THE GEOMETRY OF QUADRATIC 2×2 SYSTEMS OF CONSERVATION LAWS

J. BASTO-GONÇALVES AND H. REIS

Abstract. We consider one typical 2-parameter family of quadratic systems of 2×2 conservation laws, and study the geometry of the behaviour of the possible solutions of the Riemann problem near an umbilic point, following the geometric approach presented by Isacson, Marchesin, Palmeira, Plohr, in *A global formalism for nonlinear waves in conservation laws*, Commun. Math. Phys. (1992). The corresponding phase portraits for the rarefaction curves, shock curves and composite curves are discussed.

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1. Introduction

Contents

Conservation laws are (systems of) partial differential equations of the form:

$$\mathcal{U}_t + F(\mathcal{U})_x = 0, \qquad \mathcal{U} \in \mathbb{R}^n, \quad x, t \in \mathbb{R}$$

that arise in mathematical models of physical phenomena when dissipative effects are neglected. They usually admit discontinuous solutions, in particular shock waves and rarefaction waves, and therefore a 'solution' has to be understood in a weak sense [17].

We will deal only with the Riemann problem, where the initial condition is given at $t = 0$ by two constant states, $\mathcal{U}_l$ for $x < 0$ and $\mathcal{U}_r$ for $x > 0$, and we look for solutions involving constant states, shock waves, rarefaction waves and composite waves.

The objective here is to study the geometry of the behaviour of the possible solutions of the Riemann problem near an umbilic point, following a geometric approach presented by Isacson, Marchesin, Palmeira, Plohr [10]. They proposed an unified setting for solving that problem: it should be possible to recover all information about the solutions, their stability and bifurcations, from the geometry of a single manifold, the fundamental wave manifold, from submanifolds connected to the singularities of its projections and some curve foliations on that manifold.

The fundamental wave manifold $\mathcal{W}$ involves two different types of points: the set of points $(\mathcal{U}_-, \mathcal{U}_+, s)$ with $\mathcal{U}_- \neq \mathcal{U}_+$, satisfying the Rankine-Hugoniot condition and therefore corresponding to shocks, and those in its boundary (with $\mathcal{U}_- = \mathcal{U}_+$), the characteristic manifold $\mathcal{C}$, corresponding to rarefaction waves. In section 2 we recollect the main constructions and results of [10].

In section 3 we prove that, for generic conservation laws, the blow-up described in [10] leads to a smooth manifold, and we develop the connections of its relevant submanifolds with the critical sets of projections.

The study of the rarefaction curves leads to a implicit differential equation on the state space, that proves to be a binary equation; the general theory of implicit equations is presented by Davydov in [6], and Bruce and Tari have studied the case of binary equations and their bifurcations [3, 4]. When lifted to the characteristic manifold, that implicit equation induces a 1-dimensional foliation, the rarefaction foliation, on the characteristic manifold $\mathcal{C}$.

We prove in section 4 that the implicit equation associated to the rarefaction foliation for a generic 2×2 conservation law is generic among binary equations.

In general, 2×2 systems of conservation laws admit no umbilic points but on the other hand those points are unavoidable when considering the generic behaviour of families, even with just one parameter, of such systems, or $n \times n$ systems of conservation laws with $n \geq 3$. Umbilic points are also present in some equations that are relevant to applications, and thus the structure of the possible solutions near or at an umbilic point is important both from a theoretical and an applied point of view (see [5, 9, 10, 11, 15, 16] and references therein).

We consider generic systems, that therefore are not strictly hyperbolic: in general there exists an elliptic region, where the two eigenvalues are complex, and on the boundary of that region the eigenvalues are real but not distinct. Schaeffer and Shearer [15] classified all systems under an assumption of symmetry that forces hyperbolicity; mixed systems, near but away from an umbilic point were considered by Holden [9].

The remaining part of this work is the study of one particular, but typical in a sense to be explained later, two parameter family of quadratic systems of 2×2 conservation laws. For the family:

$$\begin{bmatrix} v_t \\ u_t \end{bmatrix} + \begin{bmatrix} (1+\gamma)v & \lambda + u \\ u & v \end{bmatrix} \begin{bmatrix} v_x \\ u_x \end{bmatrix} = 0$$

we get a precise geometric description of the fundamental wave manifold, the characteristic manifold and the sonic loci (where the shocks are no longer Lax admissible).

This family is very closely related to the normal form in [13], and in particular these geometric descriptions could be extended to it; of course all results are still valid also for systems that transform to one element of our family under a suitable change of coordinates.

For that particular family, all types of generic binary equations with umbilic points are realized: star ($\gamma < 0$), monstar ($0 < \gamma < 1$) and lemon ($\gamma > 1$); we show in section 5 that for the star type case, while the implicit equations for different values of γ are all equivalent, there is an important change of behaviour for $\gamma = -1$ concerning the orientation of the rarefaction foliation (subsection 5.2).

In section 6 we consider the shock curves; for our family, we show that the primary bifurcations correspond to the critical levels of the hyperbolic umbilic D_{4+} singularity, for $\gamma > 1$, and the critical levels of the elliptic umbilic D_{4-} singularity, for $\gamma < 1$, and that all secondary bifurcations (away from the critical points of the rarefaction foliation) are saddle points.

In section 7 we consider the composite curves and the sonic loci; again there is a bifurcation at $\gamma = -1$: if $\gamma < -1$ the left sonic locus is a double cover of the characteristic manifold, and therefore the phase portrait for the composite foliation is locally the same as that for rarefaction foliation, but for $\gamma > -1$ there are new critical points, that turn out to be always centres. There appears also a bifurcation at $\gamma = -1$ for the projection of the right sonic locus.

We also complete a characterization of the critical points of the composite foliation of [10, proposition 8.3], where some situations, that appear in particular for quadratic systems, were not considered; moreover we show that the case $\gamma = -2$ studied in [9, 16] is special even inside the family considered here (remark 13).

The objective of this paper is to present the geometry associated to quadratic, essentially 2×2, models, with umbilic points, and it is not feasible to cover the construction of the solutions for the Riemann problem from that geometric structure as well; part of this construction has already been done, as in [16] for the case $\lambda = 0$, $\gamma = -2$ in our family, corresponding to the standard star umbilic point. We also do not discuss the structural stability of the foliations involved; this aspect is discussed in [7, 8, 12, 13].

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All figures, except fig. 1, were drawn on the basis of MAPLE plots; MAPLE was also used for checking all computations.

2. Basic results and definitions

In this section we describe the geometric formalism of [10]; the main point is that all information on the solutions can be recovered from the geometric study of some manifolds and line foliations on them.

A $n \times n$-*Conservation Law* is a quasi-linear system

$$\mathcal{U}_t + F(\mathcal{U})_x = 0 \tag{1}$$

where $\mathcal{U} \in \mathbb{R}^n$, $F \in C^2(\mathcal{A}, \mathbb{R}^n)$ and $\mathcal{A} \subseteq \mathbb{R}^n$ is an open set.

The initial-value problem for (1) with initial data:

$$\mathcal{U}(x,0) = \begin{cases} \mathcal{U}_l, & \text{if } x < 0, \\ \mathcal{U}_r, & \text{if } x > 0 \end{cases}$$

is called a *Riemann problem.*

Remark 1. We assume $F \in C^2(\mathcal{A}, \mathbb{R}^n)$, but for some particular results higher regularity is needed, as for instance $F \in C^3(\mathcal{A}, \mathbb{R}^n)$ in propositions 18 and 19, even if it is not explicitly stated.

We allow discontinuous solutions, but of a very restricted type: they are formed from a finite number of classical solutions, each defined on an open set of the plane (x,t); the discontinuities appear only on the curves separating those regions.

Suppose the solution $\mathcal{U}$ is C^1 except along a line Γ given by $x = x(t)$. Then $\mathcal{U}$ must verify the *Rankine-Hugoniot condition*:

$$s(\mathcal{U}_+ - \mathcal{U}_-) = F(\mathcal{U}_+) - F(\mathcal{U}_-), \quad s = \frac{dx}{dt} \tag{2}$$

where $\mathcal{U}_+$ and $\mathcal{U}_-$ are, respectively, the values of $\mathcal{U}$ on the right and on the left of Γ.

Definition 1. A *shock wave* with constant speed s is given by :

$$\mathcal{U}(x,t) = \begin{cases} \mathcal{U}_- & \text{if } x < st, \\ \mathcal{U}_+ & \text{if } x > st \end{cases} \tag{3}$$

such that $(\mathcal{U}_-, \mathcal{U}_+, s)$ satisfies the Rankine-Hugoniot condition.

Definition 2. A *rarefaction wave* is a solution

$$\mathcal{U}(x,t) = \tilde{\mathcal{U}}\left(\frac{x}{t}\right)$$

such that for $t > 0$:

$$[DF(\tilde{\mathcal{U}}) - sI]\tilde{\mathcal{U}}' = 0, \quad s = \frac{x}{t}$$

This condition can be obtained from the Rankine-Hugoniot condition by taking the limit when $|\mathcal{U}_+ - \mathcal{U}_-| \to 0$

Triples satisfying (2) form a subset of $\mathcal{P} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Let $R \in \mathbb{R}$ and $\Omega \in S^{n-1}$ such that $R\Omega = \mathcal{U}_+ - \mathcal{U}_-$ and consider the space $\hat{\mathcal{P}} := \mathbb{R}^n \times \mathbb{R} \times S^{n-1} \times \mathbb{R}$; then $(\mathcal{U}, R, \Omega, s)$ and $(\mathcal{U}, -R, -\Omega, s)$ originate exactly the same triple $(\mathcal{U}_- = \mathcal{U}, \mathcal{U}_+ = \mathcal{U} + R\Omega, s)$. Let σ be the function that identifies those two points.

Definition 3. The *blow-up* of $\mathcal{P} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ along the set $\{\mathcal{U}_- = \mathcal{U}_+\}$ is the image $\mathcal{P}^*$ of $\hat{\mathcal{P}} := \mathbb{R}^n \times \mathbb{R} \times S^{n-1} \times \mathbb{R}$ by the identifying map σ.

In the blow-up coordinates (in $\hat{\mathcal{P}}$), the Rankine-Hugoniot condition (2) becomes:

$$sR\Omega - (F(\mathcal{U} + R\Omega) - F(\mathcal{U})) = 0$$

By Hadamard lemma [1], since $F(\mathcal{U} + R\Omega) - F(\mathcal{U}) = 0$ for $R = 0$, it can be written as $R\Phi(\mathcal{U}, R, \Omega)$ with $\Phi(\mathcal{U}, 0, \Omega) = DF(\mathcal{U})\Omega$. Then $R = 0$ is a trivial solution of Rankine-Hugoniot condition, and the non trivial solutions come from solving:

$$\mathcal{F}(\mathcal{U}, R, \Omega, s) = s\Omega - \Phi(\mathcal{U}, R, \Omega) = 0, \quad \Phi(\mathcal{U}, 0, \Omega) = DF(\mathcal{U})\Omega \tag{4}$$

Definition 4. The *fundamental wave manifold* $\mathcal{W}$ is the image, by the identifying map σ, of the zero-set of $\mathcal{F}$.

Geometrically, it represents the closure, in $\mathcal{P}^*$, of points $(\mathcal{U}_-, \mathcal{U}_+, s)$ with $\mathcal{U}_+ \neq \mathcal{U}_-$ satisfying the Rankine-Hugoniot condition (2).

Remark 2. $\hat{\mathcal{P}}$ is a two-fold covering manifold for $\mathcal{P}^*$; it is more convenient to work on and it will be used in the sequel without further comment.

A shock is represented by a point with $R \neq 0$, and the extra points, rarefaction points, satisfy:

$$R = 0, \quad [-sI + DF(\mathcal{U})]\Omega = 0 \tag{5}$$

To a rarefaction wave there corresponds a curve of rarefaction points, $s \mapsto (\tilde{\mathcal{U}}(s), \tilde{\mathcal{U}}(s), s)$ in the original coordinates and $s \mapsto (\tilde{\mathcal{U}}(s), 0, \Omega(s), s)$ after blow-up, where $\Omega(s)$ is the eigenvector of $DF(\tilde{\mathcal{U}}(s))$ corresponding to s.

Definition 5. The *characteristic manifold* is the set $\mathcal{C}$ of rarefaction points, a n-dimensional manifold when non singular.

A point $(\mathcal{U}, 0, \Omega, s)$ belongs to $\mathcal{C}$ if and only if s is an eigenvalue of $DF(\mathcal{U})$ and Ω the respective eigenvector.

Let $\rho : \mathcal{C} \to \mathcal{U}$ denote the projection $\rho(\mathcal{U}, 0, \Omega, s) = \mathcal{U}$. The hyperbolic region is the set of U such that $DF(\mathcal{U})$ has only distinct real eigenvalues, the image of $\mathcal{C}$ under ρ.

Definition 6. The *coincidence locus* is the set $\mathcal{E}$ of points $(\mathcal{U}, 0, \Omega, s) \in \mathcal{C}$ such that s is a multiple eigenvalue of $DF(\mathcal{U})$.

$\mathcal{E}$ has codimension 1 in $\mathcal{C}$, it is a $(n-1)$-dimensional submanifold of $\mathcal{P}^*$ when non singular.

When $n = 2$, the elliptic region is the set of U such that $DF(\mathcal{U})$ has no real eigenvalues. The image of the coincidence locus $\mathcal{E}$ under ρ is the parabolic region, where there is one double real eigenvalue, separating the hyperbolic and elliptic regions.

Definition 7. An *umbilic point* is a point $(\mathcal{U}, 0, \Omega, s) \in \mathcal{E}$ such that s has geometric multiplicity greater than 1.

Umbilic points form a manifold (when non singular) with codimension 3, so, in general, 2×2-conservation laws do not admit umbilic points, as there are only two variables, but 1-parameter families do, at isolated points in the 3-space of the variables and the parameter.

Fix $\mathcal{U}_0 \in \mathcal{U}$ and suppose that $DF(\mathcal{U}_0)$ has n distinct real eigenvalues. Then there exists a neighbourhood of $\mathcal{U}_0$ in which are defined functions λ_i and r_i, for each $i = 1, \ldots, n$ such that $r_i(\mathcal{U})$ is the eigenvector of $DF(\mathcal{U})$ associated to the eigenvalue $\lambda_i(\mathcal{U})$:

$$[-\lambda_i(\tilde{\mathcal{U}}) + DF(\tilde{\mathcal{U}})]r_i(\tilde{\mathcal{U}}) = 0$$

We assume that $\lambda_1(\mathcal{U}) \leq \lambda_2(\mathcal{U}) \leq \cdots \leq \lambda_n(\mathcal{U})$.

Definition 8. A *rarefaction curve* is an integral curve of the rarefaction line field, induced on $\mathcal{C}$ by the differential equation:

$$\dot{\mathcal{U}} = r_i(\mathcal{U}) \tag{6}$$

Definition 9. The *locus of rarefaction singularities*, denoted by $\mathcal{B}_0$, is the set of singularities of the rarefaction line field.

Rarefaction curves constitute a 1-dimensional foliation of $\mathcal{C} \backslash \mathcal{B}_0$. Consider a rarefaction wave with fixed left state $\mathcal{U}_-$; then the corresponding curve of rarefaction points describes the integral curve through $\mathcal{U}_-$ associated to $r_i(\mathcal{U})$, for some i, in the direction of increasing $\lambda_i(\mathcal{U})$.

Definition 10. The *inflection locus* is the set $\mathcal{I}$ of points of $\mathcal{C}$ where s has a critical point along a rarefaction curve.

$\mathcal{I}$ is an $(n-1)$-dimensional submanifold (when non singular) of $\mathcal{C}$.

Proposition 1. *The inflection locus is the set of points of $\mathcal{C}$ where the matrix $(\mathcal{F}_U\Omega, \mathcal{F}_\Omega)$ is singular. Singularities of the rarefaction line field are inflection points: $\mathcal{B}_0 = \mathcal{I} \cap \mathcal{E}$.*

Fixed the left state $\mathcal{U}_-$, the points $(\mathcal{U}_+, s)$ such that $(\mathcal{U}_-, \mathcal{U}_+, s)$ satisfy (2) are the shock waves with the same left state $\mathcal{U}_-$; they form the fixed-$\mathcal{U}_-$ shock curve. The definition of fixed-$\mathcal{U}_+$ shock curve is analogous.

Definition 11. The *left primary bifurcation locus* is the set of points where the fixed-$\mathcal{U}_+$ shock curve is singular, the map:

$$(\mathcal{U}_-, s) \mapsto F(\mathcal{U}_0) - F(\mathcal{U}_-) - s(\mathcal{U}_0 - \mathcal{U}_-), \quad \mathcal{U}_0 = \mathcal{U}_+$$

having a critical point.

The *right primary bifurcation locus* is the set of points where the fixed-$\mathcal{U}_-$ shock curve is singular.

After the blow-up, the points $(\mathcal{U}, R, \Omega, s) \in \mathcal{W}$ such that $\mathcal{U} = \mathcal{U}_0$ form a 1-dimensional manifold in a neighbourhood of each non singular point, the fixed-$\mathcal{U}_-$ shock curve. In a similar way we can define a fixed-$\mathcal{U}_+$ shock curve.

Those curves can also be regarded as integral curves of the line fields $\mathrm{d}\mathcal{U}_- = 0$ and $\mathrm{d}\mathcal{U}_+ = 0$, respectively, on the fundamental wave manifold.

Definition 12. The *right secondary bifurcation locus* $\mathcal{B}_R$ is the set of points in $\mathcal{W}\backslash\mathcal{C}$ where the fixed-$\mathcal{U}_-$ shock line field $\mathrm{d}\mathcal{U}_- = 0$ is singular. The *left secondary bifurcation locus* $\mathcal{B}_L$ is the set of points in $\mathcal{W}\backslash\mathcal{C}$ where the fixed-$\mathcal{U}_+$ shock line field $\mathrm{d}\mathcal{U}_+ = 0$ is singular.

Proposition 2. *The locus of rarefaction singularities is the set of points in the characteristic manifold that are in the closure of the secondary bifurcation loci:*

$$\overline{\mathcal{B}_R} \cap \mathcal{C} = \mathcal{B}_0 = \overline{\mathcal{B}_L} \cap \mathcal{C}$$

The fixed-$\mathcal{U}_-$ and fixed-$\mathcal{U}_+$ shock curves constitute a 1-dimensional foliation of $\mathcal{W}\backslash(\mathcal{B}_R \cup \mathcal{B}_0)$ and $\mathcal{W}\backslash(\mathcal{B}_L \cup \mathcal{B}_0)$, respectively.

All solutions considered verify the Lax criterium: a shock wave is admissible if it is a triple $(\mathcal{U}_-,\mathcal{U}_+, s)$ satisfying (2) and such that

$$\lambda_k(\mathcal{U}_+) < s < \lambda_k(\mathcal{U}_-) \quad \text{and} \quad \lambda_{k-1}(\mathcal{U}_-) < s < \lambda_{k+1}(\mathcal{U}_+) \tag{7}$$

The inequalities (7) are called Lax inequalities. If $n = 1$ this means that $DF(\mathcal{U}_-) > s > DF(\mathcal{U}_+)$.

Definition 13. The *right sonic locus* $\mathcal{S}_R$ is the set of points in $\mathcal{W}\backslash\mathcal{C}$ that are sonic on the right, i.e., such that $s = \lambda_i(\mathcal{U}_+)$ for some i. The *left sonic locus* $\mathcal{S}_L$, is the set of points in $\mathcal{W}\backslash\mathcal{C}$ that are sonic on the left, i.e. such that $s = \lambda_i(\mathcal{U}_-)$.

The sonic locus $\mathcal{S}_R$ is defined by $\det(-sI+DF(\mathcal{U}_+)) = 0$, with $R \neq 0$, in the fundamental wave manifold, and $\mathcal{S}_L$ and $\mathcal{S}_R$ are n-dimensional manifolds when non singular. Also:

$$\overline{\mathcal{S}_L} \cap \mathcal{C} = \mathcal{I} = \overline{\mathcal{S}_R} \cap \mathcal{C}, \quad \mathcal{B}_R \subseteq \mathcal{S}_R, \quad \mathcal{B}_L \subseteq \mathcal{S}_L$$

Proposition 3 ([10])**.** *The graph of s along the fixed-$\mathcal{U}_-$ shock curve, with $\mathcal{U}_- = \mathcal{U}_0$, has a critical point at $\mathcal{U}_+ \neq \mathcal{U}_0$ iff s is an eigenvalue of $DF(\mathcal{U}_+)$, i.e., iff the point $(\mathcal{U}_-,\mathcal{U}_+, s)$ is sonic, belonging to $\mathcal{S}_R$.*

Definition 14. The *exceptional inflection locus* $\mathcal{H}_0$ is the subset of $\mathcal{I}$ where $\mathcal{I}$ is not smooth or the critical point of s along a rarefaction curve is degenerate.

Proposition 4 ([10])**.** *Assume the inflection locus $\mathcal{I}$ is smooth at $p \in \mathcal{I}\backslash\mathcal{E}$; then the graph of s along a rarefaction curve through p has vanishing second derivative, i.e. p belongs to the exceptional inflection locus $\mathcal{H}_0$, if and only if the rarefaction curve is tangent to the inflection locus $\mathcal{I}$ at p.*

Definition 15. The *right hysteresis locus* $\mathcal{H}_R$ is the subset of $\mathcal{S}_R \backslash \mathcal{B}_R$ where $\mathcal{S}_R$ is not smooth or the graph of s along the fixed-U_- shock curve has vanishing second derivative. The *left hysteresis locus* $\mathcal{H}_L$ is the subset of $\mathcal{S}_L \backslash \mathcal{B}_L$ where $\mathcal{S}_L$ is not smooth or the graph of s along the fixed-U_+ shock curve has vanishing second derivative.

Proposition 5 ([10]). *The exceptional inflection locus is the set of points in the characteristic manifold that are in the closure of the hysteresis loci:*

$$\overline{\mathcal{H}_R} \cap \mathcal{C} = \mathcal{H}_0 = \overline{\mathcal{H}_L} \cap \mathcal{C}$$

Proposition 6 ([10]). *Assume the right sonic locus is smooth at $p = (U_-, U_+, s) \in \mathcal{S}_R \backslash \mathcal{B}_R$, with $U_+ \neq U_-$; then the graph of s along the fixed-U_- shock curve through p has vanishing second derivative at p, i.e., p belongs to the exceptional inflection locus $\mathcal{H}_R$, if and only if the shock curve is tangent to $\mathcal{S}_R$ at p.*

A similar result holds for $\mathcal{H}_L$, involving now fixed-U_+ shock curves and $\mathcal{S}_L$.

Definition 16. The *double sonic locus* $\mathcal{D}$ consists of the points in $\mathcal{W}$ that are simultaneously left and right sonic and for which $R \neq 0$.

The double sonic locus $\mathcal{D}$ is a $(n-1)$-dimensional submanifold, when non singular; also $\mathcal{S}_L \cap \mathcal{S}_R = \mathcal{D}$ and $\overline{\mathcal{S}_L} \cap \overline{\mathcal{S}_R} = \mathcal{I} \cup \mathcal{D}$.

Proposition 7 ([10]). *Assume $p = (\mathcal{U}_-, \mathcal{U}_+, s) \in \mathcal{S}_L \setminus \mathcal{B}_R$; the fixed-$\mathcal{U}_-$ shock curve through p is tangent to $\mathcal{S}_L$ at p if and only if $p \in \mathcal{D}$.*

Consider the function

$$\mathcal{U}(x,t) = \begin{cases} \mathcal{U}_0 \text{ if } x < s_0 t, \\ \tilde{U}\left(\frac{x}{t}\right) \text{ if } s_0 t \leq x < st, \\ \mathcal{U}_+ \text{ if } x > st \end{cases} \tag{8}$$

where $\tilde{\mathcal{U}}$ is a rarefaction wave, $\tilde{\mathcal{U}}(s_0) = \mathcal{U}_0$, and $(\mathcal{U}_- = \tilde{\mathcal{U}}(s), \mathcal{U}_+, s)$ verifies the Rankine-Hugoniot condition (2). This represents a rarefaction wave situated on the left of a shock wave, without intermediate state, and it is a solution of the Riemann problem if the shock wave is a left sonic shock wave. Those solutions are called *left composite waves.* Similarly, a rarefaction wave situated on the right of a right sonic shock wave is called a *right composite wave.*

$\tilde{\mathcal{U}}$ is the solution of the differential equation (6) with initial data $\mathcal{U}_0$, so $\tilde{\mathcal{U}}$ is uniquely determined by $\mathcal{U}_0$. $\mathcal{U}_+$ depends on s: it is such that $(\mathcal{U}_- = \tilde{\mathcal{U}}(s), \mathcal{U}_+, s)$ is a left sonic shock and belongs to the shock curve through $\mathcal{U}_- = \tilde{\mathcal{U}}(s)$.

A left composite curve is a family, depending on s, of left composite waves with fixed $\mathcal{U}_0$; it may be seen as a curve in the sonic locus:

$$s \mapsto (\mathcal{U}_- = \tilde{\mathcal{U}}(s), \mathcal{U}_+, s)$$

This curve will be an integral curve of the line field induced on $\mathcal{S}_L$ by the differential equation:

$$\dot{\mathcal{U}} = r_i(\mathcal{U})$$

The projection $(\mathcal{U}_- = \tilde{\mathcal{U}}(s), \mathcal{U}_+, s) \mapsto \mathcal{U}_-$ of the composite curve, and the projection $(\tilde{U}(s), \tilde{U}(s), s) \mapsto \mathcal{U}_- = \tilde{\mathcal{U}}(s)$ of the corresponding rarefaction curve coincide, they are integral curves of direction fields induced by the same differential equation on $\mathcal{U}_-$.

3. Singularities of the fundamental wave manifold

3.1. **Generic smoothness.**

Theorem 1. *For a generic $n \times n$ system of conservation laws, the blow-up of the fundamental wave manifold eliminates all singularities, i.e., it is a smooth manifold.*

Proof. The fundamental wave manifold is defined by n equations in $\mathbb{R}^{2n+1}$, with coordinates $(\mathcal{U}, \mathcal{V}, s)$:

$$s(\mathcal{V} - \mathcal{U}) - (F(\mathcal{V}) - F(\mathcal{U})) = 0 \tag{9}$$

Taking differentials,

$$(\mathcal{V} - \mathcal{U})\mathrm{d}s + (DF(\mathcal{U}) - s)\mathrm{d}\mathcal{U} - (DF(\mathcal{V}) - s)\mathrm{d}\mathcal{V} = 0$$

we see that the singularities of smallest codimension occur when $\mathcal{U} = \mathcal{V}$ and s is an eigenvalue of $DF(\mathcal{U})$, i.e. on the characteristic manifold.

Taking $\mathcal{V} - \mathcal{U} = R\Omega$, with $R \in \mathbb{R}$ and $\Omega \in S^{n-1}$, equations (9) become:

$$sR\Omega - (F(\mathcal{U} + R\Omega) - F(\mathcal{U})) = 0$$

By Hadamard lemma [1], and since $F(\mathcal{U} + R\Omega) - F(\mathcal{U}) = 0$ for $R = 0$, these can be written as:

$$F(\mathcal{U} + R\Omega) - F(\mathcal{U}) = R\Phi(\mathcal{U}, R, \Omega), \qquad \Phi(\mathcal{U}, 0, \Omega) = DF(\mathcal{U})\Omega$$

and, disregarding the trivial solution $R = 0$, the above equations become

$$s\Omega - \Phi(\mathcal{U}, R, \Omega) = 0$$

with differentials at $R = 0$ given by:

$$\Omega\mathrm{d}s - D^2F(\mathcal{U})(\Omega)\mathrm{d}\mathcal{U} - \frac{1}{2}D^2F(\mathcal{U})(\Omega, \Omega)\mathrm{d}R + (s - DF(\mathcal{U}))\mathrm{d}\Omega = 0$$

We now consider the subset $\tilde{\mathcal{O}}$ of $J^2(\mathbb{R}^n, \mathbb{R}^n) \times \mathbb{R} \times S^{n-1} \times \mathbb{R}$ where these differentials are not independent and the point $(\mathcal{U}, R, \Omega, s)$ belongs to the characteristic manifold; we denote by $\mathcal{O}$ its projection on $J^2(\mathbb{R}^n, \mathbb{R}^n)$, forgetting (R, Ω, s).

Belonging to $\mathcal{C}$ gives $n + 1$ equations (5):

$$R = 0, \quad (sI - DF(\mathcal{U}))\Omega = 0$$

After fixing a basis, and with standard identifications, the dependence of the differentials is equivalent to a matrix not having maximal rank.

It is clear that $-\frac{1}{2}D^2F(\mathcal{U})(\Omega, \Omega)$ is always dependent of the columns of $-D^2F(\mathcal{U})(\Omega)$ and also that, as $\Omega \neq 0$, $-D^2F(\mathcal{U})(\Omega)$ can be taken as any $n \times n$ matrix B. On $\mathcal{C}$, we see that $sI - DF(\mathcal{U})$ has not rank n, therefore at least one column is always a linear combination of the other ones; we denote by A a $n \times (n-1)$ submatrix of $sI - DF(\mathcal{U})$.

The set of (Ω, A, B) where the rank of the $n \times 2n$ matrix $[\Omega : B : A]$ is $n-1$ has codimension $n+1$ [1].

Thus the subset $\tilde{\mathcal{O}}$ is of course algebraic (defined by algebraic equations), closed and has codimension $2n+2$. Its projection $\mathcal{O}$ on $J^2(\mathbb{R}^n, \mathbb{R}^n)$, forgetting (R, Ω, s), is a closed semi-algebraic set of codimension $n+1$.

Since $\mathcal{O}$ is semialgebraic and closed, it admits a regular Whitney stratification, and from Thom transversality theorem [1], adapted to Whitney stratified sets, the set of maps F such that their 2-jets are transverse to $\mathcal{O}$ is open and dense in the Whitney topology, i.e. generic; looking at the codimension, transverse means not intersecting and the theorem follows. □

Corollary 1. *For a generic $n \times n$ system of conservation laws, the blow-up of the fundamental wave manifold eliminates all singularities of the characteristic manifold, i.e., $\mathcal{C}$ is a smooth manifold.*

Proof. Proceeding as above, smoothness of $\mathcal{C}$ follows from the independence of:

$$\begin{aligned}
&\Omega \mathrm{d}s - D^2F(\mathcal{U})(\Omega)\mathrm{d}\mathcal{U} - \frac{1}{2}D^2F(\mathcal{U})(\Omega,\Omega)\mathrm{d}R + (s - DF(\mathcal{U}))\mathrm{d}\Omega = 0 \\
&\mathrm{d}R = 0
\end{aligned}$$

or, equivalently, from the $(n+1) \times 2n$ matrix:

$$\begin{bmatrix} \Omega & B & B\Omega & A \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

having maximal rank; this is clearly equivalent to $[\Omega : B : A]$ having rank n. □

Using the same type of arguments we can prove:

Theorem 2. *After blow-up, the sonic loci $\mathcal{S}_L$ and $\mathcal{S}_R$ of a generic $n \times n$ system of conservation laws are smooth manifolds, and intersect each other transversally; the closures of the sonic loci intersect $\mathcal{C}$ transversally. Thus the inflection locus $\mathcal{I}$ and the double sonic locus $\mathcal{D}$ are also smooth manifolds.*

3.2. **Projections and their singularities.** We shall assume that the generic properties of transversality and smoothness of theorem 2 are verified.

Consider the projection:

$$\pi^- : \mathcal{W} \longrightarrow \mathbb{R}^n, \quad \pi^- : (\mathcal{U}_-, \mathcal{U}_+, s) \mapsto \mathcal{U}_-$$

and let π_L^- and π_R^- denote its restricton to $\mathcal{S}_L$ and $\mathcal{S}_R$, respectively; similarly, we define π_L^+ and π_R^+ from:

$$\pi^+ : \mathcal{W} \longrightarrow \mathbb{R}^n, \quad \pi^+ : (\mathcal{U}_-, \mathcal{U}_+, s) \mapsto \mathcal{U}_+$$

We denote by ρ the restriction of π^- to the characteristic manifold $\mathcal{C}$:

$$\rho : \mathcal{C} \longrightarrow \mathbb{R}^n, \quad \rho : (\mathcal{U}, \mathcal{U}, s) \mapsto \mathcal{U}$$

Proposition 8 ([10]). *The projection $\rho : \mathcal{C} \to \mathcal{U}$ is singular at $p \in \mathcal{C}$ if and only if $p \in \mathcal{E}$; thus the coincidence set is the set $\Sigma(\rho)$ of critical points of ρ:*

$$\mathcal{E} = \Sigma(\rho)$$

Such singularity is a fold point for ρ if and only if s has algebraic multiplicity 2 and geometric multiplicity 1.

Proposition 9 ([10]). *Let $p \in \mathcal{E} \backslash \mathcal{B}_0$ be a fold point for $\rho : \mathcal{C} \to \mathcal{U}$. Then the rarefaction curve through p is transverse to $\mathcal{E}$ and its projection on $\mathcal{U}$ is a cusp.*

Proposition 10 ([10]). *The projection π^+ restricted to a fixed-$\mathcal{U}_-$ shock curve has a critical point at $p \in \mathcal{W} \backslash (\mathcal{B}_R \cup \mathcal{B}_0)$ if and only if $p \in \mathcal{E}$; if $p \in \mathcal{E} \backslash \mathcal{B}_0$ is a fold point for $\rho : \mathcal{C} \to \mathcal{U}$, then the fixed-$\mathcal{U}_-$ shock curve through p is transverse to $\mathcal{E}$ and its projection π^+ on $\mathcal{U}$ is a cusp.*

A similar proposition is valid for π^-. It follows immediatly from the definition of secondary bifurcation loci that:

Proposition 11. *The right secondary bifurcation locus $\mathcal{B}_R$ is the set $\Sigma(\pi^-) \backslash \mathcal{C}$ of critical points of π^- outside the characteristic manifold $\mathcal{C}$, as the left secondary bifurcation locus locus $\mathcal{B}_L$ is the set $\Sigma(\pi^+) \backslash \mathcal{C}$ and the locus $\mathcal{B}_0$ of rarefaction singularities is the set of critical points in $\mathcal{C}$:*

$$\mathcal{B}_R = \Sigma(\pi^-) \backslash \mathcal{C}, \quad \mathcal{B}_L = \Sigma(\pi^+) \backslash \mathcal{C}, \quad \mathcal{B}_0 = \Sigma(\pi^-) \cap \mathcal{C} = \Sigma(\pi^+) \cap \mathcal{C}$$

Consider the projection:

$$\tau^- : \mathcal{W} \longrightarrow \mathbb{R}^{n+1}, \quad \tau^-(\mathcal{U}_-, \mathcal{U}_+, s) \mapsto (\mathcal{U}_-, s)$$

and let τ_L^- and τ_R^- denote its restriction to $\mathcal{S}_L$, and $\mathcal{S}_R$, respectively. The projections τ_L^+ and τ_R^+ are defined in the same way from:

$$\tau^+ : \mathcal{W} \longrightarrow \mathbb{R}^{n+1}, \quad \tau^-(\mathcal{U}_-, \mathcal{U}_+, s) \mapsto (\mathcal{U}_+, s)$$

Proposition 12 ([10]). *The right sonic locus $\mathcal{S}_R$ is the set $\Sigma(\tau^-) \backslash \mathcal{C}$ of critical points of τ^- outside the characteristic manifold $\mathcal{C}$, as the left sonic locus $\mathcal{S}_L$ is the set $\Sigma(\tau^+) \backslash \mathcal{C}$ of critical points of τ^+ outside $\mathcal{C}$ and the inflection locus $\mathcal{I}$ is the set of critical points in $\mathcal{C}$:*

$$\mathcal{S}_R = \Sigma(\tau^-) \backslash \mathcal{C}, \quad \mathcal{S}_L = \Sigma(\tau^+) \backslash \mathcal{C}, \quad \mathcal{I} = \Sigma(\tau^-) \cap \mathcal{C} = \Sigma(\tau^+) \cap \mathcal{C}$$

We have stated before that $\mathcal{B}_L \subset \mathcal{S}_L$ and similarly $\mathcal{B}_R \subset \mathcal{S}_R$, and this follows easily from propositions 11 and 12; we can be more precise:

Proposition 13. *The right secondary bifurcation locus $\mathcal{B}_R$ is contained in the set $\Sigma(\pi_R^-)$ of critical points of the restriction of π^- to $\mathcal{S}_R$, as the left secondary bifurcation locus locus $\mathcal{B}_L$ is contained in the set $\Sigma(\pi_L^+)$:*

$$\mathcal{B}_R \subset \Sigma(\pi_R^-), \quad \mathcal{B}_L \subset \Sigma(\pi_L^+)$$

Proof. As $\tau^\pm = (\pi^\pm, s)$, all critical points of $\pi^\pm$ are a fortiori critical points of $\tau^\pm$. $\square$

Proposition 14. *The right hysteresis locus $\mathcal{H}_R$ is the set of critical points of the restriction of π^- to $\mathcal{S}_R \backslash \mathcal{B}_R$, as the left hysteresis locus locus $\mathcal{H}_L$ is the set of critical points of the restriction of π^+ to $\mathcal{S}_R \backslash \mathcal{B}_L$:*

$$\mathcal{H}_R = \Sigma(\pi_R^-) \backslash \mathcal{B}_R, \quad \mathcal{H}_L = \Sigma(\pi_L^+) \backslash \mathcal{B}_L$$

Moreover:

$$\Sigma(\pi_R^-) = \mathcal{B}_R \cup \mathcal{H}_R, \quad \Sigma(\pi_L^+) = \mathcal{B}_L \cup \mathcal{H}_L$$

Proof. We prove $\mathcal{H}_R = \Sigma(\pi_R^-) \backslash \mathcal{B}_R$, the other statement being similar. Consider a point $p \in \mathcal{H}_R$: it follows from proposition 6 that there exists a nonzero tangent vector Ξ to $\mathcal{S}_R$ such that $\mathrm{d}\mathcal{U}(p) \cdot \Xi = 0$. Therefore $T_p\mathcal{S}_R$ and $\mathrm{d}\mathcal{U} = 0$ are not transverse, and $p \in \Sigma(\pi_R^-) \backslash \mathcal{B}_R$.

Take now a nonzero tangent vector Ξ to $\mathcal{S}_R$ at a point $p \in \mathcal{H}_R \backslash \mathcal{B}_R$ such that $\mathrm{d}\mathcal{U}(p) \cdot \Xi = 0$; such a vector always exits if $p \in \Sigma(\pi_R^-) \backslash \mathcal{B}_R$. It follows from the same proposition 6 that the graph of s along the fixed-$\mathcal{U}$ shock curve passing through p has vanishing second derivative and $p \in \mathcal{H}_R$.

Therefore $\mathcal{H}_R = \Sigma(\pi_R^-) \backslash \mathcal{B}_R$.

As $\mathcal{B}_L \subset \mathcal{S}_L$ and $\mathcal{B}_R \subset \mathcal{S}_R$, it follows that the critical points of these projections are the union of the secondary bifurcation loci and the hysteresis loci:

$$\Sigma(\pi_R^-) = \mathcal{B}_R \cup \mathcal{H}_R, \quad \Sigma(\pi_L^+) = \mathcal{B}_L \cup \mathcal{H}_L$$

$\square$

We can define maps $\Pi_R : \mathcal{S}_R \longrightarrow \mathcal{C}$ and $\Pi_L : \mathcal{S}_L \longrightarrow \mathcal{C}$ by:

$$\Pi_R(\mathcal{U}_-, \mathcal{U}_+, s) = (\mathcal{U}_+, \mathcal{U}_+, s), \quad \Pi_L(\mathcal{U}_-, \mathcal{U}_+, s) = (\mathcal{U}_-, \mathcal{U}_-, s)$$

Proposition 15. *The images of $\mathcal{H}_R$ under Π_R and of $\mathcal{H}_L$ under Π_L are the inflection locus:*

$$\Pi_R(\mathcal{H}_R) = \mathcal{I}, \quad \Pi_L(\mathcal{H}_L) = \mathcal{I}$$

Proof. We prove $\Pi_R(\mathcal{H}_R) = \mathcal{I}$, the other statement being similar. Consider a tangent vector (ξ, η, σ) to a fixed-$\mathcal{U}_-$ shock curve at a point $p = (\mathcal{U}_-, \mathcal{U}_+, s) \in \mathcal{H}_R$.

Then $(\mathcal{U}_+ - \mathcal{U}_-)\mathrm{d}s + (DF(\mathcal{U}_-) - s)\mathrm{d}\mathcal{U}_- - (DF(\mathcal{U}_+) - s)\mathrm{d}\mathcal{U}_+$ and $\mathrm{d}\mathcal{U}_-$ are zero on (ξ, η, σ), and so is $\mathrm{d}s$; thus $\sigma = 0$, s is an eigenvalue and η an eigenvector of $DF(\mathcal{U}_+)$.

We know from proposition 6 that (ξ, η, σ) is also tangent to $\mathcal{S}_R$; then there is a curve in $\mathcal{S}_R$ passing through p with velocity (ξ, η, σ), and its image under Π_R is a curve in the characteristic manifold $\mathcal{C}$ with velocity (η, η, σ). Therefore (η, η, σ) is tangent to $\mathcal{C}$ at $\Pi_R(p) = (\mathcal{U}_+, \mathcal{U}_+, s)$ and we can construct a rarefaction curve $(\mathcal{U}(t), \mathcal{U}(t), s(t))$ with velocity (η, η, σ) at $(\mathcal{U}(0), \mathcal{U}(0), s(0)) = \Pi_R(p) = (\mathcal{U}_+, \mathcal{U}_+, s)$.

As $\dot{s}(0) = \sigma = 0$, this proves that $\Pi_R(p) \in \mathcal{I}$. □

Proposition 16. *The points in the double sonic locus $\mathcal{D}$ and outside the secondary bifurcation loci are contained in the set $\Sigma(\pi_L^-)$, and similarly in $\Sigma(\pi_R^+)$:*

$$\mathcal{D} \backslash (\mathcal{B}_L \cup \mathcal{B}_R) \subset \Sigma(\pi_L^-) \cap \Sigma(\pi_R^+)$$

Proof. This is an easy consequence of proposition 7: in fact, if $p \in \mathcal{D} \backslash \mathcal{B}_R \subset \mathcal{S}_L \backslash \mathcal{B}_R$ then the fixed-$\mathcal{U}_-$ shock curve through p is tangent to $\mathcal{S}_L$ at p; therefore there exists a nonzero vector $\Xi \in T_p \mathcal{S}_L$ for which $\mathrm{d}\mathcal{U}_-(p) \cdot \Xi = 0$, and $T_p \mathcal{S}_L$ and $\mathrm{d}\mathcal{U}_- = 0$ are not transverse. It follows that $p \in \Sigma(\pi_L^-)$.

Similarly, if $p \in \mathcal{D} \backslash \mathcal{B}_L \subset \mathcal{S}_R \backslash \mathcal{B}_L$, it follows that $p \in \Sigma(\pi_R^+)$. Thus, if $\mathcal{D} \backslash (\mathcal{B}_L \cup \mathcal{B}_R)$ then $p \in \Sigma(\pi_L^-) \cap \Sigma(\pi_R^+)$. □

4. Quadratic 2×2 systems of conservation laws

4.1. **Rarefaction foliation and binary equations.** A *binary differential equation* is an implicit differential equation on two variables (x, y) of the form:

$$a(x,y)\mathrm{d}x^2 + 2b(x,y)\mathrm{d}x\mathrm{d}y + c(x,y)\mathrm{d}y^2 = 0$$

Proposition 17 ([13]). *The rarefaction foliation of a 2×2 system of conservation laws corresponds to the solutions of a binary differential equation.*

Proof. Given a 2×2 matrix $A_{i,j}$, its eigenvectors are the solutions (ξ, η) of the equation:

$$A_{1,2}\eta^2 + (A_{1,1} - A_{2,2})\xi\eta - A_{2,1}\xi^2 = 0$$

As the rarefaction curves are solutions of the differential equations:

$$\dot{\mathcal{U}} = r_i(\mathcal{U}), \quad r_i \text{ an eigenvector of } DF(\mathcal{U}),\ \mathcal{U} = (v, u)$$

they are also the solutions of the binary differential equation:

$$\frac{\partial F_1}{\partial u}\mathrm{d}u^2 + \left(\frac{\partial F_1}{\partial v} - \frac{\partial F_2}{\partial u}\right)\mathrm{d}v\mathrm{d}u - \frac{\partial F_2}{\partial v}\mathrm{d}v^2 = 0$$

□

Proposition 18. *A necessary and sufficient condition for a binary differential equation $a(x,y)\mathrm{d}x^2+2b(x,y)\mathrm{d}x\mathrm{d}y+c(x,y)\mathrm{d}y^2 = 0$ to be locally realizable as the differential equation for the rarefaction curves of a system of conservation laws is that:*

$$\frac{\partial^2 a}{\partial y^2} - 2\frac{\partial^2 b}{\partial x \partial y} + \frac{\partial^2 c}{\partial x^2} = 0 \tag{10}$$

Proof. Condition (10) is easily seen to be the integrability condition for the system of partial differential equations on F_1, F_2:

$$\begin{cases} \dfrac{\partial F_2}{\partial x}(x,y) = -a(x,y) \\[2ex] \dfrac{\partial F_1}{\partial x}(x,y) - \dfrac{\partial F_2}{\partial y}(x,y) = 2b(x,y) \\[2ex] \dfrac{\partial F_1}{\partial y}(x,y) = c(x,y) \end{cases}$$

□

Generic binary differential equations and their bifurcations near Morse singularities were studied by Bruce and Tari [3, 4]; in [4] they prove that a generic 1-parameter family of binary differential equations

$$a(x,y,\lambda)\mathrm{d}x^2 + 2b(x,y,\lambda)\mathrm{d}x\mathrm{d}y + c(x,y,\lambda)\mathrm{d}y^2 = 0$$

in a neighbourhood of a point where $a = b = c = 0$, and assuming the discriminant to have a Morse singularity at that point, is equivalent (in a precise sense explained there) to one of 8 normal forms:

(11)

$$
\begin{array}{ll}
(y+\lambda)dy^2 + 2xdxdy - ydx^2 = 0 & \textit{Lemon } (1 \textit{ saddle}) \\
(y+\lambda)dy^2 - 2xdxdy - ydx^2 = 0 & \textit{Star } (3 \textit{ saddles}) \\
(y+\lambda)dy^2 + \frac{1}{2}xdxdy - ydx^2 = 0 & \textit{Monstar } (2 \textit{ saddles} + 1 \textit{ node}) \\
(y+\lambda)dy^2 + 2xdxdy + ydx^2 = 0 & (1 \textit{saddle}) \\
(y+\lambda)dy^2 - \frac{1}{2}xdxdy + ydx^2 = 0 & (1 \textit{ node}) \\
(y+\lambda)dy^2 - 4xdxdy + ydx^2 = 0 & (3 \textit{ saddles}) \\
(y+\lambda)dy^2 + 2(y-x)dxdy + ydx^2 = 0 & (2 \textit{ saddles} + 1 \textit{ node}) \\
(y+\lambda)dy^2 - \frac{4}{3}xdxdy + ydx^2 = 0 & (1 \textit{ saddle} + 2 \textit{ nodes})
\end{array}
$$

The standard umbilic points are obtained by taking $\lambda = 0$ in the three first families.

It is easy to see that all these normal forms satisfy the integrability condition (10), and therefore it is possible to construct one-parameter families of 2×2 systems of conservation laws, that turn out to be quadratic, corresponding to these normal forms in the sense that their rarefaction foliations can be thought of as the solutions of respective family of binary differential equations.

Proposition 19. *The families (11) are generic among those satisfying the integrability condition (10).*

Proof. The conditions for the families (11) to be generic among all binary differential equations involve only their 1-jet, except one: that the critical points be normal; this condition does not involve the mixed second derivatives of $a(x, y)$, $b(x, y)$ and $c(x, y)$ [4].

The integrability condition (10) afects only the terms in their 2-jets that do not belong to the 1-jet, and in particular the mixed second derivative; they are therefore independent and the result follows easily. ☐

We will consider a 2-parameter family of conservation laws, corresponding to the existence of umbilic points, namely:

$$
F(v, u, \lambda, \gamma) = \left(\lambda u + \frac{1+\gamma}{2}v^2 + \frac{1}{2}u^2, vu\right), \quad \gamma \neq \pm 1, 0 \tag{12}
$$

The introduction of the parameter γ allows the simultaneous treatment of the first three of the above families (11), obtained for $\gamma = 2$, -2, $1/2$ respectively; from the point of view of binary equations, for any fixed $\gamma < 0$ the corresponding family is equivalent to the star family, for $0 < \gamma < 1$ to the monstar family, and for $\gamma > 1$ to the lemon family, but we will show that there is a significant change of behaviour for $\gamma = -1$.

The characteristic polynomial corresponding to (12) is $s^2-(2+\gamma)vs+(1+\gamma)v^2-u(\lambda+u)$. Thus the system is strictly hyperbolic only outside the ellipse:

$$\gamma^2 v^2 + 4\left(u+\frac{\lambda}{2}\right)^2 = \lambda^2 \tag{13}$$

and for $\lambda \neq 0$ there is an elliptical region inside that ellipse.

Remark 3. In a quadratic model the boundary of the region of strict hyperbolicity is a conic; Schaeffer and Shearer[15] only considered hyperbolic systems, and then that boundary reduces to a point, a degenerate conic.

For $\lambda \neq 0$ there are no umbilic points; for $\lambda = 0$, the origin is the unique umbilic point.

4.2. **Geometry of the characteristic manifold.** After blow-up, in the coordinates (v,u,R,ω,s) where $(v,u) = \mathcal{U}_-$, $\Omega = (\cos\omega, \sin\omega)$ and $(v+R\cos\omega, u+R\sin\omega) = \mathcal{U}_+$, the fundamental wave manifold is defined by the two equations:

(14)
$$\begin{cases} \mathcal{F}_1 = (\lambda+u)\sin\omega + ((1+\gamma)v - s)\cos\omega + \frac{1}{2}R(1+\gamma\cos^2\omega) = 0 \\ \\ \mathcal{F}_2 = (v-s)\sin\omega + u\cos\omega + R\sin\omega\cos\omega = 0 \end{cases}$$

and the characteristic manifold by (14) together with $R=0$. The sonic loci are defined by equations (14) together with:

$$f_L = s^2 - (2+\gamma)vs + (1+\gamma)v^2 - u(\lambda+u) = 0 \tag{15}$$

for the left sonic locus, and:

$$\begin{aligned} f_R = s^2 - (2+\gamma)(v+R\cos\omega)s + (1+\gamma)(v+R\cos\omega)^2 - \\ - (u+R\sin\omega)(\lambda+u+R\sin\omega) = 0 \end{aligned} \tag{16}$$

for the right sonic locus. Of course, we have to impose $R \neq 0$ to distinguish the sonic loci from the characteristic manifold.

Making the linear change of coordinates:

(17)
$$\lambda = 2\Lambda, \quad \gamma = 2\Gamma, \quad v = \frac{V}{\Gamma}, \quad u = U - \Lambda, \quad s = S + \left(1+\frac{1}{\Gamma}\right)V$$

the equations for the fundamental wave manifold become:

(18)
$$\begin{cases} -\cos\omega\ S + \cos\omega\ V + \sin\omega\ U + (\frac{1}{2} + \Gamma\cos^2\omega)\ R + \sin\omega\ \Lambda = 0 \\ -\sin\omega\ S - \sin\omega\ V + \cos\omega\ U + \sin\omega\cos\omega\ R - \cos\omega\ \Lambda = 0 \end{cases}$$

and those for the characteristic manifold become:

$$\begin{cases} -\cos\omega\ S+\cos\omega\ V+\sin\omega\ U+\sin\omega\ \Lambda=0 \\ -\sin\omega\ S-\sin\omega\ V+\cos\omega\ U-\cos\omega\ \Lambda=0 \\ R=0 \end{cases} \tag{19}$$

Proposition 20 ([11]). *The fundamental wave manifold and the characteristic manifold of a quadratic system (12) are smooth ruled surfaces.*

Proof. Equations (14) for the fundamental wave manifold are linear on the variables (v,u,R,s) with coefficients depending on (ω,λ,γ); a similar situation happens when considering equations (18), but these can always be solved for V and U:

$$\begin{aligned} V &= a_1(\omega)R+b_1(\omega)S+c_1(\omega) \\ U &= a_2(\omega)R+b_2(\omega)S+c_2(\omega) \end{aligned} \tag{20}$$

where:
(21)

$$\begin{aligned} a_1(\omega) &= \frac{1}{2}\cos\omega(1-2(\Gamma+1)\cos^2\omega) & a_2(\omega) &= -\frac{1}{2}\sin\omega(1+2(\Gamma+1)\cos^2\omega) \\ b_1(\omega) &= \cos 2\omega & b_2(\omega) &= \sin 2\omega \\ c_1(\omega) &= -\Lambda\sin 2\omega & c_2(\omega) &= \Lambda\cos 2\omega \end{aligned}$$

Therefore we see that the fundamental wave manifold is a smooth manifold generated by a 2-dimensional plane, defined by equations (20) for fixed ω, along the curve:

$$\omega\mapsto(-\Lambda\sin 2\omega,\Lambda\cos 2\omega,0,\omega,0)$$

and the characteristic manifold is a smooth manifold generated by the line

$$S\mapsto(\cos 2\omega\ S-\Lambda\sin 2\omega,\sin 2\omega\ S+\Lambda\cos 2\omega,0,\omega,S)$$

along the same curve. ☐

The geometric characterization of the characteristic manifold and sonic loci will be important in the sequel.

In the new coordinates, equation (15) becomes:

$$S^2-V^2-U^2+\Lambda^2=0 \tag{22}$$

and, for a fixed ω and $R=0$, equations (18) can be interpreted as defining one plane each in the space (V,U,S); it is easy to see that these planes are both tangent to the one leaf hyperboloid (22) at:

$$V=-\Lambda\cot\omega,\ U=-\Lambda,\ S=-\Lambda\cot\omega,$$

and

$$V=-\Lambda\tan\omega,\ U=\Lambda,\ S=\Lambda\tan\omega$$

respectively.

An one leaf hyperboloid is a bi-ruled surface: the intersection of any tangent plane with the surface gives two straight lines in the hyperboloid. Moreover these lines can be obtained as the intersection of the hyperboloid with the tangent planes through the circumference $V^2 + U^2 = \Lambda^2$, $S = 0$, for the above form (22), and they project down on the plane $S = 0$ as tangents to that circumference.

The intersection of any tangent plane with $S = 0$ is a straight line intersecting that circumference in two points, and the projection of the tangency point can be obtained as the intersection of the two tangents to the circumference at those points.

It is then clear from fig. 1 that the intersection of the planes is a straight line $l(\omega)$, tangent to the hyperboloid (22), passing through the point I with coordinates $V = -\Lambda \sin(2\omega)$, $U = \Lambda \cos(2\omega)$, $S = 0$; its projection on the (V, U)-plane is tangent to the circumference at that point, and connects the projections of the two tangency points of the planes (18), assuming $R = 0$.

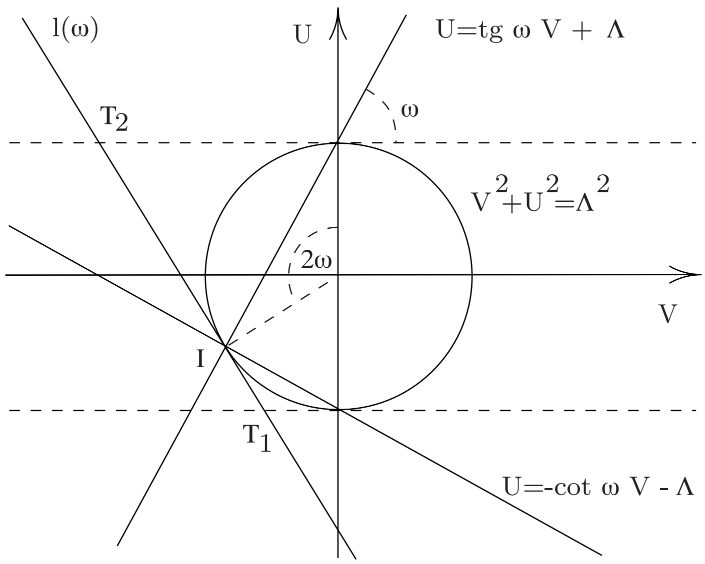

FIGURE 1. Construction of $l(\omega)$: T_1 and T_2 are the projections of the tangency points of the two planes, $l(\omega)$ the common tangent.

As ω varies, $l(\omega)$ describes twisted cylinders in $\hat{\mathcal{P}} = \mathbb{R}^2 \times \mathbb{R} \times S^1 \times \mathbb{R}$ and in $\mathcal{P}^*$, the former being a double cover of the latter.

4.3. **Geometry of the sonic loci.**

Proposition 21 ([11]). *The sonic loci of the quadratic system corresponding to (12) are smooth ruled surfaces.*

Proof. Equations (18) corresponding to the Rankine-Hugoniot condition can always be solved for the variables (V, U), as seen above, leading to:

$$\begin{aligned} V &= a_1(\omega)R + b_1(\omega)S + c_1(\omega) \\ U &= a_2(\omega)R + b_2(\omega)S + c_2(\omega) \end{aligned}$$

Substituting in the equation (16) for the right sonic locus, after the change of variables (17), gives also a linear equation on R and S:

$$a_3(\omega)R + b_3(\omega)S + c_3(\omega) = 0 \tag{23}$$

where:

$$\begin{aligned} a_3(\omega) &= \frac{1}{4}(1 - 8(\Gamma + 1)\cos^2\omega - 4(\Gamma^2 - 1)\cos^4\omega) \\ b_3(\omega) &= (3 + 2(\Gamma - 1)\cos^2\omega)\cos\omega \\ c_3(\omega) &= -\Lambda(1 + 2(\Gamma - 1)\cos^2\omega)\sin\omega \end{aligned}$$

Note that a_3 and b_3 cannot be simultaneously zero (as $\Gamma \neq -1/2$, or $\gamma \neq -1$), therefore equation (23) can always be solved for R or S, and consequently $\mathcal{S}_R$ can be smoothly parametrized by (ω, R), or (ω, S).

Proceeding as above, the substitution of the solution (V, U) of equations (18) in the equation (15) for the left sonic locus, after the change of variables (17), gives also a linear equation on R and S:

$$\bar{a}_3(\omega)R + \bar{b}_3(\omega)S + \bar{c}_3(\omega) = 0 \tag{24}$$

where:

$$\begin{aligned} \bar{a}_3(\omega) &= \frac{1}{4} + (1 + \Gamma)\cos^2\omega - (1 - \Gamma^2)\cos^4\omega \\ \bar{b}_3(\omega) &= -(3 - 2(1 - \Gamma)\cos^2\omega)\cos\omega \\ \bar{c}_3(\omega) &= \Lambda(1 - 2(1 - \Gamma)\cos^2\omega)\sin\omega \end{aligned}$$

As before, since $\bar{a}_3$ and $\bar{b}_3$ cannot be simultaneously zero (as $\Gamma \neq -1/2$), $\mathcal{S}_L$ can be smoothly parametrized by (ω, R), or (ω, S).

It is easy to see that for both sonic loci, the parametrization by (ω, R), respectively (ω, S), is linear in R, respectively S. □

It is clear that we can always solve one of the equations (18) for R, since the derivatives with respect to R cannot be simultaneously zero (as $\Gamma \neq -1/2$, or $\gamma \neq -1$). Say $R(V, U, S, \omega, \Lambda, \Gamma)$ is that solution, and substitute it into the other equation; this elimination of R gives:

$$a(\omega)S + b(\omega)V + c(\omega)U + d(\omega)\Lambda = 0 \tag{25}$$

with:

$$\begin{aligned}
a(\omega) &= (1 + 2(\Gamma - 1)\cos^2\omega)\sin\omega \\
b(\omega) &= (1 + 2(\Gamma + 1)\cos^2\omega)\sin\omega \\
c(\omega) &= (1 - 2(\Gamma + 1)\cos^2\omega)\cos\omega \\
d(\omega) &= (3 + 2(\Gamma - 1)\cos^2\omega)\cos\omega
\end{aligned}$$

Thus, for a fixed ω, the projection of the left sonic locus in the space (V, U, S) is the intersection of the plane (25) with the hyperboloid (22); in the space (V, U, R, S) it is the graph of the map $R(V, U, S, \omega, \Lambda, \Gamma)$ on that set.

A straightforward computation shows that:

$$a(\omega)^2 - b(\omega)^2 - c(\omega)^2 + d(\omega)^2 = 0$$

and therefore the plane (25) is tangent (for $\Lambda \neq 0$) to the hyperboloid (22) at the point:

$$S_{tg} = \frac{a(\omega)}{d(\omega)}\Lambda, \quad V_{tg} = -\frac{b(\omega)}{d(\omega)}\Lambda, \quad U_{tg} = -\frac{c(\omega)}{d(\omega)}\Lambda, \quad R_{tg} = 0 \tag{26}$$

The last equality follows from substituting S_{tg}, V_{tg} and U_{tg} into (18); we begin by verifying that:

$$\begin{cases} (a(\omega) + b(\omega))\cos\omega - (d(\omega) - c(\omega))\sin\omega \equiv 0 \\ (b(\omega) - a(\omega))\sin\omega - (c(\omega) + d(\omega))\cos\omega \equiv 0 \end{cases}$$

which, multiplying by $-\Lambda/d$, gives:

$$\begin{cases} (V_{tg} - S_{tg})\cos\omega + (\Lambda + U_{tg})\sin\omega = 0 \\ (V_{tg} + S_{tg})\sin\omega - (U_{tg} - \Lambda)\cos\omega = 0 \end{cases}$$

and thus $R_{tg} = 0$.

Thus, for a fixed ω, the intersection of the plane (25) with the hyperboloid (22) is given by two straight lines tangent to the hyperboloid; they project on the (V, U)-plane as tangents to the circumference $V^2 + U^2 = \Lambda^2$, with $S = 0$.

The tangency points of these lines with the circumference are obtained by solving the equations (22,25) for $S = 0$, taking in account that $a(\omega)^2 - b(\omega)^2 - c(\omega)^2 + d(\omega)^2 \equiv 0$, thus leading to:

$$T_{1,2} = (V_{1,2}, U_{1,2}) = \left(\frac{-b(\omega)d(\omega) \mp a(\omega)c(\omega)}{b(\omega)^2 + c(\omega)^2}\Lambda, \ \frac{-c(\omega)d(\omega) \pm a(\omega)b(\omega)}{b(\omega)^2 + c(\omega)^2}\Lambda \right)$$

This gives:

$$
\begin{aligned}
V_1 =& \frac{-4(1+2\Gamma\cos^2\omega)\sin\omega\cos\omega}{1+4(\Gamma+1)\cos^2\omega+4(\Gamma^2-1)\cos^4\omega}\Lambda \\
U_1 =& \frac{1+4(\Gamma-1)\cos^2\omega+4(\Gamma^2+1)\cos^4\omega}{1+4(\Gamma+1)\cos^2\omega+4(\Gamma^2-1)\cos^4\omega}\Lambda \\
R_1 =& \frac{-4\sin\omega(1+2(\Gamma-1)\cos^2\omega)}{1+4(\Gamma+1)\cos^2\omega+4(\Gamma^2-1)\cos^4\omega}\Lambda \\
S_1 =& 0
\end{aligned}
\tag{27}
$$

and

$$
\begin{aligned}
V_2 =& -2\Lambda\sin\omega\cos\omega = -\Lambda\sin 2\omega \\
U_2 =& \Lambda(2\cos^2\omega-1) = \Lambda\cos 2\omega \\
R_2 =& 0 \\
S_2 =& 0
\end{aligned}
\tag{28}
$$

The second solution (28) belongs to the characteristic manifold, in fact it gives a parametrization of the coincidence locus $\mathcal{E}$, and the corresponding tangent describes the characteristic manifold $\mathcal{C}$.

The left sonic locus is generated by the tangent to the hyperboloid passing through $(V_1, U_1, 0)$; we denote by $\mathcal{E}_L$ the corresponding curve on the sonic locus, parametrized by ω (for λ and γ fixed):

$$
\mathcal{E}_L : \omega \mapsto (V_1, U_1, R_1, \omega, 0) \tag{29}
$$

Remark 4. The denominator of V_1, U_1 and R_1 is never zero ($\Gamma \neq -1/2$): if we interpret it as a quadratic equation on Γ, its discriminant will be:

$$
-64\cos^6\omega\sin^2\omega
$$

and $\Gamma = -1/2$, when $\omega = 0$, is the only possible real root.

This way we see that the left sonic locus is formed by straight lines, tangent to the hyperboloid, on which R is not identically zero, as those where $R \equiv 0$ are exactly the ones forming the characteristic manifold, as explained above.

For $\Lambda = 0$ the hyperboloid becomes a cone and the circumference a point, and the left sonic locus is defined by the equations:

$$
S^2 - V^2 - U^2 = 0, \quad a(\omega)S + b(\omega)V + c(\omega)U = 0
$$

Now the plane defined by the second equation is not tangent to the cone. The corresponding solutions can be written as:

$$
V = \frac{-a(\omega)b(\omega) \pm c(\omega)d(\omega)}{b(\omega)^2 + c(\omega)^2}\, S, \quad U = \frac{-a(\omega)c(\omega) \mp b(\omega)d(\omega)}{b(\omega)^2 + c(\omega)^2}\, S
$$

and therefore we have the two solutions:

$$\begin{cases} V_1 = -\dfrac{1+4(\Gamma-1)\cos^2\omega+4(\Gamma^2+1)\cos^4\omega}{1+4(\Gamma+1)\cos^2\omega+4(\Gamma^2-1)\cos^4\omega}S \\ U_1 = -\dfrac{4(1+2\Gamma\cos^2\omega)\sin\omega\cos\omega}{1+4(\Gamma+1)\cos^2\omega+4(\Gamma^2-1)\cos^4\omega}S \end{cases}$$

and

$$\begin{cases} V_2 = S(2\cos^2\omega-1) = S\cos 2\omega \\ U_2 = 2S\sin\omega\cos\omega = S\sin 2\omega \end{cases}$$

Along the second one we have $R_2 \equiv 0$ (and thus is the characteristic manifold), and along the first one:

$$R_1 = \frac{4(3+2(\Gamma-1)\cos^2\omega)\cos\omega}{1+4(\Gamma+1)\cos^2\omega+4(\Gamma^2-1)\cos^4\omega}S$$

The left sonic locus can be parametrized by:

$$(\omega, S) \mapsto (V_1(\omega,S), U_1(\omega,S), R_1(\omega,S), \omega, S)$$

To study the right sonic locus we make a change of variables:

$$\lambda = 2\Lambda, \quad \gamma = 2\Gamma, \quad x = \frac{X}{\Gamma}, \quad y = Y - \Lambda, \quad s = Z + \left(1+\frac{1}{\Gamma}\right)X$$

where $x = v + R\cos\omega$, $y = u + R\sin\omega$, and follow the procedure above. It is easy to see that the only change is that the coefficients of R in the analogue of equations (18) are the symmetric of the previous ones; therefore $R(X,Y,Z,\omega,\Lambda,\Gamma)$ has a symmetric expression, and the result of the elimination of R between the two equations is the same as before.

Proposition 22. *For $\Lambda \neq 0$, the inflection locus $\mathcal{I}$ coincides with the tangency points of the plane $a(\omega)S + b(\omega)V + c(\omega)U + d(\omega)\Lambda = 0$ with the hyperboloid $S^2 - V^2 - U^2 + \Lambda^2 = 0$.*

Proof. The inflection locus is the intersection of the characteristic manifold with the left (or right) sonic locus $\mathcal{S}_L$; from the geometric characterization of those sets, it follows that the inflection locus coincides with the tangency points of the plane $a(\omega)S + b(\omega)V + c(\omega)U + d(\omega)\Lambda = 0$ with the hyperboloid $S^2 - V^2 - U^2 + \Lambda^2 = 0$, since that is the intersection of the two tangents, one in $\mathcal{C}$ and the other in $\mathcal{S}_L$. Thus $\mathcal{I}$ is given by the parametrized equations (26). □

The situation for $\Lambda = 0$ will be discussed in 5.2.

Proposition 23. *For quadratic 2×2 systems the double sonic locus is empty if $\gamma < -1$.*

Proof. A point belongs to left sonic locus if it verifies the Rankine-Hugoniot condition and if s is an eigenvalue of DF at the left state, i.e., if $|DF(\mathcal{U}_-) - sI| = 0$. In the same way, a point belongs to the right sonic locus if it verifies the Rankine-Hugoniot condition and if $|DF(\mathcal{U}_+) - sI| = 0$.

Suppose that $(\mathcal{U}_-, \mathcal{U}_+, s) \in \mathcal{D}$. Then we have, from equations (15) and (16):

$$|DF(\mathcal{U}_-) - sI| = 0$$
$$|DF(\mathcal{U}_+) - sI| = 0$$

It is also known, and verifiable by direct computation, that for quadratic systems the Hugoniot-Rankine condition implies that s is an eigenvalue of DF at the middle point between $\mathcal{U}_-$ and $\mathcal{U}_+$:

$$\left|DF\left(\frac{\mathcal{U}_- + \mathcal{U}_+}{2}\right) - sI\right| = 0$$

But if $\mathcal{U}_- \neq \mathcal{U}_+$, that would mean that the quadratic equation on t:

$$|DF((1-t)\mathcal{U}_- + t\mathcal{U}_+) - sI| = 0$$

had three different roots, $t = 0$, $1/2$ and 1, for the fixed values of $(\mathcal{U}_-, \mathcal{U}_+, s)$. Then we should have all coefficients in the quadratic equation identically zero:

$$\begin{cases} (2(1+\Gamma)\cos^2\omega - 1)R^2 = 0 \\ -2(2\Gamma V\cos\omega + (1+\Gamma)S\cos\omega - U\sin\omega)R = 0 \\ S^2 - V^2 - U^2 + \Lambda^2 = 0 \end{cases} \tag{30}$$

Thus the double sonic locus is empty whenever $\gamma < -1$ ($\Gamma < -1/2$), since the first condition cannot be fullfilled.

The third condition is again belonging to the left sonic locus, and the second one becomes an identity on $\mathcal{S}_L$ for ω solving the first condition. Therefore the double sonic locus is given by the two straight lines d_1 and d_2:

$$\begin{aligned} \omega &= \pm\arccos\frac{1}{\sqrt{2(1+\Gamma)}} = \pm\arctan(1+2\Gamma) \\ V &= -\frac{1}{2}\frac{\Gamma}{\sqrt{2(1+\Gamma)}}R \mp \frac{1+\Gamma}{\sqrt{1+2\Gamma}}\Lambda \\ U &= \mp\frac{1}{4}\sqrt{\frac{2(1+2\Gamma)}{1+\Gamma}}R \\ S &= \frac{1}{4}\sqrt{2(1+\Gamma)}\,R \pm \frac{\Gamma}{\sqrt{1+2\Gamma}}\Lambda \end{aligned} \tag{31}$$

if $\gamma > -1$, or $\Gamma > -1/2$. Their intersection with $\mathcal{C}$ is in $\mathcal{I}$ and projects on the points:

$$V = \mp\frac{1+\Gamma}{\sqrt{1+2\Gamma}}\Lambda, \quad U = 0$$

□

Remark 5. There are points simultaneously sonic on the left and on the right when $\gamma < -1$, but they are contained on the characteristic manifold, i.e., they are such that $R = 0$. More precisely, the set of those points coincides with the inflection locus.

4.4. **Rarefaction and composite foliations.** The differential equation (6) for the rarefaction curves:

$$\dot{\mathcal{U}} = r_i(\mathcal{U})$$

where r_i is an eigenvector of $DF(\mathcal{U})$, is an implicit differential equation in the space of the variables $\mathcal{U}$ whenever the eigenvalues are not always distinct. The corresponding integral curves can also be defined by a differential 1-form α:

$$\alpha = \left(\frac{\partial F_1}{\partial v} - s\right)\mathrm{d}v + \frac{\partial F_1}{\partial u}\mathrm{d}u, \quad \text{or} \quad \alpha = \frac{\partial F_2}{\partial v}\mathrm{d}v + \left(\frac{\partial F_2}{\partial u} - s\right)\mathrm{d}u$$

as s is an eigenvalue of $DF(\mathcal{U})$.

Here and subsequently we abuse notation denoting by the same symbol two 1-forms that define the same foliation.

On the characteristic manifold $\mathcal{C}$ the foliation induced by that equation can also be defined by a differential 1-form $\alpha_{\mathcal{C}} = \pi_{\mathcal{C}}^* \alpha$:

(32)

$$\alpha_{\mathcal{C}} = \left(\frac{\partial F_1}{\partial v} - s\right)\mathrm{d}v + \frac{\partial F_1}{\partial u}\mathrm{d}u, \quad \text{or} \quad \alpha_{\mathcal{C}} = \frac{\partial F_2}{\partial v}\mathrm{d}v + \left(\frac{\partial F_2}{\partial u} - s\right)\mathrm{d}u$$

where $\pi_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbb{R}^2$, $\pi_{\mathcal{C}}(\mathcal{U}, \mathcal{U}, s) = \mathcal{U}$.

Alternatively, we can think of α as a 1-form on the space (v, u, s), and then:

$$\alpha_{\mathcal{C}} = i^* \alpha$$

where i is the inclusion of $\mathcal{C}$ in that space.

As we have seen before (section 2), a left composite curve is a curve in the left sonic locus:

$$s \mapsto (\mathcal{U}_- = \tilde{\mathcal{U}}(s), \mathcal{U}_+, s)$$

that is an integral curve of the line field induced on $\mathcal{S}_L$ by the differential equation $\dot{\mathcal{U}} = r_i(\mathcal{U})$. The projection $(\mathcal{U}_- = \tilde{U}(s), \mathcal{U}_+, s) \mapsto \mathcal{U}_-$ of the composite curve, and the projection $(\tilde{\mathcal{U}}(s), \tilde{\mathcal{U}}(s), s) \mapsto \mathcal{U}_- = \tilde{\mathcal{U}}(s)$ of the corresponding rarefaction curve coincide, when they are both defined.

If τ_L^- denotes the projection $\tau^- : (\mathcal{U}_-, \mathcal{U}_+, s) \mapsto (\mathcal{U}_-, s)$ restricted to $\mathcal{S}_L$, then the left composite foliation is formed by the integral curves of $\mu = \left(\tau_L^-\right)^* \alpha$.

Similarly, the right composite foliation can be defined by a 1-form $\nu = \left(\tau_R^+\right)^* \alpha$ on $\mathcal{S}_R$, where τ_R^+ denotes the projection $\tau^+ : (\mathcal{U}_-, \mathcal{U}_+, s) \mapsto (\mathcal{U}_+, s)$ restricted to $\mathcal{S}_R$.

Proposition 24. *The left and right composite foliations are diffeomorphic.*

Proof. It is easy to see that $\Phi : \mathcal{S}_L \longrightarrow \mathcal{S}_R$, where:

$$\Phi(\mathcal{U}_-,\mathcal{U}_+,s) = (\mathcal{U}_+,\mathcal{U}_-,s)$$

is a diffeomorphism, and the following diagram commutes:

$$\begin{tikzcd}
\mathcal{S}_L \arrow[r, "\Phi"] \arrow[d, "\tau_L^-"'] & \mathcal{S}_R \arrow[d, "\tau_R^+"] \\
\mathbb{R}^{n+1} \arrow[r, "id"] & \mathbb{R}^{n+1}
\end{tikzcd}$$

Therefore $\Phi^*\nu = \mu$. $\square$

Remark 6. We also have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{S}_L \arrow[r, "\Phi"] \arrow[d, "\tau_L^+"'] & \mathcal{S}_R \arrow[d, "\tau_R^-"] \\
\mathbb{R}^{n+1} \arrow[r, "id"] & \mathbb{R}^{n+1}
\end{tikzcd}$$

where τ_L^+ and τ_R^- are the projections τ^+ and τ^- restricted to $\mathcal{S}_L$, respectively to $\mathcal{S}_R$. Thus we will study only the left composite foliation.

5. RAREFACTION CURVES

The differential equation (6) for the rarefaction curves in $\mathcal{U}$ induces a differential 1-form $\alpha_{\mathcal{C}}$ on the characteristic manifold $\mathcal{C}$, as seen before (32).

5.1. **Rarefaction foliation.** In our case, and in view of (5), the equations become:

$$\sin\omega\ \dot{v} - \cos\omega\ \dot{u} = 0 \qquad \text{or} \qquad \sin\omega\ \mathrm{d}v - \cos\omega\ \mathrm{d}u = 0$$

since $\Omega = (\cos\omega, \sin\omega)$ is an eigenvector of $DF(\mathcal{U})$ corresponding to the eigenvalue s. In the coordinates (V, U, R, ω, S) the differential equation becomes (multiplying by Γ):

$$\alpha_{\mathcal{C}} = \sin\omega\ \mathrm{d}V - \Gamma\cos\omega\ \mathrm{d}U = 0$$

and, as described before, we can parametrize the characteristic manifold $\mathcal{C}$ by $(S,\omega) \mapsto (S\cos 2\omega\ -\Lambda\sin 2\omega, S\sin 2\omega + \Lambda\cos 2\omega, 0, \omega, S)$ to obtain:

(33)

$$\begin{aligned}\alpha_{\mathcal{C}} = &-\sin\omega(1-2(1-\Gamma)\cos^2\omega)\mathrm{d}S + \\ &+\left[2\Lambda\sin\omega(1-2(1-\Gamma)\cos^2\omega) - 2S\cos\omega(\Gamma + 2(1-\Gamma)\sin^2\omega)\right]\mathrm{d}\omega\end{aligned}$$

The critical points are the solutions of:

$$w = 0 \quad \text{or} \quad \cos^2\omega = \frac{1}{2(1-\Gamma)} = \frac{1}{2-\gamma} \tag{34}$$

and

$$S = 0 \Longleftrightarrow V = -\Lambda\sin 2\omega \Longleftrightarrow v = -2\frac{\lambda}{\gamma}\sin\omega\cos\omega$$

thus they are given by:

$$P_1 = (0,0) \qquad P_{2,3} = \left(\pm 2\frac{\sqrt{1-\gamma}}{\gamma(2-\gamma)}\lambda, -\frac{1-\gamma}{2-\gamma}\lambda\right) \tag{35}$$

in the (v,u)-plane, where P_2 and P_3 are well defined only for $\gamma < 1$. The straight lines in $\mathcal{C}$ corresponding to:

$$w = 0, \quad \omega = \pm\arccos\frac{1}{\sqrt{2-\gamma}}$$

are separatrices of the respective critical points. In the coordinates (V,U) the critical points become:

$$P_1 = (0,\Lambda) \qquad P_{2,3} = \left(\pm\frac{\sqrt{1-2\Gamma}}{1-\Gamma}\Lambda, \frac{\Gamma}{1-\Gamma}\Lambda\right) \tag{36}$$

Calculating the eigenvalues of the system at critical points we can easily verify that P_2 and P_3 are saddle points independently of $\gamma < 1$, while P_1 is a node if $0 < \gamma < 1$ and a saddle when $\gamma > 1$ or $\gamma < 0$.

Thus we conclude that:

a. for $\gamma > 1$ the system has only one critical point, a saddle.
b. for $0 < \gamma < 1$ the system has 3 critical points, two saddles and a node.
c. for $\gamma < 0$ the system has 3 critical points, three saddles.

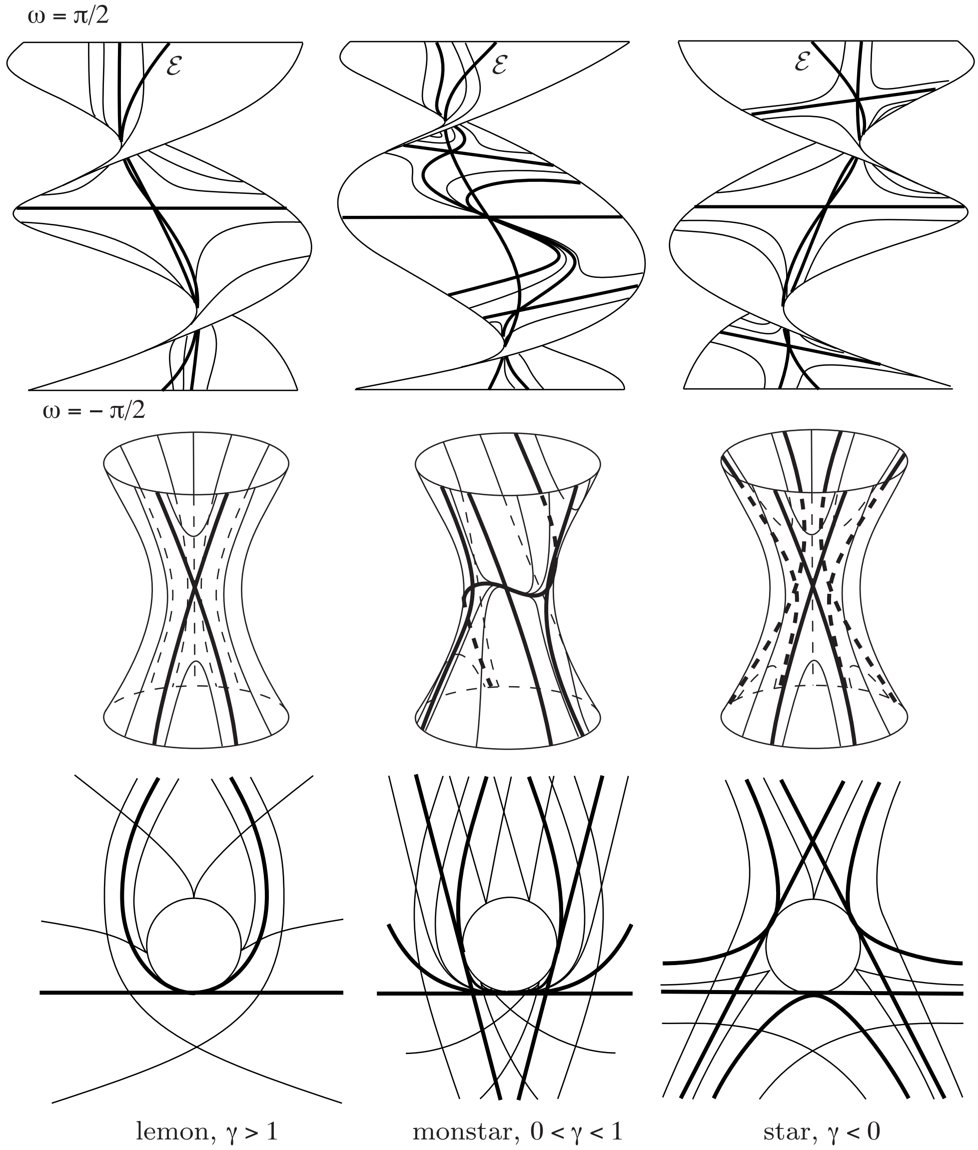

FIGURE 2. Rarefaction foliation and phase portrait of rarefaction curves for $\Lambda = -\frac{1}{2}$ ($\lambda = -1$): in the space (V, U, ω), in the space (V, U, S), and projected on (V, U)

The situation for $\lambda \neq 0$ is depicted in fig. 2, where the characteristic manifold is represented in the spaces (V, U, S) and (V, U, ω), and the rarefaction foliation is projected on the plane (V, U).

For $\lambda = 0$, the hyperboloid in the space (V, U, S) representing the characteristic manifold becomes a cone, but the characteristic manifold

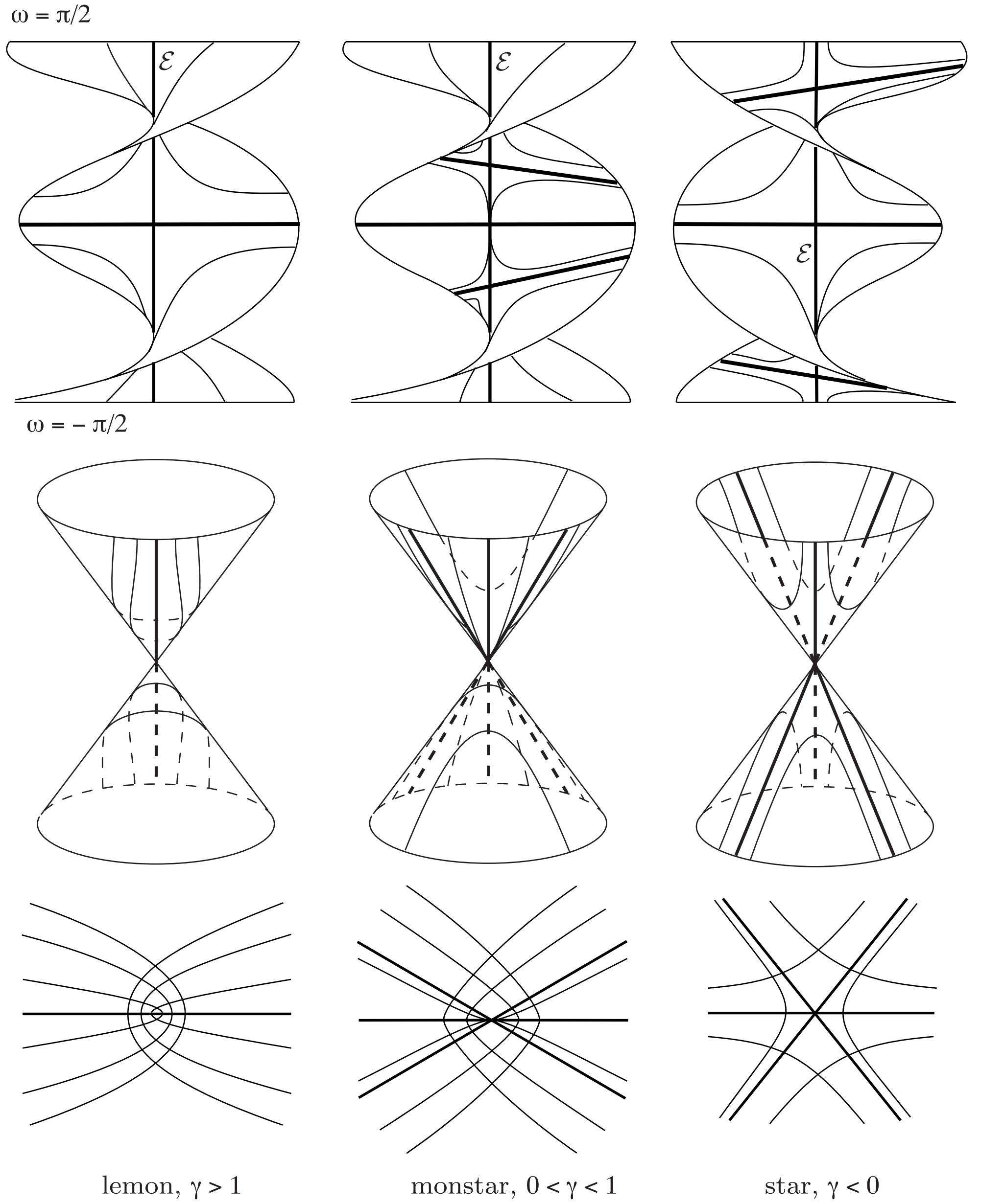

FIGURE 3. Rarefaction foliation and phase portrait of rarefaction curves, $\Lambda = 0$ ($\lambda = 0$): in the space (V, U, ω), in the space (V, U, S) rotated by $\pi/2$, and projected on (V, U)

is still smooth, as can be seen in its projection on the (V, U, ω)-space: there is a common separatrix to the existing critical points, corresponding to the vertex of the cone, that projects down on the origin.

The rarefaction foliation gives a phase portrait in the plane (V, U) corresponding to the classical umbilic points: lemon, monstar and star (fig. 3).

5.2. **Inflection Locus.** To a rarefaction wave there corresponds a rarefaction curve with an orientation: the corresponding eigenvalue should increase along it. Therefore in general that orientation changes on the inflection locus.

The variable s has a critical point along a rarefaction curve, and the corresponding point belongs to the inflection locus, if there is a nonzero vector contained simultaneously in the rarefaction line field and in the hyperplane $ds = 0$. These points can also be characterized as the left and right sonic points belonging to the characteristic manifold.

5.2.1. $\lambda \neq 0$. As remarked before, in our model and assuming $\lambda \neq 0$ (or equivalently $\Lambda \neq 0$), the inflection locus is given by (26):

$$\begin{cases} V = -\dfrac{1+2(\Gamma+1)\cos^2\omega}{3+2(\Gamma-1)\cos^2\omega}\Lambda\tan\omega \\ U = -\dfrac{1-2(\Gamma+1)\cos^2\omega}{3+2(\Gamma-1)\cos^2\omega}\Lambda \\ S = \dfrac{1+2(\Gamma-1)\cos^2\omega}{3+2(\Gamma-1)\cos^2\omega}\Lambda\tan\omega \\ R = 0 \end{cases} \tag{37}$$

In fact there is another component when $\gamma = -1$; we compute the inflection locus differently: we have seen before that it can be thought of as the intersection of the (closure of the) sonic loci, which is contained in the characteristic manifold. But here we have $f_L \equiv 0$, and instead of the equation $f_R = 0$ we use the equation:

$$\frac{1}{R}(f_L - f_R) = 0$$

(dividing by R to avoid the trivial solution given by all the characteristic manifold) and then we make $R = 0$, so finally:

$$\begin{cases} \cos\omega\, V + \sin\omega\, U - \cos\omega\, S + \Lambda\sin\omega = 0 \\ -\sin\omega\, V + \cos\omega\, U - \sin\omega\, S - \Lambda\cos\omega = 0 \\ -2\Gamma\cos\omega\, V - 2\sin\omega\, U - 2(1+\Gamma)\cos\omega\, S = 0 \\ R = 0 \end{cases} \tag{38}$$

The matrix of the coefficients of the system for (V, U, R, S) has rank 4, except for:

$$\omega = \frac{\pi}{2} \quad \vee \quad \omega = \pm\arctan\left(\frac{1}{3}\sqrt{-3(1+2\Gamma)}\right) \tag{39}$$

where the last values make sense only for $\Gamma \leq -\frac{1}{2}$ (or $\gamma \leq -1$).

When $\lambda \neq 0$, there are no solutions corresponding to these singular values unless $\Gamma = -1/2$ (or $\gamma = -1$), and then:

$$\Gamma = -\frac{1}{2}, \quad \omega = 0, \quad V = S, \quad U = \Lambda$$

The singular points of the rarefaction curves belong to the inflection locus, more specifically, $\mathcal{B}_0 = \mathcal{I} \cap \mathcal{E}$ (Proposition 1); so there should be a change of behaviour at $\gamma = 1$, for $\lambda \neq 0$.

The asymptotes of the projection of the inflection locus (37) are as follows:

For $\omega = \pi/2$, we have a horizontal asymptote:

$$U = -\frac{\Lambda}{3}, \text{ or equivalently, } u = -\frac{2\lambda}{3}$$

For $\omega = \pm \arctan\left(\sqrt{-3(1+\gamma)}/3\right)$ and $\gamma \leq -1$, we have two other asymptotes:

$$U = \frac{1-\Gamma}{3(2+\Gamma)}\Lambda \pm \frac{\sqrt{-3(1+2\Gamma)}}{\Gamma+2}V \text{ or } u = -\frac{(2\gamma+5)\lambda}{3(\gamma+4)} \pm \frac{\gamma\sqrt{-3(1+\gamma)}}{\gamma+4}\, v$$

At $\gamma = -1$, the projection of the asymptotes of the curve given by (37) is:

$$|V| \geq \frac{2\sqrt{2}}{3}|\Lambda|, \quad U = -\frac{\Lambda}{3} \quad \text{ or } \quad |v| \geq \frac{2\sqrt{2}}{3}|\lambda|, \quad u = -\frac{2\lambda}{3}$$

Finally, we can easily verify that if $-4 < \gamma < -3$ or $-1 < \gamma < 0$ there are 4 points whose tangent is vertical, while there are only two points in those conditions if $-3 < \gamma < -1$ and none for $\gamma > 0$ or $\gamma < -4$. This property distinguishes the cases $-1 < \gamma < 0$ and $0 < \gamma < 1$.

Therefore, for $\lambda \neq 0$, the projection of the inflection locus has relevant changes at $\gamma = -4$, -3, -1, 0 and 1, i.e., at $\Gamma = -2$, $-\frac{3}{2}$, $-\frac{1}{2}$, 0 and $\frac{1}{2}$ (fig. 4); the asymptotes:

$$U = -\frac{1-\Gamma}{3(\Gamma+2)}\Lambda \pm \frac{\sqrt{-3(1+2\Gamma)}}{\Gamma+2}\, V$$

become vertical and then change the sign of their slopes at $\Gamma = -2$ ($\gamma = -4$), and at that value of Γ there are 1 horizontal asymptote and 2 vertical asymptotes:

$$U = -\frac{1}{3}\Lambda, \quad V = -\Lambda\Gamma, \quad V = \Lambda\Gamma$$

Consequently, for $\gamma < -3$ ($\gamma \neq -4$), the projection of the inflection locus has a self-intersection, but different as $\gamma > -4$ or $\gamma < -4$, while if $\gamma = -3$ there is a cusp at the origin.

Those self-intersections and the cusp exist only on the projection, and thus are noticeable in the phase portrait on the (v, u)-plane. On the characteristic manifold only the changes at $\gamma = -1$, 0 and 1 affect the behaviour of the rarefaction curves. Figure 5 shows that bifurcation for $\gamma = -1$ in the (V, U)-plane.

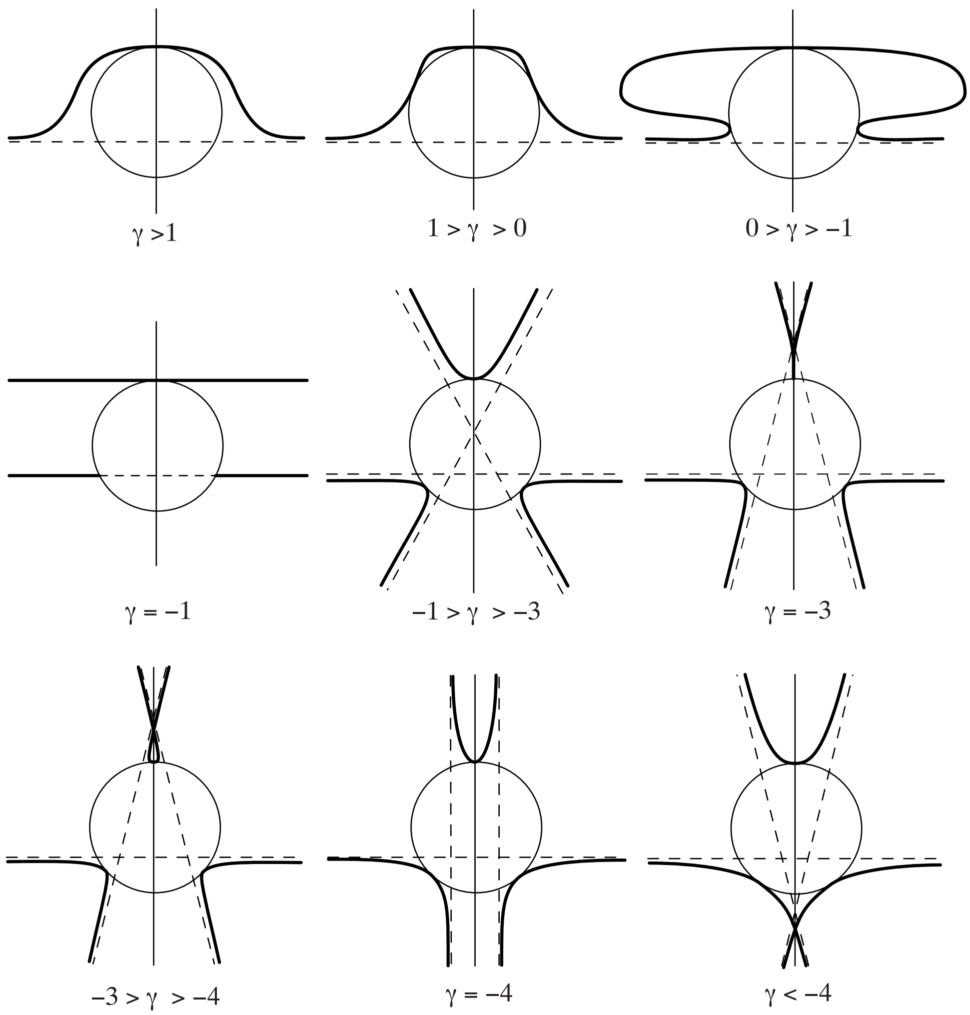

FIGURE 4. Bifurcation of the inflection locus in the (V, U)-plane, $\Lambda \neq 0$

5.2.2. $\lambda = 0$. Making $\Lambda = 0$ in equations (38) we obtain:

$$
\begin{cases}
\cos\omega\, V + \sin\omega\, U - \cos\omega\, S = 0 \\
-\sin\omega\, V + \cos\omega\, U - \sin\omega\, S = 0 \\
-2\Gamma\cos\omega\, V - 2\sin\omega\, U - 2(1+\Gamma)\cos\omega\, S = 0 \\
R = 0
\end{cases} \tag{40}
$$

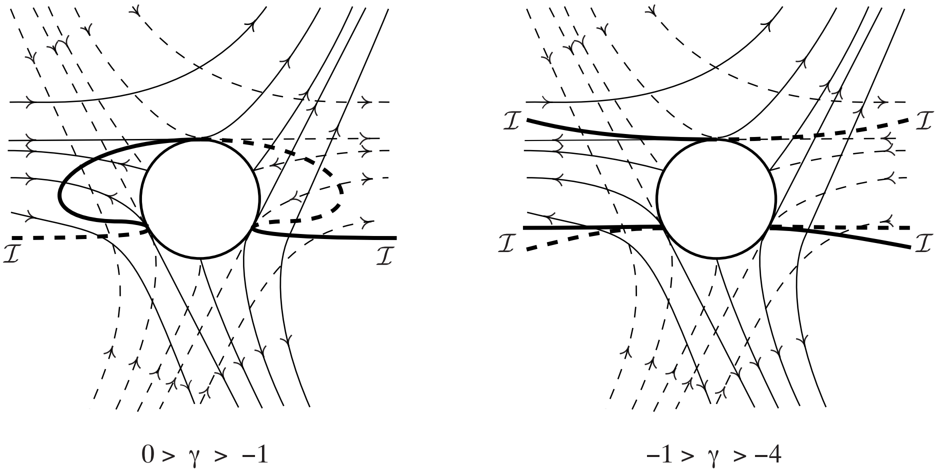

FIGURE 5. Orientation of the rarefaction curves, $\lambda \neq 0$

Now the only nontrivial solutions are the singular ones (39), with $\Lambda = 0$:

$$
\begin{cases} \omega = \dfrac{\pi}{2} \\ V = -S \\ U = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} \omega = \pm \arctan\left(\dfrac{1}{3}\sqrt{-3(1+2\Gamma)}\right) \\ V = \dfrac{2+\Gamma}{1-\Gamma} S \\ U = \pm \dfrac{\sqrt{-3(1+2\Gamma)}}{1-\Gamma} S \end{cases} \tag{41}
$$

We are interested in the projection of the inflection locus in the plane (V, U), or (v, u), in order to study the orientation of the rarefaction curves.

Thus for $\lambda = 0$ the points of bifurcation are $\gamma = -1$ and $\gamma = -4$ (fig. 6), as the projection of the inflection locus in the (v, u)-plane gives the straight lines:

$$
u = 0, \qquad u = \pm \frac{\gamma\sqrt{-3(1+\gamma)}}{\gamma + 4} v
$$

In the (V, U)-plane the projection is given by:

$$
U = 0, \qquad U = \pm \frac{\sqrt{-3(1+2\Gamma)}}{\Gamma + 2} V
$$

Thus, even if the phase portrait for the rarefaction curves in the plane (v, u) is the same for all values $\gamma < 0$, there is a change in the orientation of the rarefaction curves when $\gamma = -1$ (fig. 7), for instance.

5.3. **Exceptional inflection locus.** We can easily verify that, in our model, $\mathcal{I}$ is smooth if $\Lambda \neq 0$. A point $p = (U, 0, \Omega, s) \in \mathcal{I} \backslash \mathcal{E}$ is a degenerate critical point for the graph of s along the rarefaction curve through p if and only if the rarefaction curve is tangent to $\mathcal{I}$ at p (proposition 4).

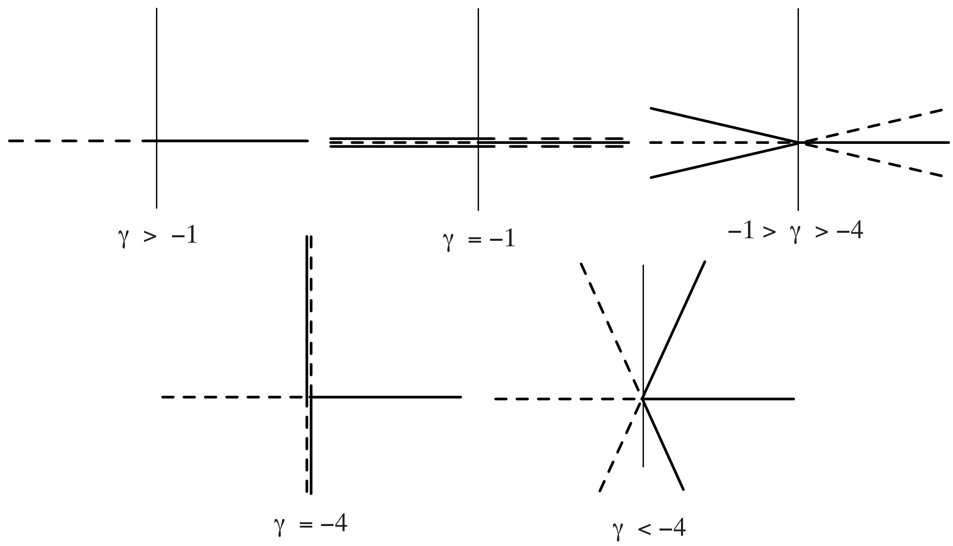

FIGURE 6. Bifurcation of the inflection locus in the (v, u)-plane, $\lambda = 0$

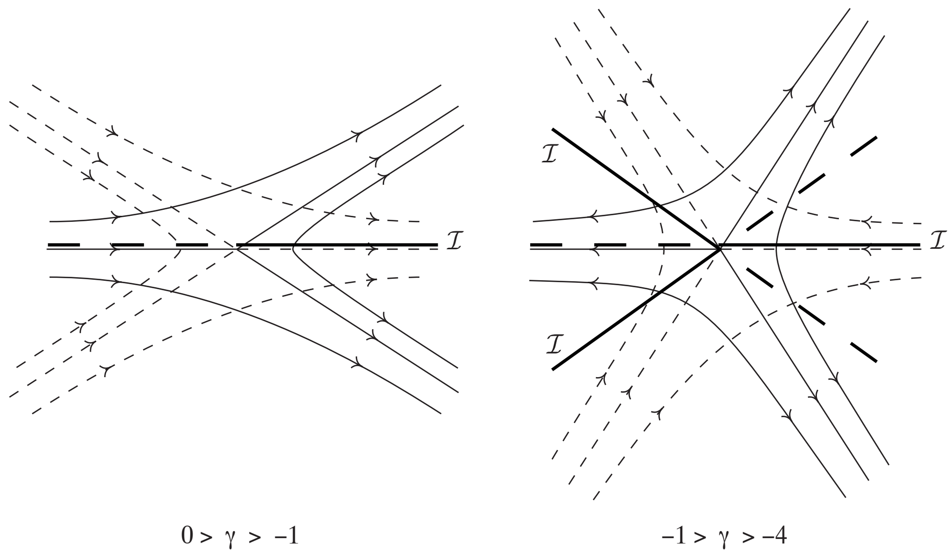

FIGURE 7. Orientation of the rarefaction curves, $\lambda = 0$

A rarefaction curve is an integral curve of the rarefaction line field $\alpha_{\mathcal{C}} = 0$. So, the rarefaction curve is tangent to $\mathcal{I}$ at $p = p(\omega)$ if and only if

$$\sin\omega \frac{dV}{d\omega} - \Gamma \cos\omega \frac{dU}{d\omega} = 0$$

with V and U given by equations (37), the parametrization of the inflection locus for $\lambda \neq 0$. This leads to:

$$\frac{(4(\Gamma^2-1)\cos^4\omega + 4\cos^2\omega - 1)\sin\omega}{(3-2(1-\Gamma)\cos^2\omega)^2\cos^2\omega}\Lambda = 0$$

with solutions:

$$\omega = 0, \quad \omega = \pm\arctan(\sqrt{1+2\Gamma}), \quad \omega = \pm\arctan(\sqrt{1-2\Gamma})$$

Remark 7. We can easily verify that, for the singularities of the parametrization:

$$\omega = \pm\frac{\pi}{2}, \quad \omega = \pm\arccos\sqrt{\frac{3}{2(1-\Gamma)}}$$

the condition for being an inflection point is never satisfied unless $\lambda = 0$.

The solution $\omega = 0$ corresponds to the critical point of the rarefaction line field $P_1 = (0, \Lambda)$ (in coordinates (V, U)).

The other critical points, $P_{2,3} = \left(\pm\frac{\sqrt{1-2\Gamma}}{1-\Gamma}\Lambda, \frac{\Gamma}{1-\Gamma}\Lambda\right)$, of the rarefaction line field are obtained for $\omega = \mp\arctan(\sqrt{1-2\Gamma})$, respectively.

Those points are in $\mathcal{E}$, but the points $P_{4,5} = \left(\pm\frac{1+\Gamma}{\sqrt{1+2\Gamma}}\Lambda, 0\right)$, corresponding to $\omega = \mp\arctan(\sqrt{1+2\Gamma})$, respectively, are not. These points are well defined only for $\Gamma > -\frac{1}{2}$ ($\gamma > -1$), and then:

$$\begin{aligned}\mathcal{H}_0 &= \{P_0, P_{1,2}\} \cup \left\{\left(\pm\frac{1+\Gamma}{\sqrt{1+2\Gamma}}\Lambda, 0, 0, \mp\arctan\sqrt{1+2\Gamma}, \mp\frac{\Gamma}{\sqrt{1+2\Gamma}}\Lambda\right)\right\} \\ &= \{P_0, P_{1,2}\} \cup \left(\overline{\mathcal{D}} \cap \mathcal{C}\right)\end{aligned}$$

The inflection locus for $\lambda = 0$ is given by the solutions of equations (41) together with the line $\{V = U = S = R = 0\}$; the rarefaction curve is never tangent to $\mathcal{I}$ except along the trivial solution $\{V = 0, U = 0, S = 0, R = 0\}$. Thus, for $\Lambda = 0$:

$$\mathcal{H}_0 = \{(0,0,0,\omega,0)\}$$

6. Shock curves

To study the shock curves, let $(v, u) = \mathcal{U}_-$ and $(x, y) = \mathcal{U}_+$; the fundamental wave manifold is defined by two equations:

$$s(x - v, y - u) = F(x, y, \lambda, \gamma) - F(v, u, \lambda, \gamma)$$

and eliminating s between them, we arrive at:

$$\begin{aligned} y^3 + (\gamma - 1)yx^2 - (\gamma + 1)yv^2 - (\gamma + 1)ux^2 + 2vux + 2vyx - \\ -uy^2 + 2\lambda(y - u)^2 - yu^2 + u^3 + (\gamma - 1)uv^2 = 0 \end{aligned} \tag{42}$$

The projection of the characteristic manifold, forgeting s, is given by:

$$x = v, \quad y = u, \quad \gamma^2 v^2 + (2u + \lambda)^2 \geq \lambda^2 \tag{43}$$

the projection of the coincidence locus corresponding to $\gamma^2 v^2 + (2u + \lambda)^2 = \lambda^2$.

If we consider (x, y) as variables and (v, u, λ, γ) as parameters, the shock curves can be seen as the solutions of a bifurcation problem:

$$y^3 + (\gamma - 1)x^2 y + r(x, y, v, u, \lambda, \gamma) = 0, \quad r(x, y, 0, 0, 0, \gamma) = 0 \tag{44}$$

(where the terms of r in x and y are of order at most two) with core the $D_{4\pm}$ singularity $y^3 + (\gamma - 1)x^2 y$, the elliptic umbilic for $\gamma < 1$ and the hyperbolic umbilic for $\gamma > 1$. The structure of the shock curves can be obtained from the knowledge of that singularity and its unfolding.

Remark 8. The singularities of the bifurcation locus can be simpler, in general, than those of the function (44): when the projection $(v, u, x, y, s) \mapsto (v, u, x, y)$ has singularities along a level set of $\mathcal{F}$, the singularities in the smaller space are more complex than in the space above.

6.1. Primary Bifurcation Locus.

Here we deal with the singularities of the shock curves before blow-up, and only those that are affected by it: at rarefaction points.

We begin by studying the universal unfolding of the umbilics, in the form:

$$H = Y^3 + AX^2 Y + aX + bY + cY^2$$

with parameters (a, b, c); the elliptic, respectively hyperbolic, case corresponds to $A < 0$, respectively $A > 0$. We reproduce the main points in the analysis of the respective catastrophes from [14], where different but equivalent unfoldings are used.

The catastrophe set, where H has critical points, is defined by:

$$a = -2AXY, \quad b = -3Y^2 - AX^2 - 2cY \tag{45}$$

These critical points are Morse except when they belong to the cone:

$$-12AX^2 + (6Y + c)^2 = c^2 \tag{46}$$

The image of this set under the map:

$$\Phi : (X, Y, c) \mapsto (a = -2AXY, b = -3Y^2 - AX^2 - 2cY, c = c) \tag{47}$$

is the bifurcation set.

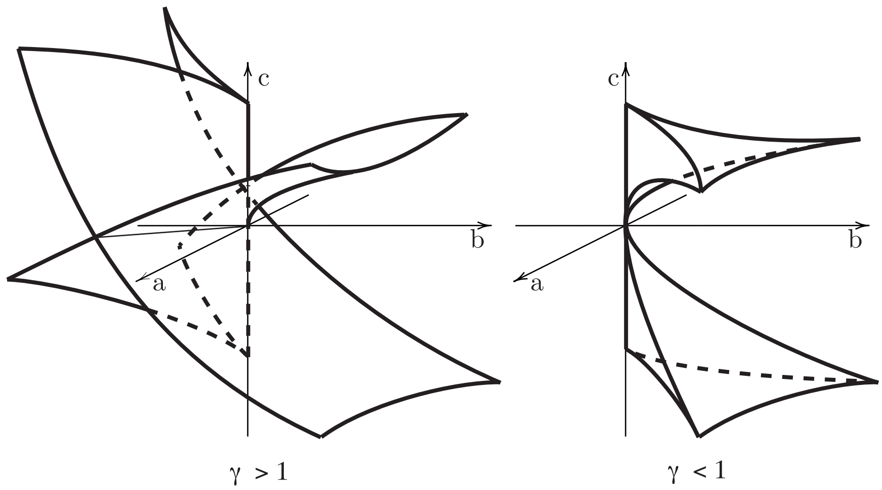

FIGURE 8. Bifurcation set: hyperbolic umbilic ($\gamma > 1$) and elliptic umbilic ($\gamma < 1$)

In general the direction along which the second derivative of H is degenerate does not coincide with the directions:

$$Y = 0, \ \sqrt{-A}X, \ -\sqrt{-A}X$$

along which the cubic part is zero; but this happens on the lines:

$$\begin{aligned} X &= 0, & Y &= 0 \\ X &= -\frac{\sqrt{-A}c}{4A}, & Y &= \frac{c}{4} \\ X &= \frac{\sqrt{-A}c}{4A}, & Y &= -\frac{c}{4} \end{aligned} \tag{48}$$

In the (X, Y, c) space, we can think of these lines as generatrices on the cone (46). Their image under Φ in the parameter space is the cuspidal edges of the bifurcation set.

A complete description of the critical points and critical levels of H appears in [14, chapter 9]; the main result is that the nature of the critical point is completely determined by its position in the catastrophe set relatively to the bifurcation set and the special lines on it. We will just recall that the critical points in the catastrophe set but not on the bifurcation set, and outside the cone, i.e. where $-12AX^2+(6Y+c)^2 > c^2$, are saddle points; there are fold points on the bifurcation set outside the cuspidal edges, and cusp points on those.

Going back to our equation (42), we see that after the change of coordinates:

$$X = x + \frac{\xi}{\gamma - 1}, \quad Y = y - \frac{\gamma+1}{\gamma-1}\eta, \quad \xi = v, \quad \eta = u$$

it becomes

$$\begin{aligned} Y^3 + (\gamma-1)X^2Y + &a(\xi,\eta,\lambda,\gamma)X + b(\xi,\eta,\lambda,\gamma)Y + \\ &+ c(\xi,\eta,\lambda,\gamma)Y^2 + d(\xi,\eta,\lambda,\gamma) = 0 \end{aligned} \tag{49}$$

with:

$$\begin{aligned} a(\xi,\eta,\lambda,\gamma) &= \frac{4\gamma}{\gamma-1}\xi\eta \\ b(\xi,\eta,\lambda,\gamma) &= -\frac{\gamma^2}{\gamma-1}\xi^2 + \frac{8\lambda}{\gamma-1}\eta + \frac{8\gamma+4}{(\gamma-1)^2}\eta^2 \\ c(\xi,\eta,\lambda,\gamma) &= 2\lambda + 2\frac{\gamma+2}{\gamma-1}\eta \\ d(\xi,\eta,\lambda,\gamma) &= 4\left(-\frac{\gamma^2}{(\gamma-1)^2}\xi^2 + \frac{2\lambda}{(\gamma-1)^2}\eta + \frac{2\gamma}{(\gamma-1)^3}\eta^2\right)\eta \end{aligned} \tag{50}$$

Thus the nature of the critical points of (42) is determined studying the intersection of the image of the map:

$$\Psi : (\xi,\eta,\lambda,\gamma) \mapsto (a(\xi,\eta,\lambda,\gamma), b(\xi,\eta,\lambda,\gamma), c(\xi,\eta,\lambda,\gamma))$$

with the bifurcation set. It is easy to see that, for fixed $(\lambda, \gamma \neq -2)$, that image is (diffeomorphic to) a Whitney umbrella, a surface in 3-space given by:

$$x = st, \quad y = s^2, \quad z = t$$

Remark 9. When $\gamma = -2$, the image of Ψ is a subset of the plane $c = 2\lambda$, containing the origin in its interior.

We have already seen how to describe the characteristic manifold in the variables (v, u, x, y); in the coordinates (X, Y, ξ, η), (43) becomes:

$$X = \frac{\gamma}{\gamma-1}\xi, \quad Y = -\frac{2}{\gamma-1}\eta, \quad \gamma^2\xi^2 + (2\eta+\lambda)^2 \geq \lambda^2 \tag{51}$$

or equivalently

(52)
$$\xi = \frac{\gamma-1}{\gamma}X, \quad \eta = -\frac{\gamma-1}{2}Y, \quad (\gamma-1)^2X^2 + ((\gamma-1)Y - \lambda)^2 \geq \lambda^2$$

Lemma 1. *In the space (X, Y, a, b, c), the characteristic manifold is contained in the catastrophe set, the coincidence locus in the bifurcation set and the critical points of the rarefaction foliation in the cusp lines.*

Proof. In the space (X, Y, a, b, c) we view the characteristic manifold as a surface $\mathcal{S}$, parametrized by (ξ, η) such that $\gamma^2\xi^2 + (2\eta + \lambda)^2 \geq \lambda^2$ (for fixed λ and γ):

$$(\xi, \eta) \mapsto (\frac{\gamma}{\gamma - 1}\xi, -\frac{2}{\gamma - 1}\eta, a(\xi, \eta), b(\xi, \eta), c(\xi, \eta))$$

The projection of $\mathcal{S}$ on the space (X, Y, c) is given by:

$$c = 2\lambda - (\gamma + 2)Y, \quad (\gamma - 1)^2X^2 + ((\gamma - 1)Y - \lambda)^2 \geq \lambda^2$$

writing $c(\xi, \eta)$ in terms of $X$and Y. We can rewrite the inequality, solving $c = 2\lambda - (\gamma + 2)Y$ for λ, to obtain:

$$c = 2\lambda - (\gamma + 2)Y, \quad -12(\gamma - 1)X^2 + (6Y + c)^2 \geq c^2$$

when $\gamma - 1 < 0$, otherwise:

$$c = 2\lambda - (\gamma + 2)Y, \quad (6Y + c)^2 \leq 12(\gamma - 1)X^2 + c^2$$

Thus, in any case, the projection of $\mathcal{S}$ on the space (X, Y, c) is the part of the plane $c = 2\lambda - (\gamma + 2)Y$ outside and on the cone of non-Morse singularities (Fig. 9).

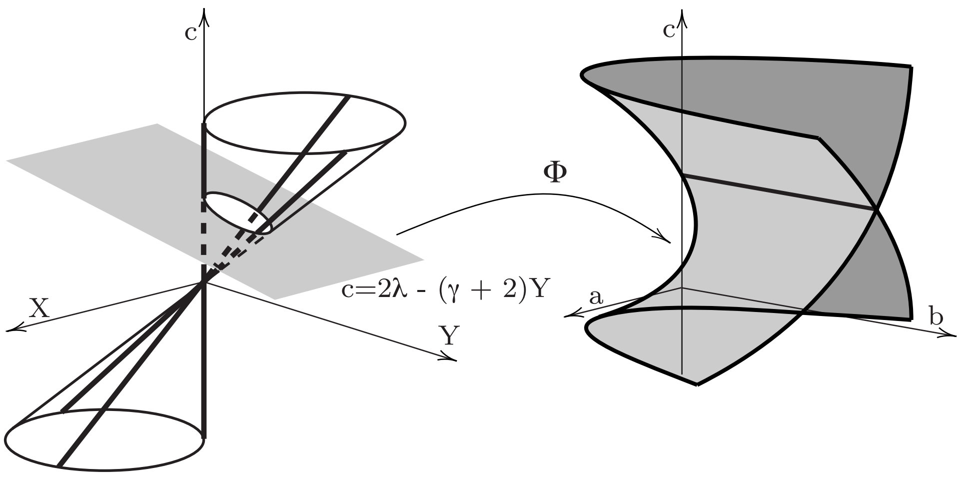

Figure 9. Characteristic manifold in (X, Y, a, b, c) space

Writing the equality $(\gamma - 1)^2X^2 + ((\gamma - 1)Y - \lambda)^2 = \lambda^2$ in terms of ξ and η we obtain the equation of the coincidence locus, and similarly we see that the intersection of the cusp lines with that plane corresponds to the critical points.

Therefore the proof will be finished if we show that $\mathcal{S}$ is contained in the catastrophe set:

$$\{(X, Y, a, b, c) : (a, b, c) = \Phi(X, Y, c)\}$$

or equivalently, that:

$$\begin{aligned}
a(\xi,\eta) =& -2(\gamma-1)\frac{\gamma}{\gamma-1}\xi\left(-\frac{2}{\gamma-1}\eta\right)\\
b(\xi,\eta) =& -(\gamma-1)\left(\frac{\gamma}{\gamma-1}\xi\right)^2-3\left(-\frac{2}{\gamma-1}\eta\right)^2-\\
&-2\left(2\lambda-(\gamma+2)\left(-\frac{2}{\gamma-1}\eta\right)\right)\left(-\frac{2}{\gamma-1}\eta\right)
\end{aligned}$$

This is easily verified by a straightforward computation. □

Remark 10. If $\lambda = 0$, the plane $c = 2\lambda - (\gamma+2)Y$ intersects the cone just at the origin; all points correspond to saddles, except the origin: for $\gamma < 1$ the critical point is equivalent to the monkey saddle $Y^3 - X^2Y$, and for $\gamma > 1$ the critical point is equivalent $Y^3 + X^2Y$. The respective critical levels are: three straight lines through the origin, $Y = 0$ and $Y = \pm X$, or just one line $Y = 0$.

We are interested in the bifurcations of the level sets of (49); their connection with the critical points of H is given by:

Lemma 2. *For fixed $(\xi, \eta, \lambda, \gamma)$, the zero set level of (49) is the critical level of H for the corresponding values of (a, b, c).*

Proof. The critical values of H are obtained when $(a, b, c) = \Phi(X, Y, c)$, on the catastrophe set; we have proved above that $(a(\xi,\eta), b(\xi,\eta), c(\xi,\eta)) = \Phi(X(\xi,\eta), Y(\xi,\eta), c(\xi,\eta))$. Again the proof is just a computation, as we only need to show that:

$$H(X(\xi,\eta), Y(\xi,\eta), b(\xi,\eta), c(\xi,\eta), a(\xi,\eta)) + d(\xi,\eta) \equiv 0$$

and that is easily verified. □

From the lemmas we get a complete description of the primary bifurcations:

Theorem 3. *All points in the characteristic manifold are in the primary bifurcation set; the bifurcations of the shock curves correspond to saddle points outside the coincidence locus, to fold points on the coincidence locus and to cusp points at the critical points of the rarefaction foliation. At $\lambda = 0$ there are only saddle points and a critical level at the origin, either three straight lines through the origin if $\gamma < 1$ or one straight line if $\gamma > 1$.*

6.2. **Secondary Bifurcation Locus.** Here we only consider singularities of the shock curves outside the characteristic manifold; these are not affected by the blow-up process. On the other hand, as it will become clear, all primary bifurcations but those at the critical points disappear in that process.

We have seen before that the fundamental wave manifold can be written as the graph of the map:

$$W : (R, \omega, S) \mapsto (V, U)$$

defined by:

$$\begin{cases} V = a_1(\omega)R + b_1(\omega)S + c_1(\omega) \\ U = a_2(\omega)R + b_2(\omega)S + c_2(\omega) \end{cases}$$

with the coefficients given in (21).

Then it is clear that:

Proposition 25. *The right secondary bifurcation locus $\mathcal{B}_R$ is the subset of the graph of the map W corresponding to its critical points.*

Therefore, the set $\mathcal{B}_R$ is the subset of the fundamental wave manifold where:

$$\text{rank } DW(R, \omega, S) < 2 \quad \text{and} \quad R \neq 0$$

This leads to the equations:

$$\begin{cases} \sin\omega\ (1 - 2(1-\Gamma)\cos^2\omega) = 0 \\ -\cos\omega\ (1 + 4\Gamma + 2(1-\Gamma)\cos^2\omega)R + 4S = 0 \end{cases} \tag{53}$$

The solutions of the first equation are exactly the same ones we found in (34) as the values of ω corresponding to critical points of the rarefaction foliation:

$$w = 0, \quad \omega = \pm \arccos \frac{1}{\sqrt{2(1-\Gamma)}}$$

Solving the second equation for S, with the above values for ω, and using the expression for W, gives:

(54)

$$\begin{cases} \omega = 0 \\ V = \dfrac{1-2\Gamma}{4} R \\ U = \Lambda \\ S = \dfrac{3+2\Gamma}{4} R \end{cases}, \text{ and } \begin{cases} \omega = \pm \arccos \dfrac{1}{\sqrt{2(1-\Gamma)}} \\ V = \mp \dfrac{\sqrt{1-2\Gamma}}{1-\Gamma}\Lambda + \dfrac{\Gamma(2\Gamma-1)\sqrt{2(1-\Gamma)}}{4(\Gamma-1)^2} R \\ U = \dfrac{\Gamma}{1-\Gamma}\Lambda \mp \dfrac{(2\Gamma-1)\sqrt{2(\Gamma-1)(2\Gamma-1)}}{4(\Gamma-1)^2} R \\ S = \dfrac{(2\Gamma+1)\sqrt{2(1-\Gamma)}}{4(1-\Gamma)} R \end{cases}$$

These are three (one, for $\Gamma \geq 1/2$) straight lines whose projections on the (V, U)-plane are tangent to the projection of the coincidence locus $V^2 + U^2 = \Lambda^2$ at the critical points of the rarefaction foliation, and given by:

$$U = \Lambda, \quad U = \frac{1-\Gamma}{\Gamma}\Lambda \pm \frac{\sqrt{1-2\Gamma}}{\Gamma}\ V \tag{55}$$

where the two last solutions make sense only for $\gamma \leq 1$ (or $\Gamma \leq 1/2$).

The important point here is that R does not vanish identically along these lines, therefore they do not belong to the characteristic manifold; in particular there are no secondary bifurcations on the characteristic manifold outside the critical points of the rarefaction foliation.

Returning to coordinates (v,u), (54) become:

$$\begin{cases} u = 0 \\ R = 2\dfrac{\gamma}{1-\gamma} v \end{cases} , \quad \begin{cases} u = \dfrac{1-\gamma}{\gamma}\lambda \pm \sqrt{1-\gamma}\; v \\ R = \mp 2\dfrac{\sqrt{(1-\gamma)(2-\gamma)}}{\gamma(\gamma-1)}\lambda - \dfrac{(2-\gamma)\sqrt{2-\gamma}}{\gamma-1}\; v \end{cases}$$

Each solution is also a straight line in the (v,u)-plane passing through the critical points of the rarefaction curves, and its slope in the (v,u)-plane is defined by $\tan\omega$ for the corresponding value of ω.

It has been remarked in [11] that they coincide with the projection of separatrices on that plane.

This can also be seen as follows: the second (or third) equation in (53) can be solved for R; as R is not involved in the first equation, we see that the lines of secondary bifurcations project on the ruled surface $\mathcal{C}$ as one of its lines, with fixed ω; comparing with the corresponding values for the separatrix $\omega = \mathit{constant}$ (34) through the same critical point, the result follows.

To sudy the left bifurcation locus $\mathcal{B}_L$ we have to consider the critical points of the map:

$$W_R : (R,\omega,S) \mapsto W(R,\omega,S) + (\Gamma R\cos\omega, R\sin\omega)$$

for $R \neq 0$ and, as before, we see that the condition: rank $DW_R(R,\omega,S) < 2$ leads to two independent equations:

$$\begin{cases} \sin\omega\; (1 - 2(1-\Gamma)\cos^2\omega) = 0 \\ -\cos\omega\; (3 - 2(1-\Gamma)\cos^2\omega)R + 4S = 0 \end{cases} \tag{56}$$

The first equation is the same as the first in (53), and solving the second one for S, with the above values for ω, and using the expression for W_R, gives:

(57)

$$\begin{cases} \omega = 0 \\ V = -\dfrac{1-2\Gamma}{4}R \\ U = \Lambda \\ S = \dfrac{1+2\Gamma}{4}R \end{cases} , \text{ and } \begin{cases} \omega = \pm \arccos \dfrac{1}{\sqrt{2(1-\Gamma)}} \\ V = \mp\dfrac{\sqrt{1-2\Gamma}}{1-\Gamma}\Lambda + \dfrac{\Gamma\sqrt{2(1-\Gamma)}}{4(\Gamma-1)^2}R \\ U = \dfrac{\Gamma}{1-\Gamma}\Lambda \mp \dfrac{\sqrt{2(1-\Gamma)(1-2\Gamma)}}{4(1-\Gamma)^2}R \\ S = \dfrac{\sqrt{2(1-\Gamma)}}{4(1-\Gamma)}R \end{cases}$$

These lines belong to $\mathcal{S}_L$ and their projections on the (V,U)-plane coincide with those of $\mathcal{B}_R$:

$$U = \Lambda, \quad U = \frac{1-\Gamma}{\Gamma}\Lambda \pm \frac{\sqrt{1-2\Gamma}}{\Gamma}\, V \tag{58}$$

Theorem 4. *All secondary bifurcations correspond to Morse critical points of the function (44), in fact to saddle points.*

Proof. We verified that the secondary bifurcation loci are formed by straight lines passing through the critical points of the rarefaction foliation.

Going back to the coordinates (X, Y, ξ, η), these lines in the space (X, Y, c) are still straight lines passing through the points corresponding to the critical points, are tangent to the cone of non-Morse singularities at those points, but they do not belong to that cone; therefore all points corresponding to $R \neq 0$ are outside that cone, and correspond to saddle points of the function (44).

Along those lines, apart from the critical points, the singularities of the level sets of $\mathcal{F}$ and those of the function (44) are the same, as explained in remark 8. $\square$

7. Composite curves

In order to study the behaviour of the composite foliation, and since a composite curve may be seen as a curve in the sonic locus, the geometry of the sonic locus has to be analysed, specially its projection on the space of the variables (V, U).

7.1. **Geometry of the sonic loci.** The right sonic locus and the left sonic locus are similar, as seen in the proof of proposition 21, so it should be sufficient to study the behaviour of the left sonic locus, and both projections on $\mathcal{U}_-$ and $\mathcal{U}_+$. Instead, we consider both loci separately and their projection on the same space, $\mathcal{U}_-$.

Remark 11. We will always assume $\Lambda \geq 0$ (or $\lambda \geq 0$) from now on; the results for $\Lambda \leq 0$ are absolutely analogous.

We consider the curve $\mathcal{E}_L$ on the sonic locus, parametrized by ω (for λ and γ fixed):

$$\mathcal{E}_L : \omega \mapsto (V_1, U_1, R_1, \omega, 0)$$

as in (29).

Proposition 26. *The image of the restriction of π_L^- to $\mathcal{E}_L$ is contained in the circumference $V^2+U^2 = \Lambda^2$ and moreover the image of $\mathcal{E}_L$ under $\Pi_L : \mathcal{S}_L \longrightarrow \mathcal{C}$, defined by $\Pi_L(\mathcal{U}_-,\mathcal{U}_+, s) = (\mathcal{U}_-,\mathcal{U}_-, s)$ is contained in the coincidence locus:*

$$\Pi_L(\mathcal{E}_L) \subset \mathcal{E}$$

If $\gamma < -1$ that restriction is a double cover of $V^2 + U^2 = \Lambda^2$, if $\gamma > -1$ its image is the part of that circumference where $U \geq \frac{\Gamma}{1+\Gamma}\Lambda$, and $Q_{1,2} = \mathcal{E}_L\left(\pm \arctan \sqrt{1 + 2\Gamma}\right)$ are fold points.

Proof. It is clear that the image of $\mathcal{E}_L$ by π_L^- is contained in the circumference $V^2 + U^2 = \Lambda^2$. This also leads to $\Pi_L(\mathcal{E}_L) \subset \mathcal{E}$.

If we consider $U_1(\omega)$, we see that:

$$\frac{d}{d\omega}U_1(\omega) = 0 \Longrightarrow \omega = 0, \pm\pi/2, \pm \arctan \sqrt{-1 - 2\Gamma}, \pm \arctan \sqrt{1 + 2\Gamma}$$

but only the last values are also zeros for $dV_1(\omega)/d\omega$. They are well defined only when $\Gamma > -1/2$, or $\gamma > -1$, and then correspond to nondegenerate minima of U_1, assuming $\Lambda > 0$.

Thus, when $\Gamma > -\frac{1}{2}$ and $\Lambda > 0$, there are two fold points:

$$Q_{1,2} = \left(\mp\frac{\sqrt{1 + 2\Gamma}}{1 + \Gamma}\Lambda, \frac{\Gamma}{1 + \Gamma}\Lambda, \mp 2\Gamma\frac{\sqrt{2(1 + 2\Gamma)(1 + \Gamma)}}{(1 + 2\Gamma)(1 + \Gamma)}\Lambda, \pm \arctan \sqrt{1 + 2\Gamma}, 0\right)$$

otherwise (with $\lambda \neq 0$) $\pi_L^-(\mathcal{E}_L) = (V_1(\omega), U_1(\omega))$ has nonzero velocity everywhere and passes twice at any point of $V^2 + U^2 = \Lambda^2$. $\square$

Proposition 27. *The critical set of the projection $\pi_L^- : \mathcal{S}_L \to \mathbb{R}^2$ is the union of $\mathcal{E}_L$, as defined in (29), and the double sonic locus $\mathcal{D} = d_{1,2}$:*

$$\Sigma(\pi_L^-) = \mathcal{E}_L \cup \mathcal{D}$$

The points in $(\mathcal{E}_L \cup d_{1,2}) - \{Q_{1,2}\}$ are the fold points for π_L, and $Q_{1,2} = d_{1,2} \cap \mathcal{E}_L$ are degenerate cusp points, for $\gamma > -1$.

Proof. As we have seen before, for fixed $(\omega, \Lambda, \Gamma)$, the projection of the left sonic locus on the (V, U, S)-space is the intersection of the hyperboloid (22) with a tangent plane (25); the tangency point (26)

$$P_{tg}(\omega) = \left(-\frac{b(\omega)}{d(\omega)}\Lambda, -\frac{c(\omega)}{d(\omega)}\Lambda, 0, \omega, \frac{a(\omega)}{d(\omega)}\Lambda\right)$$

belongs to the inflection locus, and that intersection is given by the two straight lines $l_\pm$ passing through that point defined by the directions

$$\nu_\pm = \left(\frac{c^2 - a^2}{ab \pm cd}, \frac{-bc \mp ad}{ab \pm cd}, 1\right)$$

Using the relation $a^2 - b^2 - c^2 + d^2 \equiv 0$, we get:

$$\left\| \left(\frac{c^2 - a^2}{ab \pm cd}, \frac{-bc \mp ad}{ab \pm cd}\right) \right\| = 1$$

therefore $\nu_\pm$ are well defined directions, the two first components giving a unit tangent vector of $V^2 + U^2 = \Lambda^2$.

It is easy to see that $\mathcal{E}_L(\omega) \in l_+$; the tangent vector $L(\omega)$ to $\mathcal{S}_L$ corresponding to ν_+, has components $(T(\omega), \rho(\omega), 0, \sigma(\omega))$ such that:

$$\begin{aligned} T(\omega) &= \frac{1}{ab + cd}(c^2 - a^2, -bc - ad) \\ \rho(\omega) &= \frac{1}{ab + cd} \frac{(ab + cd - c^2 + a^2)\cos\omega + (bc + ad)\sin\omega}{1/2 + \Gamma\cos^2\omega} \\ \sigma(\omega) &= 1 \end{aligned} \tag{59}$$

as R on $\mathcal{S}_L$ is given by:

$$R = \frac{(S - V)\cos\omega - (U + \Lambda)\sin\omega}{1/2 + \Gamma\cos^2\omega} \quad \text{or} \quad R = \frac{V + S}{\cos\omega} + \frac{\Lambda - U}{\sin\omega}$$

.

We can parametrize $\mathcal{S}_L$ by:

$$\Psi : (\omega, \xi) \mapsto \mathcal{E}_L(\omega) + \xi\ L(\omega), \quad L(\omega) = (T(\omega), \rho(\omega), 0, \sigma(\omega)) \tag{60}$$

It is clear that $L(\omega)$ is never tangent to $\mathcal{E}_L$, thus the parametrization is well defined.

To study the singularities of the projection π_L^-, defined on $\mathcal{S}_L$, we only have to consider the (V, U) coordinates of $\mathcal{S}_L$; these we can think of as parametrized as follows:

$$\Phi : (\omega, \xi) \mapsto (V_1(\omega), U_1(\omega)) + \xi T(\omega)$$

At the singular points we must have:

$$\frac{d}{d\omega}\left[P(\omega)+\xi T(\omega)\right] \parallel T(\omega) \tag{61}$$

where $P(\omega) = (V_1(\omega), U_1(\omega))$. This is verified on $\xi = 0$, and also for arbitrary ξ at values of ω such that:

$$\frac{d}{d\omega}P(\omega) = \frac{d}{d\omega}(V_1(\omega), U_1(\omega)) = 0$$

The first condition, $\xi = 0$, gives the points in $\mathcal{E}_L$, and, taking in account the proof of proposition 26, the second condition gives the points on the straight lines $d_{1,2}$ through $Q_{1,2}$, the double sonic locus as defined in (31).

The points in $\mathcal{E}_L - \{Q_{1,2}\}$ are fold points, since for $\xi = 0$ the two vectors in (61) cross each other with nonzero velocity, fixing ω and varying ξ:

$$\left[P(\omega)\cdot T(\omega) \equiv 0 \text{ and } \frac{d}{d\omega}P(\omega) = \varphi(\omega)T(\omega) \neq 0\right] \Longrightarrow \frac{dT}{d\omega}(\omega) \nparallel T(\omega)$$

where $\varphi(\omega) \neq 0$ follows from proposition 26. Thus:

$$\frac{d}{d\xi}\left\{\frac{d}{d\omega}\left[P(\omega)+\xi T(\omega)\right]\wedge T(\omega)\right\} = \frac{dT}{d\omega}(\omega)\wedge T(\omega) \neq 0$$

The same is true at the points on $d_{1,2} - \{Q_{1,2}\}$; now the two vectors cross each other with nonzero velocity, fixing $\xi \neq 0$ and varying ω: as proved in proposition 26, $\{Q_{1,2}\}$ correspond to nondegenerate minima of U_1, therefore

$$\left[\frac{d^2P}{d\omega^2}(\omega) \neq 0 \text{ and } \frac{dT}{d\omega}(\omega) = 0\right] \Longrightarrow$$

$$\Longrightarrow 0 \neq \frac{d\varphi}{d\omega}(\omega) = \frac{d^2P}{d\omega^2}(\omega)\cdot T(\omega) = -\frac{d^2T}{d\omega^2}(\omega)\cdot P(\omega) \Longrightarrow \frac{d^2T}{d\omega^2}(\omega) \nparallel T(\omega)$$

and

$$\frac{d}{d\omega}\left\{\frac{d}{d\omega}\left[P(\omega)+\xi T(\omega)\right]\wedge T(\omega)\right\} = \xi\frac{d^2T}{d\omega^2}(\omega)\wedge T(\omega) \neq 0$$

For the points $\{Q_{1,2}\}$ both derivatives computed above are zero, but:

$$\frac{d}{d\xi}\frac{d}{d\omega}\left\{\frac{d}{d\omega}\left[P(\omega)+\xi T(\omega)\right]\wedge T(\omega)\right\} = \frac{d^2T}{d\omega^2}(\omega)\wedge T(\omega) \neq 0$$

and therefore they are cusp points for π_L^-. Their degeneracy comes from the fact that they are the intersection of two lines of critical points, $Q_{1,2} = \mathcal{E}_L \cap d_{1,2}$. □

The change of behaviour at $\gamma = -1$ for the projection of $\mathcal{E}_L$ can be seen in fig. 10.

The image of the projection π_L^- on the plane (V, U) can be thought of as the set of tangents of the circumference $V^2 + U^2 = \Lambda^2$ at points

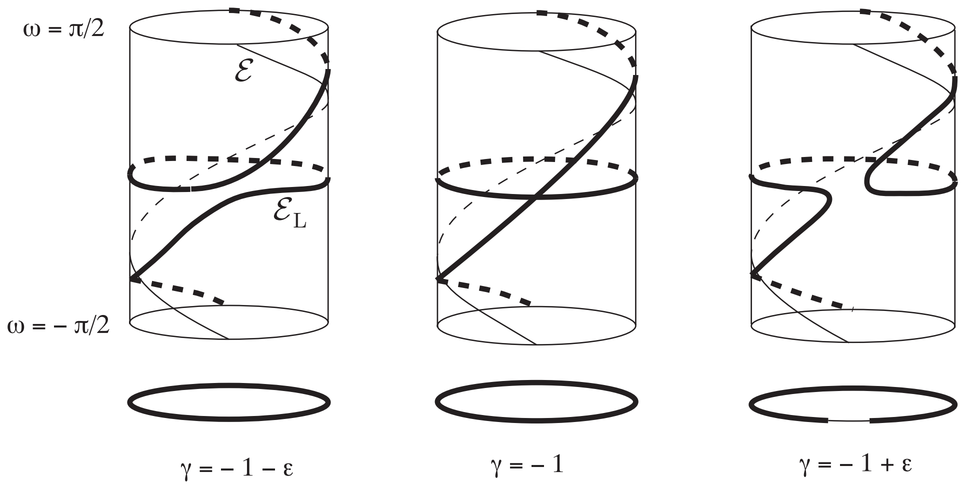

FIGURE 10. Bifurcation of the fold curve in the sonic locus, $\lambda \neq 0$

$(V_1(\omega), U_1(\omega))$ for which $U_1(\omega) \geq \Lambda\Gamma/(1+\Gamma)$; for $\gamma \leq -1$ this restriction is irrelevant.

All points in the circumference above $\Lambda\Gamma/(1+\Gamma)$ appear twice, and the terminal points:

$$\pi_L^-(Q_{1,2}) = \left(\mp\frac{\sqrt{1+2\Gamma}}{1+\Gamma}\Lambda, \frac{\Gamma}{1+\Gamma}\Lambda\right)$$

appear once; the tangents at those points, corresponding to lines $d_{1,2}$ on $\mathcal{S}_L$, are:

$$U = \pm\frac{\sqrt{1+2\Gamma}}{\Gamma}\,V + \frac{1+\Gamma}{\Gamma}\Lambda \tag{62}$$

In the original coordinates:

$$u = \pm\sqrt{1+\gamma}\;v + \frac{\lambda}{\gamma}$$

are the terminal tangents, at the points $(\mp\frac{\sqrt{1+2\Gamma}}{1+\Gamma}\Lambda, -\frac{\Gamma}{1+\Gamma}\Lambda)$ respectively, and the part of the ellipse covered is defined by $U \geq \frac{1}{1+\Gamma}\Lambda$; this is shown in fig. 11, for $\Lambda \neq 0$, and in fig. 12 for $\Lambda = 0$, along with the number of points of the sonic locus that project in one point of the plane (V, U).

Proposition 28. *For $\gamma < -1$ $(\Gamma < -1/2)$, the (closure of the) left sonic locus is a double cover of the characteristic manifold.*

Proof. The left sonic locus $\mathcal{S}_L$ is a ruled surface, it can be parametrized as before by $\Psi(\omega, \xi)$, ω beeing the parameter along the curve $\mathcal{E}_L$ and ξ the parameter along the straight line corresponding to a tangent to the hyperboloid at $(V_1(\omega), U_1(\omega), 0)$ in the (V, U, S)-space.

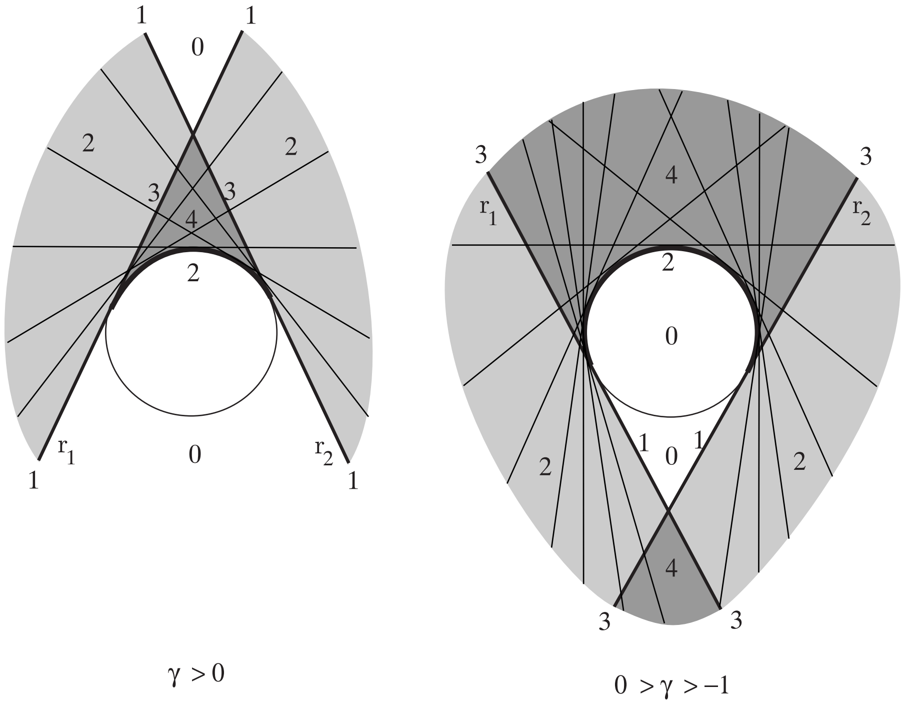

FIGURE 11. Projection of the left sonic locus, $\lambda > 0$

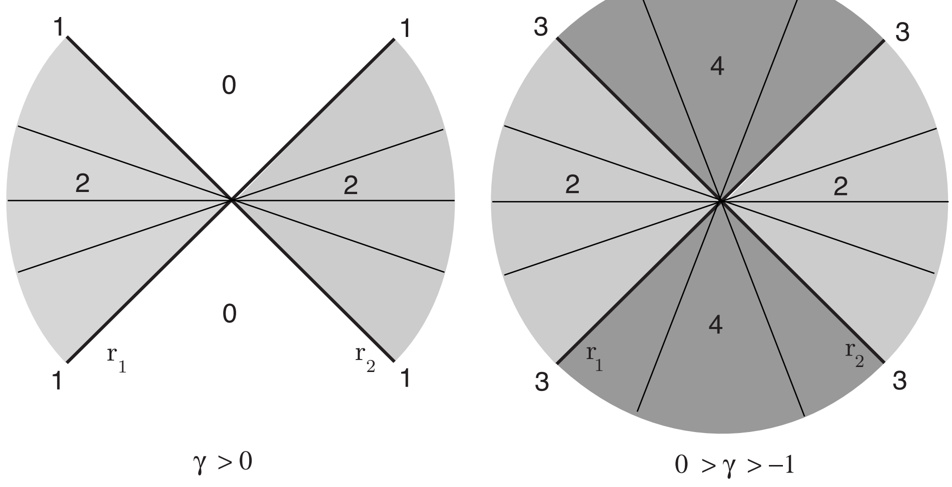

FIGURE 12. Projection of the left sonic locus, $\lambda = 0$

Similarly, the characteristic manifold can be parametrized by $(\theta, \xi) \in [-\pi/2, \pi/2] \times \mathbb{R}$, θ being the parameter along the coincidence locus $\mathcal{E}$ and ξ the parameter along the line corresponding to a tangent to the hyperboloid at $(-\Lambda \sin 2\theta, \Lambda \cos 2\theta, 0)$.

The coincidence locus $\mathcal{E}$ is clearly diffeomorphic to the circumference $V^2 + U^2 = \Lambda^2$ with the standard parametrization, and we proved above (proposition 26) that $\mathcal{E}_L$ is a double cover of that circumference. Thus $\mathcal{E}_L$ is a double cover of $\mathcal{E}$, and extending that by the identity along the lines, we get a double cover of the characteristic manifold $\mathcal{C}$ by the left sonic locus $\mathcal{S}_L$. □

Remark 12. The case $\gamma = -2$ is specially simple, since the map:

$$(v, u, R, \omega, s) \in \mathcal{S}_L \mapsto (v, u, 0, -2\omega, -s) \in \mathcal{C}$$

induces a double cover of the characteristic manifold by the left sonic locus.

The characteristic manifold is a twisted cylinder, and the sonic loci are also cylinders (twisted in the opposite direction, when $\gamma < -1$).

Proposition 29. *The set $\Sigma(\pi_R^-)$ of critical points of the projection of the right sonic locus on the (V, U)-plane (or on the (v, u)-plane) is formed by a curve $\mathcal{I}_R$ and the lines r_0, $r_{1,2}$, with $\mathcal{H}_R = \mathcal{I}_R$ and $\mathcal{B}_R = \{r_0,\ r_{1,2}\}$; all the critical points are fold points, except for the two degenerate critical points $X_{1,2} = \mathcal{I}_R \cap r_{1,2}$.*

Moreover, the image of $\mathcal{I}_R$ under $\Pi_R : \mathcal{S}_R \longrightarrow \mathcal{C}$, defined by

$$\Pi_R(\mathcal{U}_-, \mathcal{U}_+, s) = (\mathcal{U}_+, \mathcal{U}_+, s)$$

is the inflection locus:

$$\Pi_R(\mathcal{I}_R) = \mathcal{I}$$

Proof. The projection of $\mathcal{S}_R$ in (V, U)-plane is the image of $\pi_R^- : \mathcal{S}_R \to \mathbb{R}^2$ defined by:

$$\pi_R^-(V, U, R, \omega, S) = (V, U)$$

As seen before, equations (18) corresponding to the Rankine-Hugoniot condition and equation (16) for the right sonic locus can be written as:

$$\begin{aligned} V &= a_1(\omega)R + b_1(\omega)S + c_1(\omega) \\ U &= a_2(\omega)R + b_2(\omega)S + c_2(\omega) \end{aligned}$$

and

$$a_3(\omega)R + b_3(\omega)S + c_3(\omega) = 0 \tag{63}$$

The map π_R^- being singular gives another linear equation on R and S:

$$a_4(\omega)R + b_4(\omega)S + c_4(\omega) = 0 \tag{64}$$

where:

$$a_4(\omega) = \frac{1}{8}\Big(11 + 20\Gamma - 2(9 + 7\Gamma - 8\Gamma^2)\cos^2\omega + \\ + 4(3 - 4\Gamma)(1 - \Gamma^2)\cos^4\omega - 8(1 - \Gamma)^2(1 + \Gamma)\cos^6\omega\Big)\cos\omega$$

$$b_4(\omega) = -2 + \left(\frac{5}{2} - 2\Gamma\right)\cos^2\omega - 2(1 - 3\Gamma + 2\Gamma^2)\cos^4\omega + 2(1 - \Gamma)^2\cos^6\omega$$

$$c_4(\omega) = \frac{1}{2}\Lambda\Big(1 + 4\Gamma - 8\Gamma(1 - \Gamma)\cos^2\omega - 4(1 - \Gamma)^2\cos^4\omega\Big)\sin\omega\cos\omega$$

These two last equations (63,64) form a linear system of equations on (R, S) whose determinant is:

$$\frac{1}{2}\big(1 + (8\Gamma - 1)\cos^2\omega + 4(3\Gamma^2 - 4\Gamma - 1)\cos^4\omega - 4(3\Gamma^2 - 2\Gamma - 1)\cos^6\omega\big)$$

with zeros:

$$\omega_0 = 0, \quad \omega_{1,2} = \pm\arccos\frac{1}{\sqrt{2(1 - \Gamma)}}, \quad \pm\arccos\frac{1}{\sqrt{-2(1 + 3\Gamma)}}$$

Solving the system for (R, S) and substituting in the expressions for V and U leads to the line of critical points:

$$\mathcal{I}_R : \omega \mapsto (\bar{V}_c(\omega), \bar{U}_c(\omega), \bar{R}_c(\omega), \omega, \bar{S}_c(\omega))$$

where:

$$V_c(\omega) = -2\frac{(1 + 2\Gamma)(3 - 2(1 + \Gamma)\cos^2\omega)\cos\omega}{(1 + 2(3\Gamma + 1)\cos^2\omega)\sin\omega}\Lambda$$

$$U_c(\omega) = -3\frac{1 - 2(\Gamma + 1)\cos^2\omega}{1 + 2(3\Gamma + 1)\cos^2\omega}\Lambda$$

$$R_c(\omega) = 4\frac{1 - 2(\Gamma + 1)\cos^2\omega}{(1 + 2(3\Gamma + 1)\cos^2\omega)\sin\omega}\Lambda$$

$$S_c(\omega) = \frac{(1 + 2\Gamma)(3 - 2(3 + \Gamma)\cos^2\omega)\cos\omega}{(1 + 2(3\Gamma + 1)\cos^2\omega)\sin\omega}\Lambda \tag{65}$$

To $\omega = 0$ and to the last pair of zeros of the determinant there correspond the asymptotes of $\mathcal{I}_R$ inside the surface $\mathcal{S}_R$.

To the special values $\omega_0 = 0$ and $\omega_{1,2}$ there correspond solutions that are straight lines:

$$r_0 : R \mapsto \left(-\frac{2\Gamma - 1}{4}R, \Lambda, R, 0, \frac{3 + 2\Gamma}{4}R\right) \tag{66}$$

and:

$$r_{1,2} : R \mapsto (V_{1,2}(R), U_{1,2}(R), R, \omega_{1,2}, S_{1,2}(R)) \tag{67}$$

where:

$$
\begin{aligned}
&V_{1,2}(R) = \pm\frac{\sqrt{1-2\Gamma}}{\Gamma-1}\Lambda + \frac{\Gamma(2\Gamma-1)\sqrt{2(1-\Gamma)}}{4(\Gamma-1)^2}R\\
&U_{1,2}(R) = -\frac{\Gamma}{\Gamma-1}\Lambda \mp \frac{(2\Gamma-1)\sqrt{2(\Gamma-1)(2\Gamma-1)}}{4(\Gamma-1)^2}R\\
&S_{1,2}(R) = -\frac{(2\Gamma+1)\sqrt{2(1-\Gamma)}}{4(\Gamma-1)}R \qquad (68)
\end{aligned}
$$

These are three (one, for $\Gamma \geq 1/2$) straight lines whose projections on the (V,U)-plane are tangent to the projection of the coincidence locus $V^2+U^2=\Lambda^2$ at the critical points of the rarefaction foliation. These three lines r_0 and $r_{1,2}$ form the secondary bifurcation locus $\mathcal{B}_R$ (54); their projections on the (V,U)-plane coincide with separatrices of the critical points.

For $\Gamma < 1/2$, the straight lines $r_{1,2}$ intersect transversally (inside the surface $\mathcal{S}_R$) the curve $\mathcal{I}_R$ in two points $X_{1,2}$, which are the images of the intersection of the lines $\omega = \omega_{1,2}$ with the graph of $R_c(\omega)$ in the (ω,R)-plane; if $\Gamma=-1$ the two lines $r_{1,2}$ are asymptotes of $\mathcal{I}_R$, and there is no intersection. These two critical points $X_{1,2}$ are not folds nor cusps: they are degenerate critical points in the sense that their type of singularity does not appear in generic projections of surfaces on a plane.

Equation (23) defines a surface in the (R,S,ω)-space, as does equation (64); they intersect transversally at all points except $X_{1,2}$: the matrix

$$
\begin{bmatrix}
a_3(\omega) & b_3(\omega) & a_3'(\omega)R + b_3'(\omega)S + c_3'(\omega)\\
a_4(\omega) & b_4(\omega) & a_4'(\omega)R + b_4'(\omega)S + c_4'(\omega)
\end{bmatrix}
$$

can only have characteristic one on the lines $r_{1,2}$, and that gives a linear condition on R and S, independent of (23) and (64); since clearly there is no transversality at $X_{1,2}$, those are the only points where the loss of transversality occurs.

It is easy to see that $\pi_R \circ \mathcal{I}_R$, $\pi_R \circ r_0$ and $\pi_R \circ r_{1,2}$ have no critical points, assuming $\Gamma \notin \{1/2, 0, -1/2\}$ or $\gamma \notin \{1,0,-1\}$, so then there are no cusp (or more complex) critical points: except for $X_{1,2}$, all other critical points are fold points. $\square$

Remark 13. The points $X_{1,2}$ are well defined when $\Gamma < 1/2$ and $\Gamma \neq -1$ (or $\gamma < 1$ and $\gamma \neq -2$); for $\Gamma = -1$ there is a bifurcation: for that value of Γ the zeros $\omega_{1,2}$ have multiplicity 2, and the lines r_0, $r_{1,2}$ coincide with the asymptotes of $\mathcal{I}_R$ and $\mathcal{I}_R \cap r_{1,2} = \emptyset$.

Remark 14. The projection of the intersection of $\mathcal{H}_R$ with $\mathcal{C}$ gives the points P_4 and P_5, as we can easily verify. By proposition 2, the points P_4 and P_5 are also the projection of the intersection of $\mathcal{H}_L$ with $\mathcal{C}$.

The projection of the right sonic locus on the (V, U)-plane, when $\Lambda \neq 0$, is depicted in fig. 13.

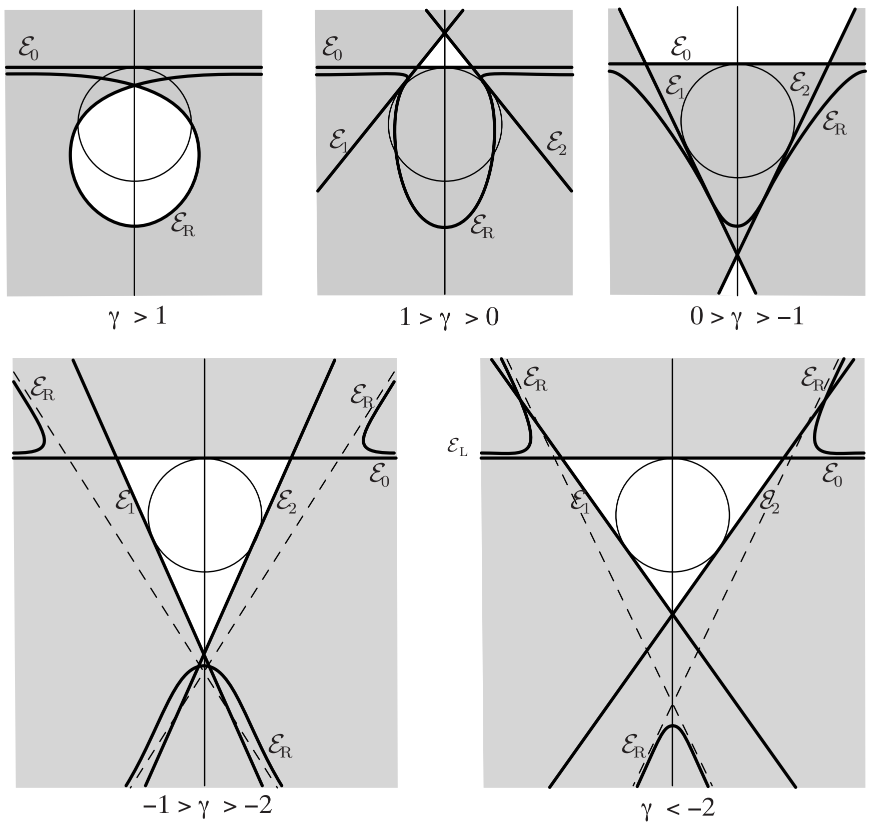

Figure 13. Projection of the right sonic locus on the (V, U)-plane, $\Lambda > 0$

When $\Lambda = 0$, the projection of the right sonic locus on the (V, U)-plane covers all the plane, except for $0 > \gamma > -1$ (fig. 14).

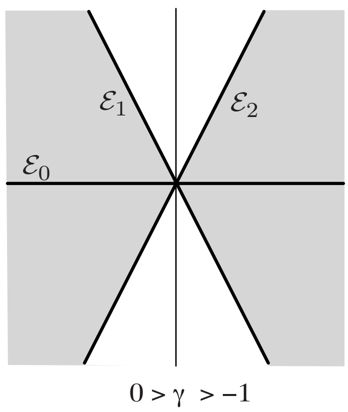

Figure 14. Projection of the right sonic locus on the (V, U)-plane, $\Lambda = 0$

If a point $\mathcal{U}$ does not belong to the projection $(\mathcal{U}_-, \mathcal{U}_+, s) \mapsto \mathcal{U}_-$ of the left and right sonic loci, and if we fix it as the left state, then any shock verifying the Rankine-Hugoniot condition is not sonic, and it is always Lax admissible.

7.2. **Composite foliation.** We begin by finding the critical points for the composite foliation. Proceeding as before, in the proof of the proposition 29, we have:

$$\begin{aligned} V &= a_1(\omega)R + b_1(\omega)S + c_1(\omega) \\ U &= a_2(\omega)R + b_2(\omega)S + c_2(\omega) \end{aligned}$$

and

$$\bar{a}_3(\omega)R + \bar{b}_3(\omega)S + \bar{c}_3(\omega) = 0$$

as equations for the left sonic locus, and the map π_L^- being singular gives another linear equation on R and S:

$$\bar{a}_4(\omega)R + \bar{b}_4(\omega)S + \bar{c}_4(\omega) = 0 \tag{69}$$

where:

$$\begin{aligned} \bar{a}_4(\omega) = &-\frac{1}{8}\Big(1 + 4\Gamma + 2\left(3 + 9\Gamma + 8\Gamma^2\right)\cos^2\omega + 4(1-4\Gamma)(1-\Gamma^2)\cos^4\omega - \\ &- 8(1-\Gamma)(1-\Gamma^2)\cos^6\omega\Big)\cos\omega \\ \bar{b}_4(\omega) = &\ 1 + \left(4\Gamma - \frac{1}{2}\right)\cos^2\omega + 2(1 - 3\Gamma + 2\Gamma^2)\cos^4\omega - 2(1-\Gamma)^2\cos^6\omega \\ \bar{c}_4(\omega) = &-\frac{1}{2}\Lambda\Big(1 + 4\Gamma - 8\Gamma(1-\Gamma)\cos^2\omega - 4(1-\Gamma)^2\cos^4\omega\Big)\sin\omega\cos\omega \end{aligned}$$

The last two equations form a linear system of equations on (R, S) whose determinant is:

$$-\frac{1}{4} - \frac{1}{2}(1+\Gamma)\cos^2\omega + (3 + 4\Gamma + \Gamma^2)\cos^4\omega - 2(1+\Gamma)(1-\Gamma^2)\cos^6\omega$$

with zeros (modulo π, and with $\Gamma \neq \pm 1$):

$$\omega_{1,2} = \pm\arccos\frac{1}{\sqrt{2(1+\Gamma)}}, \quad \pm\arccos\sqrt{\frac{1 + \Gamma \pm \sqrt{2(1+\Gamma)}}{2(1-\Gamma^2)}}$$

Solving the system for (R, S) and substituting in the expressions for V and U leads to the line of critical points:

$$\mathcal{E}_L : \omega \mapsto (\bar{V}_c(\omega), \bar{U}_c(\omega), \bar{R}_c(\omega), \omega, \bar{S}_c(\omega))$$

where:

$$\bar{V}_c(\omega) = -4\frac{1+2\Gamma\cos^2\omega}{1+4(1+\Gamma)\cos^2\omega - 4(1-\Gamma^2)\cos^4\omega}\Lambda\sin\omega\cos\omega$$
$$\bar{U}_c(\omega) = \frac{1-4(1-\Gamma)\cos^2\omega + 4(1+\Gamma^2)\cos^4\omega}{1+4(1+\Gamma)\cos^2\omega - 4(1-\Gamma^2)\cos^4\omega}\Lambda$$
$$\bar{R}_c(\omega) = -4\frac{1-2(1-\Gamma)\cos^2\omega}{1+4(1+\Gamma)\cos^2\omega - 4(1-\Gamma^2)\cos^4\omega}\Lambda\sin\omega$$
$$\bar{S}_c(\omega) = 0$$

To $\omega_{1,2}$ there correspond solutions that are the straight lines $d_{1,2}$:

$$\mathcal{E}_{L_{1,2}} = d_{1,2} : R \mapsto (V_{1,2}(R), U_{1,2}(R), R, \omega_{1,2}, S_{1,2}(R)) \tag{70}$$

where:

$$V_{1,2}(R) = \mp\frac{1+\Gamma}{\sqrt{1+2\Gamma}}\Lambda - \frac{1}{2}\frac{\Gamma}{\sqrt{2(1+\Gamma)}}R$$
$$U_{1,2}(R) = \mp\frac{1}{4}\sqrt{\frac{2(1+2\Gamma)}{1+\Gamma}}R$$
$$S_{1,2}(R) = \pm\frac{\Gamma}{\sqrt{1+2\Gamma}}\Lambda + \frac{1}{4}\sqrt{2(1+\Gamma)}R \tag{71}$$

The other formulæ for the zeros of the determinant do not correspond to any ω (for $\Gamma \neq -1/2$).

Remark 15. The lines $d_{1,2}$ form the double sonic locus $\mathcal{D}$ (section 4).

For our quadratic model, we can describe more precisely the critical points:

Theorem 5. *The critical ponts of the composite foliation are singular points of the projection π_L^-, and therefore they all belong to $\mathcal{E}_L \cup \mathcal{E}_{L_1} \cup \mathcal{E}_{L_2}$:*

- *For $\gamma > -1$, the points in $(\mathcal{E}_L \cup \mathcal{E}_{L_1} \cup \mathcal{E}_{L_2}) \cap \mathcal{I}$ that project down on:*
$$\left(\pm\frac{2+\gamma}{2\gamma\sqrt{1+\gamma}}\lambda, -\frac{1}{2}\lambda\right)$$
which are centres.
- *All points whose image by π_L^- is also the projection of some critical point of the rarefaction foliation; they have the same type as the corresponding ones for the rarefaction foliation.*

Remark 16. The second type of critical points was not considered in [10, proposition 8.3]: the existence of a left eigenvector l and a right eigenvector r of $DF(\mathcal{U})$ such that $lr \neq 0$ is not verified whenever they correspond to an eigenvalue with algebraic multiplicity two and geometric multiplicity one.

Proof. We consider the 1-form:

$$\alpha = \frac{\partial F_2}{\partial v}\mathrm{d}v + \left(\frac{\partial F_2}{\partial u} - s\right)\mathrm{d}u$$

on the (v, u, s) space; on the (V, U, S) space, and after multiplication by Γ, it becomes:

$$\alpha = (U - \Lambda)\mathrm{d}V - \Gamma(V + S)\mathrm{d}U$$

The composite foliation is defined by ${\tau_L^-}^*\alpha = 0$ on $\mathcal{S}_L$, as described in 4.4. Its critical points are:

- the points whose image under π_L^- is a critical point of α on the (V, U)-plane, i.e. a critical point of the projection of the rarefaction foliation:

$$(0, \Lambda), \qquad \left(\pm\frac{\sqrt{1-2\Gamma}}{1-\Gamma}\Lambda, \frac{\Gamma}{1-\Gamma}\Lambda\right)$$

  Note that for these points we have $S = 0$;
- the points in the critical set for π_L^-, the curve $\mathcal{E}_L$ and the lines $d_{1,2}$, at whose projection by π_L^- the 1-form α is zero on the discriminant set of π_L^- (the image by π_L^- of its critical set).

Let us consider the first case:

The vector field $b(\omega, R)\partial/\partial\omega - a(\omega, R)\partial/\partial R$, where $a(\omega, R)$ and $b(\omega, R)$ are given by:

$$a(\omega, R)\mathrm{d}\omega + b(\omega, R)\mathrm{d}R = \frac{3 - 2(1-\Gamma)\cos^2\omega}{\sin\omega}\,{\tau_L^-}^*\alpha,$$

defines the composite foliation near these critical points; the eigenvalues of its linear part at $(0, \Lambda, 0, 0, 0)$ are:

$$4\Gamma\Lambda, \quad 2\Lambda(1 - 2\Gamma)$$

and therefore the critical point is a saddle, for $\Gamma < 0$ and $\Gamma > 1/2$, and a node for $0 < \Gamma < 1/2$.

At the other critical points:

$$\left(\pm\frac{\sqrt{1-2\Gamma}}{1-\Gamma}\Lambda, \frac{\Gamma}{1-\Gamma}\Lambda, 0, \mp\arccos\frac{1}{\sqrt{2(1-\Gamma)}}, 0\right)$$

the eigenvalues of the linear part of the same vector field are:

$$4\Gamma\Lambda, \quad -4\Gamma\Lambda\frac{1-2\Gamma}{1-\Gamma}$$

and therefore they are saddles (when defined, $\Gamma < 1/2$).

For the second case, we remark that the discriminant set is part of the circumference $V^2 + U^2 = \Lambda^2$ together with its tangents $d_{1,2}$.

We have $S \equiv 0$ on the circumference $V^2 + U^2 = \Lambda^2$, and the conditions:

$$\begin{aligned} \alpha(U, -V) &= VU - \Gamma(U + \Lambda)V = 0 \\ \text{or} \quad \alpha(U, -V) &= (U - \Lambda)U + \Gamma V^2 = 0 \end{aligned}$$

have as solutions exactly the critical points (36), i.e. the critical points of the rarefaction curves.

The projections on the (V, U)-plane of the lines $d_{1,2}$ have the equations:

$$\begin{aligned} V_{1,2}(R) &= \mp \frac{1+\Gamma}{\sqrt{1+2\Gamma}}\Lambda - \frac{1}{2}\frac{\Gamma}{\sqrt{2(1+\Gamma)}}R \\ U_{1,2}(R) &= \mp \frac{1}{4}\sqrt{\frac{2(1+2\Gamma)}{1+\Gamma}}R \end{aligned}$$

and along them we have:

$$S_{1,2}(R) = \pm \frac{\Gamma}{\sqrt{1+2\Gamma}}\,\Lambda + \frac{1}{4}\sqrt{2(1+\Gamma)}R$$

as we have seen before (71).

At a critical point we should have:

$$\alpha(V_{1,2}(R), U_{1,2}(R), S_{1,2}(R))\left(-\frac{1}{2}\frac{\Gamma}{\sqrt{2(1+\Gamma)}}, \mp\frac{1}{4}\sqrt{\frac{2(1+2\Gamma)}{1+\Gamma}}\right) = 0$$

These conditions lead to linear equations on R, whose solutions are $R = 0$ and therefore the critical points are:

$$V = \mp \frac{1+\Gamma}{\sqrt{1+2\Gamma}}\,\Lambda, \quad U = 0, \quad R = 0, \quad \omega = \omega_{1,2} \quad S = \pm \frac{\Gamma}{\sqrt{1+2\Gamma}}\,\Lambda$$

Again the vector field $b(\omega, R)\partial/\partial\omega - a(\omega, R)\partial/\partial R$ defines the composite foliation near these critical points; the eigenvalues of its linear part are:

$$\pm 2\sqrt{2}\,\Gamma\Lambda\, i$$

and therefore the critical points are centres. It also follows that these extra critical points are in the inflection locus $\mathcal{I}$, where $R = 0$, and moreover their projection on the $\mathcal{U}$-plane is:

$$\left(\mp \frac{1+\Gamma}{\sqrt{1+2\Gamma}}\,\Lambda, 0\right)$$

i.e., the points $P_{4,5}$ in $\mathcal{H}_0$. □

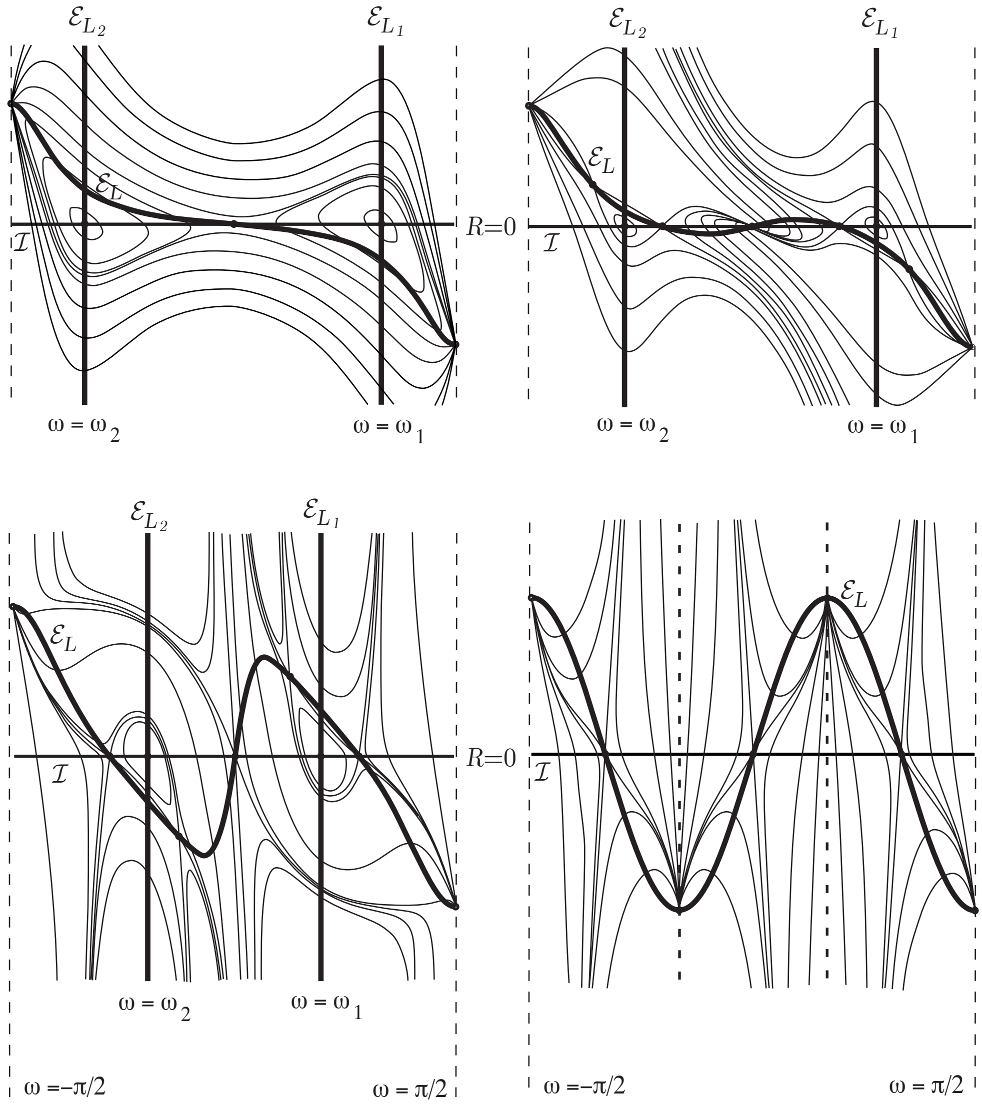

FIGURE 15. Composite foliation on (ω, R) coordinates: $\lambda > 0$, and from left to right and from the top down, $\gamma > 1$, $1 > \gamma > 0$, $0 > \gamma > -1$, $\gamma < -1$; on the dashed lines only the critical points belong to the domain of the parametrization.

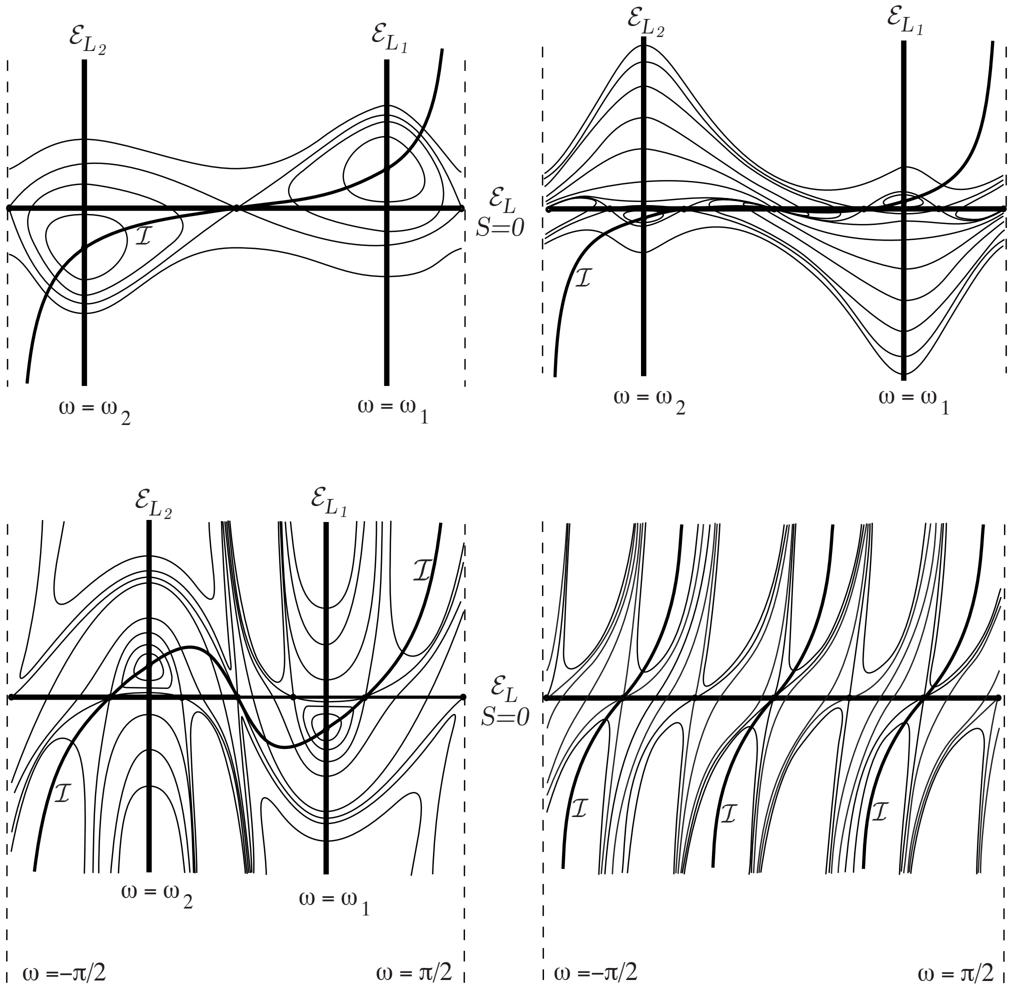

FIGURE 16. Composite foliation on (ω, S) coordinates: $\lambda > 0$, and from left to right and from the top down, $\gamma > 1$, $1 > \gamma > 0$, $0 > \gamma > -1$, $\gamma < -1$.

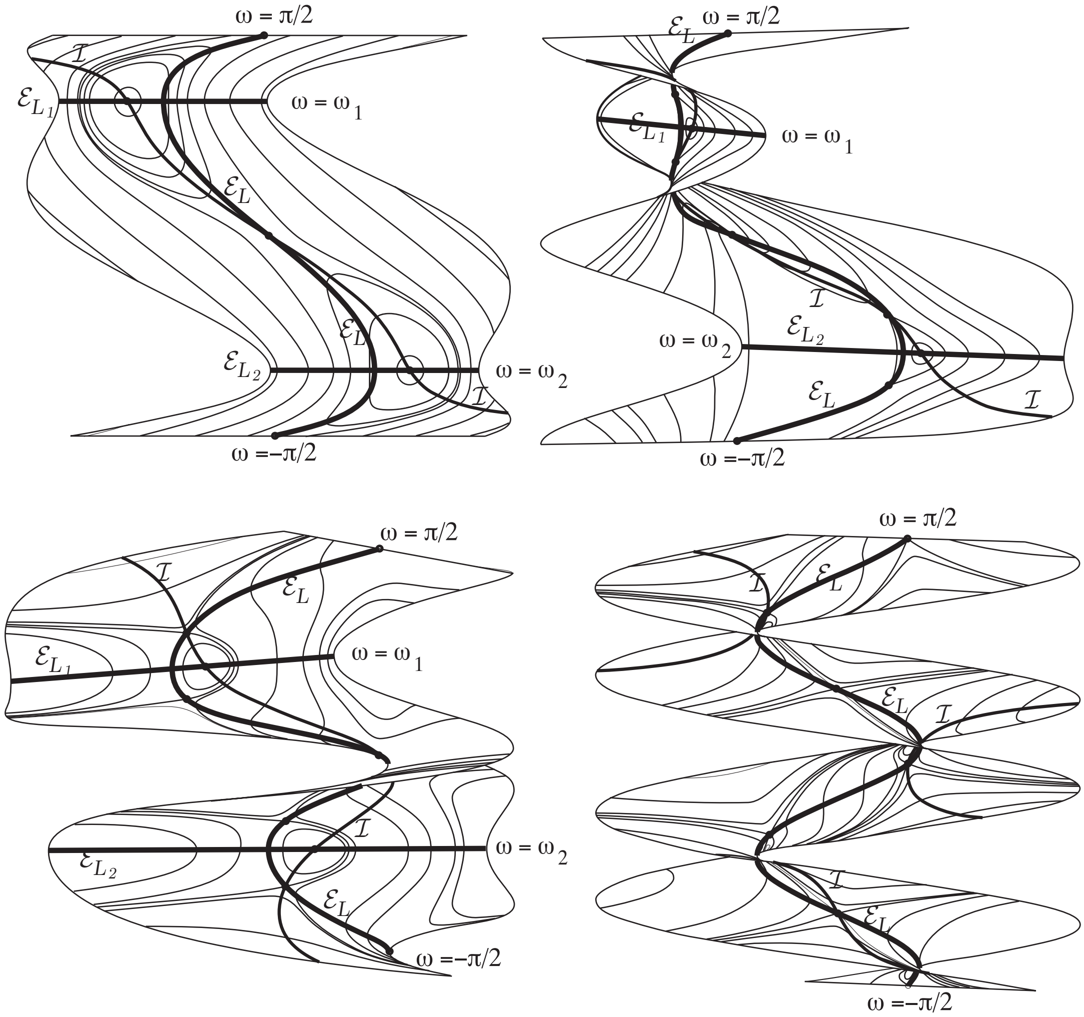

FIGURE 17. Composite foliation on (V, U, ω) coordinates: $\lambda > 0$, and from left to right and from the top down, $\gamma > 1$, $1 > \gamma > 0$, $0 > \gamma > -1$, $\gamma < -1$.

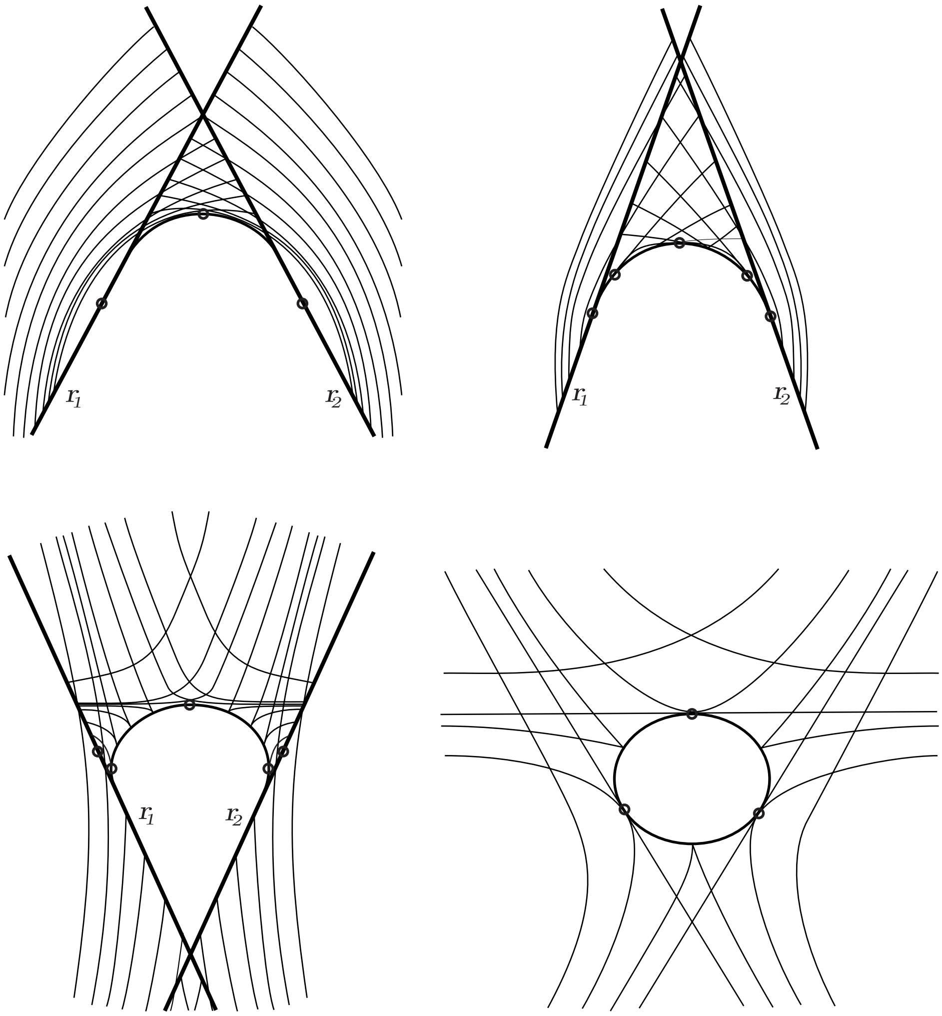

FIGURE 18. Projection of composite foliation on (V, U)-plane: $\lambda > 0$, and from left to right and from the top down, $\gamma > 1$, $1 > \gamma > 0$, $0 > \gamma > -1$, $\gamma < -1$.

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Centro de Matemática da Universidade do Porto, Portugal
E-mail address: jbgoncal@fc.up.pt

E-mail address: hreis@fep.up.pt