(V_0 \times V_1)$ and $x_n \in W_\delta^{cs}(w_n)$, and also $\Delta_n = W_\delta^{cs}(w)$ is a E-disk and a graph through x_n, i.e. $\Delta_n \in \mathcal{G}_E(\delta)$. Then applying Lemma 4.5 several times we get $\Delta_0 = (f_\omega^n)^{-1}(\Delta_n) \cap \psi_{w_0}(V_0 \times V_1) \in \mathcal{G}_E(\delta)$.

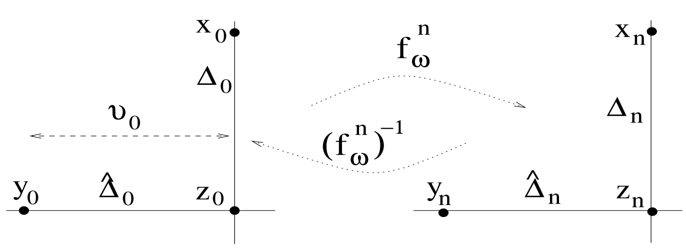

FIGURE 2. The construction of $\Delta_0, \hat{\Delta}_0, \Delta_n$ and $\hat{\Delta}_n$.

Now there exists $(y_0^u, y_0^s) \in V_0 \times V_1$ such that $y_0 = \psi_{w_0}(y_0^u, y_0^s)$. Let $\hat{\Delta}_0 = \psi_{w_0}(V_0 \times \{y_0^s\}) \in \mathcal{G}_F(\delta)$ be a F-disk through y_0. Then we get a F-disk $\hat{\Delta}_n = f_\omega^n(\hat{\Delta}_0) \in \mathcal{G}_F(\delta)$ which is a graph through y_n. Since both $\Delta_n, \hat{\Delta}_n \subset \psi_{w_n}(V_0 \times V_1)$ are graphs we know there exists a unique intersection z_n and thus there is $z_0 = (f_\omega^n)^{-1}(z_n) = \Delta_0 \cap \hat{\Delta}_0$, see Figure 2.

Let $\upsilon_0 = \operatorname{dist}_{\hat{\Delta}_0}(y_0, z_0)$. We note that if $\upsilon_0 = 0$, then $y_0 \in \Delta_0$ and we can proceed as in the beginning. Hence we assume $\upsilon_0 > 0$ and get $\rho > \operatorname{dist}(y_n, x_n) \geq \operatorname{dist}(y_n, z_n) - \operatorname{dist}(z_n, x_n) \geq \operatorname{dist}(y_n, z_n) - \rho$ i.e. $\operatorname{dist}(y_n, z_n) < 2\rho$. But by construction and applying Lemma 4.5

$$\upsilon_0 \leq \operatorname{dist}_{\hat{\Delta}_0}(y_0, z_0) \leq \sigma_F(\upsilon_0)^{-n} \cdot \operatorname{dist}_{\hat{\Delta}_n}(y_n, z_n) \leq \sigma_F(\upsilon_0)^{-n} \cdot K_0 \cdot \operatorname{dist}(y_n, z_n) \leq 2\rho K_0 \cdot \sigma_F(\upsilon_0)^{-n},$$

where K_0 is a constant relating distances in M with distances along F-disks and depending of the curvature $\kappa(\hat{\Delta}_n)$, which is globally bounded, see Subsection 4.2.

This shows that υ_0 can be made as small as we please. Then for $\upsilon_0 > 0$ small enough there exists $\Delta \in \mathcal{G}_E(\delta)$ with $x_0, y_0 \in \Delta$, e.g. take the image by ψ_{w_0} of any d_E-plane intersected with $V_0 \times V_1 \subset \mathbb{R}^d$ through $(y_0^u, y_0^s), (0, y_0^s)$ and $\psi_{w_0}^{-1}(x_0)$. Thus we can always reduce to the first case above. The proof is complete. □

Lemma 6.4 implies that ξ is a random generating partition Lebesgue modulo zero, hence μ^ε modulo zero for all $\varepsilon > 0$, as in the statement of the Random Kolmogorov-Sinai Theorem 3.6. We conclude that $h_{\mu^{\varepsilon_k}}((\hat{f}, \theta_{\varepsilon_k}), \xi) = h_{\mu^{\varepsilon_k}}(\hat{f}, \theta_{\varepsilon_k})$ for all $k \geq 1$.

6.3. **Semicontinuity of entropy on zero-noise.** Now we start the proof of Theorem 6.1. We need to construct a sequence of partitions of $\hat{\Omega} \times M$ according to the following result — see Subsection 3.3 for the definitions of $\hat{\Omega}$ and entropy. For a partition $\mathcal{P}$ of a given space Y and $y \in Y$ we denote by $\mathcal{P}(y)$ the element (atom) of $\mathcal{P}$ containing y. We set $\omega_0 = (\dots, t_0, t_0, t_0, \dots) \in \hat{\Omega}$ in what follows.

Lemma 6.5. *There exists a sequence of measurable partitions $(\hat{\mathcal{B}}_\ell)_{\ell \geq 1}$ of $\hat{\Omega}$ such that*

1. $\omega_0 \in \operatorname{int} \hat{\mathcal{B}}_\ell(\omega_0)$ *for all* $\ell \geq 1$*;*
2. $\hat{\mathcal{B}}_\ell \nearrow \hat{\mathcal{B}}$, $\hat{\theta}^{\varepsilon_j}$ mod 0 *for all* $j \geq 1$ *when* $n \to \infty$*;*
3. $\lim_{n\to\infty} H_\rho(\xi \mid \hat{\mathcal{B}}_n) = H_\rho(\xi \mid \hat{\mathcal{B}})$ *for every measurable finite partition* ξ *and any G-invariant probability measure* ρ.

Proof. For the first two items we let $\mathcal{C}_n$ be a finite $\hat{\theta}_{\varepsilon_j}$ mod 0 partition of X such that $t_0 \in \operatorname{int} \mathcal{C}_n(t_0)$ with $\operatorname{diam} \mathcal{C}_n \to 0$ when $n \to \infty$, for any fixed $j \geq 1$. Example: take a cover $(B(t, 1/n))_{t \in X}$ of X by $1/n$-balls and take a subcover $U_1, \dots, U_l$ of $X \setminus B(t_0, 2/n)$ together with $U_0 = B(t_0, 3/n)$; then let $\mathcal{C}_n = U_0 \vee \dots \vee U_l$.

We observe that we may assume that the boundary of these balls has null $\hat{\theta}_{\varepsilon_j}$-measure for all $j \geq 1$, since $(\hat{\theta}_{\varepsilon_j})_{j \geq 1}$ is a denumerable family of non-atomic probability measures on X (see Remark 2). Now we set

$$\hat{\mathcal{B}}_n = X^{\mathbb{N}} \times \mathcal{C}_n \times \overset{2n+1}{\cdots} \times \mathcal{C}_n \times X^{\mathbb{N}} \quad \text{for all } n \geq 1,$$

meaning that $\hat{\mathcal{B}}_n$ is the family of all sets containing points $\omega \in \hat{\Omega}$ such that $\omega_i \in X$ for all $|i| > n$ and $\omega_i \in C_i$ for some $C_i \in \mathcal{C}_n, |i| \leq n$. Then since $\operatorname{diam} \mathcal{C}_n \leq 2/n$ for all $n \geq 1$ we have $\operatorname{diam} \hat{\mathcal{B}}_n \leq 2/n$ and so tends to zero when $n \to \infty$. Then $\hat{\mathcal{B}}_n$ is an increasing sequence of partitions and $\vee_{n \geq 1} \hat{\mathcal{B}}_n$ generates the σ-algebra $\hat{\mathcal{B}}$, $\hat{\theta}^{\varepsilon_j}$ mod 0 (see e.g. [9, Lemma 3, Chpt. 2]) for all $j \geq 1$. This proves items (1) and (2). Item (3) is Theorem 12.1 of Billingsley [9]. □

Now we deduce the right inequalities from known properties of the conditional entropy. First we get from Theorem 3.5 and [28, Thm. 0.5.3]

$$\begin{aligned} h_{\mu^{\varepsilon_j}}(\hat{f}, \theta_{\varepsilon_j}) &= h_{\hat{\mu}^{\varepsilon_j}}^{\hat{\mathcal{B}} \times M}(G) = h_{\hat{\mu}^{\varepsilon_j}}^{\hat{\mathcal{B}} \times M}(G, \hat{\Omega} \times \xi) \\ &= \inf \frac{1}{n} H_{\hat{\mu}^{\varepsilon_j}} \left(\bigvee_{i=0}^{n-1} (G^i)^{-1} (\hat{\Omega} \times \xi) \mid \hat{\mathcal{B}} \times M \right), \end{aligned}$$

where $\hat{\Omega}\times\xi=\{\hat{\Omega}\times A : A\in\xi\}$. Then for any given fixed $N\geq 1$ and for every $\ell\geq 1$

$$\begin{aligned} h_{\mu^{\varepsilon_j}}(\hat f,\theta_{\varepsilon_j}) &\leq \frac{1}{N}H_{\hat\mu^{\varepsilon_j}}\left(\bigvee_{i=0}^{N-1}(G^i)^{-1}(\hat\Omega\times\xi)\mid \hat{\mathcal{B}}\times M\right)\\ &\leq \frac{1}{N}H_{\hat\mu^{\varepsilon_j}}\left(\bigvee_{i=0}^{N-1}(G^i)^{-1}(\hat\Omega\times\xi)\mid \hat{\mathcal{B}}_\ell\times M\right)\end{aligned}$$

because $\hat{\mathcal{B}}_\ell\times M\subset\hat{\mathcal{B}}\times M$. Now we fix N and ℓ, let $j\to\infty$ and note that since

$$\mu^0(\partial\xi)=0=\delta_{\omega_0}(\partial\hat{\mathcal{B}}_m)\quad\text{then}\quad(\delta_{\omega_0}\times\mu^0)(\partial(B_i\times\xi_l))=0$$

for all $B_i\in\hat{\mathcal{B}}_m$ and $\xi_l\in\xi$, where δ_{ω_0} is the point mass concentrated at ω_0. By weak* convergence $\theta^{\varepsilon_l}\to\delta_{\omega_0}$ and $\mu^{\varepsilon_l}\to\mu^0$ we get $\hat\mu^{\varepsilon_l}\to\hat\mu^0=\delta_{\omega_0}\times\mu^0$ when $l\to\infty$, see Lemma 3.3. Hence

$$\limsup_{j\to\infty}h_{\mu^{\varepsilon_j}}(\hat f,\theta_{\varepsilon_k})\leq\frac{1}{N}H_{\delta_{\omega_0}\times\mu^0}\left(\bigvee_{i=0}^{N-1}(G^i)^{-1}(\hat\Omega\times\xi)\mid\hat{\mathcal{B}}_\ell\times M\right)=\frac{1}{N}H_{\mu^0}\Big(\bigvee_{i=0}^{N-1}f_0^{-i}\xi\Big).\tag{6.2}$$

Here it is easy to see that the middle conditional entropy of (6.2) (involving only finite partitions) equals $N^{-1}\sum_i\mu^0(P_i)\log\mu^0(P_i)$, where $P_i=\xi_0\cap f^{-1}\xi_1\cap\cdots\cap f^{-(N-1)}\xi_{N-1}$ ranges over every sequence of possible atoms $\xi_0,\dots,\xi_{N-1}\in\xi$.

Finally, since N was an arbitrary integer, Theorem 6.1 follows from the inequality in (6.2). As already explained, this completes the proof of Theorem C.

7. Stochastic stability

Here we prove Theorem A. Let $f_0: M\to M$ be as in the statement of Theorem C and let μ be an equilibrium state for $-\varphi$, as in (1.3) — recall the definition of φ in Section 5.

Condition (3) in the statement of Theorem A ensures that the only possibility for the ergodic decomposition of μ is the one given by item (1) in statement of Theorem C. In fact, after Remark 8, every weak* accumulation point μ of μ^ε when $\varepsilon\to 0$ will be not only an equilibrium state for $-\varphi$, as shown in Section 6, but will also have strictly positive entropy $h_\mu(f_0)\geq c>0$, after the statement of Theorem 6.2. Hence combining the statements in Section 6 with Theorem C we see that *every weak* accumulation point μ of μ^ε when $\varepsilon\to 0$ is a finite convex linear combination of the ergodic equilibrium states for $-\varphi$, which are physical measures*.

This shows that the family of equilibrium states for $-\varphi$ in the setting of Theorem A is stochastically stable.

In addition, if f_0 is transitive, then there is only one equilibrium state μ for $-\varphi$ which is ergodic and whose basin covers U_0 Lebesgue almost everywhere, by the last part of the statement of Theorem 5.1. Then *every weak* accumulation point of μ^ε when $\varepsilon\to 0$ necessarily equals μ*. This finishes the proof of Theorem A.

References

[1] J. F. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.*, 140(2):351–398, 2000.

[2] J.F. Alves. *Statistical Analysis of non-uniformly expanding dynamical systems*. 2003. XXIV Braz. Math. Colloq. IMPA, Rio de Janeiro.
[3] Jose F. Alves and Vitor Araujo. Random perturbations of nonuniformly expanding maps. *Astérisque*, 286:25–62, 2003.
[4] V. Araújo. Attractors and time averages for random maps. *Annales de l'Inst. Henri Poincaré - Analyse Non-linéaire*, 17:307–369, 2000.
[5] V. Araújo and A. Tahzibi. Stochastic stability at the boundary of expanding maps. *Preprint arXiv.org/math.DS/0404368*, 2004.
[6] Ludwig Arnold. *Random dynamical systems*. Springer-Verlag, Berlin, 1998.
[7] V. Baladi and M. Viana. Strong stochastic stability and rate of mixing for unimodal maps. *Ann. Sci. École Norm. Sup. (4)*, 29(4):483–517, 1996.
[8] M. Benedicks and M. Viana. Random perturbations and statistical properties of certain Hénon-like maps. To appear.
[9] P. Billingsley. *Ergodic theory and information*. J. Wiley & Sons, New York, 1965.
[10] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.*, 29:181–202, 1975.
[11] M. Brin and Yu. Kifer. Dynamics of markov chains and stable manifolds for random diffeomorphisms. *Ergodic Theory and Dynamical Systems*, 7:351–374, 1987.
[12] M. Carvalho. Sinai-Ruelle-Bowen measures for *n*-dimensional derived from Anosov diffeomorphisms. *Ergod. Th. & Dynam. Sys.*, 13:21–44, 1993.
[13] E Catsigeras and E. Enrich. SRB measures of certain almost hyperbolic diffeomorphisms with a tangency. *Discrete and Continuous Dynamical Systems*, 7(1):177–202, 2001.
[14] W. Cowieson and L.-S. Young. SRB measures as zero-noise limits. *Preprint*, 2004.
[15] Kenneth Falconer. *Fractal Geometry: mathematical foundations and applications*. John Wiley & Sons, New York, USA, 1990.
[16] A. Fathi, M. Herman, and J.-C. Yoccoz. A proof of Pesin's stable manifold theorem. In *Geometric dynamics (Rio de Janeiro 1981)*, volume 1007 of *Lect. Notes in Math.*, pages 177–215. Springer Verlag, 1983.
[17] V. Guillemin and A. Pollack. *Differential Topology*. Prentice Hall, New Jersey, 1974.
[18] M. Hirsch, C. Pugh, and M. Shub. *Invariant manifolds*, volume 583 of *Lect. Notes in Math.* Springer Verlag, New York, 1977.
[19] Morris Hirsch. *Differential Topology*. Springer-Verlag, New-York, 1976.
[20] H. Hu. Conditions for the existence of SBR measures for "almost Anosov" diffeomorphisms. *Trans. Amer. Math. Soc.*, 352(5):2331–2367, 2000.
[21] H. Hu. Statistical properties of some almost hyperbolic systems. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, volume 69 of *Proc. Sympos. Pure Math.*, pages 367–384. Amer. Math. Soc., Providence, RI, 2001.
[22] H. Hu and L.-S. Young. Nonexistence of SBR measures for some diffeomorphisms that are "almost Anosov". *Ergodic Theory Dynam. Systems*, 15(1):67–76, 1995.
[23] Yu. Kifer. *Ergodic theory of random perturbations*. Birkhäuser, 1986.
[24] Yu. Kifer. General random perturbations of hyperbolic and expanding transformations. *J. Analyse Math.*, 47:11–150, 1986.
[25] Yu. Kifer. *Random perturbations of dynamical systems*. Birkhäuser, 1988.
[26] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms i. characterization of measures satisfying pesin's entropy formula. *Ann. of Math*, 122:509–539, 1985.
[27] F Ledrappier and L.-S. Young. Entropy formula for random transformations. *Probab. Theory and Related Fields*, 80(2):217–240, 1988.
[28] P.-D. Liu and M. Qian. *Smooth ergodic theory of random dynamical systems*, volume 1606 of *Lect. Notes in Math.* Springer Verlag, 1995.
[29] R. Mañé. *Ergodic theory and differentiable dynamics*. Springer Verlag, New York, 1987.

[30] V. I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.
[31] Ya. Pesin. Families of invariant manifolds corresponding to non-zero characteristic exponents. *Math. USSR. Izv.*, 10:1261–1302, 1976.
[32] D. Ruelle. A measure associated with Axiom A attractors. *Amer. J. Math.*, 98:619–654, 1976.
[33] Ruelle, David. An inequality for the entropy of differentiable maps. *Bol. Soc. Bras. Mat.*, 9:83–87, 1978.
[34] Ya. Sinai. Gibbs measures in ergodic theory. *Russian Math. Surveys*, 27:21–69, 1972.
[35] M. Thaler. Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel Journal of Mathematics*, 37(4):303–314, 1980.
[36] M. Thaler. Transformations on $[0,1]$ with infinite invariant measures. *Israel Journal of Mathematics*, 46(1-2):67–96, 1983.
[37] M. Viana. *Stochastic dynamics of deterministic systems*. Lecture Notes XXI Braz. Math. Colloq. IMPA, Rio de Janeiro, 1997.
[38] Carlos Vásquez. *Statistical stability for diffeomorphisms with dominated splitting*. PhD thesis, IMPA, Rio de Janeiro, 2003.
[39] R. F. Williams. The "DA" maps of smale and structural stability. *Proc. Symp. Pure Math., Amer. Math. Soc.*, 14:329–334, 1970.
[40] L.-S. Young. Stochastic stability of hyperbolic attractors. *Ergod. Th. & Dynam. Sys.*, 6:311–319, 1986.

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