

# BETTI NUMBERS OF THE MODULI SPACE OF RANK 3 PARABOLIC HIGGS BUNDLES

O. GARCÍA-PRADA, P.B. GOTHEN, AND V. MUÑOZ

ABSTRACT. Parabolic Higgs bundles on a Riemann surface are of interest for many reasons, one of them being their importance in the study of representations of the fundamental group of the punctured surface in the complex general linear group. In this paper we calculate the Betti numbers of the moduli space of rank 3 parabolic Higgs bundles, using Morse theory. A key point is that certain critical submanifolds of the Morse function can be identified with moduli spaces of parabolic triples. These moduli spaces come in families depending on a real parameter and we carry out a careful analysis of them by studying their variation with this parameter. Thus we obtain in particular information about the topology of the moduli spaces of parabolic triples for the value of the parameter relevant to the study of parabolic Higgs bundles. The remaining critical submanifolds are also described: one of them is the moduli space of parabolic bundles, while the remaining ones have a description in terms of symmetric products of the Riemann surface. As another consequence of our Morse theoretic analysis, we obtain a proof of the parabolic version of a theorem of Laumon, which states that the nilpotent cone (the preimage of zero under the Hitchin map) is a Lagrangian subvariety of the moduli space of parabolic Higgs bundles.

## 1. INTRODUCTION

Let  $X$  be a connected, smooth projective complex algebraic curve of genus  $g$  and let  $D = p_1 + p_2 + \dots + p_n$  be a divisor, with distinct points  $p_1, \dots, p_n$ . Let  $K$  be the canonical bundle of  $X$ . A *parabolic Higgs bundle* is a pair  $(E, \Phi)$ , where  $E$  is a parabolic bundle, that is a holomorphic bundle over  $X$  together with a weighted flag in the fibre of  $E$  over each  $p \in D$ , and  $\Phi : E \rightarrow E \otimes K(D)$  is a strongly parabolic homomorphism. This means that  $\Phi$  is a meromorphic endomorphism valued one-form with simple poles along  $D$  whose residue at  $p$  is nilpotent with respect to the flag.

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Like in the non-parabolic case, there is a stability criterion allowing the construction of moduli spaces of semistable parabolic Higgs bundles [33]. For generic weights, semistability and stability coincide and the moduli space is a smooth quasiprojective algebraic manifold. The goal of this paper is to compute the Betti numbers of this moduli space in the case in which the rank of the bundle is 3. The computation for rank 2 was carried out by Boden and Yokogawa [5], and Nasatyr and Steer [27] in the case of rational weights. In the non-parabolic case the Betti numbers had been previously computed by Hitchin [18] in rank 2 and Gothen [13] in rank 3. We have been informed by T. Hausel of a conjecture for the Betti numbers of the moduli space of parabolic Higgs bundles of any rank and degree, analogous to his conjecture for the case of non-parabolic Higgs bundles [17]. His formula gives the same result as ours in the cases that we have checked (cf. Remark 11.3), thus providing support for his conjecture.

Similarly to the non-parabolic case, the moduli space of parabolic Higgs bundles has an extremely rich geometric structure. It can be identified with the moduli space of solutions to the parabolic version of Hitchin's equations:

$$F(A)^\perp + [\Phi, \Phi^*] = 0 \quad \text{and} \quad \bar{\partial}_A \Phi = 0,$$

where  $A$  is a singular connection unitary with respect to a singular hermitian metric on  $E$  adapted to the parabolic structure (see Section 2.3 for details). The moduli space of parabolic Higgs bundles contains the total space of the cotangent bundle of the moduli space of parabolic bundles, whose natural holomorphic symplectic form can be naturally extended to the whole moduli space. This form can be combined with the real symplectic form coming from the gauge-theoretic interpretation to endow the moduli space with a hyperkähler structure [21, 27].

An important motivation to study ordinary Higgs bundles comes from their relation with complex representations of the fundamental group of the curve. This is established by identifying the moduli space of solutions to Hitchin's equations with the moduli space of Higgs bundles [18, 30] as well as with the moduli space of complex connections with constant central curvature [9, 10]. In the parabolic case, there is a similar correspondence proved by Simpson [29]. This involves meromorphic complex connections with simple poles at the points and parabolic weights. At the topological side one has to consider filtered local systems. The natural context for the correspondence is a class larger than parabolic Higgs bundles in which the Higgs field  $\Phi$  is allowed to be parabolic and not necessarily strongly parabolic. In other words, at a parabolic point the residue of  $\Phi$  is parabolic with respect to the flag. Under this correspondence, parabolic Higgs bundles (those for which the Higgs field is strongly parabolic) are in bijection with meromorphic flat connections whose holonomy around each parabolic point defines a conjugacy class of an element in the unitary group. These, in turn, correspond to representations of the fundamental group of the punctured surface in the general linear group, which send a small loop around each parabolic point to an element conjugated to a unitary element.

The main tool for our computation of the Betti numbers, as in the previously studied cases, is the use of the Morse-theoretic techniques introduced by Hitchin [18]: the  $L^2$ -norm of the Higgs field defines a perfect Bott–Morse function on the moduli space. We have to compute the Poincaré series and the indices of the various critical subvarieties. In fact the calculation of the indices can be carried out for any rank, whereas the calculation of the Poincaré series of the critical subvarieties depends crucially on the rank 3 assumption. Here is a description of the paper.

In Section 2 we review the basic definitions and basic facts of parabolic Higgs bundles. In Section 3 we consider the Bott–Morse function on the moduli space and identify the critical subvarieties. These coincide with the fixed subvarieties under the action of  $S^1$  on the moduli space given by multiplying the Higgs field. These in turn correspond, as shown by Simpson [29], to variations of Hodge structures, in particular the bundle has to be a direct sum of subbundles. We then compute the indices — this can be done for any rank and leads to a parabolic version of the theorem of Laumon, that the nilpotent cone in the moduli space of parabolic Higgs bundles is a Lagrangian subvariety. In the rank 3 case, the possible decompositions of the vector bundle in a sum of subbundles are of two types: a sum of three line bundles or a sum of a line bundle and a rank 2 vector bundle. The latter case gives rise to so-called parabolic triples. These have been introduced in [3] and generalise the triples studied in [7] and [8]. Through Sections 4, 5 and 6 we study the moduli spaces of parabolic triples. They depend on a real parameter, relating to parabolic Higgs bundles when the value of this parameter is  $2g - 2$ . To compute the Betti numbers for a given value of the parameter (in particular for  $2g - 2$ ) we follow the strategy introduced by Thaddeus in [31]. After characterising the moduli space for the largest value of the parameter, we need to analyse the changes when we cross a finite number of values until we get to the one we want. In Section 7 we compute the Poincaré polynomial and indices for the critical subvarieties for which the vector bundle is a sum of three line bundles. In Sections 8 and 9 we do the other cases using the previous computations for the moduli space of parabolic triples. An important technical point is that the Betti numbers of the moduli space of parabolic Higgs bundles do not depend on the degree and the weights (certainly if the weights are generic). So we can choose the degree coprime with the rank and the weights as small as convenient, to facilitate our computations. In Section 10, based on the computations by Nitsure [28] and Holla [20] of the Betti numbers of the moduli space of parabolic bundles, we work out the formula for the rank 3 case. Finally, in Section 11 we collect all the computations, to give the Poincaré polynomial of the rank 3 moduli space of parabolic Higgs bundles.

## 2. PARABOLIC HIGGS BUNDLES

**2.1. Definitions and basic facts.** Let  $X$  be a connected, smooth projective complex algebraic curve of genus  $g$  together with a finite (non-zero) number of *marked* distinct points  $p_1, \dots, p_n$ . We will denote the divisor  $D = p_1 + p_2 + \dots + p_n$ .

Let  $E$  be a holomorphic bundle over  $X$ . A parabolic structure on  $E$  consists of weighted flags

$$\begin{aligned} E_p &= E_{p,1} \supset E_{p,2} \supset \cdots \supset E_{p,s_p} \supset E_{p,s_p+1} = 0, \\ 0 &\leq \alpha_1(p) < \cdots < \alpha_{s_p}(p) < 1, \end{aligned}$$

over each  $p \in D$ . A holomorphic map  $\phi : E \rightarrow F$  between parabolic bundles is called parabolic if  $\alpha_i^E(p) > \alpha_j^F(p)$  (the superindex indicating to which parabolic bundle the weight corresponds) implies  $\phi(E_{p,i}) \subset F_{p,j+1}$  for all  $p \in D$ . We call  $\phi$  strongly parabolic if  $\alpha_i^E(p) \geq \alpha_j^F(p)$  implies  $\phi(E_{p,i}) \subset F_{p,j+1}$  for all  $p \in D$ .

We will abuse notation by simply writing  $E$  for a bundle with a parabolic structure.

This notion can be generalised to higher dimensions. In this case, a parabolic bundle consists of a holomorphic vector bundle together with a weighted holomorphic filtration of the restriction of the bundle to a fixed divisor. This generalisation is also relevant for us in the particular case in which the manifold is the product of the curve  $X$  and a higher dimensional manifold  $Y$ , and the divisor is  $D \times Y$ . Parabolic and strongly parabolic homomorphisms are defined in a similar way.

Also  $\text{ParHom}(E, F)$  and  $\text{SParHom}(E, F)$  will denote respectively the sheaves of parabolic and strongly parabolic homomorphisms from  $E$  to  $F$ .

Let  $m_i(p) = \dim E_{p,i}/E_{p,i+1}$  be the multiplicity of  $\alpha_i(p)$ . It will sometimes be convenient to repeat each weight according to its multiplicity, i.e., we set  $\tilde{\alpha}_1(p) = \dots = \tilde{\alpha}_{m_1(p)}(p) = \alpha_1(p)$ , etc. We then have weights  $0 \leq \tilde{\alpha}_1(p) \leq \cdots \leq \tilde{\alpha}_r(p) < 1$ , where  $r = \text{rk}(E)$  is the rank of  $E$ . Define the parabolic degree and parabolic slope of  $E$  by

$$\begin{aligned} \text{pardeg}(E) &= \deg(E) + \sum_{p \in D} \sum_{i=1}^{s_p} m_i(p) \alpha_i(p) = \deg(E) + \sum_{p \in D} \sum_{i=1}^r \tilde{\alpha}_i(p), \\ \text{par}\mu(E) &= \frac{\text{pardeg}(E)}{\text{rk}(E)}. \end{aligned}$$

If  $F$  is a subbundle of  $E$ , then  $F$  inherits a parabolic structure by setting  $F_{p,i} = F_p \cap E_{p,i}$  and discarding those weights of multiplicity zero. We call this the induced parabolic structure on  $F$ . In a similar manner, one can give a parabolic structure to the quotient  $E/F$ .

A parabolic bundle  $E$  is said to be stable if  $\text{par}\mu(F) < \text{par}\mu(E)$  for all proper parabolic subbundles  $F \subset E$ . Semistability is defined by replacing the strict inequality by the weak inequality. For generic weights, stability and semistability are equivalent.

For parabolic bundles  $E$  there is a well-defined notion of dual  $E^*$ . This is done by considering the bundle  $\text{Hom}(E, \mathcal{O}(-D))$ , and at each  $p \in D$ , defining the filtration

$$E_p^* = E_{p,1}^* \supset \cdots \supset E_{p,s_p}^* \supset 0,$$

with  $E_{p,i}^* = \text{Hom}(E_p/E_{p,s_p+2-i}, \mathcal{O}(-D)_p)$  and weights  $1 - \alpha_s(p) < \cdots < 1 - \alpha_1(p)$ . It is easy to prove that  $E^{**} = E$ , and  $\text{pardeg}(E^*) = -\text{pardeg}(E)$ .

There is also a notion of tensor product  $\otimes^P$  of two parabolic bundles [34], which is best understood in terms of  $\mathbb{R}$ -filtered sheaves. Here we shall only use the case of tensoring a parabolic bundle  $E$  with a parabolic line bundle  $L$ . Let  $\alpha_i(p)$  be the weights of  $E$  and  $\beta(p)$  be the weights of  $L$ . Then the parabolic bundle  $F = E \otimes^P L$  is, as a bundle, the kernel of

$$E \otimes L(D) \twoheadrightarrow \bigoplus_{p \in D} ((E_p/E_{p,i_p}) \otimes L(D)_p),$$

where  $i_p = \min\{s_p + 1, i \mid \alpha_i(p) + \beta(p) \geq 1\}$ ,  $p \in D$ . The weights of  $F$  are

$$(2.1) \quad \alpha_{i_p}(p) + \beta(p) - 1 < \cdots < \alpha_{s_p}(p) + \beta(p) - 1 < \alpha_1(p) + \beta(p) < \cdots < \alpha_{i_{p-1}}(p) + \beta(p),$$

with multiplicities  $m_{i_p}, \dots, m_{s_p}, m_1, \dots, m_{i_{p-1}}$ . It is then easy to see that  $\text{par}\mu(E \otimes^P L) = \text{par}\mu(E) + \text{par}\mu(L)$ , whereas

$$\deg(E \otimes^P L) = \deg(E) + \text{rk}(E) \deg(L) + \sum_p \dim E_{p,i_p}.$$

We denote by  $K$  the canonical bundle on  $X$ . A *parabolic Higgs bundle* is a pair  $(E, \Phi)$ , where  $E$  is a parabolic bundle and  $\Phi \in H^0(\text{SParEnd}(E) \otimes K(D))$ , i.e.  $\Phi$  is a meromorphic endomorphism valued one-form with simple poles along  $D$  whose residue at  $p$  is nilpotent with respect to the flag. We shall sometimes denote a parabolic Higgs bundle by  $\mathbf{E} = (E, \Phi)$ .

The notion of stability is extended to parabolic Higgs bundles in the usual way:

$$\text{par}\mu(F) < \text{par}\mu(E)$$

for all proper parabolic subbundles  $F \subset E$  which are preserved by  $\Phi$ . Semistability is defined by replacing the strict inequality by the weak inequality.

The standard properties of stable bundles also apply to parabolic Higgs bundles; for example, if  $\mathbf{E}$  and  $\mathbf{F}$  are stable parabolic Higgs bundles of the same parabolic slope, then there are no parabolic maps between them unless they are isomorphic, and the only parabolic endomorphisms of a stable parabolic Higgs bundle are the scalar multiples of the identity.

We shall say that the weights are *generic* when every semistable parabolic Higgs bundle is automatically stable, i.e., when there are no properly semistable Higgs bundles. Let us fix (generic) weights  $\alpha_i(p)$  and topological invariants  $\text{rk}(E)$  and  $\deg(E)$ . The moduli space  $\mathcal{M}$  of stable parabolic Higgs bundles was constructed using Geometric Invariant Theory by Yokogawa [33, 34], who also showed that it is a smooth irreducible complex variety. The moduli space  $\mathcal{M}$  contains the cotangent bundle of the moduli space of stable parabolic bundles.

The following result will facilitate the computation of the Betti numbers of  $\mathcal{M}$ .

**Proposition 2.1.** *Fix the rank  $r$ . For different choices of degrees and (generic) weights, the moduli spaces of parabolic Higgs bundles are diffeomorphic to each other.*

*Proof.* For fixed degree, the moduli spaces of Higgs bundles for different choices of weights are *diffeomorphic* (when the weights are generic), as has been proved by Thaddeus [32].

On the other hand, fix any any parabolic line bundle  $L$  with degree  $d_L$  and weights  $\beta(p)$ . Then the map

$$(E, \Phi) \mapsto (E \otimes^P L, \Phi)$$

gives an *isomorphism* between the moduli space of Higgs bundles of rank  $r$ , degree  $\Delta$  and weights  $\alpha_i(p)$  and the moduli space of Higgs bundles of rank  $r$ , degree  $\Delta + rd_L + \sum_p \sum_{i \geq i_p} m_i(p)$ , and weights given by (2.1). Choosing weights of multiplicity one and a suitable parabolic line bundle  $L$  we see that moduli spaces of parabolic Higgs bundles for different degrees are isomorphic.

Therefore all moduli spaces of Higgs bundles of given rank  $r$  are diffeomorphic to each other. □

**2.2. Deformation theory.** The deformation theory of parabolic Higgs bundles was worked out by Yokogawa [34]; see also Thaddeus [32] and Biswas and Ramanan [4]. Everything in this section is essentially contained in these references but we shall find it convenient to give an exposition tailored to our purposes. Let  $\mathbf{E} = (E, \Phi)$  and  $\mathbf{F} = (F, \Psi)$  be parabolic Higgs bundles. We define a complex of sheaves

$$\begin{aligned} C^\bullet(\mathbf{E}, \mathbf{F}) : \quad \text{ParHom}(E, F) &\rightarrow \text{SParHom}(E, F) \otimes K(D) \\ f &\mapsto (f \otimes 1)\Phi - \Psi f, \end{aligned}$$

and write  $C^\bullet(\mathbf{E}) = C^\bullet(\mathbf{E}, \mathbf{E})$ .

**Proposition 2.2.** (i) *The space of infinitesimal deformations of a parabolic Higgs bundle  $\mathbf{E}$  is naturally isomorphic to the first hypercohomology group of the complex*

$$\begin{aligned} C^\bullet(\mathbf{E}) : \quad \text{ParEnd}(E) &\xrightarrow{[-, \Phi]} \text{SParEnd}(E) \otimes K(D) \\ f &\longmapsto (f \otimes 1)\Phi - \Phi f. \end{aligned}$$

*Thus the tangent space to  $\mathcal{M}$  at a point represented by a stable parabolic Higgs bundle  $\mathbf{E}$  is isomorphic to  $\mathbb{H}^1(C^\bullet(\mathbf{E}))$ .*

- (ii) *The space of homomorphisms between parabolic Higgs bundles  $\mathbf{E}$  and  $\mathbf{F}$  is naturally isomorphic to the zeroth hypercohomology group  $\mathbb{H}^0(C^\bullet(\mathbf{E}, \mathbf{F}))$ .*
- (iii) *The space of extensions  $0 \rightarrow \mathbf{E}' \rightarrow \mathbf{E} \rightarrow \mathbf{E}'' \rightarrow 0$  of parabolic Higgs bundles  $\mathbf{E}'$  and  $\mathbf{E}''$  is naturally isomorphic to the first hypercohomology  $\mathbb{H}^1(C^\bullet(\mathbf{E}'', \mathbf{E}'))$ .*

(iv) *There is a long exact sequence*

$$\begin{aligned}
 (2.2) \quad & 0 \rightarrow \mathbb{H}^0(C^\bullet(\mathbf{E}, \mathbf{F})) \rightarrow H^0(\text{ParHom}(E, F)) \rightarrow H^0(\text{SParHom}(E, F) \otimes K(D)) \\
 & \rightarrow \mathbb{H}^1(C^\bullet(\mathbf{E}, \mathbf{F})) \rightarrow H^1(\text{ParHom}(E, F)) \rightarrow H^1(\text{SParHom}(E, F) \otimes K(D)) \\
 & \rightarrow \mathbb{H}^2(C^\bullet(\mathbf{E}, \mathbf{F})) \rightarrow 0.
 \end{aligned}$$

*Proof.* For proofs of (i) – (iii) see Thaddeus [32]. The proof of (iv) follows by a standard argument in Higgs bundle theory (see, e.g., Biswas and Ramanan [4]).  $\square$

As for ordinary Higgs bundles, duality plays an important role for parabolic Higgs bundles. The results of the following proposition are consequences of the theory developed by Yokogawa (cf. (3.1) and Proposition 3.7 of [34], see also § 3 of Thaddeus [32] and § 5 of Bottacin [6]).

**Proposition 2.3.** (i) *Let  $E$  and  $F$  be parabolic bundles. The sheaves  $\text{ParHom}(E, F)$  and  $\text{SParHom}(F, E(D))$  are naturally dual.*

(ii) *Let  $\mathbf{E}$  and  $\mathbf{F}$  be parabolic Higgs bundles. Then there is a natural isomorphism*

$$\mathbb{H}^i(C^\bullet(\mathbf{E}, \mathbf{F})) \cong \mathbb{H}^{2-i}(C^\bullet(\mathbf{F}, \mathbf{E}))^*.$$

*In particular we obtain a natural isomorphism  $T_{\mathbf{E}}\mathcal{M} \cong T_{\mathbf{E}}^*\mathcal{M}$  for a stable parabolic Higgs bundle  $\mathbf{E}$ .*

*Proof.* We just show how (i) implies (ii). From (i) it follows that the dual complex  $C^\bullet(\mathbf{E}, \mathbf{F})^*$  is related to the original one by

$$C^\bullet(\mathbf{E}, \mathbf{F})^* \otimes K \cong C^\bullet(\mathbf{F}, \mathbf{E}).$$

Thus, Serre duality for hypercohomology and Proposition 2.2 (i) give statement (ii) of the present proposition.  $\square$

Next we shall show how these results can be used to calculate the dimension of the moduli space of parabolic Higgs bundles. The result is well known but it seems worthwhile to include the calculation here, since we shall use the same ideas again below. First we introduce some convenient notation for parabolic bundles  $E$  and  $F$  as follows. We denote by  $P_p(E, F)$  the subspace of  $\text{Hom}(E_p, F_p)$  consisting of parabolic maps and by  $N_p(E, F)$  the subspace of strictly parabolic maps. We also write  $P_D(E, F) = \bigoplus P_p(E, F)$  and  $N_D(E, F) = \bigoplus N_p(E, F)$ . When there is no risk of confusion we shall omit the parabolic bundles  $E$  and  $F$  from the notation. We then have short exact sequences of sheaves

$$0 \rightarrow \text{ParHom}(E, F) \rightarrow \text{Hom}(E, F) \rightarrow \text{Hom}(E_D, F_D)/P_D(E, F) \rightarrow 0,$$

and

$$0 \rightarrow \text{SParHom}(E, F) \rightarrow \text{Hom}(E, F) \rightarrow \text{Hom}(E_D, F_D)/N_D(E, F) \rightarrow 0.$$

Thus we can calculate the Euler characteristics of  $\text{ParHom}(E, F)$  and  $\text{SParHom}(E, F)$  as follows:

$$(2.3) \quad \begin{aligned} \chi(\text{ParHom}(E, F)) &= \chi(\text{Hom}(E, F)) + \sum_{p \in D} (\dim P_p - \text{rk}(E) \text{rk}(F)), \\ \chi(\text{SParHom}(E, F)) &= \chi(\text{Hom}(E, F)) + \sum_{p \in D} (\dim N_p - \text{rk}(E) \text{rk}(F)). \end{aligned}$$

With these preliminaries in place we can calculate the dimension of the moduli space.

**Proposition 2.4.** *The complex dimension of the moduli space  $\mathcal{M}$  of stable rank  $r$  parabolic Higgs bundles is*

$$r^2(2g - 2) + 2 + 2 \sum_p f_p,$$

where  $r = \text{rk}(E)$  and  $f_p = \frac{1}{2}(r^2 - \sum_i m_i(p)^2)$ .

*Proof.* Since  $\mathbf{E}$  is stable, its only endomorphisms are the scalars. Hence, using Proposition 2.2 (ii) and the duality statement Proposition 2.3 (ii), we have that  $\dim \mathbb{H}^0(C^\bullet(\mathbf{E})) = \dim \mathbb{H}^2(C^\bullet(\mathbf{E})) = 1$ . It follows that the dimension of the moduli space is

$$\begin{aligned} \dim \mathcal{M} &= \dim \mathbb{H}^1(C^\bullet(\mathbf{E})) \\ &= 2 - \chi(C^\bullet(\mathbf{E})) \\ &= 2 - \chi(\text{ParEnd}(E)) + \chi(\text{SParEnd}(E) \otimes K(D)), \end{aligned}$$

where in the last equality we have used the long exact sequence (2.2). From this we obtain the result by using equations (2.3), the fact that  $\dim P_p - \dim N_p = \sum_i m_i(p)^2$  and the Riemann–Roch formula.  $\square$

**2.3. Parabolic Higgs bundles and gauge theory.** Our main goal is to study the topology of  $\mathcal{M}$ . To do this we need the gauge-theoretic interpretation of this moduli space in terms of solutions to Hitchin’s equations due to Simpson [29]. The construction of the moduli space from this point of view is due to Konno [21]. Let  $E$  be a smooth parabolic vector bundle of rank  $r$  and fix a hermitian metric  $h$  in  $E$  which is smooth in  $X \setminus D$  and whose (degenerate) behaviour around the punctures is given as follows. We say that a local frame  $\{e_1, \dots, e_r\}$  for  $E$  around  $p$  respects the flag at  $p$  if  $E_{p,i}$  is spanned by the vectors  $\{e_{M_i+1}(p), \dots, e_r(p)\}$ , where  $M_i = \sum_{j \leq i} m_j$ . Let  $z$  be a local coordinate around  $p$  such that  $z(p) = 0$ . We require that  $h$  be of the form

$$h = \begin{pmatrix} |z|^{2\tilde{\alpha}_1} & & 0 \\ & \ddots & \\ 0 & & |z|^{2\tilde{\alpha}_r} \end{pmatrix}$$

with respect to some local frame around  $p$  which respects the flag at  $p$ . We denote the space of smooth  $\bar{\partial}$ -operators on  $E$  by  $\mathcal{C}$  and the space of associated  $h$ -unitary connections by  $\mathcal{A}$ . Note that the unitary connection associated to a smooth  $\bar{\partial}_A$  via the hermitian



metric  $h$  is singular around the punctures: if we write  $z = \rho \exp(\mathbf{i}\theta)$  and  $\{e_i\}$  is the local frame used in the definition of  $h$ , then with respect to the local frame  $\{\epsilon_i = e_i/|z|^{\tilde{\alpha}_i}\}$ , the connection is of the form

$$d_A = d + \mathbf{i} \begin{pmatrix} \tilde{\alpha}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{\alpha}_r \end{pmatrix} d\theta + A',$$

where  $A'$  is regular.

We denote the space of Higgs fields by  $\Omega = \Omega^{1,0}(\text{SParEnd}(E) \otimes \mathcal{O}(D))$ , the group of complex parabolic gauge transformations by  $\mathcal{G}_{\mathbb{C}}$  and the subgroup of  $h$ -unitary parabolic gauge transformations by  $\mathcal{G}$ .

Following Biquard [2], Konno introduces certain weighted Sobolev norms; we denote the corresponding Sobolev completions of the spaces defined above by  $\mathcal{C}_1^p$ ,  $\Omega_1^p$ ,  $(\mathcal{G}_{\mathbb{C}})_2^p$  and  $\mathcal{G}_2^p$  (the detailed definitions are not important to us so we refer to [21] for them). Let

$$\mathcal{H} = \{(\bar{\partial}_A, \Phi) \in \mathcal{C} \times \Omega \mid \bar{\partial}_A \Phi = 0\}$$

and let  $\mathcal{H}_1^p$  be the corresponding subspace of  $\mathcal{C}_1^p \times \Omega_1^p$ . Then  $\mathcal{H}_1^p$  carries a hyper-Kähler metric induced by  $h$ . Let  $F(A)^\perp$  denote the trace-free part of the curvature of the  $h$ -unitary connection  $A$  corresponding to  $\bar{\partial}_A$  and let  $\Phi^*$  be the adjoint with respect to  $h$ . One can then consider the moduli space  $\mathcal{S}$  defined by the subspace of  $\mathcal{H}_1^p$  satisfying *Hitchin's equation* (modulo  $\mathcal{G}_2^p$ ),

$$\mathcal{S} = \{(\bar{\partial}_A, \Phi) \in \mathcal{H}_1^p \mid F(A)^\perp + [\Phi, \Phi^*] = 0\} / \mathcal{G}_2^p,$$

where the equation is only defined on  $X \setminus D$ . Konno proves that  $\mathcal{S}$  is a hyper-Kähler quotient and that it can be naturally identified with the moduli space

$$\mathcal{M} = \mathcal{H}_1^p / (\mathcal{G}_{\mathbb{C}})_2^p.$$

Furthermore, Konno proves that the natural map  $\mathcal{H} / \mathcal{G}_{\mathbb{C}} \rightarrow \mathcal{H}_1^p / (\mathcal{G}_{\mathbb{C}})_2^p$  is a diffeomorphism.

### 3. MORSE THEORY ON THE MODULI SPACE

**3.1. The Morse function.** The non-zero complex numbers  $\mathbb{C}^*$  act on the moduli space  $\mathcal{M}$  via the map  $\lambda \cdot (E, \Phi) = (E, \lambda\Phi)$ . However, to have an action on the set of solutions to Hitchin's equations, one must restrict to the action of  $S^1 \subset \mathbb{C}^*$ . Obviously the identification  $\mathcal{S} \cong \mathcal{M}$  respects the circle action and thus we have a circle action on this hyper-Kähler manifold. With respect to one of the complex structures (coinciding with the one on  $\mathcal{M}$ ) this is a Hamiltonian action and the associated moment map is

$$(3.1) \quad [(A, \Phi)] \mapsto -\frac{1}{2} \|\Phi\|^2 = -\mathbf{i} \int_X \text{tr}(\Phi\Phi^*).$$

We shall, however, prefer to consider the positive function

$$(3.2) \quad f([A, \Phi]) = \frac{1}{2} \|\Phi\|^2.$$

In the case of non-parabolic Higgs bundles, Hitchin [18] proved that this is a proper map, using Uhlenbeck's compactness theorem. It was observed by Boden and Yokogawa [5] that the same argument works in the parabolic case, by using the parabolic analogue of Uhlenbeck's theorem, proved by Biquard [2]. Thus we have the following result.

**Proposition 3.1.** *The map  $f: \mathcal{M} \rightarrow \mathbb{R}$  is proper.* □

Next we recall a general result of Frankel [11], which was first used in the context of moduli spaces of Higgs bundles by Hitchin [18].

**Theorem 3.2.** *Let  $\tilde{f}: M \rightarrow \mathbb{R}$  be a proper moment map for a Hamiltonian circle action on a Kähler manifold  $M$ . Then  $\tilde{f}$  is a perfect Bott–Morse function.* □

The following result on the Morse indices of such a Morse function is implicit in Frankel's paper.

**Proposition 3.3.** *In the situation of Theorem 3.2, the critical points of  $\tilde{f}$  are exactly the fixed points of the circle action. Moreover, the Morse index of  $\tilde{f}$  at a critical point equals the dimension of the negative weight space of the circle action on the tangent space.*

*Proof.* The condition for  $\tilde{f}$  to be a moment map is that

$$\text{grad}(\tilde{f}) = IX,$$

where  $X$  is the vector field generating the circle action and  $I$  is the complex structure on  $M$ . Hence  $p$  is a critical point of  $\tilde{f}$  if and only if it is fixed under the circle action. Let  $\nabla$  be the Levi-Civita connection on  $M$ , then the Hessian  $H_{\tilde{f}}$  of  $\tilde{f}$  at  $p$  is the quadratic form associated to the symmetric endomorphism  $\nabla(\text{grad}(\tilde{f}))_p$  of  $T_pM$ . Let  $Y_p \in T_pM$  and let the vector field  $Y$  be an extension of  $Y_p$  around  $p$ . Then we have

$$\begin{aligned} H_{\tilde{f}}(Y_p) &= \nabla_{Y_p}(IX) \\ &= \nabla_{IX_p}(Y) - [IX, Y]_p \\ &= -[IX, Y]_p, \end{aligned}$$

where we have used that  $X_p = 0$ . On the other hand it is easy to see (cf. [11]) that the infinitesimal circle action on  $T_pM$  is given by  $Y_p \mapsto [Y, X]_p$ . It follows that the eigenvalues of  $H_{\tilde{f}}$  are exactly the weights of the circle action on  $T_pM$ . □

Thus we must identify the fixed point set of the action of  $S^1 \subset \mathbb{C}^*$  on  $\mathcal{M}$ . This was done by Simpson and is analogous to what happens for ordinary Higgs bundles.

### 3.2. Fixed points of the $S^1$ action on the moduli space.

**Proposition 3.4** ([29, Theorem 8]). *The equivalence class of a stable parabolic Higgs bundle  $(E, \Phi)$  is fixed under the action of  $S^1$  if and only if it is a parabolic complex variation of Hodge structure. This means that  $E$  has a direct sum decomposition*

$$E = E_0 \oplus \cdots \oplus E_m$$

*as parabolic bundles, such that  $\Phi$  is strongly parabolic and of degree one with respect to this decomposition, in other words the restriction  $\Phi_l = \Phi|_{E_l}$  belongs to*

$$H^0(\text{SParHom}(E_l, E_{l+1}) \otimes K(D)).$$

*Furthermore, stability implies that  $\Phi_l \neq 0$  for  $l = 0, \dots, m-1$ . The type of the parabolic complex variation of Hodge structure is the vector  $(\text{rk}(E_0), \dots, \text{rk}(E_m))$ .  $\square$*

*Remark 3.5.* If  $m = 0$ , then  $E = E_0$  and  $\Phi = 0$ , corresponding to the obvious fixed points  $(E, 0)$ , with  $E$  a stable parabolic bundle.

The following important fact was also noted by Simpson. For a proof see [1, Proposition 3.11] (in fact, this deals with the ordinary case but the argument can easily be adapted to the parabolic case).

**Proposition 3.6.** *A parabolic complex variation of Hodge structure  $(E = \bigoplus E_l, \Phi)$  is stable as a parabolic Higgs bundle if and only if the stability condition is satisfied for subbundles of  $E$  which respect the decomposition  $E = \bigoplus E_l$ .  $\square$*

Next we need to calculate the weights of the circle action on the tangent space to  $\mathcal{M}_\alpha$  at a critical point of  $f$ , represented by  $\mathbf{E} = (\bigoplus E_l, \Phi)$ . By the characterization of the critical points provided by Propositions 3.3 and 3.4, we have decompositions

$$\text{ParEnd}(E) = \bigoplus_{l=-m}^m U_l, \quad \text{SParEnd}(E) = \bigoplus_{l=-m}^m \hat{U}_l,$$

where we use the notation

$$U_l = \bigoplus_{j-i=l} \text{ParHom}(E_i, E_j), \quad \hat{U}_l = \bigoplus_{j-i=l} \text{SParHom}(E_i, E_j).$$

We get a corresponding decomposition of the deformation complex

$$C^\bullet(\mathbf{E}) = \bigoplus_{l=-m-1}^m C^\bullet(\mathbf{E})_l,$$

where  $C^\bullet(\mathbf{E})_l$  denotes the subcomplex

$$C^\bullet(\mathbf{E})_l : U_l \rightarrow \hat{U}_{l+1} \otimes K(D).$$

With this notation we have the following result.

**Proposition 3.7.** *Let  $\mathbf{E} = (\bigoplus E_l, \Phi)$  represent a fixed point of the circle action on  $\mathcal{M}$ . Then the weight  $l$  subspace of  $T_{\mathbf{E}}\mathcal{M}$  is isomorphic to the first hypercohomology  $\mathbb{H}^1(C^\bullet(\mathbf{E})_{-l})$ .*

*Proof.* It is clear that the derivative of the circle action at  $\mathbf{E} = (E, \Phi)$  is induced by the following map of deformation complexes  $C^\bullet(E, \Phi) \rightarrow C^\bullet(E, e^{i\theta}\Phi)$ :

$$\begin{array}{ccc} C^\bullet(E, \Phi) : & \bigoplus U_l & \xrightarrow{[-, \Phi]} \bigoplus \hat{U}_{l+1} \otimes K(D) \\ & \downarrow e^{i\theta} & \downarrow 1 & \downarrow e^{i\theta} \\ C^\bullet(E, e^{i\theta}\Phi) : & \bigoplus U_l & \xrightarrow{[-, e^{i\theta}\Phi]} \bigoplus \hat{U}_{l+1} \otimes K(D). \end{array}$$

In order to work out the circle action on  $T_{\mathbf{E}}\mathcal{M}$  from this we need to determine the identification  $\mathbb{H}^1(C^\bullet(E, \Phi)) \cong \mathbb{H}^1(C^\bullet(E, e^{i\theta}\Phi))$  induced by the isomorphism between  $(E, \Phi)$  and  $(E, e^{i\theta}\Phi)$ . But it is easy to write down such an isomorphism  $f_\theta$ : with respect to the decomposition  $E = \bigoplus E_l$  we can define  $f_\theta$  to be multiplication by  $e^{il\theta}$  on  $E_l$ . The corresponding isomorphism between the complexes  $C^\bullet(E, \Phi)$  and  $C^\bullet(E, e^{i\theta}\Phi)$  is given by the adjoint  $\text{Ad}(f_\theta): \psi \mapsto f_\theta \psi f_\theta^{-1}$ . Note that  $f_\theta$  is unique up to multiplication by scalars and hence  $\text{Ad}(f_\theta)$  is unique. Since  $\text{Ad}(f_\theta)$  is multiplication by  $e^{il\theta}$  on both  $U_l$  and  $\hat{U}_l$ , we can write down the induced isomorphism of complexes; the piece in degree  $l$  is given by

$$\begin{array}{ccc} C^\bullet(E, \Phi)_l : & U_l & \xrightarrow{[-, \Phi]} \hat{U}_{l+1} \otimes K(D) \\ & \downarrow \text{Ad}(f_\theta) & \downarrow e^{i\theta} & \downarrow e^{i(l+1)\theta} \\ C^\bullet(E, e^{i\theta}\Phi)_l : & U_l & \xrightarrow{[-, e^{i\theta}\Phi]} \hat{U}_{l+1} \otimes K(D). \end{array}$$

It follows that the derivative of the action of  $e^{i\theta}$  is the endomorphism of  $\mathbb{H}^1(C^\bullet(E, \Phi))$  induced by the composite map of complexes

$$C^\bullet(E, \Phi) \xrightarrow{e^{i\theta}} C^\bullet(E, e^{i\theta}\Phi) \xrightarrow{\text{Ad}(f_\theta)^{-1}} C^\bullet(E, \Phi),$$

whose degree  $l$  piece is

$$\begin{array}{ccc} C^\bullet(E, \Phi)_l : & U_l & \xrightarrow{[-, \Phi]} \hat{U}_{l+1} \otimes K(D) \\ & \downarrow & \downarrow e^{-il\theta} & \downarrow e^{-i\theta} \\ C^\bullet(E, \Phi)_l : & U_l & \xrightarrow{[-, \Phi]} \hat{U}_{l+1} \otimes K(D). \end{array}$$

Thus  $\mathbb{H}^1(C^\bullet(E, \Phi)_l)$  is isomorphic to the weight  $-l$  subspace of  $\mathbb{H}^1(C^\bullet(E, \Phi)) \cong T_{\mathbf{E}}\mathcal{M}$ .  $\square$

Summarizing the results of this section so far, we obtain the following.

**Theorem 3.8.** *The function  $f: \mathcal{M} \rightarrow \mathbb{R}$  defined by  $f([A, \Phi]) = \frac{1}{2}\|\Phi\|^2$  is a perfect Bott–Morse function. A parabolic Higgs bundle  $(E, \Phi)$  represents a critical point of  $f$  if and only if it is a parabolic complex variation of Hodge structure, i.e.,  $E = \bigoplus_{l=0}^m E_l$  with  $\Phi: E_l \rightarrow E_{l+1} \otimes K(D)$  strongly parabolic (where  $\Phi = 0$  if and only  $m = 0$ ). The*

tangent space to  $\mathcal{M}$  at a critical point  $\mathbf{E}$  decomposes as

$$T_{\mathbf{E}}\mathcal{M} = \bigoplus_{l=-m-1}^m T_{\mathbf{E}}\mathcal{M}_l,$$

where the eigenvalue  $l$  subspace of the Hessian of  $f$  is

$$T_{\mathbf{E}}\mathcal{M}_l \cong \mathbb{H}^1(C^\bullet(E, \Phi)_l).$$

*Proof.* Immediate from Propositions 3.3 and 3.7. Note that since our Morse function  $f$  is *minus* the moment map  $\tilde{f}$  (cf. (3.1) and (3.2)), the eigenvalue of the Hessian coincides with the weight  $l$  hypercohomology (with a *positive* sign).  $\square$

### 3.3. Morse indices.

**Proposition 3.9.** (i) *There is a natural isomorphism*

$$\mathbb{H}^1(C^\bullet(\mathbf{E})_l) \cong \mathbb{H}^1(C^\bullet(\mathbf{E})_{-l-1})^*$$

*and hence a natural isomorphism*

$$T_{\mathbf{E}}\mathcal{M}_l \cong (T_{\mathbf{E}}\mathcal{M}_{1-l})^*.$$

(ii) *If  $\mathbf{E}$  is stable, then we have*

$$\mathbb{H}^0(C^\bullet(\mathbf{E})_l) = \begin{cases} \mathbb{C} & \text{if } l = 0, \\ 0 & \text{otherwise,} \end{cases}$$

*and*

$$\mathbb{H}^2(C^\bullet(\mathbf{E})_l) = \begin{cases} \mathbb{C} & \text{if } l = -1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* (i) It follows from Proposition 2.3 (i) that there is an isomorphism of complexes

$$(C^\bullet(\mathbf{E})_l)^* \otimes K \cong C^\bullet(\mathbf{E})_{-l-1}.$$

Hence Serre duality for hypercohomology gives the first isomorphism of the statement. The second isomorphism is now immediate from the last statement of Theorem 3.8.

(ii) When  $\mathbf{E}$  is stable we have that  $\mathbb{H}^0(C^\bullet(\mathbf{E})) \cong \mathbb{C}$ , generated by the identity endomorphism of  $\mathbf{E}$ , and hence the first statement follows. For the same reason as in the proof of (i) we have the isomorphism  $\mathbb{H}^0(C^\bullet(\mathbf{E})_l) \cong \mathbb{H}^2(C^\bullet(\mathbf{E})_{-l-1})^*$  and thus the second statement follows from the first.  $\square$

**Corollary 3.10.** *Let  $\mathbf{E}$  represent a critical point of  $f$ , let  $T_{\mathbf{E}}\mathcal{M}_{\leq 0}$  be the subspace of the tangent space on which the Hessian of  $f$  has eigenvalues less than or equal to zero and*

let  $T_{\mathbf{E}}\mathcal{M}_{>0}$  be the subspace on which the Hessian of  $f$  has eigenvalues greater than zero. Then

$$T_{\mathbf{E}}\mathcal{M}_{\leq 0} \cong (T_{\mathbf{E}}\mathcal{M}_{>0})^*$$

under the isomorphism of Proposition 2.3 (ii). It follows that the dimension of  $T_{\mathbf{E}}\mathcal{M}_{\leq 0}$  is half the dimension of the moduli space, i.e.,

$$\dim T_{\mathbf{E}}\mathcal{M}_{\leq 0} = r^2(g-1) + 1 + \sum_p f_p.$$

*Proof.* Immediate from Propositions 3.9 and 2.4.  $\square$

**Proposition 3.11.** *Let the parabolic Higgs bundle  $\mathbf{E} = (E, \Phi)$  represent a critical point of  $f$ . Then the Morse index of  $f$  at this point is*

$$\begin{aligned} \lambda_{\mathbf{E}} &= r^2(2g-2) + 2 \sum_p f_p + 2\chi(C^\bullet(\mathbf{E})_0) \\ &= r^2(2g-2) + 2 \sum_p f_p + 2 \sum_{l=0}^m \chi(\text{ParEnd}(E_l)) - 2 \sum_{l=0}^{m-1} \chi(\text{SParHom}(E_l, E_{l+1}) \otimes K(D)) \\ &= r^2(2g-2) + 2 \sum_p f_p + 2 \sum_{l=0}^m \left[ (1-g-n) \text{rk}(E_l)^2 + \sum_p \dim P_p(E_l, E_l) \right] \\ &\quad + 2 \sum_{l=0}^{m-1} \left[ (1-g) \text{rk}(E_l) \text{rk}(E_{l+1}) - \text{rk}(E_l) \deg(E_{l+1}) + \text{rk}(E_{l+1}) \deg(E_l) \right. \\ &\quad \left. - \sum_p \dim N_p(E_l, E_{l+1}) \right], \end{aligned}$$

where  $E = \bigoplus_{l=0}^m E_l$  with  $\Phi \in H^0(\text{SParHom}(E_l, E_{l+1}) \otimes K(D))$ .

*Proof.* Since we are calculating real dimensions, the Morse index is twice the dimension of  $T_{\mathbf{E}}\mathcal{M}_{<0}$ , the subspace on which the Hessian of  $f$  has negative eigenvalues. Hence Corollary 3.10 shows that

$$\begin{aligned} \frac{1}{2}\lambda_{\mathbf{E}} &= \dim T_{\mathbf{E}}\mathcal{M}_{<0} \\ &= \dim T_{\mathbf{E}}\mathcal{M}_{\leq 0} - \dim T_{\mathbf{E}}\mathcal{M}_0 \\ &= r^2(g-1) + 1 + \sum_p f_p - \dim T_{\mathbf{E}}\mathcal{M}_0. \end{aligned}$$

On the other hand from Proposition 3.9 (ii) we have that  $\mathbb{H}^0(C^\bullet(\mathbf{E})_0) = \mathbb{C}$ , while  $\mathbb{H}^2(C^\bullet(\mathbf{E})_0) = 0$ . Hence Theorem 3.8 shows that we have

$$\begin{aligned} \dim T_{\mathbf{E}}\mathcal{M}_0 &= \dim \mathbb{H}^1(C^\bullet(\mathbf{E})_0) \\ &= 1 - \chi(C^\bullet(\mathbf{E})_0), \end{aligned}$$

and this finishes the proof of the first identity of the statement of the Proposition. The rest can be deduced from the long exact sequence in hypercohomology for the complex  $C^\bullet(\mathbf{E})_0$ , analogous to (2.2), and using the same method as in the proof of Proposition 2.4.  $\square$

*Remark 3.12.* Obviously, the absolute minima is for  $m = 0$ , for which the computation in Proposition 3.11 naturally gives

$$\begin{aligned} \lambda_{(E,0)} &= r^2(2g-2) + 2 \sum_p f_p + 2\chi(\text{ParEnd}(E)) \\ &= r^2(2g-2) + 2 \sum_p f_p + 2r^2(1-g) + 2 \sum_p (r^2 - f_p - r^2) \\ &= 0. \end{aligned}$$

**3.4. Rank three parabolic Higgs bundles.** Now we turn our attention to the moduli space  $\mathcal{M}$  of parabolic Higgs bundles of rank three. Let  $(E, \Phi)$  be a critical point of  $f$ . By Theorem 3.8, the only possibilities that we have in this situation are:

- (a)  $E$  is a stable rank three parabolic Higgs bundle and  $\Phi = 0$ .
- (b)  $E = E_0 \oplus E_1 \oplus E_2$  where  $E_l$  are parabolic line bundles. These are line bundles with weights at each  $p \in D$ . The map  $\Phi$  decomposes as strongly parabolic maps  $\Phi_1 : E_0 \rightarrow E_1 \otimes K(D)$  and  $\Phi_2 : E_1 \rightarrow E_2 \otimes K(D)$ .
- (c)  $E = E_0 \oplus E_1$  where  $E_0$  is a parabolic line bundle  $L$  and  $E_1$  is a rank 2 parabolic bundle. Here  $\Phi$  gives a strongly parabolic map  $\Phi : L \rightarrow E_1 \otimes K(D)$ .
- (d)  $E = E_0 \oplus E_1$  where  $E_0$  is a rank 2 parabolic bundle and  $E_1$  is a parabolic line bundle  $L$ . Here  $\Phi$  gives a strongly parabolic map  $\Phi : E_0 \rightarrow L \otimes K(D)$ .

In case (a) the corresponding critical subvariety can obviously be identified with the moduli space of ordinary parabolic bundles. Its Betti numbers can be computed from a formula given by Nitsure [28] and Holla [20]. In Section 10 below we work out explicitly what their formula gives for the Poincaré polynomial in our situation of rank three parabolic bundles. Case (b) involves basically line bundles and divisors and can be dealt with easily [24]. The other two cases, (c) and (d), are more involved. They are particular cases of objects called *parabolic triples*, which have been introduced and studied from a gauge-theoretic point of view in [3], and will be studied in Section 4 below.

**3.5. Laumon's Theorem for parabolic Higgs bundles.** At this point we make a small digression in order to deduce, following Hausel, a parabolic version of a Theorem of Laumon from the analysis leading to our calculation of the Morse indices.

As in the non-parabolic case studied by Hitchin [18, 19], there is a *Hitchin map*

$$\chi : \mathcal{M} \rightarrow \mathbb{C}^{r^2(g-1)+1+\sum_p f_p}$$

defined by taking the parabolic Higgs bundle  $(E, \Phi)$  to the characteristic polynomial of  $\Phi$  (see [27]). The Hitchin map defines an algebraic completely integrable system.

The pre-image of 0 under the Hitchin map,

$$N = \chi^{-1}(0),$$

is called the *nilpotent cone*. The main result of Laumon [23], proved for the moduli stack of Higgs bundles, is that the nilpotent cone is Lagrangian.

In the non-parabolic case, Hausel [16, Theorem 5.2] proved that the downwards Morse flow on the moduli space of Higgs bundles coincides with the nilpotent cone. His proof goes over word by word to the parabolic case, so we have the following Theorem.

**Theorem 3.13.** *The downwards Morse flow on the moduli space of parabolic Higgs bundles coincides with the nilpotent cone  $N$ .*  $\square$

As pointed out by Hausel, the nilpotent cone is isotropic because the Hitchin map is a completely integrable system. (To be precise, the nilpotent cone being isotropic means that its tangent space at any non-singular point of the nilpotent cone is an isotropic subspace of the tangent space to  $\mathcal{M}$ , cf. Ginzburg [12].) Hence the nilpotent cone is Lagrangian if its dimension equals half that of the moduli space  $\mathcal{M}$ . But this fact follows at once from our Corollary 3.10. Thus we have the following version of Laumon's theorem for the moduli space of parabolic Higgs bundles.

**Theorem 3.14.** *The nilpotent cone  $N$  is a Lagrangian subvariety of the moduli space of parabolic Higgs bundles.*  $\square$

#### 4. PARABOLIC TRIPLES

**4.1. Definitions and basic facts.** A *parabolic triple*  $T = (E_1, E_2, \phi)$  on  $X$  consists of two parabolic vector bundles  $E_1$  and  $E_2$  on  $X$  and a  $\phi \in H^0(\text{SParHom}(E_2, E_1(D)))$ . A homomorphism from  $T' = (E'_1, E'_2, \phi')$  to  $T = (E_1, E_2, \phi)$  is a commutative diagram

$$\begin{array}{ccc} E'_2 & \xrightarrow{\phi'} & E'_1(D) \\ \downarrow & & \downarrow \\ E_2 & \xrightarrow{\phi} & E_1(D), \end{array}$$

where the vertical arrows are parabolic sheaf homomorphisms. A triple  $T' = (E'_1, E'_2, \phi')$  is a subtriple of  $T = (E_1, E_2, \phi)$  if the sheaf homomorphisms  $E'_1 \rightarrow E_1$  and  $E'_2 \rightarrow E_2$  are injective. A subtriple  $T' \subset T$  is called *proper* if  $T' \neq 0$  and  $T' \neq T$ .

**Definition 4.1.** For any  $\sigma \in \mathbb{R}$  the  $\sigma$ -degree and  $\sigma$ -slope of  $T$  are defined to be

$$\begin{aligned} \deg_\sigma(T) &= \text{pardeg}(E_1) + \text{pardeg}(E_2) + \sigma \text{rk}(E_2), \\ \mu_\sigma(T) &= \frac{\deg_\sigma(T)}{\text{rk}(E_1) + \text{rk}(E_2)} \\ &= \text{par}\mu(E_1 \oplus E_2) + \sigma \frac{\text{rk}(E_2)}{\text{rk}(E_1) + \text{rk}(E_2)}. \end{aligned}$$



We say  $T = (E_1, E_2, \phi)$  is  $\sigma$ -stable if

$$\mu_\sigma(T') < \mu_\sigma(T),$$

for any proper subtriple  $T' = (E'_1, E'_2, \phi')$ . We define  $\sigma$ -semistability by replacing the above strict inequality with a weak inequality. A triple is called  $\sigma$ -polystable if it is the direct sum of  $\sigma$ -stable triples of the same  $\sigma$ -slope.

Let us fix the topological and parabolic types of  $E_1$  and  $E_2$ . We denote by  $\mathcal{N}_\sigma$  the moduli space of  $\sigma$ -stable triples  $T = (E_1, E_2, \phi)$  of the given type.

Given a triple  $T = (E_1, E_2, \phi)$  one has the dual triple  $T^* = (E_2^*, E_1^*, \phi^*)$ , where  $E_i^*$  is the parabolic dual of  $E_i$  and  $\phi^*$  is the transpose of  $\phi$ . The following is not difficult to prove.

**Proposition 4.2.** *The  $\sigma$ -(semi)stability of  $T$  is equivalent to the  $\sigma$ -(semi)stability of  $T^*$ . The map  $T \mapsto T^*$  defines an isomorphism of moduli spaces.  $\square$*

This can be used to restrict our study to  $\text{rk}(E_1) \geq \text{rk}(E_2)$  and appeal to duality to deal with the case  $\text{rk}(E_1) < \text{rk}(E_2)$ .

There are certain necessary conditions in order for  $\sigma$ -semistable triples to exist. Let  $r_1 = \text{rk}(E_1)$ ,  $r_2 = \text{rk}(E_2)$ ,  $\text{par}\mu_1 = \text{par}\mu(E_1)$  and  $\text{par}\mu_2 = \text{par}\mu(E_2)$  be the ranks and parabolic degrees of  $E_1$  and  $E_2$ , and define

$$(4.1) \quad \sigma_m = \text{par}\mu_1 - \text{par}\mu_2,$$

$$(4.2) \quad \sigma_M = \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right) (\text{par}\mu_1 - \text{par}\mu_2) + n \frac{r_1 + r_2}{|r_1 - r_2|}, \quad \text{if } r_1 \neq r_2.$$

**Proposition 4.3.** *A necessary condition for  $\mathcal{N}_\sigma$  to be non-empty is*

- (i)  $\sigma_m \leq \sigma \leq \sigma_M$ , if  $r_1 \neq r_2$ ,
- (ii)  $\sigma_m \leq \sigma$ , if  $r_1 = r_2$ .

*Proof.* The proof is similar to the one given in [7, Proposition 3.18] for ordinary triples.  $\square$

*Remark 4.4.* The upper bound given for  $\sigma$  is not optimal. A better one can be found, as will be seen later in Section 6.

Using the dimensional reduction construction given in [3], the moduli space  $\mathcal{N}_\sigma$  can be realised as a subvariety of a certain moduli space of parabolic bundles on  $X \times \mathbb{P}^1$ . Such moduli spaces have been constructed by Maruyama and Yokogawa [25] in arbitrary dimensions using GIT methods.

Another important aspect that follows also from the dimensional reduction point of view is the existence of a correspondence between stability and the existence of solutions

to certain gauge-theoretic equations on a parabolic triple  $T = (E_1, E_2, \phi)$ , known as the *parabolic vortex equations* [3]. The parabolic vortex equations

$$(4.3) \quad \begin{aligned} \mathbf{i}\Lambda F(E_1) + \phi\phi^* &= \tau_1 \text{Id}_{E_1}, \\ \mathbf{i}\Lambda F(E_2) - \phi^*\phi &= \tau_2 \text{Id}_{E_2}, \end{aligned}$$

are equations for Hermitian metrics on  $E_1$  and  $E_2$  adapted to the parabolic structure. Here  $\Lambda$  is contraction by the Kähler form of a metric on  $X$  (normalized so that  $\text{vol}(X) = 2\pi$ ),  $F(E_i)$  is the curvature of the unique connection on  $E_i$  compatible with the Hermitian metric and the holomorphic structure of  $E_i$ , and  $\tau_1$  and  $\tau_2$  are real parameters satisfying  $\text{pardeg}(E_1) + \text{pardeg}(E_2) = r_1\tau_1 + r_2\tau_2$ . Also, here  $\phi^*$  is the adjoint of  $\phi$  with respect to the Hermitian metrics. One has the following.

**Theorem 4.5.** [3, Theorem 3.4] *A solution to (4.3) exists if and only if  $T$  is  $\sigma$ -polystable for  $\sigma = \tau_1 - \tau_2$ .  $\square$*

**4.2. Parabolic Higgs bundles and parabolic triples.** The relation between parabolic Higgs bundles and parabolic triples is given by the following.

**Proposition 4.6.** *Suppose that  $(E, \Phi)$  is a stable parabolic Higgs bundle such that  $E = E_1 \oplus E_2$  and*

$$\Phi = \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix}$$

*with  $\phi : E_2 \rightarrow E_1 \otimes K(D)$  a strongly parabolic map. Then  $(E, \Phi)$  is stable if and only if the parabolic triple  $(E_1 \otimes K, E_2, \phi)$  is  $\sigma$ -stable for  $\sigma = 2g - 2$ .*

*Proof.* Take a sub-object  $E' \subset E$  with  $\Phi(E') \subset E' \otimes K(D)$ . This can be assumed to be of the form  $E' = E'_1 \oplus E'_2$  and hence it defines a subtriple  $(E'_1 \otimes K, E'_2, \phi')$  where  $\phi' = \phi|_{E'_2}$ . The result follows now from the equivalence between

$$\begin{aligned} \frac{\text{pardeg}(E'_1) + \text{pardeg}(E'_2)}{r'_1 + r'_2} &< \frac{\text{pardeg}(E_1) + \text{pardeg}(E_2)}{r_1 + r_2}, \quad \text{and} \\ \frac{\text{pardeg}(E'_1) + r'_1(2g - 2) + \text{pardeg}(E'_2)}{r'_1 + r'_2} + \sigma \frac{r'_2}{r'_1 + r'_2} &< \\ \frac{\text{pardeg}(E_1) + r_1(2g - 2) + \text{pardeg}(E_2)}{r_1 + r_2} + \sigma \frac{r_2}{r_1 + r_2}, \end{aligned}$$

which is the  $\sigma$ -stability of the triple  $(E_1 \otimes K, E_2, \Phi)$ , for  $\sigma = 2g - 2$ .  $\square$

**4.3. Extensions and deformations of parabolic triples.** In order to analyse the differences between the moduli spaces  $\mathcal{N}_\sigma$  as  $\sigma$  changes, as well as the smoothness properties of the moduli space for a given value of  $\sigma$ , we need to study the homological algebra of parabolic triples. This is done by considering the hypercohomology of a certain complex of sheaves, in an analogous way to the case of holomorphic triples studied in [8], and the parabolic Higgs bundle case studied in Section 2.2.

Let  $T' = (E'_1, E'_2, \phi')$  and  $T'' = (E''_1, E''_2, \phi'')$  be two parabolic triples. Let  $\text{Hom}(T'', T')$  denote the linear space of homomorphisms from  $T''$  to  $T'$ , and let  $\text{Ext}^1(T'', T')$  denote the linear space of equivalence classes of extensions of the form

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0,$$

where by this we mean a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 & \longrightarrow & 0 \\ & & \phi' \downarrow & & \phi \downarrow & & \phi'' \downarrow & & \\ 0 & \longrightarrow & E'_1(D) & \longrightarrow & E_1(D) & \longrightarrow & E''_1(D) & \longrightarrow & 0. \end{array}$$

Hence, to analyse  $\text{Ext}^1(T'', T')$  one considers the complex of sheaves

$$(4.4) \quad C^\bullet(T'', T') : \text{ParHom}(E''_1, E'_1) \oplus \text{ParHom}(E''_2, E'_2) \xrightarrow{c} \text{SParHom}(E''_2, E'_1(D)),$$

where the map  $c$  is defined by

$$c(\psi_1, \psi_2) = \phi' \psi_2 - \psi_1 \phi''.$$

**Proposition 4.7.** *There are natural isomorphisms*

$$\begin{aligned} \text{Hom}(T'', T') &\cong \mathbb{H}^0(C^\bullet(T'', T')), \\ \text{Ext}^1(T'', T') &\cong \mathbb{H}^1(C^\bullet(T'', T')), \end{aligned}$$

and a long exact sequence associated to the complex  $C^\bullet(T'', T')$ :

$$(4.5) \quad \begin{aligned} 0 &\rightarrow \mathbb{H}^0 \rightarrow H^0(\text{ParHom}(E''_1, E'_1) \oplus \text{ParHom}(E''_2, E'_2)) \rightarrow H^0(\text{SParHom}(E''_2, E'_1(D))) \\ &\rightarrow \mathbb{H}^1 \rightarrow H^1(\text{ParHom}(E''_1, E'_1) \oplus \text{ParHom}(E''_2, E'_2)) \rightarrow H^1(\text{SParHom}(E''_2, E'_1(D))) \\ &\rightarrow \mathbb{H}^2 \rightarrow 0, \end{aligned}$$

where  $\mathbb{H}^i = \mathbb{H}^i(C^\bullet(T'', T'))$ . □

We introduce the following notation:

$$\begin{aligned} h^i(T'', T') &= \dim \mathbb{H}^i(C^\bullet(T'', T')), \\ \chi(T'', T') &= h^0(T'', T') - h^1(T'', T') + h^2(T'', T'). \end{aligned}$$

**Proposition 4.8.** *For any parabolic triples  $T'$  and  $T''$  we have*

$$\chi(T'', T') = \chi(\text{ParHom}(E''_1, E'_1)) + \chi(\text{ParHom}(E''_2, E'_2)) - \chi(\text{SParHom}(E''_2, E'_1(D)))$$

where  $\chi(E) = \dim H^0(E) - \dim H^1(E)$  is the Euler characteristic of  $E$ .

*Proof.* Immediate from the long exact sequence (4.5) and the Riemann–Roch formula. □

**Corollary 4.9.** *For any extension  $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$  of parabolic triples,*

$$\chi(T, T) = \chi(T', T') + \chi(T'', T'') + \chi(T'', T') + \chi(T', T'').$$

□

**Proposition 4.10.** *Suppose that  $T'$  and  $T''$  are  $\sigma$ -semistable.*

- (i) *If  $\mu_\sigma(T') < \mu_\sigma(T'')$  then  $\mathbb{H}^0(C^\bullet(T'', T')) \cong 0$ .*
- (ii) *If  $\mu_\sigma(T') = \mu_\sigma(T'')$  and  $T'$  and  $T''$  are both  $\sigma$ -stable, then*

$$\mathbb{H}^0(C^\bullet(T'', T')) \cong \begin{cases} \mathbb{C}, & \text{if } T' \cong T'', \\ 0, & \text{if } T' \not\cong T''. \end{cases}$$

□

**Corollary 4.11.** *Let  $T'$  and  $T''$  be  $\sigma$ -semistable parabolic triples with  $\mu_\sigma(T') = \mu_\sigma(T'')$ , and suppose that  $\mathbb{H}^2(C^\bullet(T'', T')) = 0$ . Then*

$$\dim \text{Ext}^1(T'', T') = h^0(T'', T') - \chi(T'', T').$$

□

Since the space of infinitesimal deformations of  $T$  is isomorphic to  $\mathbb{H}^1(C^\bullet(T, T))$ , the considerations of the previous sections also apply to studying deformations of a parabolic triple  $T$  (the proofs are analogous to the non-parabolic case [8]). To be precise, one has the following.

**Theorem 4.12.** *Let  $T = (E_1, E_2, \phi)$  be a  $\sigma$ -stable parabolic triple.*

- (i) *The Zariski tangent space at the point defined by  $T$  in the moduli space of stable triples is isomorphic to  $\mathbb{H}^1(C^\bullet(T, T))$ .*
- (ii) *If  $\mathbb{H}^2(C^\bullet(T, T)) = 0$ , then the moduli space of  $\sigma$ -stable parabolic triples is smooth in a neighbourhood of the point defined by  $T$ .*
- (iii)  *$\mathbb{H}^2(C^\bullet(T, T)) = 0$  if and only if the homomorphism*

$$H^1(\text{ParEnd}(E_1)) \oplus H^1(\text{ParEnd}(E_2)) \longrightarrow H^1(\text{SParHom}(E_2, E_1(D)))$$

*in the corresponding long exact sequence is surjective.*

- (iv) *At a smooth point  $T \in \mathcal{N}_\sigma$  the dimension of the moduli space of  $\sigma$ -stable parabolic triples is*

$$(4.6) \quad \begin{aligned} \dim \mathcal{N}_\sigma &= h^1(T, T) = 1 - \chi(T, T) \\ &= \chi(\text{ParEnd}(E_1, E_1)) + \chi(\text{ParEnd}(E_2, E_2)) - \chi(\text{SParHom}(E_2, E_1(D))) \end{aligned}$$

- (v) *If  $\phi$  is injective or surjective then  $T = (E_1, E_2, \phi)$  defines a smooth point in the moduli space.*

□

5.1. **Critical values.** A parabolic triple  $T = (E_1, E_2, \phi)$  of fixed topological and parabolic type is strictly  $\sigma$ -semistable if and only if it has a proper subtriple  $T' = (E'_1, E'_2, \phi')$  such that  $\mu_\sigma(T') = \mu_\sigma(T)$ , i.e.

$$(5.1) \quad \text{par}\mu(E'_1 \oplus E'_2) + \sigma \frac{r'_2}{r'_1 + r'_2} = \text{par}\mu(E_1 \oplus E_2) + \sigma \frac{r_2}{r_1 + r_2},$$

where  $r_1, r_2, r'_1, r'_2$  are the ranks of  $E_1, E_2, E'_1, E'_2$ . There are two ways in which this can happen. The first one is if there exists a subtriple  $T'$  such that

$$\frac{r'_2}{r'_1 + r'_2} = \frac{r_2}{r_1 + r_2}, \quad \text{and} \\ \text{par}\mu(E'_1 \oplus E'_2) = \text{par}\mu(E_1 \oplus E_2).$$

In this case the terms containing  $\sigma$  drop from (5.1) and  $T$  is strictly  $\sigma$ -semistable for all values of  $\sigma$ . We refer to this phenomenon as  $\sigma$ -independent semistability.

The other way in which strict  $\sigma$ -semistability can happen is if equality holds in (5.1) but

$$\frac{r'_2}{r'_1 + r'_2} \neq \frac{r_2}{r_1 + r_2}.$$

The values of  $\sigma$  for which this happens are called critical values.

From now on we shall make the following assumption on the weights.

**Assumption 5.1.** Let  $\alpha_i(p)$  be the collection of all the weights of  $E_1$  and  $E_2$  together. We assume that they are all of multiplicity one and that, for a large integer  $N$  depending only on the ranks, they satisfy the following property:

$$\sum_{1 \leq i \leq r, p \in D} n_{i,p} \alpha_i(p) \in \mathbb{Z}, \quad n_{i,p} \in \mathbb{Z}, |n_{i,p}| \leq N \implies n_{i,p} = 0, \text{ for all } p, i.$$

The weights failing this genericity condition are a finite union of hyperplanes in  $[0, 1]^{nr}$ , where  $nr$  is the total number of weights,  $r = r_1 + r_2$ .

**Proposition 5.2.** (i) *Under Assumption 5.1, there are no  $\sigma$ -independent semistable triples (by taking  $N$  larger than  $r_1 + r_2$ ).*

(ii) *The critical values of  $\sigma$  form a discrete subset of  $[\sigma_m, \infty)$ , where  $\sigma_m$  is as in (4.1).*

(iii) *If  $r_1 \neq r_2$  the number of critical values is finite and they lie in the interval  $[\sigma_m, \sigma_M]$ , where  $\sigma_M$  is as in (4.2).*

(iv) *The stability criteria for two values of  $\sigma$  lying between two consecutive critical values are equivalent; thus the corresponding moduli spaces coincide.*

(v) *If  $\sigma_c$  is a critical value and  $T'$  is a subtriple of a  $\sigma_c$ -semistable triple  $T$  such that  $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T)$ , then  $T'$  and the quotient triple  $T'' = T/T'$  are  $\sigma_c$ -stable (for this, it may be necessary to take a larger value of  $N$  in Assumption 5.1).*

□

**5.2. Crossing critical values and universal extensions.** In this section we study the differences between the moduli spaces  $\mathcal{N}_\sigma$ , for fixed type but different values of  $\sigma$ .

We begin with a set theoretic description of the differences between two spaces  $\mathcal{N}_\sigma$  and  $\mathcal{N}_{\sigma'}$  when  $\sigma$  and  $\sigma'$  are separated by a single critical value (as defined in Section 4.1). For the rest of this section we adopt the following notation: when  $r_1 \neq r_2$  the bounds  $\sigma_m$  and  $\sigma_M$  are as in (4.1) and (4.2). When  $r_1 = r_2$  we adopt the convention that  $\sigma_M = \infty$ . Let  $\sigma_c \in \mathbb{R}$  be a critical value such that

$$\sigma_m \leq \sigma_c \leq \sigma_M.$$

Set

$$\sigma_c^+ = \sigma_c + \epsilon, \quad \sigma_c^- = \sigma_c - \epsilon,$$

where  $\epsilon > 0$  is small enough so that  $\sigma_c$  is the only critical value in the interval  $(\sigma_c^-, \sigma_c^+)$ .

**Definition 5.3.** Let  $\sigma_c$  be a critical value. We define the *flip loci*  $\mathcal{S}_{\sigma_c^\pm} \subset \mathcal{N}_{\sigma_c^\pm}$  by the conditions that the points in  $\mathcal{S}_{\sigma_c^+}$  represent triples which are  $\sigma_c^+$ -stable but  $\sigma_c^-$ -unstable, while the points in  $\mathcal{S}_{\sigma_c^-}$  represent triples which are  $\sigma_c^-$ -stable but  $\sigma_c^+$ -unstable.

**Lemma 5.4.** *In the above notation,*

$$\mathcal{N}_{\sigma_c^+} - \mathcal{S}_{\sigma_c^+} = \mathcal{N}_{\sigma_c} = \mathcal{N}_{\sigma_c^-} - \mathcal{S}_{\sigma_c^-}.$$

□

As a consequence of Proposition 5.2 (v) we have the following.

**Proposition 5.5.** *Let  $\sigma_c$  be a critical value. Let  $T = (E_1, E_2, \phi)$  be a triple of this type which is  $\sigma_c$ -semistable. Then  $T$  has a (unique) description as the middle term in an extension*

$$(5.2) \quad 0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$$

*in which  $T'$  and  $T''$  are  $\sigma_c$ -stable and  $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$ .*

□

We thus have the following.

**Proposition 5.6.** *The set  $\mathcal{S}_{\sigma_c^+}$  coincides with the set of equivalence classes of extensions (5.2), in which  $T'$  and  $T''$  are  $\sigma_c$ -stable,  $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$ , and  $r'_2/r' < r''_2/r''$ .*

*Similarly,  $\mathcal{S}_{\sigma_c^-}$  coincides with the set of equivalence classes of extensions (5.2), in which  $T'$  and  $T''$  are  $\sigma_c$ -stable,  $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$ , and  $r'_2/r' > r''_2/r''$ ; or, equivalently, extensions*

$$0 \rightarrow T'' \rightarrow T \rightarrow T' \rightarrow 0$$

*where  $T'$  and  $T''$  are as above, but  $r'_2/r' < r''_2/r''$ .*

□

To construct the locus  $\mathcal{S}_{\sigma_c^\pm}$ , we first observe that, by the genericity of the weights, the moduli spaces  $\mathcal{N}'_{\sigma_c}$  and  $\mathcal{N}''_{\sigma_c}$  are fine moduli spaces (cf. [33]), i.e., there are universal parabolic triples  $\mathcal{T}' = (\mathcal{E}'_1, \mathcal{E}'_2, \Phi')$  and  $\mathcal{T}'' = (\mathcal{E}''_1, \mathcal{E}''_2, \Phi'')$  over  $\mathcal{N}'_{\sigma_c} \times X$  and  $\mathcal{N}''_{\sigma_c} \times X$

respectively. Let  $B = \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$  and let pull back  $\mathcal{T}'$  and  $\mathcal{T}''$  to  $B \times X$ . Considering the complex  $C^\bullet(\mathcal{T}'', \mathcal{T}')$  as defined in (4.4), taking relative hypercohomology  $\mathbb{H}_\pi(C^\bullet(\mathcal{T}'', \mathcal{T}'))$  with respect to the projection  $\pi : B \times X \rightarrow B$ , and putting

$$(5.3) \quad W^+ := \mathbb{H}_\pi^1(C^\bullet(\mathcal{T}'', \mathcal{T}')),$$

we have the following exact sequence of sheaves over  $B$ :

$$(5.4) \quad \begin{aligned} 0 &\rightarrow \mathbb{H}_\pi^0(C^\bullet(\mathcal{T}'', \mathcal{T}')) \rightarrow \pi_* \text{ParHom}(\mathcal{E}_2'', \mathcal{E}_2') \oplus \pi_* \text{ParHom}(\mathcal{E}_1'', \mathcal{E}_1') \rightarrow \pi_* \text{SParHom}(\mathcal{E}_2'', \mathcal{E}_1'(D)) \\ &\rightarrow W^+ \rightarrow R^1 \pi_* \text{ParHom}(\mathcal{E}_2'', \mathcal{E}_2') \oplus R^1 \pi_* \text{ParHom}(\mathcal{E}_1'', \mathcal{E}_1') \rightarrow R^1 \pi_* \text{SParHom}(\mathcal{E}_2'', \mathcal{E}_1'(D)) \\ &\rightarrow \mathbb{H}_\pi^2(C^\bullet(\mathcal{T}'', \mathcal{T}')) \rightarrow 0. \end{aligned}$$

Analogously, we can consider the complex  $C^\bullet(\mathcal{T}', \mathcal{T}'')$  and define

$$W^- := \mathbb{H}_\pi^1(C^\bullet(\mathcal{T}', \mathcal{T}'')).$$

**Proposition 5.7.** *If  $\mathbb{H}^2(C^\bullet(\mathcal{T}'', \mathcal{T}')) = 0$  for every  $(\mathcal{T}', \mathcal{T}'') \in \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ , then  $W^+$  defined in (5.3) is locally free. Similarly for  $W^-$ .*

*Proof.* By Proposition 4.10,  $\mathbb{H}_\pi^0(C^\bullet(\mathcal{T}'', \mathcal{T}')) = 0$  for every  $(\mathcal{T}', \mathcal{T}'') \in \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$  and hence  $\mathbb{H}_\pi^0(C^\bullet(\mathcal{T}'', \mathcal{T}')) = 0$ . By assumption  $\mathbb{H}_\pi^2(C^\bullet(\mathcal{T}'', \mathcal{T}')) = 0$  and the result thus follows from (5.4).  $\square$

*Remark 5.8.* In our applications, the vanishing assumption in Proposition 5.7 will always be satisfied due to the small rank of the bundles involved. In fact, the vanishing is probably true in general for  $\sigma \geq 2g - 2$ , as in the non-parabolic case [8].

Clearly, from Proposition 5.6, we have the following.

**Proposition 5.9.** *If  $\mathbb{H}^2(C^\bullet(\mathcal{T}'', \mathcal{T}')) = 0$  and  $\mathbb{H}^2(C^\bullet(\mathcal{T}', \mathcal{T}'')) = 0$  for every  $(\mathcal{T}', \mathcal{T}'') \in \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ , then*

$$\mathcal{S}_{\sigma_c^\pm} = \mathbb{P}W^\pm.$$

$\square$

The following will be important to study the relation between  $\mathcal{N}_{\sigma_c^-}$  and  $\mathcal{N}_{\sigma_c^+}$ .

**Proposition 5.10.** *Over  $\mathbb{P}W^+ \times X$  there is a universal extension*

$$(5.5) \quad 0 \rightarrow \mathcal{T}' \otimes \mathcal{O}_{\mathbb{P}W^+}(1) \rightarrow \mathcal{T}^+ \rightarrow \mathcal{T}'' \rightarrow 0,$$

where  $\mathcal{T}' \otimes \mathcal{O}_{\mathbb{P}W^+}(1) := (\mathcal{E}_1' \otimes \mathcal{O}_{\mathbb{P}W^+}(1), \mathcal{E}_2' \otimes \mathcal{O}_{\mathbb{P}W^+}(1), \Phi')$  (we omit pull-backs for clarity). Similarly, on  $\mathbb{P}W^- \times X$  there is a universal extension

$$0 \rightarrow \mathcal{T}'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1) \rightarrow \mathcal{T}^- \rightarrow \mathcal{T}' \rightarrow 0,$$

where  $\mathcal{T}'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1) := (\mathcal{E}_1'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1), \mathcal{E}_2'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1), \Phi'')$ .

*Proof.* The proof is analogous to the one given by Lange [22] for extensions of sheaves. In fact, our result could be derived from that one by making use of the correspondence between parabolic triples over  $X$  and  $\mathrm{SL}(2, \mathbb{C})$ -invariant parabolic vector bundles over  $X \times \mathbb{P}^1$  (cf. [3]). Hence we only give the main ingredients of the proof.

Let  $(T', T'') \in B = \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ . Let  $\mathbb{W} = \mathbb{H}^1(X, C^\bullet(T'', T'))$ , and let  $\mathbb{P} = \mathbb{P}(\mathbb{W})$ . Over  $\mathbb{P} \times X$  there is a universal extension

$$(5.6) \quad 0 \rightarrow T'(1) \rightarrow T \rightarrow T'' \rightarrow 0,$$

where  $T'(1) := (E'_1 \otimes \mathcal{O}_{\mathbb{P}}(1), E'_2 \otimes \mathcal{O}_{\mathbb{P}}(1), \phi')$  and we are omitting pull-backs. By the universal property of this extension we mean that  $T$  restricted to  $\{p\} \times X$  is a triple whose corresponding equivalence class is precisely  $p \in \mathbb{P}$ . Extensions like (5.6) are parametrised by  $\mathbb{H}^1(\mathbb{P} \times X, C^\bullet(T'', T'(1)))$  which by the Künneth formula is isomorphic to

$$\mathbb{H}^1(X, C^\bullet(T'', T')) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \cong \mathbb{W} \otimes \mathbb{W}^* \cong \mathrm{End}(\mathbb{W}).$$

One can show that the identity element in  $\mathrm{End}(\mathbb{W})$  defines the universal extension.

To prove the relative version stated in the proposition, we consider the spectral sequence

$$H^p(B, \mathbb{H}^q_{\pi}(C^\bullet(T'', T'))) \Rightarrow \mathbb{H}^{p+q}(B \times X, C^\bullet(T'', T'))$$

relating relative and global hypercohomology groups. Since  $\mathbb{H}^0_{\pi}(C^\bullet(T'', T')) = 0$ , the induced map

$$\mathbb{H}^1(B \times X, C^\bullet(T'', T')) \rightarrow H^0(B, \mathbb{H}^1_{\pi}(C^\bullet(T'', T')))$$

is an isomorphism. Similarly, if  $P := \mathbb{P}W^+$ , we have an isomorphism

$$\mathbb{H}^1(P \times X, C^\bullet(T'', T' \otimes \mathcal{O}_P(1))) \cong H^0(P, \mathbb{H}^1_{\nu}(C^\bullet(T'', T' \otimes \mathcal{O}_P(1))),$$

where  $\nu : P \times X \rightarrow P$  is the canonical projection.

Now, write  $p : P \rightarrow B$  for the projection. Again omitting pull-backs when convenient, we have that the image of the identity of  $W^+$  under the canonical isomorphisms

$$\begin{aligned} H^0(B, \mathrm{End} W^+) &= H^0(B, W^+ \otimes (W^+)^*) = H^0(B, W^+ \otimes p_* \mathcal{O}_P(1)) \\ &= H^0(B, p_*(p^* W^+ \otimes \mathcal{O}_P(1))) \\ &= H^0(P, p^*(\mathbb{H}^1_{\pi}(C^\bullet(T'', T')))) \otimes \mathcal{O}_P(1) \\ &= H^0(P, \mathbb{H}^1_{\nu}(C^\bullet(T'', T' \otimes \mathcal{O}_P(1)))) \end{aligned}$$

is a nonvanishing section defining the universal extension (5.5). A technical ingredient in proving the universal property is the commutation of  $\mathbb{H}^1(B \times X, C^\bullet(T'', T'))$  with base change (see [22] for details on the analogous situation of extensions of sheaves).  $\square$



5.3. **Flips.** Now we assume that  $\mathcal{N}_{\sigma_c^+}$ ,  $\mathcal{N}_{\sigma_c^-}$ ,  $\mathcal{N}'_{\sigma_c}$  and  $\mathcal{N}''_{\sigma_c}$  are smooth. Also assume that  $\mathbb{H}^2(C^\bullet(T'', T')) = 0$  and  $\mathbb{H}^2(C^\bullet(T', T'')) = 0$  for every  $(T', T'') \in \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ . This will always be the case in our applications. In order to relate  $\mathcal{N}_{\sigma_c^-}$  and  $\mathcal{N}_{\sigma_c^+}$  we have to blow up  $\mathcal{N}_{\sigma_c^\pm}$  along  $\mathcal{S}_{\sigma_c^\pm}$ . For this it is necessary to study the normal bundle to  $\mathcal{S}_{\sigma_c^\pm} = \mathbb{P}W^\pm$  in  $\mathcal{N}_{\sigma_c^\pm}$ .

**Proposition 5.11.** *Let  $p : \mathbb{P}W^\pm \rightarrow B$  be the natural projection and  $j : \mathbb{P}W^\pm \hookrightarrow \mathcal{N}_{\sigma_c^\pm}$  be the natural inclusion. Then there is an exact sequence*

$$0 \longrightarrow T\mathbb{P}W^\pm \longrightarrow j^*T\mathcal{N}_{\sigma_c^\pm} \longrightarrow p^*W^\mp \otimes \mathcal{O}_{\mathbb{P}W^\pm}(-1) \longrightarrow 0,$$

and hence, the normal bundle to  $\mathcal{S}_{\sigma_c^\pm} = \mathbb{P}W^\pm$  in  $\mathcal{N}_{\sigma_c^\pm}$  is isomorphic to  $p^*W^\mp \otimes \mathcal{O}_{\mathbb{P}W^\pm}(-1)$ .

*Proof.* We consider the case of  $\mathcal{N}_{\sigma_c^+}$  — the case of  $\mathcal{N}_{\sigma_c^-}$  is analogous. Over  $\mathcal{N}_{\sigma_c^+} \times X$  there is a universal triple  $\mathcal{T} = (\mathcal{E}_1, \mathcal{E}_2, \Phi)$ , whose restriction to  $\mathcal{S}_{\sigma_c^+} = \mathbb{P}W^+$  is the universal extension  $\mathcal{T}^+$  in (5.5). The tangent bundle of  $\mathcal{N}_{\sigma_c^+}$  is given by the relative  $\mathbb{H}^1$  of the complex

$$\text{ParHom}(\mathcal{E}_1, \mathcal{E}_1) \oplus \text{ParHom}(\mathcal{E}_2, \mathcal{E}_2) \longrightarrow \text{SParHom}(\mathcal{E}_2, \mathcal{E}_1(D))$$

over  $\mathcal{N}_{\sigma_c^+} \times X$  with respect to the natural projection  $\mathcal{N}_{\sigma_c^+} \times X \rightarrow \mathcal{N}_{\sigma_c^+}$ .

Denote  $\mathcal{E}'_i(1) = \mathcal{E}'_i \otimes \mathcal{O}_{\mathbb{P}W^+}(1)$  and define

$$\text{ParHom}_U(\mathcal{E}_i, \mathcal{E}_i) := \ker(\text{ParHom}(\mathcal{E}_i, \mathcal{E}_i) \rightarrow \text{ParHom}(\mathcal{E}'_i(1), \mathcal{E}''_i))$$

and

$$\text{SParHom}_U(\mathcal{E}_2, \mathcal{E}_1(D)) := \ker(\text{SParHom}(\mathcal{E}_2, \mathcal{E}_1(D)) \rightarrow \text{SParHom}(\mathcal{E}'_2(1), \mathcal{E}''_1(D))).$$

The tangent bundle of  $\mathbb{P}W^+$  is the relative  $\mathbb{H}^1$  with respect to the projection  $\mathbb{P}W^+ \times X \rightarrow \mathbb{P}W^+$  of the middle complex in the following exact sequence of complexes

$$\begin{array}{ccc} \text{ParHom}(\mathcal{E}''_1, \mathcal{E}'_1(1)) \oplus \text{ParHom}(\mathcal{E}''_2, \mathcal{E}'_2(1)) & \longrightarrow & \text{SParHom}(\mathcal{E}''_2, \mathcal{E}'_1(1)(D)) \\ \downarrow & & \downarrow \\ \text{ParHom}_U(\mathcal{E}_1, \mathcal{E}_1) \oplus \text{ParHom}_U(\mathcal{E}_2, \mathcal{E}_2) & \longrightarrow & \text{SParHom}_U(\mathcal{E}_2, \mathcal{E}_1(D)) \\ \downarrow & & \downarrow \\ \text{ParHom}(\mathcal{E}'_1(1), \mathcal{E}'_1(1)) \oplus \text{ParHom}(\mathcal{E}'_2(1), \mathcal{E}'_2(1)) \oplus & \longrightarrow & \text{SParHom}(\mathcal{E}'_2(1), \mathcal{E}'_1(1)(D)) \oplus \\ \oplus \text{ParHom}(\mathcal{E}''_1, \mathcal{E}''_1) \oplus \text{ParHom}(\mathcal{E}''_2, \mathcal{E}''_2) & & \oplus \text{SParHom}(\mathcal{E}''_2, \mathcal{E}''_1(D)). \end{array}$$

Note that when passing to cohomology, this gives us the exact sequence

$$0 \longrightarrow T_V\mathbb{P}W^+ \longrightarrow T\mathbb{P}W^+ \longrightarrow T(\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}) \longrightarrow 0,$$

where  $T_V\mathbb{P}W^+ \cong p^*W^+(1)/\mathcal{O}_{\mathbb{P}W^+}$  is the vertical tangent bundle.

Therefore the normal bundle to  $\mathbb{P}W^+$  is the relative  $\mathbb{H}^1$  of the quotient complex in the following exact sequence of complexes

$$\begin{array}{ccc} \text{ParHom}_U(\mathcal{E}_1, \mathcal{E}_1) \oplus \text{ParHom}_U(\mathcal{E}_2, \mathcal{E}_2) & \longrightarrow & \text{SParHom}_U(\mathcal{E}_2, \mathcal{E}_1(D)) \\ \downarrow & & \downarrow \\ \text{ParHom}(\mathcal{E}_1, \mathcal{E}_1) \oplus \text{ParHom}(\mathcal{E}_2, \mathcal{E}_2) & \longrightarrow & \text{SParHom}(\mathcal{E}_2, \mathcal{E}_1(D)) \\ \downarrow & & \downarrow \\ \text{ParHom}(\mathcal{E}'_1(1), \mathcal{E}'_1'') \oplus \text{ParHom}(\mathcal{E}'_2(1), \mathcal{E}'_2'') & \longrightarrow & \text{SParHom}(\mathcal{E}'_2(1), \mathcal{E}'_1''(D)), \end{array}$$

and it is hence isomorphic to  $p^*W^- \otimes \mathcal{O}_{\mathbb{P}W^+}(-1)$ .  $\square$

In particular, we conclude that the embedding  $\mathbb{P}W^\pm \hookrightarrow \mathcal{N}_{\sigma_c^\pm}$  is smooth. So we can blow-up  $\mathcal{N}_{\sigma_c^\pm}$  along  $\mathbb{P}W^\pm$  to get  $\tilde{\mathcal{N}}_{\sigma_c^\pm}$  with exceptional divisor  $E_\pm \subset \tilde{\mathcal{N}}_{\sigma_c^\pm}$  such that

$$E_\pm = \mathbb{P}W^\pm \times_B \mathbb{P}W^\mp.$$

Note that  $\mathcal{O}_{E_\pm}(E_\pm) = \mathcal{O}_{\mathbb{P}W^+}(-1) \otimes \mathcal{O}_{\mathbb{P}W^-}(-1)$ , by Proposition 5.11.

**Proposition 5.12.** *There is a natural isomorphism  $\tilde{\mathcal{N}}_{\sigma_c^+} \cong \tilde{\mathcal{N}}_{\sigma_c^-}$ .*

*Proof.* Let  $\mathcal{T}$  be the universal triple over  $\mathcal{N}_{\sigma_c^+} \times X$ . By Proposition 5.10, the restriction of  $\mathcal{T}$  to  $\mathbb{P}W^+ \times X$  lies in the universal extension

$$(5.7) \quad 0 \rightarrow \mathcal{T}' \otimes \mathcal{O}_{\mathbb{P}W^+}(1) \rightarrow \mathcal{T}|_{\mathbb{P}W^+ \times X} \rightarrow \mathcal{T}'' \rightarrow 0.$$

Now pull back  $\mathcal{T}$  by the blow-up map  $q : \tilde{\mathcal{N}}_{\sigma_c^+} \rightarrow \mathcal{N}_{\sigma_c^+}$ . Consider the composition  $q^*\mathcal{T} \rightarrow q^*\mathcal{T}|_{E_+ \times X} \rightarrow q^*i_*\mathcal{T}''$ , where  $i : \mathbb{P}W^+ \times X \hookrightarrow \mathcal{N}_{\sigma_c^+} \times X$ . Since  $q^*i_*\mathcal{T}''$  is a triple formed by two bundles supported on a divisor, the kernel is a triple (i.e. it is formed by two bundles, and not just coherent sheaves). Define the triple  $\hat{\mathcal{T}}$  on  $\tilde{\mathcal{N}}_{\sigma_c^+} \times X$  by the exact sequence

$$(5.8) \quad 0 \rightarrow \hat{\mathcal{T}} \otimes \mathcal{O}_{\mathbb{P}W^+}(1) \rightarrow q^*\mathcal{T} \rightarrow q^*i_*\mathcal{T}'' \rightarrow 0.$$

(This is called an *elementary transformation*.)

Let us see that all the triples in the family  $\hat{\mathcal{T}}$  are  $\sigma_c^-$ -stable. Therefore this defines a map  $\tilde{\mathcal{N}}_{\sigma_c^+} \rightarrow \mathcal{N}_{\sigma_c^-}$ . Obviously, off  $E_+ \times X$ ,  $\hat{\mathcal{T}} \cong \mathcal{T}$  is the family parametrising triples which are  $\sigma_c^+$ -stable and  $\sigma_c^-$ -stable at the same time. Tensoring the exact sequence (5.8) with  $\mathcal{O}_{E_+ \times X}$ , we obtain, over  $E_+ \times X$ ,

$$0 \rightarrow \underline{\text{Tor}}(q^*i_*\mathcal{T}'', \mathcal{O}_{E_+ \times X}) \rightarrow \hat{\mathcal{T}}|_{E_+ \times X} \otimes \mathcal{O}_{\mathbb{P}W^+}(1) \rightarrow q^*\mathcal{T}|_{E_+ \times X} \rightarrow q^*i_*\mathcal{T}'' \rightarrow 0.$$

Since  $\underline{\text{Tor}}(q^*i_*\mathcal{T}'', \mathcal{O}_{E_+ \times X}) = \mathcal{T}'' \otimes \mathcal{O}_{E_+ \times X}(-E_+ \times X) = \mathcal{T}'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1) \otimes \mathcal{O}_{\mathbb{P}W^+}(1)$ , and also using (5.7), we get a triple

$$(5.9) \quad 0 \rightarrow \mathcal{T}'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1) \rightarrow \hat{\mathcal{T}}|_{E_+ \times X} \rightarrow \mathcal{T}' \rightarrow 0.$$

We have to check that all extensions in the family (5.9) are non-trivial. For this, restrict to the fibre  $\mathbb{P}W^+ \times \mathbb{P}W^-$  over a point  $b \in B$ . This corresponds to fixing some specific

triples  $T'$  and  $T''$ . We have an exact sequence

$$0 \rightarrow T'' \otimes \mathcal{O}_{\mathbb{P}W_b^-}(1) \rightarrow \hat{\mathcal{T}}|_{\mathbb{P}W_b^+ \times \mathbb{P}W_b^- \times X} \rightarrow T' \rightarrow 0.$$

This extension class is parametrised by

$$\mathrm{Ext}^1(T', T'' \otimes \mathcal{O}_{\mathbb{P}W_b^-}(1)) = W_b^- \otimes H^0(\mathcal{O}_{\mathbb{P}W_b^-}(1)) = \mathrm{End}(W_b^-).$$

Moreover the linear group  $GL(W_b^-)$  acts on  $W_b^-$ . The extension class is invariant by this action, therefore it is a linear multiple of the identity. Letting  $b$  move in  $B$  we have a section of  $\mathrm{End}(W^-)$ . Since this is a multiple of the identity, it lives in  $\mathcal{O} \cdot \mathrm{Id} \subset \mathrm{End}(W)$ , therefore it is a *constant* multiple of the identity. This cannot be constantly zero for, otherwise, it would be  $\hat{\mathcal{T}}|_{E_+ \times X} = T' \oplus (T'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1))$ . Then  $\mathrm{Hom}(\hat{\mathcal{T}}, q^*i_*T'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1)) \neq 0$ . Hence (5.8) would imply that the map

$$(5.10) \quad \mathrm{Ext}^1(T'' \otimes \mathcal{O}_{\mathbb{P}W^+}(-1), T'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1)) \rightarrow \mathrm{Ext}^1(T \otimes \mathcal{O}_{\mathbb{P}W^+}(-1), T'' \otimes \mathcal{O}_{\mathbb{P}W^-}(1))$$

is not injective. On the other hand, using the projection  $\pi : \mathcal{N}_{\sigma_c^+} \times X \rightarrow \mathcal{N}_{\sigma_c^+}$  in (5.7) we have that

$$\begin{aligned} \mathrm{Hom}_\pi(T'', T'') &\cong \mathrm{Hom}_\pi(T, T''), \\ \mathrm{Ext}_\pi^1(T'', T'') &\hookrightarrow \mathrm{Ext}_\pi^1(T, T''), \end{aligned}$$

as bundles over  $\mathcal{N}_{\sigma_c^+}$ . Twisting by  $\mathcal{O}_{\mathbb{P}W^+}(1) \otimes \mathcal{O}_{\mathbb{P}W^-}(1)$  and using the spectral sequence

$$H^p(\mathcal{N}_{\sigma_c^+}, \mathrm{Ext}_\pi^q(\cdot, \cdot)) \Rightarrow \mathbb{H}^{p+q}(\mathcal{N}_{\sigma_c^+} \times X, C^\bullet(\cdot, \cdot))$$

we have that (5.10) is injective, giving a contradiction.

Hence the extension class of (5.9) is a non-zero multiple of the identity. This gives a map  $\tilde{\mathcal{N}}_{\sigma_c^+} \rightarrow \mathcal{N}_{\sigma_c^-}$  which restricts to  $E_+$  as the natural projection on the second factor  $E_+ \cong \mathbb{P}W^+ \times_B \mathbb{P}W^- \rightarrow \mathbb{P}W^-$ . Analogously we obtain a map  $\tilde{\mathcal{N}}_{\sigma_c^-} \rightarrow \mathcal{N}_{\sigma_c^+}$ . So there are two *injective* maps  $\tilde{\mathcal{N}}_{\sigma_c^\pm} \rightarrow \mathcal{N}_{\sigma_c^+} \times \mathcal{N}_{\sigma_c^-}$ . Their images are both the closures of the image of  $\tilde{\mathcal{N}}_{\sigma_c^\pm} \setminus E_\pm$ , which are the same. So they coincide.  $\square$

*Remark 5.13.* Let  $\sigma_c$  be a critical value. If  $w^+ = \mathrm{rk}(W^+) > 0$  and  $w^- = \mathrm{rk}(W^-) > 0$  then the moduli spaces  $\mathcal{N}_{\sigma_c^-}$  and  $\mathcal{N}_{\sigma_c^+}$  are birational. This is true because  $w^+ + w^- + \dim B - 1 = \dim \mathcal{N}_{\sigma_c^\pm}$  by Corollary 4.9, and the flip loci  $\mathcal{S}_{\sigma_c^\pm}$  have dimension  $w^\pm + \dim B - 1$ .

## 6. PARABOLIC TRIPLES WITH $r_1 = 2$ AND $r_2 = 1$

In this section we use the results of Section 5 to compute the Poincaré polynomial of the moduli space of parabolic triples  $\mathcal{N}_\sigma$  for the case  $r_1 = 2$  and  $r_2 = 1$  and for non-critical values of  $\sigma$ . We are studying triples of the form  $\phi : L \rightarrow E_1(D)$ , where  $L$  is a parabolic line bundle with  $\mathrm{deg}(L) = d_2$  and weights  $\alpha(p)$ , and  $E_1$  is a parabolic rank 2 bundle with  $\mathrm{deg}(E_1) = d_1$  and weights  $\beta_1(p) < \beta_2(p)$ . By Theorem 4.12 (v), the moduli space of stable elements in  $\mathcal{N}_\sigma$  is smooth. Moreover, applying the exact sequence (4.5), one can easily show that we are in the situation given in Proposition 5.7.

**6.1. Flips.** By Subsection 5.1, there are the following three possibilities for the existence of critical values:

- $r'_1 = 1$  and  $r'_2 = 0$ . Then the subtriple  $T'$  is of the form  $0 \rightarrow M(D)$ , where  $M$  is a line bundle of degree  $d_M$ . Since  $M$  inherits weights from  $E_1$ , there is a function  $\varepsilon = \{\varepsilon(p)\}_{p \in D}$ , which assigns to each  $p \in D$  a number  $\varepsilon(p) \in \{1, 2\}$  such that the weight of  $M$  at  $p$  is  $\beta_{\varepsilon(p)}(p)$ . We have an exact sequence of triples

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \longrightarrow & L \\ \downarrow & & \downarrow & & \downarrow \\ M(D) & \longrightarrow & E_1(D) & \longrightarrow & F(D). \end{array}$$

The quotient triple is of the form  $L \rightarrow F(D)$ , where  $F$  is a parabolic line bundle of degree  $d_1 - d_M$  and weights  $\beta_{\varsigma(p)}(p)$ , with  $\varsigma(p) = 3 - \varepsilon(p)$ . Note that  $\{\beta_{\varepsilon(p)}(p), \beta_{\varsigma(p)}(p)\} = \{\beta_1(p), \beta_2(p)\}$ . By (5.1), the critical value is

$$(6.1) \quad \sigma_c = 3d_M - d_1 - d_2 + \sum_p (2\beta_{\varepsilon(p)}(p) - \alpha(p) - \beta_{\varsigma(p)}(p)).$$

As described in Subsection 5.3, this defines the subspace  $\mathcal{S}_{\sigma_c+} = \mathbb{P}W_{\sigma_c}^+$ , where

$$W_{\sigma_c}^+ \longrightarrow B_{\sigma_c} = \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$$

(we shall make the dependence on  $\sigma_c$  explicit in this section, since we shall be working with various flip loci simultaneously). The moduli space parametrizing the possible parabolic line bundles  $M$  with fixed weights  $\beta_{\varepsilon(p)}(p)$  is  $\mathcal{N}'_{\sigma_c} = \text{Jac}^{d_M} X$ . The moduli space parametrizing triples of the form  $L \rightarrow F(D)$ , which are parabolic line bundles with fixed weights is  $\text{Jac}^{d_2} X \times S^N X$ , where  $N = \deg \text{SParHom}(L, F(D))$ . To compute this we use the following.

**Lemma 6.1.** *Let  $L_1, L_2$  be two parabolic line bundles with weights  $\alpha_{L_1}(p)$  and  $\alpha_{L_2}(p)$ , respectively. Then*

$$\text{SParHom}(L_1, L_2 \otimes K(D)) \cong \text{Hom}(L_1, L_2 \otimes K(S)),$$

where  $S = \{p \in D \mid \alpha_{L_1}(p) < \alpha_{L_2}(p)\}$ .

*Proof.* By definition, a strongly parabolic map  $\Phi : L_1 \rightarrow L_2$  satisfies

$$\text{Res}_p \Phi_l = 0 \iff \alpha_{L_1}(p) \geq \alpha_{L_2}(p).$$

From this the result is clear.  $\square$

In our case, we introduce the following notations:

$$(6.2) \quad \begin{aligned} S_1 &= \{p \in D \mid \alpha(p) < \beta_{\varsigma(p)}(p)\}, \\ S_2 &= \{p \in D \mid \alpha(p) < \beta_{\varepsilon(p)}(p)\}, \\ S_3 &= \{p \in D \mid \beta_{\varepsilon(p)}(p) < \beta_{\varsigma(p)}(p)\}, \\ s_1 &= \#S_1, \quad s_2 = \#S_2, \quad s_3 = \#S_3. \end{aligned}$$

Then

$$(6.3) \quad \begin{aligned} N &= \deg \text{SParHom}(L, F(D)) = \deg \text{Hom}(L, F(S_1)) = \\ &= \deg(F) - \deg(L) + s_1 = d_1 - d_2 - d_M + s_1. \end{aligned}$$

Now  $\mathbb{P}W_{\sigma_c}^+$  is a projective fibration over  $B_{\sigma_c}$  with fibres projective spaces of dimension  $w_{\sigma_c}^+ - 1$ . By Proposition 4.8 and Corollary 4.11,

$$(6.4) \quad \begin{aligned} w_{\sigma_c}^+ &= \dim \text{Ext}^1(T'', T') = -\chi(T'', T') \\ &= -\chi(\text{ParHom}(F, M)) + \chi(\text{SParHom}(L, M(D))) \\ &= -\chi(\text{Hom}(F, M(-S_3))) + \chi(\text{Hom}(L, M(S_2))) \\ &= d_1 - d_2 - d_M + s_2 + s_3. \end{aligned}$$

- $r'_1 = 1, r'_2 = 1$ . Then the subtriple  $T'$  is of the form  $L \rightarrow F(D)$  and the quotient triple is of the form  $0 \rightarrow M(D)$ , yielding an exact sequence

$$\begin{array}{ccccc} L & \longrightarrow & L & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ F(D) & \longrightarrow & E(D) & \longrightarrow & M(D), \end{array}$$

where  $M$  is a line bundle of degree  $d_M$  and weights  $\beta_{\varepsilon(p)}(p)$ , for some  $\varepsilon = \{\varepsilon(p)\}_{p \in D}$ , and  $F$  is a parabolic line bundle of degree  $d_1 - d_M$  and weights  $\beta_{\zeta(p)}(p)$ , with  $\zeta(p) = 3 - \varepsilon(p)$ . The critical value is again given by (6.1). These extensions define the subspace  $\mathcal{S}_{\sigma_c^-} = \mathbb{P}W_{\sigma_c}^-$ , where

$$W_{\sigma_c}^- \longrightarrow B_{\sigma_c} = \mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c} = \text{Jac}^{d_M} X \times \text{Jac}^{d_2} X \times S^N X,$$

with  $N$  as in (6.3). Now  $\mathbb{P}W_{\sigma_c}^-$  is a projective fibration over  $B_{\sigma_c}$  with fibres projective spaces of dimension  $w_{\sigma_c}^- - 1$  where

$$(6.5) \quad \begin{aligned} w_{\sigma_c}^- &= h^1(T', T'') = -\chi(T', T'') = -\chi(\text{ParHom}(M, F)) = \\ &= -\chi(\text{Hom}(M, F(-(D - S_3)))) = 2d_M - d_1 + g - 1 + n - s_3. \end{aligned}$$

- $r'_1 = 2, r'_2 = 0$ . Then the triples  $T$  are extensions of  $0 \rightarrow E_1$  by  $L \rightarrow 0$ . The critical value is  $\sigma_c = \text{par}\mu_1 - \text{par}\mu_2 = \sigma_m$ , which is the minimum possible value for the parameter  $\sigma$ . At this value, the moduli space  $\mathcal{N}_{\sigma_m^-} = \emptyset$  and  $\mathcal{N}_{\sigma_m^+} = \mathcal{S}_{\sigma_m^+}$ . This can be described explicitly as a projective fibration over a product of a Jacobian and a moduli space of rank 2 stable parabolic bundles, but we will not go into this since we shall not use it.

*Remark 6.2.* Since we are taking generic values for the weights, the values of  $\sigma_c$  are distinct for the different choices of  $d_M$  and  $\varepsilon$ . The genericity condition was necessary in Section 5 to have smooth flip loci, and this is essentially due to the fact that at a critical value, the Jordan-Hölder filtrations are of length at most two. In the case we treat here,  $r_1 = 2$  and  $r_2 = 1$ , the Jordan-Hölder filtrations are of length at most two even with non-generic weights, because the ranks are too small. Therefore the computations of

this section work as well for the case of *distinct* non-generic weights. Of course, we shall need the genericity of weights at many other places in the coming sections.

*Remark 6.3.* Let  $\sigma_L$  be the largest critical value. This means that the moduli space  $\mathcal{N}_{\sigma_L^+} = \emptyset$ , i.e.,  $\mathcal{S}_{\sigma_L^-} = \mathcal{N}_{\sigma_L^-}$ . Since

$$\dim B_{\sigma_c} + w_{\sigma_c}^+ + w_{\sigma_c}^- - 1 = \dim \mathcal{N}_{\sigma_c^\pm},$$

by Corollary 4.9, we have that  $w_{\sigma_L}^+ = 0$ . By (6.4), this corresponds to the case  $s_3 = 0$  and  $d_1 - d_2 - d_M + s_2 = 0$ , i.e., when  $L \cong F(S_2) \subset F(D)$ . In this case  $\mathcal{N}_{\sigma_L^-}$  equals  $\mathbb{P}W_{\sigma_L}^- \rightarrow B_{\sigma_L}$ .

Also  $\sigma_L < \sigma_M$ ; in general they are not equal. The value of  $\sigma_L$  obtained in (6.1) is always slightly smaller than that of  $\sigma_M$  in (4.2).

**6.2. Poincaré polynomial of moduli of triples.** Let  $\sigma_c$  be a critical value as in Subsection 6.1 with the only condition  $\sigma_c \neq \sigma_m$ . Then we have that

$$(6.6) \quad P_t(\mathcal{N}_{\sigma_c^-}) - P_t(\mathcal{N}_{\sigma_c^+}) = P_t(\mathbb{P}W_{\sigma_c}^-) - P_t(\mathbb{P}W_{\sigma_c}^+).$$

Note that this formula also holds when  $w_{\sigma_c}^+ = 0$  or  $w_{\sigma_c}^- = 0$ . For instance, if  $w_{\sigma_c}^+ = 0$  then  $\mathcal{S}_{\sigma_c^+} = \emptyset$  and  $\mathcal{S}_{\sigma_c^-} = \mathbb{P}W_{\sigma_c}^-$  is of the same dimension as  $\mathcal{N}_{\sigma_c^-}$ , hence it is a component of it. So (6.6) holds. In particular we can use (6.6) for  $\sigma_c = \sigma_L$  (see Remark 6.3). But we cannot use it for  $\sigma_c = \sigma_m$  (see Remark 6.5).

**Theorem 6.4.** *Let  $\sigma > \sigma_m$  be a non-critical value. For any  $\varepsilon = \{\varepsilon(p)\}_{p \in D}$ ,  $\varepsilon(p) \in \{1, 2\}$ , let  $s_1, s_2, s_3$  be given by (6.2) and*

$$\bar{d}_M = \left\lceil \frac{1}{3} \left( d_1 + d_2 + \sum (\alpha(p) + \beta_{\varsigma(p)}(p) - 2\beta_{\varepsilon(p)}(p)) + \sigma \right) \right\rceil + 1,$$

where  $\varsigma(p) = 3 - \varepsilon(p)$  and  $[x]$  is the integer part of  $x$ . Then  $P_t(\mathcal{N}_\sigma)$  equals

$$\sum_{\varepsilon} \text{Coeff}_{x^0} \left( \frac{(1+t)^{4g}(1+xt)^{2g}t^{2d_1-2d_2+2s_2+2s_3-2\bar{d}_M}x^{\bar{d}_M}}{(1-t^2)(1-x)(1-xt^2)(1-t^{-2}x)x^{d_1-d_2+s_1}} \right. \\ \left. - \frac{(1+t)^{4g}(1+xt)^{2g}t^{-2d_1+2g-2+2n-2s_3+4\bar{d}_M}x^{\bar{d}_M}}{(1-t^2)(1-x)(1-xt^2)(1-t^4x)x^{d_1-d_2+s_1}} \right).$$

*Proof.* From (6.6), we have that

$$\begin{aligned}
 P_t(\mathcal{N}_\sigma) &= \sum_{\sigma_c > \sigma} (P_t(\mathbb{P}W_{\sigma_c}^-) - P_t(\mathbb{P}W_{\sigma_c}^+)) \\
 &= \sum_{\sigma_c > \sigma} \left( P_t(\mathbb{P}w_{\sigma_c}^-) - P_t(\mathbb{P}w_{\sigma_c}^+) \right) P_t(B_{\sigma_c}) \\
 &= \sum_{\sigma_c > \sigma} \left( \frac{1 - t^{2w_{\sigma_c}^-}}{1 - t^2} - \frac{1 - t^{2w_{\sigma_c}^+}}{1 - t^2} \right) P_t(\text{Jac } X)^2 P_t(\text{Sym}^N X) \\
 &= \sum_{\sigma_c > \sigma} \frac{t^{2w_{\sigma_c}^+} - t^{2w_{\sigma_c}^-}}{1 - t^2} (1 + t)^{4g} \text{Coeff}_{x^0} \left( \frac{(1 + xt)^{2g}}{(1 - x)(1 - xt^2)x^N} \right) \quad \text{by [24]} \\
 &= \sum_{\sigma_c > \sigma} \frac{t^{2d_1 - 2d_2 - 2d_M + 2s_2 + 2s_3} - t^{4d_M - 2d_1 + 2g - 2 + 2n - 2s_3}}{1 - t^2} (1 + t)^{4g} \cdot \\
 &\quad \cdot \text{Coeff}_{x^0} \left( \frac{(1 + xt)^{2g}}{(1 - x)(1 - xt^2)x^{d_1 - d_2 - d_M + s_1}} \right) \quad \text{by (6.3), (6.4) and (6.5)} \\
 &= \sum_{\varepsilon} \text{Coeff}_{x^0} \left( \frac{(1 + t)^{4g} (1 + xt)^{2g} t^{2d_1 - 2d_2 + 2s_2 + 2s_3}}{(1 - t^2)(1 - x)(1 - xt^2)x^{d_1 - d_2 + s_1}} \sum_{d_M | \sigma_c > \sigma} t^{-2d_M} x^{d_M} \right. \\
 &\quad \left. - \frac{(1 + t)^{4g} (1 + xt)^{2g} t^{-2d_1 + 2g - 2 + 2n - 2s_3}}{(1 - t^2)(1 - x)(1 - xt^2)x^{d_1 - d_2 + s_1}} \sum_{d_M | \sigma_c > \sigma} t^{4d_M} x^{d_M} \right).
 \end{aligned}$$

The condition for  $d_M$  is

$$\sigma_c = 3d_M - d_1 - d_2 + \sum (2\beta_{\varepsilon(p)}(p) - \alpha(p) - \beta_{\varsigma(p)}(p)) > \sigma,$$

which translates into

$$d_M > \frac{1}{3} \left( d_1 + d_2 + \sum (\alpha(p) + \beta_{\varsigma(p)}(p) - 2\beta_{\varepsilon(p)}(p)) + \sigma \right).$$

Since  $\sigma$  is not a critical value, we cannot have equality, so the right hand side is not an integer. The inequality becomes  $d_M \geq \bar{d}_M$ , with  $\bar{d}_M$  as in the statement. Now

$$\begin{aligned}
 \sum_{d_M = \bar{d}_M}^{\infty} t^{-2d_M} x^{d_M} &= \frac{t^{-2\bar{d}_M} x^{\bar{d}_M}}{1 - t^{-2}x}, \\
 \sum_{d_M = \bar{d}_M}^{\infty} t^{4d_M} x^{d_M} &= \frac{t^{4\bar{d}_M} x^{\bar{d}_M}}{1 - t^4x}.
 \end{aligned}$$

So finally

$$\begin{aligned}
 P_t(\mathcal{N}_\sigma) &= \sum_{\varepsilon} \text{Coeff}_{x^0} \left( \frac{(1 + t)^{4g} (1 + xt)^{2g} t^{2d_1 - 2d_2 + 2s_2 + 2s_3} t^{-2\bar{d}_M} x^{\bar{d}_M}}{(1 - t^2)(1 - x)(1 - xt^2)(1 - t^{-2}x)x^{d_1 - d_2 + s_1}} \right. \\
 &\quad \left. - \frac{(1 + t)^{4g} (1 + xt)^{2g} t^{-2d_1 + 2g - 2 + 2n - 2s_3} t^{4\bar{d}_M} x^{\bar{d}_M}}{(1 - t^2)(1 - x)(1 - xt^2)(1 - t^4x)x^{d_1 - d_2 + s_1}} \right).
 \end{aligned}$$

□

*Remark 6.5.* The formula in this theorem only works for  $\sigma > \sigma_m$ . For  $\sigma < \sigma_m$ ,  $\mathcal{N}_\sigma$  is empty, but the formula above does not give zero for such values.

## 7. CRITICAL SUBMANIFOLDS OF TYPE (1, 1, 1)

**7.1. Description of the critical submanifolds.** In this section we consider the critical points of the Bott–Morse function  $f$  represented by parabolic Higgs bundles  $(E, \Phi)$  of type (1, 1, 1), i.e., of the form  $E = L_1 \oplus L_2 \oplus L_3$  where  $L_l$  are parabolic line bundles, i.e., line bundles with weights at the points  $p \in D$ . We denote the (fixed) weights of  $(E, \Phi)$  at  $p \in D$  by  $0 \leq \alpha_1(p) < \alpha_2(p) < \alpha_3(p) < 1$ . Each possible choice of the distribution of these weights among the line bundles  $L_l$  is given by a permutation  $\varpi_p \in S_3$  such that the weight on the fibre  $L_{l,p}$  at  $p$  is  $\alpha_{\varpi_p(l)}(p) = \alpha_{\varpi(l)}(p)$  for  $l = 1, 2, 3$ . The map  $\Phi$  decomposes as strongly parabolic maps  $\Phi_1 : L_1 \rightarrow L_2 \otimes K(D)$  and  $\Phi_2 : L_2 \rightarrow L_3 \otimes K(D)$ .

We define

$$\begin{aligned} d_l &= \deg(L_l) \quad \text{for } l = 1, 2, 3, \\ m &= d_1 + d_2. \end{aligned}$$

We shall choose to describe the topological data  $(d_1, d_2, d_3)$  using the parameters  $(d_1, m, \Delta)$ , where  $\Delta = d_1 + d_2 + d_3$ . In terms of this data we have  $d_2 = m - d_1$  and  $d_3 = \Delta - m$ . We also introduce the notation

$$(7.1) \quad \begin{aligned} F(\alpha, \varpi) &= \sum_{p \in D} (\alpha_1(p) + \alpha_2(p) + \alpha_3(p) - 3\alpha_{\varpi(3)}(p)), \\ G(\alpha, \varpi) &= \sum_{p \in D} (2\alpha_1(p) + 2\alpha_2(p) + 2\alpha_3(p) - 3\alpha_{\varpi(2)}(p) - 3\alpha_{\varpi(3)}(p)). \end{aligned}$$

**Proposition 7.1.** *A parabolic Higgs bundle  $(L_1 \oplus L_2 \oplus L_3, \Phi)$  of type (1, 1, 1) is stable if and only if the maps  $\Phi_1$  and  $\Phi_2$  are non-zero and, furthermore,*

$$\begin{aligned} 3m &> 2\Delta - F(\alpha, \varpi), \\ 3d_1 &> \Delta - G(\alpha, \varpi). \end{aligned}$$

*Proof.* It is immediate from Propositions 3.4 and 3.6 that a parabolic Higgs bundle of type (1, 1, 1) is stable if and only if the conditions  $\text{par}\mu(L_3) < \text{par}\mu(E)$  and  $\text{par}\mu(L_2 \oplus L_3) < \text{par}\mu(E)$  hold and  $\Phi_l \neq 0$  for  $l = 1, 2$ . From this we obtain the characterization given in the Proposition by calculating the relevant parabolic slopes. □

Denote by  $\mathcal{N}_{(1,1,1)}(d_1, m, \varpi)$  the critical submanifold of parabolic Higgs bundles of type (1, 1, 1) with invariants  $(d_1, m)$  and weights given by  $\varpi = \{\varpi_p\}_{p \in D}$ . Introduce the



following notation

$$(7.2) \quad \begin{aligned} S_1 &= \{p \in D \mid \alpha_{\varpi(1)}(p) > \alpha_{\varpi(2)}(p)\}, \\ S_2 &= \{p \in D \mid \alpha_{\varpi(2)}(p) > \alpha_{\varpi(3)}(p)\}, \\ s_1 &= \#S_1, \quad s_2 = \#S_2. \end{aligned}$$

There is no risk of confusion with the sets  $S_1, S_2$  defined in Section 6, since this notation will only apply to this section. By Lemma 6.1,

$$\text{SParHom}(L_l, L_{l+1} \otimes K(D)) \cong \text{Hom}(L_l, L_{l+1} \otimes K(D - S_l))$$

for  $l = 1, 2$ . We let for  $l = 1, 2$ ,

$$m_l = \deg(\text{Hom}(L_l, L_{l+1} \otimes K(D - S_l))).$$

Then

$$(7.3) \quad \begin{aligned} m_1 &= d_2 - d_1 + 2g - 2 + n - s_1 \\ &= m - 2d_1 + n - s_1 + 2g - 2, \\ m_2 &= d_3 - d_2 + 2g - 2 + n - s_2 \\ &= \Delta - 2m + d_1 + n - s_2 + 2g - 2. \end{aligned}$$

**Proposition 7.2.** *The critical submanifold  $\mathcal{N}_{(1,1,1)}(d_1, m, \varpi)$  is non-empty if and only if the following conditions are satisfied*

$$\begin{aligned} & 3m > 2\Delta - F(\alpha, \varpi), \\ & 3d_1 > \Delta - G(\alpha, \varpi), \\ m_1 \geq 0 & \iff 2d_1 - m \leq n - s_1 + 2g - 2, \\ m_2 \geq 0 & \iff 2m - d_1 \leq \Delta + n - s_2 + 2g - 2, \end{aligned}$$

where  $F$  and  $G$  were defined in (7.1). Moreover, the map

$$\begin{aligned} \mathcal{N}_{(1,1,1)}(d_1, m, \varpi) &\rightarrow \text{Jac}^{d_1}(X) \times S^{m_1}X \times S^{m_2}X \\ (L_1 \oplus L_2 \oplus L_3, \Phi_1, \Phi_2) &\mapsto (L_1, \text{div}(\Phi_1), \text{div}(\Phi_2)) \end{aligned}$$

is an isomorphism.

*Proof.* Proposition 7.1 shows that the conditions in the statement are necessary for  $\mathcal{N}_{(1,1,1)}(d_1, m, \varpi)$  to be non-empty.

Assume now that we are given  $(d_1, m, \varpi)$  satisfying this conditions. For any line bundle  $L_1$  in  $\text{Jac}^{d_1}(X)$  and effective divisors  $D_1 \in S^{m_1}X$  and  $D_2 \in S^{m_2}X$  we get line bundles  $M_l = \mathcal{O}(D_l)$  with non-zero sections  $\Phi_l$  determined up to multiplication by nonzero scalars for  $l = 1, 2$ . We then obtain a parabolic Higgs bundle of type  $(1, 1, 1)$  by letting

$$\begin{aligned} L_2 &= L_1 K^{-1}(S_1 - D) M_1, \\ L_3 &= L_2 K^{-1}(S_2 - D) M_2, \end{aligned}$$

and defining  $\Phi$  to have components  $\Phi_1$  and  $\Phi_2$ . Clearly this parabolic Higgs bundle has the desired invariants  $(d_1, m, \varpi)$  and, if the conditions in the statement are satisfied, then Proposition 7.1 shows that it is indeed stable.

It follows from this construction that the map given in the statement of the Proposition is surjective. To see that it is injective, we note that taking non-zero scalar multiples of the Higgs fields  $\Phi_1 \in H^0(L_1^{-1}L_2K(D - S_1))$  and  $\Phi_2 \in H^0(L_2^{-1}L_3K(D - S_2))$  gives rise to isomorphic parabolic Higgs bundles of type  $(1, 1, 1)$ . Thus the map given is, in fact, an isomorphism.  $\square$

**Corollary 7.3.** *The Poincaré polynomial of the critical submanifold  $\mathcal{N}_{(1,1,1)}(d_1, m, \varpi)$  is*

$$P_t(\mathcal{N}_{(1,1,1)}(d_1, m, \varpi)) = (1+t)^{2g} \underset{x^0y^0}{\text{Coeff}} \left( \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)x^{m_1}} \cdot \frac{(1+yt)^{2g}}{(1-y)(1-yt^2)y^{m_2}} \right).$$

*Proof.* Immediate from MacDonal's formula [24] for the Poincaré polynomial of a symmetric product of  $X$ .  $\square$

The total contribution to the Poincaré polynomial of  $\mathcal{M}$  from submanifolds of type  $(1, 1, 1)$  is

$$(7.4) \quad P_t(\Delta, (1, 1, 1)) := \sum_{d_1, m, \varpi} t^{\lambda_{(d_1, m, \varpi)}} P_t(\mathcal{N}_{(1,1,1)}(d_1, m, \varpi)),$$

where  $\lambda_{(d_1, m, \varpi)}$  is the index of the critical submanifold  $\mathcal{N}_{(1,1,1)}(d_1, m, \varpi)$  and the sum is over all permutations  $\varpi = \{\varpi_p\}_{p \in D} \in (S_3)^n$  and pairs of integers  $(d_1, m)$  such that the bounds of Proposition 7.2 are satisfied.

**Lemma 7.4.** *The index of the critical submanifold  $\mathcal{N}_{(1,1,1)}(d_1, m, \varpi)$  is*

$$\lambda_{(d_1, m, \varpi)} = 2(4g - 4 + n + s_1 + s_2 - \Delta + d_1 + m).$$

*Proof.* The formula for the Morse index is given in Proposition 3.11. For  $p \in D$ , the space  $P_p(L_l, L_l)$  consists of the parabolic endomorphisms of  $L_{l,p}$ , so  $\dim P_p(L_l, L_l) = 1$ . The space  $N_p(L_l, L_{l+1})$  is the space of strictly parabolic maps from  $L_{l,p}$  to  $L_{l+1,p}$  and hence

$$\dim N_p(L_l, L_{l+1}) = \begin{cases} 0, & \text{if } \alpha_{\varpi(l)}(p) > \alpha_{\varpi(l+1)}(p), \\ 1, & \text{otherwise.} \end{cases}$$

Recalling the definition (7.2) of  $S_l$ , it follows that

$$(7.5) \quad \sum_p \dim N_p(L_l, L_{l+1}) = n - s_l.$$

Now recall (from Proposition 2.4) that  $f_p = (1/2)(r^2 - \sum_i m_i(p)^2)$ . Thus  $f_p = 3$  in this case, and hence  $\sum_p f_p = 3n$ . Substituting this and (7.5) in the formula for the Morse index we obtain the result.  $\square$

**7.2. The sum for fixed  $\varpi$ .** We shall now calculate the total contribution (7.4) to the Poincaré polynomial of  $\mathcal{M}$  from submanifolds of type  $(1, 1, 1)$  in several stages. We begin by doing the sum over  $(d_1, m)$  for a fixed permutation  $\varpi$ .

**Lemma 7.5.** *Let  $\varpi = \{\varpi_p\}_{p \in D} \in (S_3)^n$  be fixed. Then*

$$\sum_{d_1, m} t^{\lambda(d_1, m, \varpi)} P_t(\mathcal{N}_{(1,1,1)}(d_1, m, \varpi)) = \text{Coeff}_{x^0 y^0} \Psi(\varpi),$$

where we have defined

$$\Psi(\varpi) = \sum_{\substack{d_1 \geq \bar{d}_1 \\ m \geq \bar{m}}} t^{2(4g-4+n+s_1+s_2-\Delta+d_1+m)} (1+t)^{2g} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)x^{m_1}} \cdot \frac{(1+yt)^{2g}}{(1-y)(1-yt^2)y^{m_2}}$$

with

$$(7.6) \quad \begin{aligned} \bar{m} &= [(2/3)\Delta - (1/3)F(\alpha, \varpi) + 1], \\ \bar{d}_1 &= [(1/3)\Delta - (1/3)G(\alpha, \varpi) + 1]. \end{aligned}$$

*Proof.* The identity would be clear from Corollary 7.3 and Lemma 7.4 if the latter sum were over  $(d_1, m)$  satisfying the conditions of Proposition 7.2. Now, from these equations we see that we need to sum over the closed region in the  $(m, d_1)$ -plane bounded by the lines

$$\begin{aligned} m &= \bar{m}, \\ d_1 &= \bar{d}_1, \\ 2d_1 - m &= n - s_1 + 2g - 2, \\ 2m - d_1 &= \Delta + n - s_2 + 2g - 2. \end{aligned}$$

Thus, summing over the semi-infinite region defined by  $d_1 \geq \bar{d}_1$  and  $m \geq \bar{m}$ , we introduce in the sum extra terms. But, since the lines given by the third and fourth equations represent the conditions  $m_1 \geq 0$  and  $m_2 \geq 0$ , these extra terms have strictly positive powers of  $x$  or  $y$  and hence this does not change the coefficient of  $x^0 y^0$ .  $\square$

Using (7.3) we have that  $x^{m_1} = x^{n-s_1+2g-2} x^{-2d_1} x^m$  and  $y^{m_2} = y^{\Delta+n-s_2+2g-2} y^{d_1} y^{-2m}$ , hence

$$(7.7) \quad \begin{aligned} \Psi(\varpi) &= t^{2(4g-4+n+s_1+s_2-\Delta)} (1+t)^{2g} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \cdot \frac{(1+yt)^{2g}}{(1-y)(1-yt^2)} \\ &\quad \cdot \frac{1}{x^{n-s_1+2g-2} y^{\Delta+n-s_2+2g-2}} \sum_{d_1=\bar{d}_1}^{\infty} \frac{t^{2d_1} x^{2d_1}}{y^{d_1}} \sum_{m=\bar{m}}^{\infty} \frac{t^{2m} y^{2m}}{x^m} \\ &= t^{2(s_1+s_2)} x^{s_1} y^{s_2} (x^2 y^{-1} t^2)^{\bar{d}_1} (x^{-1} y^2 t^2)^{\bar{m}} \\ &\quad \cdot \frac{t^{2(4g-4+n-\Delta)} (1+t)^{2g} (1+xt)^{2g} (1+yt)^{2g}}{x^{n+2g-2} y^{\Delta+n+2g-2} (1-x)(1-xt^2)(1-y)(1-yt^2)(1-x^2 y^{-1} t^2)(1-x^{-1} y^2 t^2)}, \end{aligned}$$

where we have separated powers of  $x$ ,  $y$  and  $t$  which potentially depend on  $\varpi$ .

**7.3. The sum over  $\varpi$ .** In order to proceed with the calculation we need to sum the contribution (7.7) over all permutations  $\varpi = (\varpi_p) \in (S_3)^n$ . For this we need to understand the dependence of  $s_1$ ,  $s_2$ ,  $\bar{m}$  and  $\bar{d}_1$  on  $\varpi$ . Now, looking at the definitions (7.6), we see that  $\bar{d}_1$  and  $\bar{m}$  also depend on the weights. In order to deal with this dependence, we shall take advantage of Proposition 2.1 which allows us to do the computation in the case where the degree  $\Delta$  satisfies that  $\Delta \not\equiv 0 \pmod{3}$ . Hence we can choose the weights so as to facilitate the computations, as long as we keep the same choice throughout. Now, if  $\Delta \not\equiv 0 \pmod{3}$  and we choose the weights  $\alpha_i(p)$  sufficiently small, then  $\bar{d}_1$  and  $\bar{m}$  are independent of  $\varpi$ . For future reference, we state here our assumptions.

**Assumption 7.6.** Write  $D = p_1 + \dots + p_n$ . In addition to Assumption 5.1, we shall from now on assume that  $\Delta \not\equiv 0 \pmod{3}$  and that the weights  $\alpha_i(p)$  are chosen to satisfy

$$\alpha_i(p_j) \ll 1 \quad \text{for all } i, j.$$

Next we consider the dependence of  $s_1$  and  $s_2$  on  $\varpi$ . We can write

$$\begin{aligned} s_1 &= \sum_{p \in D} s_1(p), \\ s_2 &= \sum_{p \in D} s_2(p), \end{aligned}$$

where  $s_1(p)$  and  $s_2(p)$  are defined in the obvious way:

$$s_1(p) = \begin{cases} 1 & \text{if } \varpi_p(1) > \varpi_p(2), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$s_2(p) = \begin{cases} 1 & \text{if } \varpi_p(2) > \varpi_p(3), \\ 0 & \text{otherwise.} \end{cases}$$

We give the values of  $s_1(p)$ ,  $s_2(p)$  and  $s_1(p) + s_2(p)$  as a function of  $\varpi_p$  in Table 1, using the notation  $\varpi = (\varpi(1) \varpi(2) \varpi(3))$  for a permutation  $\varpi \in S_3$ .

TABLE 1.  $s_1(p)$  and  $s_2(p)$  as a function of  $\varpi_p$

$\varpi_p$	(123)	(231)	(312)	(213)	(132)	(321)
$s_1(p)$	0	0	1	1	0	1
$s_2(p)$	0	1	0	0	1	1
$s_1(p) + s_2(p)$	0	1	1	1	1	2

Under Assumption 7.6,  $\bar{m}$  and  $\bar{d}_1$  are independent of  $\varpi$ : in fact we have from the definitions (7.6) of  $\bar{m}$  and  $\bar{d}_1$  that

$$(7.8) \quad \begin{aligned} \bar{m} &= \left[ \frac{2\Delta}{3} \right] + 1, \\ \bar{d}_1 &= \left[ \frac{\Delta}{3} \right] + 1. \end{aligned}$$

Therefore to do the sum  $\sum_{\varpi} \Psi(\varpi)$ , we only need to do  $\sum_{\varpi \in (S_3)^n} t^{2(s_1+s_2)} x^{s_1} y^{s_2}$  in (7.7). For this we use Table 1 and obtain:

$$(7.9) \quad \begin{aligned} \sum_{\varpi \in (S_3)^n} t^{2(s_1+s_2)} x^{s_1} y^{s_2} &= \prod_{p \in D} (1 + 2t^2x + 2t^2y + t^4xy) \\ &= (1 + 2t^2x + 2t^2y + t^4xy)^n. \end{aligned}$$

Combining (7.9) with (7.7) we finally obtain:

$$(7.10) \quad \begin{aligned} \sum_{\varpi} \Psi(\varpi) &= (1 + 2t^2x + 2t^2y + t^4xy)^n \cdot \\ &\cdot \frac{t^{2(4g-4+n-\Delta)} (1+t)^{2g} (1+xt)^{2g} (1+yt)^{2g} (x^2y^{-1}t^2)^{\bar{d}_1} (x^{-1}y^2t^2)^{\bar{m}}}{x^{n+2g-2} y^{\Delta+n+2g-2} (1-x)(1-xt^2)(1-y)(1-yt^2)(1-x^2y^{-1}t^2)(1-x^{-1}y^2t^2)}. \end{aligned}$$

Note that this expression has arbitrarily large positive and negative powers of  $x$  and  $y$ . Therefore it is not very suitable for extracting the coefficient to  $x^0y^0$ . However, to facilitate this task we can make the following change of variable:

$$x = u^2v, \quad y = uv^2.$$

Then we have  $x^2y^{-1} = u^3$ ,  $x^{-1}y^2 = v^3$  and  $xy = u^3v^3$ . Substituting in (7.10), and using (7.8), we finally obtain the formula for the contribution to the Poincaré polynomial from critical submanifolds of type  $(1, 1, 1)$ :

**Proposition 7.7.** *Under Assumption 7.6, let  $\Delta_0 \in \{1, 2\}$  be the remainder modulo 3 of  $\Delta$ . Then*

$$\begin{aligned} P_t(\Delta, (1, 1, 1)) &= \text{Coeff}_{u^0v^0} \left( (1 + 2u^2vt^2 + 2uv^2t^2 + u^3v^3t^4)^n \cdot \right. \\ &\cdot \left. \frac{t^{2(4g-3+n)} (1+t)^{2g} (1+u^2vt)^{2g} (1+uv^2t)^{2g}}{u^{3n+6g-9+\Delta_0} v^{3n+6g-6-\Delta_0} (1-u^2v)(1-uv^2)(1-u^2vt^2)(1-uv^2t^2)(1-v^3t^2)(1-u^3t^2)} \right). \end{aligned}$$

□

## 8. CRITICAL SUBMANIFOLDS OF TYPE $(1, 2)$

**8.1. Description of the critical submanifolds.** In this section, we consider the critical points of the Bott–Morse function  $f$  represented by Higgs bundles  $(E, \Phi)$  which are of the form  $E = E_0 \oplus E_1$  where  $E_0 = L$  is a parabolic line bundle,  $E_1$  is a rank 2 parabolic bundle and  $\Phi : L \rightarrow E_1 \otimes K(D)$  is a strongly parabolic homomorphism. This

defines a parabolic triple  $(E_1 \otimes K, L)$  of type  $(1, 2)$ . By Proposition 4.6, the triple is  $\sigma$ -stable exactly for the value  $\sigma = 2g - 2$ .

In order to do the computations, let us introduce some notation. Recall that we keep our assumption of generic weights. The (fixed) weights of  $E$  at each  $p \in D$  are  $0 < \alpha_1(p) < \alpha_2(p) < \alpha_3(p) < 1$ . Each possible choice of distribution of these weights is given by a function  $\varpi = \{\varpi_p\}_{p \in D}$  that assigns to each  $p \in D$  a number  $\varpi(p) \in \{1, 2, 3\}$  such that the weight of  $L$  is  $\alpha(p) = \alpha_{\varpi(p)}(p)$ . The weights of  $E_1$  are  $\beta_1(p) < \beta_2(p)$ , so that  $\{\beta_1(p), \beta_2(p), \alpha(p)\} = \{\alpha_1(p), \alpha_2(p), \alpha_3(p)\}$ . In the decomposition  $E = L \oplus E_1$ , we define

$$\begin{aligned} d_1 &= \deg(E_1 \otimes K) = \deg(E_1) + 4g - 4, \\ d_2 &= \deg(L), \end{aligned}$$

so that  $\Delta = \deg(E) = d_1 + d_2 + 4 - 4g$ .

Denote by  $\mathcal{N}_{(1,2)}(d_1, \varpi)$  the critical submanifold of parabolic Higgs bundles of type  $(1, 2)$  with topological invariants given by  $(d_1, d_2 = \Delta - d_1 + 4g - 4)$  and weights determined by  $\varpi$ . The contribution of all critical submanifolds of type  $(1, 2)$  is given as

$$P_t(\Delta, (1, 2)) := \sum_{d_1, \varpi} t^{\lambda(d_1, \varpi)} P_t(\mathcal{N}_{(1,2)}(d_1, \varpi)),$$

where  $\lambda_{(d_1, \varpi)}$  is the index of  $\mathcal{N}_{(1,2)}(d_1, \varpi)$ .

**Lemma 8.1.** *The index of the critical submanifold  $\mathcal{N}_{(1,2)}(d_1, \varpi)$  is*

$$\lambda_{(d_1, \varpi)} = 12g - 12 + 4n - 2d_1 + 4d_2 - 2s_0,$$

where  $s_0 = \#\{\beta_i(p) \mid \beta_i(p) > \alpha(p)\}$ .

*Proof.* From Proposition 2.4,  $f_p = \frac{1}{2}(r^2 - \sum_i m_i(p)^2) = 3$ , since the multiplicities are all 1. For  $p \in D$ , the space  $P_p(L, L)$  consists of endomorphisms of  $L_p$ , so  $\dim P_p(L, L) = 1$  and  $P_p(E_1 \otimes K, E_1 \otimes K)$  consists of parabolic endomorphisms of  $(E_1 \otimes K)_p$ , so it has dimension 3. The dimension of the space of strongly parabolic homomorphisms from  $L_p$  to  $(E_1 \otimes K)_p$  is given by

$$\dim N_p(L, E_1 \otimes K) = \begin{cases} 2, & \text{if } \alpha(p) < \beta_1(p), \\ 1, & \text{if } \beta_1(p) < \alpha(p) < \beta_2(p), \\ 0, & \text{if } \beta_2(p) < \alpha(p). \end{cases}$$

Therefore

$$s_0 = \sum_p \dim N_p(L, E_1 \otimes K) = \#\{\beta_i(p) \mid \beta_i(p) > \alpha(p)\}.$$

Substituting this in the formula for the Morse index in Proposition 3.11, we have

$$\begin{aligned} \lambda_{(d, \varpi)} &= 9(2g - 2) + 6n + 2(5(1 - g - n) + 2(1 - g) - (d_1 - 4g + 4) + 2d_2 + 4n - s_0) \\ &= 12g - 12 + 4n - 2d_1 + 4d_2 - 2s_0. \end{aligned}$$

□

By Proposition 4.6,  $\mathcal{N}_{(1,2)}(d_1, \varpi)$  is isomorphic to the moduli space of  $\sigma$ -stable triples of the appropriate type with  $\sigma = 2g - 2$ . By the genericity of weights, such  $\sigma$  is not a critical value. Its Poincaré polynomial is given by Theorem 6.4. So, for each  $\varepsilon = \{\varepsilon(p)\}_{p \in D}$ , let  $s_1, s_2, s_3$  be defined by (6.2),  $\varsigma(p) = 3 - \varepsilon(p)$  and

$$\begin{aligned} \bar{d}_M &= \left[ \frac{1}{3} \left( d_1 + d_2 + \sum (\alpha(p) + \beta_{\varsigma(p)}(p) - 2\beta_{\varepsilon(p)}(p)) + 2g - 2 \right) \right] + 1 \\ &= \left[ \frac{1}{3} \left( \Delta + \sum (\alpha(p) + \beta_{\varsigma(p)}(p) - 2\beta_{\varepsilon(p)}(p)) \right) \right] + 2g - 1. \end{aligned}$$

Then Theorem 6.4, Lemma 8.1 and the fact that  $s_0 = s_1 + s_2$  yield that

$$(8.1) \quad P_t(\Delta, (1, 2)) =$$

$$\begin{aligned} \sum_{d_1, \varpi} t^{12g-12+4n-2d_1+4d_2-2s_1-2s_2} \sum_{\varepsilon} \text{Coeff}_{x^0} \left( \frac{(1+t)^{4g}(1+xt)^{2g}t^{2d_1-2d_2+2s_2+2s_3-2\bar{d}_M}x^{\bar{d}_M}}{(1-t^2)(1-x)(1-xt^2)(1-t^{-2}x)x^{d_1-d_2+s_1}} \right. \\ \left. - \frac{(1+t)^{4g}(1+xt)^{2g}t^{-2d_1+2g-2+2n-2s_3+4\bar{d}_M}x^{\bar{d}_M}}{(1-t^2)(1-x)(1-xt^2)(1-t^4x)x^{d_1-d_2+s_1}} \right). \end{aligned}$$

**8.2. The sum for fixed  $(\varpi, \varepsilon)$ .** We shall compute the contribution (8.1) to the Poincaré polynomial of  $\mathcal{M}$  from submanifolds of type  $(1, 2)$ . We start by performing the sum over all possibilities of  $d_1$  for each choice of  $(\varpi, \varepsilon)$ . The condition that the moduli space  $\mathcal{N}_{(1,2)}(d_1, \varpi)$  be non-empty is  $2g - 2 > \text{par}\mu_1 - \text{par}\mu_2$  (see Remark 6.5). This means that

$$2g - 2 > d_1/2 - d_2 + \sum (\beta_1(p) + \beta_2(p) - 2\alpha(p))/2.$$

Using that  $\Delta = d_1 + d_2 + 4 - 4g$ , this is translated into

$$d_2 - d_1 > 4 - 4g - \frac{\Delta}{3} + \frac{2}{3} \sum (\beta_1(p) + \beta_2(p) - 2\alpha(p)).$$

But  $d_2 - d_1 \equiv \Delta \pmod{2}$ . So  $d_2 - d_1 = \bar{d}_0 + 2k$ ,  $k \geq 0$ , and

$$\bar{d}_0 = 4 - 4g + 2 \left[ \frac{1}{2} \left( \left[ -\frac{\Delta}{3} + \frac{2}{3} \sum (\beta_1(p) + \beta_2(p) - 2\alpha(p)) \right] + \Delta \right) \right] - \Delta + 2.$$

This gives the range for the summation in (8.1) for  $d_1$  for fixed  $(\varpi, \varepsilon)$ . Looking at (8.1), one sees that we need to compute

$$\begin{aligned} \sum t^{2d_2} x^{d_2-d_1} &= t^{\Delta+4g-4} \sum t^{d_2-d_1} x^{d_2-d_1} = \frac{t^{\bar{d}_0+\Delta+4g-4} x^{\bar{d}_0}}{1-t^2x^2}, \\ \sum t^{-4d_1+4d_2} x^{d_2-d_1} &= \frac{t^{4\bar{d}_0} x^{\bar{d}_0}}{1-t^8x^2}. \end{aligned}$$

Substituting into (8.1), we get that

(8.2)

$$P_t(\Delta, (1, 2)) = \sum_{\varpi, \varepsilon} \text{Coeff}_{x^0} \left( \frac{(1+t)^{4g}(1+xt)^{2g}t^{16g-16+4n-2s_1+2s_3-2\bar{d}_M+\bar{d}_0+\Delta}x^{\bar{d}_M+\bar{d}_0-s_1}}{(1-t^2)(1-x)(1-xt^2)(1-t^{-2}x)(1-t^2x^2)} \right. \\ \left. - \frac{(1+t)^{4g}(1+xt)^{2g}t^{14g-14+6n-2s_1-2s_2-2s_3+4\bar{d}_M+4\bar{d}_0}x^{\bar{d}_M+\bar{d}_0-s_1}}{(1-t^2)(1-x)(1-xt^2)(1-t^4x)(1-t^8x^2)} \right).$$

**8.3. The sum over  $\varpi$  and  $\varepsilon$ .** Now we need to perform the sum in (8.2) for all choices of  $(\varpi, \varepsilon)$ . For this we arrange the degree and the weights to satisfy Assumption 7.6. Write  $\Delta = 3k + \Delta_0$ ,  $\Delta_0 \in \{1, 2\}$ . Since  $\alpha_i(p)$  are sufficiently small, we have that

$$(8.3) \quad \bar{d}_M = \left\lfloor \frac{\Delta}{3} \right\rfloor + 2g - 1 = k + 2g - 1, \\ \bar{d}_0 = 4 - 4g + 2 \left\lfloor \frac{1}{2} \left( \left\lfloor -\frac{\Delta}{3} \right\rfloor + \Delta \right) \right\rfloor - \Delta + 2 = 6 - 4g - k - \Delta_0$$

are independent of  $(\varpi, \varepsilon)$ . Therefore to do the sum (8.2), we only need to do

$$\sum_{\varpi, \varepsilon} t^{-2s_1+2s_3}x^{-s_1} \quad \text{and} \quad \sum_{\varpi, \varepsilon} t^{-2s_1-2s_2-2s_3}x^{-s_1}.$$

We have to write down the dependence of  $s_1, s_2, s_3$  on  $(\varpi, \varepsilon)$ . Note that we can write

$$s_1 = \sum_{p \in D} s_1(p), \\ s_2 = \sum_{p \in D} s_2(p), \\ s_3 = \sum_{p \in D} s_3(p),$$

where  $s_1(p), s_2(p)$  and  $s_3(p)$  are defined in the obvious way:

$$s_1(p) = \begin{cases} 1 & \text{if } \alpha(p) < \beta_{\varsigma(p)}(p), \\ 0 & \text{otherwise,} \end{cases}$$

$$s_2(p) = \begin{cases} 1 & \text{if } \alpha(p) < \beta_{\varepsilon(p)}(p), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$s_3(p) = \begin{cases} 1 & \text{if } \beta_{\varepsilon(p)}(p) < \beta_{\varsigma(p)}(p), \\ 0 & \text{otherwise.} \end{cases}$$

We give the values of  $s_1(p), s_2(p)$  and  $s_3(p)$  as a function of  $(\varpi(p), \varepsilon(p)) \in \{1, 2, 3\} \times \{1, 2\}$  in Table 2,



TABLE 2.  $s_1(p)$ ,  $s_2(p)$  and  $s_3(p)$  as a function of  $(\varpi(p), \varepsilon(p))$ 

$(\varpi(p), \varepsilon(p))$	(1, 1)	(1, 2)	(2, 1)	(2, 2)	(3, 1)	(3, 2)
$s_1(p)$	1	1	1	0	0	0
$s_2(p)$	1	1	0	1	0	0
$s_3(p)$	1	0	1	0	1	0

We obtain

$$\begin{aligned}
 \sum_{\varpi, \varepsilon} t^{-2s_1+2s_3} x^{-s_1} &= \prod_{p \in D} \sum_{\varpi(p), \varepsilon(p)} t^{-2s_1(p)+2s_3(p)} x^{-s_1(p)} \\
 &= \prod_{p \in D} (t^{-2}x^{-1} + 2 + 2x^{-1} + t^2) \\
 &= (t^{-2}x^{-1} + 2 + 2x^{-1} + t^2)^n \\
 &= t^{-2n} x^{-n} (1 + 2t^2 + 2t^2x + t^4x)^n, \\
 \sum_{\varpi, \varepsilon} t^{-2s_1-2s_2-2s_3} x^{-s_1} &= \prod_{p \in D} \sum_{\varpi(p), \varepsilon(p)} t^{-2s_1(p)-2s_2(p)-2s_3(p)} x^{-s_1(p)} \\
 &= (2t^{-4}x^{-1} + x^{-1}t^{-6} + 1 + 2t^{-2})^n \\
 &= t^{-6n} x^{-n} (1 + 2t^2 + 2t^4x + t^6x)^n.
 \end{aligned}$$

Combining this with (8.2) and using that  $\bar{d}_M + \bar{d}_0 = 5 - 2g - \Delta_0$  and  $-3\bar{d}_M + \Delta = 3 - 6g + \Delta_0$ , which follow from (8.3), we have the following.

**Proposition 8.2.** *Under Assumption 7.6, let  $\Delta_0 \in \{1, 2\}$  be the remainder modulo 3 of  $\Delta$ . Then*

$$\begin{aligned}
 P_t(\Delta, (1, 2)) &= \text{Coeff}_{x^0} \left( \frac{(1+t)^{4g}(1+xt)^{2g}t^{8g-8+2n}x^{5-2g-\Delta_0-n}(1+2t^2+2t^2x+t^4x)^n}{(1-t^2)(1-x)(1-xt^2)(1-t^{-2}x)(1-t^2x^2)} \right. \\
 &\quad \left. - \frac{(1+t)^{4g}(1+xt)^{2g}t^{6g+6-4\Delta_0}x^{5-2g-\Delta_0-n}(1+2t^2+2t^4x+t^6x)^n}{(1-t^2)(1-x)(1-xt^2)(1-t^4x)(1-t^8x^2)} \right).
 \end{aligned}$$

□

## 9. CRITICAL SUBMANIFOLDS OF TYPE (2, 1)

**9.1. Description of the critical submanifolds.** In this section, we consider the critical points of the Bott–Morse function  $f$  represented by Higgs bundles  $(E, \Phi)$  which are of the form  $E = E_0 \oplus E_1$  where  $E_1 = L$  is a parabolic line bundle,  $E_0$  is a rank 2 parabolic bundle and  $\Phi : E_0 \rightarrow L \otimes K(D)$  is a strongly parabolic homomorphism. This defines a parabolic triple  $(L \otimes K, E_0)$  of type (2, 1). By Proposition 4.6, the triple is  $\sigma$ -stable exactly for the value  $\sigma = 2g - 2$ .

As in Section 8, the (fixed) weights of  $E$  at each  $p \in D$  are  $0 < \alpha_1(p) < \alpha_2(p) < \alpha_3(p) < 1$ . Each possible choice of distribution of these weights is given by some  $\varpi =$

$\{\varpi(p)\}_{p \in D}$  where  $\varpi(p) \in \{1, 2, 3\}$ ,  $p \in D$  such that the weight of  $L$  is  $\alpha(p) = \alpha_{\varpi(p)}(p)$ . The weights of  $E_0$  are  $\beta_1(p) < \beta_2(p)$ , so that  $\{\beta_1(p), \beta_2(p), \alpha(p)\} = \{\alpha_1(p), \alpha_2(p), \alpha_3(p)\}$ . In the decomposition  $E = E_0 \oplus L$ , we define

$$\begin{aligned} d_1 &= \deg(L \otimes K) = \deg(L) + 2g - 2, \\ d_2 &= \deg(E_0), \end{aligned}$$

so that  $\Delta = \deg(E) = d_1 + d_2 + 2 - 2g$ .

Denote by  $\mathcal{N}_{(2,1)}(d_1, \varpi)$  the critical submanifold of parabolic Higgs bundles of type  $(2, 1)$  with topological invariants given by  $(d_1, d_2 = \Delta - d_1 + 2g - 2)$  and where the weights are determined by  $\varpi$ .

**Lemma 9.1.**  *$\mathcal{N}_{(2,1)}(d_1, \varpi)$  is isomorphic to the moduli space of  $\sigma$ -stable parabolic triples of type  $(1, 2)$  with degrees  $d'_2 = -n - d_1$ ,  $d'_1 = -2n - d_2$  and weights  $1 - \alpha(p)$  for the line bundle and  $1 - \beta_2(p) < 1 - \beta_1(p)$  for the rank 2-bundle, for  $\sigma = 2g - 2$ .*

*Proof.* The result follows by dualizing and applying Proposition 4.2. And by the definition of dual of a parabolic bundle.  $\square$

Note that by the genericity of weights, the value  $\sigma = 2g - 2$  is not a critical value for the moduli space of parabolic triples. Now let  $\varepsilon = \{\varepsilon(p)\}_{p \in D}$ ,  $\varepsilon(p) \in \{1, 2\}$ , and  $\varsigma(p) = 3 - \varepsilon(p)$ . We introduce the following sets:

$$\begin{aligned} S_1 &= \{p \in D \mid 1 - \alpha(p) < 1 - \beta_{\varsigma(p)}(p)\} = \{p \in D \mid \alpha(p) > \beta_{\varsigma(p)}(p)\}, \\ S_2 &= \{p \in D \mid 1 - \alpha(p) < 1 - \beta_{\varepsilon(p)}(p)\} = \{p \in D \mid \alpha(p) > \beta_{\varepsilon(p)}(p)\}, \\ S_3 &= \{p \in D \mid 1 - \beta_{\varepsilon(p)}(p) < 1 - \beta_{\varsigma(p)}(p)\} = \{p \in D \mid \beta_{\varepsilon(p)}(p) > \beta_{\varsigma(p)}(p)\}. \end{aligned}$$

and denote  $s_1 = \#S_1$ ,  $s_2 = \#S_2$  and  $s_3 = \#S_3$ . Applying Theorem 6.4, we have

$$\begin{aligned} \bar{d}_M &= \left[ \frac{1}{3} \left( -n - d_1 - 2n - d_2 + \sum (1 - \alpha(p) + 1 - \beta_{\varsigma(p)}(p) - 2 + 2\beta_{\varepsilon(p)}(p)) + 2g - 2 \right) \right] + 1 \\ &= -n + \left[ \frac{1}{3} \left( -\Delta - \sum (\alpha(p) + \beta_{\varsigma(p)}(p) - 2\beta_{\varepsilon(p)}(p)) \right) \right] + 1. \end{aligned}$$

Then

$$(9.1) \quad P_t(\mathcal{N}_{(2,1)}(d_1, \varpi)) = \sum_{\varepsilon} \text{Coeff}_{x^0} \left( \frac{(1+t)^{4g}(1+xt)^{2g}t^{2d_1-2d_2-2n+2s_2+2s_3-2\bar{d}_M}x^{\bar{d}_M}}{(1-t^2)(1-x)(1-xt^2)(1-t^2x)x^{d_1-d_2-n+s_1}} \right. \\ \left. - \frac{(1+t)^{4g}(1+xt)^{2g}t^{2d_2+2g-2+6n-2s_3+4\bar{d}_M}x^{\bar{d}_M}}{(1-t^2)(1-x)(1-xt^2)(1-t^4x)x^{d_1-d_2-n+s_1}} \right).$$

Similarly to Lemma 8.1, we can prove

**Lemma 9.2.** *The index of the critical submanifold  $\mathcal{N}_{(2,1)}(d_1, \varpi)$  is*

$$\lambda_{(d_1, \varpi)} = 12g - 12 + 4n - 4d_1 + 2d_2 - 2s_0,$$

where  $s_0 = \#\{\beta_i(p) \mid \beta_i(p) < \alpha(p)\} = s_1 + s_2$ .  $\square$

Therefore the contribution of all critical submanifolds of type  $(2, 1)$  is given as

$$\begin{aligned}
 P_t(\Delta, (2, 1)) &:= \sum_{d_1, \varpi} t^{\lambda(d_1, \varpi)} P_t(\mathcal{N}_{(2,1)}(d_1, \varpi)) \\
 (9.2) \quad &= \sum_{d_1, \varpi} \sum_{\varepsilon} \text{Coeff}_{x^0} \left( \frac{(1+t)^{4g}(1+xt)^{2g} t^{12g-12+2n-2d_1-2s_1+2s_3-2\bar{d}_M} x^{\bar{d}_M}}{(1-t^2)(1-x)(1-xt^2)(1-t^{-2}x)x^{d_1-d_2-n+s_1}} \right. \\
 &\quad \left. - \frac{(1+t)^{4g}(1+xt)^{2g} t^{14g-14+10n-4d_1+4d_2-2s_1-2s_2-2s_3+4\bar{d}_M} x^{\bar{d}_M}}{(1-t^2)(1-x)(1-xt^2)(1-t^4x)x^{d_1-d_2-n+s_1}} \right),
 \end{aligned}$$

obtained by using Lemma 9.2 and (9.1).

**9.2. The sum for fixed  $(\varpi, \varepsilon)$ .** Now we shall compute the contribution (9.2) to the Poincaré polynomial of  $\mathcal{M}$  from submanifolds of type  $(2, 1)$ . As before, we do first the sum over all possibilities of  $d_1$  for each choice of  $(\varpi, \varepsilon)$ . The condition that the moduli space  $\mathcal{N}_{(2,1)}(d_1, \varpi)$  be non-empty is  $2g - 2 > \text{par}\mu_1 - \text{par}\mu_2$ . This means that

$$2g - 2 > d_1 - d_2/2 + \sum (2\alpha(p) - \beta_1(p) - \beta_2(p))/2,$$

Using that  $\Delta = d_1 + d_2 + 2 - 2g$ , this is translated into

$$d_2 - d_1 > 2 - 2g + \frac{\Delta}{3} + \frac{4}{3} \sum (2\alpha(p) - \beta_1(p) - \beta_2(p)).$$

But  $d_2 - d_1 \equiv \Delta \pmod{2}$ . So  $d_2 - d_1 = \bar{d}_0 + 2k$ ,  $k \geq 0$ , and

$$\bar{d}_0 = 2 - 2g + 2 \left[ \frac{1}{2} \left( \left[ \frac{\Delta}{3} + \frac{4}{3} \sum (2\alpha(p) - \beta_1(p) - \beta_2(p)) \right] - \Delta \right) \right] + \Delta.$$

In (9.2) we need to compute the terms

$$\begin{aligned}
 \sum t^{-2d_1} x^{d_2-d_1} &= t^{-\Delta-2g+2} \sum t^{d_2-d_1} x^{d_2-d_1} = \frac{t^{\bar{d}_0-\Delta-2g+2} x^{\bar{d}_0}}{1-t^2x^2}, \\
 \sum t^{-4d_1+4d_2} x^{d_2-d_1} &= \frac{t^{4\bar{d}_0} x^{\bar{d}_0}}{1-t^8x^2}.
 \end{aligned}$$

Plugging this into (9.2) we get

$$\begin{aligned}
 (9.3) \quad P_t(\Delta, (2, 1)) &= \sum_{\varpi, \varepsilon} \text{Coeff}_{x^0} \left( \frac{(1+t)^{4g}(1+xt)^{2g} t^{10g-10+2n-2s_1+2s_3-2\bar{d}_M+\bar{d}_0-\Delta} x^{\bar{d}_M+\bar{d}_0+n-s_1}}{(1-t^2)(1-x)(1-xt^2)(1-t^{-2}x)(1-t^2x^2)} \right. \\
 &\quad \left. - \frac{(1+t)^{4g}(1+xt)^{2g} t^{14g-14+10n-2s_1-2s_2-2s_3+4\bar{d}_M+4\bar{d}_0} x^{\bar{d}_M+\bar{d}_0+n-s_1}}{(1-t^2)(1-x)(1-xt^2)(1-t^4x)(1-t^8x^2)} \right).
 \end{aligned}$$

**9.3. The sum over  $\varpi$  and  $\varepsilon$ .** To perform the sum in (9.3) for all choices of  $(\varpi, \varepsilon)$ , we arrange the degree and the weights to satisfy Assumption 7.6. Write  $\Delta = 3k + \Delta_0$ ,  $\Delta_0 \in \{1, 2\}$ . Since  $\alpha_i(p)$  are sufficiently small, we have that

$$\begin{aligned}\bar{d}_M &= -n + \left\lceil -\frac{\Delta}{3} \right\rceil + 1 = -n - k, \\ \bar{d}_0 &= 2 - 2g + 2 \left\lceil \frac{1}{2} \left( \left\lceil \frac{\Delta}{3} \right\rceil - \Delta \right) \right\rceil + \Delta + 2 = 2 - 2g + k + \Delta_0,\end{aligned}$$

are independent of  $(\varpi, \varepsilon)$ . Therefore to do the sum in (9.3), we only need to compute

$$\sum_{\varpi, \varepsilon} t^{-2s_1+2s_3} x^{-s_1} \quad \text{and} \quad \sum_{\varpi, \varepsilon} t^{-2s_1-2s_2-2s_3} x^{-s_1}.$$

As before, we write  $s_1 = \sum_{p \in D} s_1(p)$ ,  $s_2 = \sum_{p \in D} s_2(p)$  and  $s_3 = \sum_{p \in D} s_3(p)$ , where

$$s_1(p) = \begin{cases} 1 & \text{if } \alpha(p) > \beta_{\varsigma(p)}(p), \\ 0 & \text{otherwise,} \end{cases}$$

$$s_2(p) = \begin{cases} 1 & \text{if } \alpha(p) > \beta_{\varepsilon(p)}(p), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$s_3(p) = \begin{cases} 1 & \text{if } \beta_{\varepsilon(p)}(p) > \beta_{\varsigma(p)}(p), \\ 0 & \text{otherwise.} \end{cases}$$

The values of  $s_1(p)$ ,  $s_2(p)$  and  $s_3(p)$  as a function of  $(\varpi(p), \varepsilon(p))$  are in Table 3,

TABLE 3.  $s_1(p)$ ,  $s_2(p)$  and  $s_3(p)$  as a function of  $(\varpi(p), \varepsilon(p))$

$(\varpi(p), \varepsilon(p))$	(1, 1)	(1, 2)	(2, 1)	(2, 2)	(3, 1)	(3, 2)
$s_1(p)$	0	0	0	1	1	1
$s_2(p)$	0	0	1	0	1	1
$s_3(p)$	0	1	0	1	0	1

We obtain

$$\begin{aligned}\sum_{\varpi, \varepsilon} t^{-2s_1+2s_3} x^{-s_1} &= \prod_{p \in D} \sum_{\varpi(p), \varepsilon(p)} t^{-2s_1(p)+2s_3(p)} x^{-s_1(p)} \\ &= (t^{-2}x^{-1} + 2 + 2x^{-1} + t^2)^n \\ &= t^{-2n} x^{-n} (1 + 2t^2 + 2t^2x + t^4x)^n, \\ \sum_{\varpi, \varepsilon} t^{-2s_1-2s_2-2s_3} x^{-s_1} &= (2t^{-4}x^{-1} + t^{-6}x^{-1} + 1 + 2t^{-2})^n \\ &= t^{-6n} x^{-n} (1 + 2t^2 + 2t^4x + t^6x)^n.\end{aligned}$$

Plugging this into (9.3) and using that  $\bar{d}_M + \bar{d}_0 = 2 - 2g - n + \Delta_0$  and  $-2\bar{d}_M + \bar{d}_0 - \Delta = 2 - 2g + 2n$ , we obtain the following.

**Proposition 9.3.** *Under Assumption 7.6, let  $\Delta_0 \in \{1, 2\}$  be the remainder modulo 3 of  $\Delta$ . Then*

$$P_t(\Delta, (2, 1)) = \text{Coeff}_{x^0} \left( \frac{(1+t)^{4g}(1+xt)^{2g}t^{8g-8+2n}x^{2-2g+\Delta_0-n}(1+2t^2+2t^2x+t^4x)^n}{(1-t^2)(1-x)(1-xt^2)(1-t^{-2}x)(1-t^2x^2)} \right. \\ \left. - \frac{(1+t)^{4g}(1+xt)^{2g}t^{6g-6+4\Delta_0}x^{2-2g+\Delta_0-n}(1+2t^2+2t^4x+t^6x)^n}{(1-t^2)(1-x)(1-xt^2)(1-t^4x)(1-t^8x^2)} \right).$$

□

## 10. BETTI NUMBERS OF THE MODULI SPACE OF RANK THREE PARABOLIC BUNDLES

The Betti numbers of the moduli space of parabolic vector bundles were computed by Nitsure [28] and Holla [20]. Here we work out Holla's formula for the special case when the rank is 3 and all flags at the parabolic points are full. We also continue to work with the choice of weights made in Assumptions 5.1 and 7.6.

**10.1. Notation.** Given a parabolic bundle  $E$ , the corresponding *quasi-parabolic data*,  $R$ , gives the multiplicity of each step of the flag at the parabolic points:

$$R_i^p = \dim E_{p,i} - \dim E_{p,i+1}$$

where  $E_p = E_{p,1} \supset \cdots \supset E_{p,s_p+1} = 0$  is the parabolic filtration at  $p$ . Thus  $R_i^p = m_i(p)$  in the notation of Subsection 2.1. We choose to keep Holla's original notation in this section because it is better suited for the calculations to be carried out. The *rank* of  $R$  is just the rank of  $E$ ,  $n(R) = \sum_i R_i^p$ . One defines  $\alpha(R) = \sum_{p,i} \alpha_i(p)R_i^p$ , so that

$$\text{pardeg}(E) = \deg(E) + \alpha(R).$$

Given a parabolic bundle  $E$  with Harder–Narasimhan filtration

$$0 = G_0 \subsetneq G_1 \subsetneq \cdots \subsetneq G_r = E,$$

each subbundle  $G_j$  is a parabolic bundle with the induced parabolic structure. The induced quasi-parabolic data  $R_{\leq k}^I$  is defined by

$$(R_{\leq k}^I)_i^p = \dim(G_{k,p} \cap E_{p,i}) - \dim(G_{k,p} \cap E_{p,i+1}).$$

Thus  $(R_{\leq k}^I)_i^p$  is the multiplicity of the  $i$ -th step of the induced parabolic structure on  $G_k$  at  $p$  (note that this may be zero). Each subquotient  $G_k/G_{k-1}$  is also a parabolic bundle and the corresponding parabolic data is  $R_k^I$ , given by

$$(R_k^I)_i^p = (R_{\leq k}^I)_i^p - (R_{\leq k-1}^I)_i^p.$$

The *intersection matrix*  $I$  is defined by letting

$$I_{i,k}^p = (R_k^I)_i^p,$$

in other words,  $I_{i,k}^p$  is the multiplicity of the  $i$ -th step of the induced parabolic structure on  $G_k/G_{k-1}$  at  $p$ . The rank of the subquotient  $G_k/G_{k-1}$  is  $n(R_k^I) = \sum_i I_{i,k}^p$  and hence the Harder–Narasimhan type of  $E$  can be written as

$$\mathbf{n} = (n_1, \dots, n_r) = (n(R_1^I), \dots, n(R_r^I)).$$

**10.2. Holla’s formula.** The formula [20, Theorem 5.23] for the Poincaré polynomial of the moduli space of parabolic bundles of degree  $\Delta$  and rank  $n(R)$  is

$$(10.1) \quad P_t(\Delta, n(R)) = (1 - t^2) \sum_{r=1}^{n(R)} \sum_I \frac{t^{2\{\sigma'(I) - \Delta(n(R) - n(R_r^I)) + M_g(I, \alpha)\}}}{(t^{2n(R_1^I) + 2n(R_2^I)} - 1) \dots (t^{2n(R_{r-1}^I) + 2n(R_r^I)} - 1)} \prod_{k=1}^r P_{R_k^I}(t),$$

where the sum is over intersection matrices  $I$  of all possible Harder–Narasimhan filtrations of parabolic bundles. Here  $r$  is the length of the Harder–Narasimhan filtration corresponding to  $I$ , the number  $\sigma'(I)$  is defined by  $\sigma'(I) = \sum_{p \in D} \sigma'_p(I)$ , where

$$\sigma'_p(I) = \sum_{k>l, i<j} I_{i,k}^p I_{j,l}^p,$$

the number  $M_g(I, \alpha)$  is defined by

$$M_g(I, \alpha) = \sum_{k=1}^{r-1} (n(R_k^I) + n(R_{k+1}^I)) \left( \left[ n(R_{\leq k}^I) \frac{\Delta + \alpha(R)}{n(R)} - \alpha(R_{\leq k}^I) \right] + 1 \right) + (g-1) \sum_{i<j} n(R_i^I) n(R_j^I)$$

and

$$P_R(t) = \left( \frac{\prod_{i=1}^{n(R)} (1 - t^{2i})^n}{\prod_{p \in D} \prod_{\{i \mid R_i^p \neq 0\}} \prod_{l=1}^{R_i^p} (1 - t^{2l})} \right) \left( \frac{\prod_{i=1}^{n(R)} (1 + t^{2i-1})^{2g}}{(1 - t^{2n(R)}) \prod_{i=1}^{n(R)-1} (1 - t^{2i})^2} \right).$$

This formula is valid for all choices of weights such that a parabolically semistable bundle is automatically parabolically stable. In particular, it is valid under our Assumption 5.1 on genericity of the weights.

*Remark 10.1.* This is the formula for the non-fixed determinant case, whereas Holla states the formula for the fixed determinant case. The two formulas differ by a factor of  $(1 + t)^{2g}$  coming from the Poincaré polynomial of the Jacobian (see [28]).

**10.3. The rank 3 case.** We now work out explicitly Holla’s formula for the case of rank 3 parabolic bundles under Assumptions 5.1 and 7.6. This implies in particular that all parabolic flags are full. For full flags we have the following simplification of the expressions  $P_{R_k^I}(t)$ .

**Proposition 10.2.** *Assume that all parabolic flags are full. Then*

$$P_{R_k^I}(t) = \frac{\prod_{i=1}^{n(R_k^I)} (1 - t^{2i})^{n-1} (1 + t^{2i-1})^{2g}}{(1 - t^2)^{n(R_k^I)n} \prod_{i=1}^{n(R_k^I)-1} (1 - t^{2i})}.$$

*Proof.* Since all flags are full, we have  $\#\{i \mid (R_k^I)_i^p \neq 0\} = n(R_k^I)$ . □

Note that  $P_{R_k^I}(t)$  only depends on  $I$  through the rank  $n(R_k^I)$  of  $G_k/G_{k-1}$ . For  $n_k = n(R_k^I)$  we shall therefore write

$$P_{n_k}(t) = P_{R_k^I}(t).$$

We calculate  $P_{n_k}$  for  $n_k = 1, 2, 3$ . We obtain

$$(10.2) \quad \begin{aligned} P_1(t) &= \frac{(1+t)^{2g}}{(1-t^2)}, \\ P_2(t) &= \frac{(1+t^2)^{n-1} (1+t)^{2g} (1+t^3)^{2g}}{(1-t^2)^3}, \\ P_3(t) &= \frac{(1+2t^2+2t^4+t^6)^{n-1} (1+t)^{2g} (1+t^3)^{2g} (1+t^5)^{2g}}{(1-t^2)^4 (1-t^4)}. \end{aligned}$$

Now rewrite (10.1) as

$$(10.3) \quad P_t(\Delta, n(R)) = (1-t^2) \sum_{\mathbf{n}} \sum_{I \text{ of type } \mathbf{n}} t^{2\sigma'(I)} Q_I(t) \prod_{k=1}^r P_{n_k}(t),$$

where

$$Q_I(t) = \frac{t^{2\{M_g(I, \alpha) - \Delta(n(R) - n(R_r^I))\}}}{(t^{2n(R_1^I) + 2n(R_2^I)} - 1) \cdots (t^{2n(R_{r-1}^I) + 2n(R_r^I)} - 1)}.$$

For rank  $n(R) = 3$ , the possible Harder–Narasimhan types  $\mathbf{n}$  are  $(3)$ ,  $(1, 2)$ ,  $(2, 1)$  and  $(1, 1, 1)$ . In the following, we list all the possible intersection matrices  $I$  according to the various types for rank 3. We also give the corresponding values of  $\sigma'_p(I)$ .

*Intersection matrix for type (3).*

$I_{i,k}^p$	$k = 1$
$i = 1$	1
$i = 2$	1
$i = 3$	1
$\sigma'_p(I)$	0

Intersection matrices for type (1, 2).

$I_{i,k}^p$	$k = 1$	$k = 2$
$i = 1$	1	0
$i = 2$	0	1
$i = 3$	0	1
$\sigma'_p(I)$	0	

$I_{i,k}^p$	$k = 1$	$k = 2$
$i = 1$	0	1
$i = 2$	1	0
$i = 3$	0	1
$\sigma'_p(I)$	1	

$I_{i,k}^p$	$k = 1$	$k = 2$
$i = 1$	0	1
$i = 2$	0	1
$i = 3$	1	0
$\sigma'_p(I)$	2	

Intersection matrices for type (2, 1).

$I_{i,k}^p$	$k = 1$	$k = 2$
$i = 1$	1	0
$i = 2$	1	0
$i = 3$	0	1
$\sigma'_p(I)$	0	

$I_{i,k}^p$	$k = 1$	$k = 2$
$i = 1$	1	0
$i = 2$	0	1
$i = 3$	1	0
$\sigma'_p(I)$	1	

$I_{i,k}^p$	$k = 1$	$k = 2$
$i = 1$	0	1
$i = 2$	1	0
$i = 3$	1	0
$\sigma'_p(I)$	2	

Intersection matrices for type (1, 1, 1).

$I_{i,k}^p$	$k = 1$	$k = 2$	$k = 3$
$i = 1$	1	0	0
$i = 2$	0	1	0
$i = 3$	0	0	1
$\sigma'_p(I)$	0		

$I_{i,k}^p$	$k = 1$	$k = 2$	$k = 3$
$i = 1$	1	0	0
$i = 2$	0	0	1
$i = 3$	0	1	0
$\sigma'_p(I)$	1		

$I_{i,k}^p$	$k = 1$	$k = 2$	$k = 3$
$i = 1$	0	1	0
$i = 2$	1	0	0
$i = 3$	0	0	1
$\sigma'_p(I)$	1		

$I_{i,k}^p$	$k = 1$	$k = 2$	$k = 3$
$i = 1$	0	1	0
$i = 2$	0	0	1
$i = 3$	1	0	0
$\sigma'_p(I)$	2		

$I_{i,k}^p$	$k = 1$	$k = 2$	$k = 3$
$i = 1$	0	0	1
$i = 2$	1	0	0
$i = 3$	0	1	0
$\sigma'_p(I)$	2		

$I_{i,k}^p$	$k = 1$	$k = 2$	$k = 3$
$i = 1$	0	0	1
$i = 2$	0	1	0
$i = 3$	1	0	0
$\sigma'_p(I)$	3		

Now we compute the exponent of the power of  $t^2$  in the numerator of  $Q_I(t)$ . This is greatly simplified thanks to our assumption on the degree and weights.

**Proposition 10.3.** *Let  $n(R) = 3$ . Under Assumption 7.6, we have*

$$M_g(I, \alpha) - \Delta(n(R) - n(R_r^I)) = \begin{cases} 3 \left[ \frac{\Delta}{3} \right] - \Delta + 2g + 1, & \text{for } I \text{ of type } (1, 2), \\ 3 \left[ \frac{2\Delta}{3} \right] - 2\Delta + 2g + 1, & \text{for } I \text{ of type } (2, 1), \\ 3g - 1, & \text{for } I \text{ of type } (1, 1, 1), \end{cases}$$



*Proof.* As  $\Delta \not\equiv 0 \pmod{3}$ , we have that  $n(R_{\leq k}^I) \frac{\Delta}{n(R)}$  is non-integer. Since the weights are small, we have that, for  $k \leq r-1$ ,

$$\left[ n(R_{\leq k}^I) \frac{\Delta + \alpha(R)}{n(R)} - \alpha(R_{\leq k}^I) \right] + 1 = \left[ n(R_{\leq k}^I) \frac{\Delta}{n(R)} \right] + 1.$$

Substituting into the definition of  $M_g(I, \alpha)$ , we get the result. For type  $(1, 1, 1)$ , we have used that  $\left[ \frac{\Delta}{3} \right] + \left[ \frac{2\Delta}{3} \right] - \Delta = -1$ , since  $\Delta \not\equiv 0 \pmod{3}$ .  $\square$

Note that, in particular, Proposition 10.3 implies that  $Q_I(t)$  only depends on  $I$  through its Harder–Narasimhan type. We shall therefore need to calculate  $\sum_I t^{2\sigma'(I)}$  for each type. This is an easy task using the tables given for the intersection matrices and the fact that  $\sum_I t^{2\sigma'(I)} = \prod_{p \in D} \sum_{I^p} t^{2\sigma_p'(I)}$ , as is easily seen by induction on the number of points in  $D$ . The result is:

$$(10.4) \quad \begin{aligned} \sum_I t^{2\sigma'(I)} &= (1 + t^2 + t^4)^n, & \text{for } I \text{ of type } (1, 2) \text{ and } (2, 1), \\ \sum_I t^{2\sigma'(I)} &= (1 + 2t^2 + 2t^4 + t^6)^n, & \text{for } I \text{ of type } (1, 1, 1). \end{aligned}$$

We can now calculate the contribution to  $P_t(\Delta, 3)$  from  $I$  of type  $(1, 1, 1)$ :

$$(10.5) \quad \sum_{I \text{ of type } (1, 1, 1)} t^{2\sigma'(I)} Q_I(t) \prod_{k=1}^r P_{n_k}(t) = \frac{(1 + 2t^2 + 2t^4 + t^6)^n t^{6g-2}}{(t^4 - 1)^2} P_1(t)^3.$$

Similarly, the contributions to  $P_t(\Delta, 3)$  from  $I$  of type  $(1, 2)$  and  $(2, 1)$  are:

$$\begin{aligned} \sum_{I \text{ of type } (1, 2)} t^{2\sigma'(I)} Q_I(t) \prod_{k=1}^r P_{n_k}(t) &= \frac{(1 + t^2 + t^4)^n t^{2\{3\lceil \frac{\Delta}{3} \rceil - \Delta + 2g + 1\}}}{t^6 - 1} P_1(t) P_2(t), \\ \sum_{I \text{ of type } (2, 1)} t^{2\sigma'(I)} Q_I(t) \prod_{k=1}^r P_{n_k}(t) &= \frac{(1 + t^2 + t^4)^n t^{2\{3\lceil \frac{2\Delta}{3} \rceil - 2\Delta + 2g + 1\}}}{t^6 - 1} P_1(t) P_2(t). \end{aligned}$$

Summing the contributions of type  $(1, 2)$  and type  $(2, 1)$  some simplification results because, whenever  $\Delta \not\equiv 0 \pmod{3}$ , one has

$$t^{2\{3\lceil \frac{\Delta}{3} \rceil - \Delta + 2g + 1\}} + t^{2\{3\lceil \frac{2\Delta}{3} \rceil - 2\Delta + 2g + 1\}} = t^{4g-2}(1 + t^2).$$

Hence we obtain

$$(10.6) \quad \sum_{I \text{ of type } (1, 2) \text{ or } (2, 1)} t^{2\sigma'(I)} Q_I(t) \prod_{k=1}^r P_{n_k}(t) = \frac{(1 + t^2 + t^4)^n t^{4g-2}(1 + t^2)}{t^6 - 1} P_1(t) P_2(t).$$

Since for  $\mathbf{n} = (3)$  we clearly have

$$(10.7) \quad t^{2\sigma'(I)} Q_I(t) = 1,$$

we are now in a position to put everything together and calculate  $P_t(\Delta, 3)$  for  $\Delta \not\equiv 0 \pmod{3}$ .

**Proposition 10.4.** *Under Assumption 7.6, the Poincaré polynomial of the moduli space of stable parabolic bundles of rank 3 is given by*

$$P_t(\Delta, 3) = (1+t)^{2g}(1+2t^2+2t^4+t^6)^{n-1} \cdot \frac{(1+t^3)^{2g}(1+t^5)^{2g} + (1+t)^{4g}(1+t^2+t^4)t^{6g-2} - (1+t)^{2g}(1+t^3)^{2g}(1+t^2)^{2t^{4g-2}}}{(1-t^2)^3(1-t^4)}.$$

*Proof.* Substituting (10.5), (10.6) and (10.7) in (10.3) and using (10.2) we obtain the stated formula.  $\square$

## 11. BETTI NUMBERS OF THE MODULI SPACE OF RANK THREE PARABOLIC HIGGS BUNDLES

In this final section we put everything together to obtain the Poincaré polynomial of the moduli space of rank three parabolic Higgs bundles.

### 11.1. Poincaré polynomial.

**Theorem 11.1.** *Let  $\mathcal{M}$  be the moduli space of rank three parabolic Higgs bundles of some fixed degree and weights, over a connected, smooth projective complex algebraic curve of genus  $g$ . If the weights are generic (in the sense that there are no properly semistable parabolic Higgs bundles), then the Poincaré polynomial of  $\mathcal{M}$  is given by*

$$P_t(\mathcal{M}) = \text{Coeff}_{u^0v^0} \left( (1+2u^2vt^2+2uv^2t^2+u^3v^3t^4)^n \cdot \frac{t^{2(4g-3+n)}(1+t)^{2g}(1+u^2vt)^{2g}(1+uv^2t)^{2g}}{u^{3n+6g-8}v^{3n+6g-7}(1-u^2v)(1-uv^2)(1-u^2vt^2)(1-uv^2t^2)(1-v^3t^2)(1-u^3t^2)} \right) \\ + \text{Coeff}_{x^0} \left( \frac{(1+t)^{4g}(1+xt)^{2g}t^{6g-6}x^{2-2g-n}}{(1-t^2)(1-x)(1-xt^2)} \cdot \left( \frac{t^{2g-2+2n}(x+x^2)(1+2t^2+2t^2x+t^4x)^n}{(1-t^2x)(1-t^2x^2)} - \frac{(t^4x+t^8x^2)(1+2t^2+2t^4x+t^6x)^n}{(1-t^4x)(1-t^8x^2)} \right) \right) \\ \cdot \frac{(1+t)^{2g}(1+2t^2+2t^4+t^6)^{n-1} \cdot (1+t^3)^{2g}(1+t^5)^{2g} + (1+t)^{4g}(1+t^2+t^4)t^{6g-2} - (1+t)^{2g}(1+t^3)^{2g}(1+t^2)^{2t^{4g-2}}}{(1-t^2)^3(1-t^4)}.$$

*Proof.* It follows from Morse theory, as explained in Section 3, that  $P_t(\mathcal{M}) = P_t(\Delta, 3) + P_t(\Delta, (1, 2)) + P_t(\Delta, (2, 1)) + P_t(\Delta, (1, 1, 1))$ , where the polynomials on the right hand side are given in Propositions 7.7, 8.2, 9.3 and 10.4. In order to apply these formulas we need to choose  $\Delta_0 \in \{1, 2\}$ . However, the contribution  $P_t(\Delta, (1, 1, 1))$  of Proposition 7.7 is independent of this choice, by using the duality  $(u, v) \mapsto (v, u)$ . Also, the contribution  $P_t(\Delta, (1, 2)) + P_t(\Delta, (2, 1))$  from Propositions 8.2 and 9.3 is independent of the choice of

$\Delta_0$  by using that for  $\Delta_0 = 1, 2$  we have  $x^{\Delta_0} + x^{3-\Delta_0} = x + x^2$  and  $x^{\Delta_0} t^{4\Delta_0} + x^{3-\Delta_0} t^{12-4\Delta_0} = t^4 x + t^8 x^2$ .

Even though the various contributions to the Poincaré polynomial were calculated for a specific choice of degree and weights (cf. Assumptions 5.1 and 7.6), we know by Proposition 2.1 that the final result is independent of this choice.  $\square$

*Remark 11.2.* Obviously, it is possible to do the computation of  $P_t(\mathcal{M})$  under different choices of degree and weights. Most of the calculations in Sections 7–11 are carried out in general, and we have always introduced our Assumption 7.6 as late as possible in each section. Of course, the final answer will be the same as the one given in Theorem 11.1, though the partial contributions of the critical submanifolds of different types may differ.

**11.2. Special low genus cases.** We can calculate the Poincaré polynomial of the moduli space of parabolic Higgs bundles for specific values of  $n$  and  $g$  by using a computer algebra system. For instance, if  $n = 1$  and  $g = 2$  then Theorem 11.1 gives

$$\begin{aligned} P_t(\mathcal{M}) = & 36 t^{26} + 324 t^{25} + 1368 t^{24} + 3620 t^{23} + 6810 t^{22} + 9860 t^{21} + 11670 t^{20} + 11876 t^{19} \\ & + 10860 t^{18} + 9224 t^{17} + 7408 t^{16} + 5688 t^{15} + 4216 t^{14} + 3036 t^{13} + 2134 t^{12} + 1464 t^{11} \\ & + 981 t^{10} + 640 t^9 + 401 t^8 + 244 t^7 + 144 t^6 + 80 t^5 + 42 t^4 + 20 t^3 + 9 t^2 + 4 t + 1, \end{aligned}$$

and when  $n = 2$  and  $g = 2$  we obtain

$$\begin{aligned} P_t(\mathcal{M}) = & 252 t^{32} + 2416 t^{31} + 10848 t^{30} + 30540 t^{29} + 61178 t^{28} + 94368 t^{27} \\ & + 119187 t^{26} + 129952 t^{25} + 127737 t^{24} + 116656 t^{23} + 100849 t^{22} \\ & + 83564 t^{21} + 66925 t^{20} + 52100 t^{19} + 39605 t^{18} + 29504 t^{17} + 21572 t^{16} \\ & + 15472 t^{15} + 10884 t^{14} + 7496 t^{13} + 5043 t^{12} + 3312 t^{11} + 2113 t^{10} \\ & + 1308 t^9 + 782 t^8 + 448 t^7 + 247 t^6 + 128 t^5 + 62 t^4 + 28 t^3 + 11 t^2 + 4 t + 1. \end{aligned}$$

Another example is the Poincaré polynomial of  $\mathcal{M}$  for  $g = 1$  and  $n = 1$ ,

$$P_t(\mathcal{M}) = 6 t^8 + 18 t^7 + 24 t^6 + 20 t^5 + 13 t^4 + 8 t^3 + 4 t^2 + 2 t + 1.$$

*Remark 11.3.* In [17] Hausel conjectured a formula for the Poincaré polynomial of the moduli space of stable Higgs bundles of any rank. Hausel has informed us of an analogous conjecture for the the Poincaré polynomial of the moduli space of parabolic Higgs bundles of any rank. The formulas are difficult to compare in general, but in computer calculations Hausel’s formula provides the same result as ours in all the cases that we have checked. Thus our results provide evidence for Hausel’s conjecture.

We finish by considering the case when  $X$  has genus zero. It is easy to see that under our Assumption 7.6 of small weights, the moduli space of stable parabolic bundles on  $X$  is empty, because any parabolically stable bundle would have to be stable. However,

our results show that there are non-empty critical submanifolds of the moduli space of parabolic Higgs bundles for  $n \geq 3$ : for example, if  $g = 0$  and  $n = 3$ , our calculations show that

$$P_t(\mathcal{M}) = 7t^2 + 1 ,$$

where the only contribution is from critical submanifolds of type  $(1, 1, 1)$ . This means that stable parabolic Higgs bundles exist and hence the moduli space is non-empty. Following our general description of the critical submanifolds of Section 7.1 one can explicitly describe the critical submanifolds of type  $(1, 1, 1)$  and thus get examples of stable parabolic Higgs bundles. One such example is given as follows. Define parabolic line bundles  $L_1 = \mathcal{O}(1)$ ,  $L_2 = \mathcal{O}$  and  $L_3 = \mathcal{O}$  with small weights  $\alpha_i(p)$  on  $L_i$ , such that  $\alpha_1(p) < \alpha_2(p) < \alpha_3(p)$  at each marked point. Let  $E = L_1 \oplus L_2 \oplus L_3$ . Then any map from  $L_i$  to  $L_{i+1}$  is strongly parabolic and we can define a Higgs field  $\Phi$  with non-zero components in  $\text{SParHom}(L_1, L_2K(D)) = \text{Hom}(L_1, L_2K(D)) = \Gamma(\mathcal{O}(n-3))$  and  $\text{SParHom}(L_2, L_3K(D)) = \text{Hom}(L_2, L_3K(D)) = \Gamma(\mathcal{O}(n-2))$ . Clearly the resulting parabolic Higgs bundle is stable. When  $n = 3$  this parabolic Higgs bundle is a minimum of the Morse function, as follows from Lemma 7.4, and the critical submanifold consisting of such parabolic Higgs bundles is easily seen to be isomorphic to  $\mathbb{P}^1$ . From our description of the critical submanifolds of Section 7.1 one sees that there are six other critical submanifolds, all consisting of parabolic Higgs bundles with the same underlying vector bundle but with different distributions of the weights. All these other critical submanifolds consist of one point. Of course these observations check with our calculation of the Poincaré polynomial.

For higher values of  $n$ , there are also contributions from critical submanifolds of type  $(1, 2)$  and  $(2, 1)$ . For instance, for  $g = 0$  and  $n = 4$  our formula gives

$$P_t(\mathcal{M}) = 271t^8 + 144t^6 + 43t^4 + 9t^2 + 1 ,$$

with a non-zero contributions from critical submanifolds of type  $(1, 2)$ . For  $g = 0$  and  $n = 5$  critical submanifolds of both type  $(1, 2)$  and  $(2, 1)$  contribute and one obtains

$$P_t(\mathcal{M}) = 4645t^{14} + 3791t^{12} + 1926t^{10} + 762t^8 + 249t^6 + 63t^4 + 11t^2 + 1 .$$

## REFERENCES

- [1] L. Álvarez-Cónsul and O. García-Prada, Dimensional reduction,  $\text{SL}(2, \mathbb{C})$ -equivariant bundles and stable holomorphic chains, *Internat. J. Math.* **12** (2001) 159–201.
- [2] O. Biquard, Fibrés paraboliques stables et connexions singulières plates, *Bull. Soc. Math. Fr.* **119** (1991) 231–257.
- [3] O. Biquard and O. García-Prada, Parabolic vortex equations and instantons of infinite energy, *J. Geom. Physics* **21** (1997) 238–254.
- [4] I. Biswas and S. Ramanan, An infinitesimal study of the moduli of Hitchin pairs, *J. London Math. Soc. (2)* **49** (1994) 219–231.
- [5] H.U. Boden and K. Yokogawa, Moduli spaces of parabolic Higgs bundles and parabolic  $K(D)$  pairs over smooth curves. I, *Internat. J. Math.* **7** (1996) 573–598.

- [6] F. Bottacin, Symplectic geometry on moduli spaces of stable pairs, *Ann. Sci. Éc. Norm. Supér., IV. Sér.* **28** (1995) 391–433.
- [7] S.B. Bradlow and O. García-Prada, Stable triples, equivariant bundles and dimensional reduction, *Math. Ann.* **304** (1996) 225–252.
- [8] S.B. Bradlow, O. García-Prada and P.B. Gothen, Moduli spaces of holomorphic triples over compact Riemann surfaces, *Math. Ann.* **328** (2004) 299–351.
- [9] K. Corlette, Flat  $G$ -bundles with canonical metrics, *J. Differential Geom.* **28** (1988) 361–382.
- [10] S.K. Donaldson, Twisted harmonic maps and the self-duality equations, *Proc. London Math. Soc.* **55** (1987) 127–131.
- [11] T. Frankel, Fixed points and torsion on Kähler manifolds, *Ann. Math.* **70** (1959) 1–8.
- [12] V. Ginzburg, The global nilpotent variety is Lagrangian, *Duke Math. J.* **109** 511–519.
- [13] P.B. Gothen, The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface, *Internat. J. Math.* **5** (1994) 861–875.
- [14] P.B. Gothen, Components of spaces of representations and stable triples, *Topology* **40** (2001) 823–850.
- [15] P.B. Gothen and A.D. King, Homological algebra of twisted quiver bundles, to appear in *J. London Math. Soc.*, [arXiv:math.AG/0202033](https://arxiv.org/abs/math/0202033)
- [16] T. Hausel, Compactification of moduli of Higgs bundles, *J. Reine Angew. Math.* **503** (1998) 169–192.
- [17] T. Hausel, Mirror symmetry and Langlands duality in the non-Abelian Hodge theory of a curve, [arXiv:math.AG/0406380](https://arxiv.org/abs/math/0406380)
- [18] N.J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* (3) **55** (1987) 59–126.
- [19] N.J. Hitchin, Stable bundles and integrable systems, *Duke Math. J.* **54** (1987) 91–114.
- [20] Y.I. Holla, Poincaré polynomial of the moduli space of parabolic bundles, *Proc. Indian Acad. Sci* **110** (2000) 233–261.
- [21] H. Konno, Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface, *J. Math. Soc. Japan* **45** (1993) 253–276.
- [22] H. Lange, Universal families of extensions, *J. Algebra* **83** (1983) 101–112.
- [23] G. Laumon, Un analogue global du cône nilpotent, *Duke Math. J.* **57**(1988) 647–671.
- [24] I.G. Macdonald, Symmetric products of an algebraic curve, *Topology* **1** (1962) 319–343.
- [25] M. Maruyama and K. Yokogawa, Moduli of parabolic stable sheaves, *Math. Ann.* **293** (1992) 77–99.
- [26] H. Nakajima, Hyper-Kähler structures on moduli spaces of parabolic Higgs bundles on Riemann surfaces, *Moduli of vector bundles*. Marcel Dekker. Lect. Notes Pure Appl. Math. 179 (1996) 199–208. Edited by Maruyama and Masaki.
- [27] B. Nasatyr and B. Steer, Orbifold Riemann surfaces and the Yang–Mills–Higgs equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995) 595–643.
- [28] N. Nitsure, Cohomology of the moduli space of parabolic vector bundles, *Proc. Indian Acad. Sci* **95** (1986) 61–77.
- [29] C.T. Simpson, Harmonic bundles on noncompact curves, *J. Amer. Math. Soc.* **3** (1990) 713–770.
- [30] C.T. Simpson, Higgs bundles and local systems *Inst. Hautes Études Sci. Publ. Math.* **75** (1992) 5–95.
- [31] M. Thaddeus, Stable pairs, linear systems and the Verlinde formula, *Invent. Math.* **117** (1994) 317–353.
- [32] M. Thaddeus, Variation of moduli of parabolic Higgs bundles, *J. Reine Angew. Math.* **547** (2002) 1–14.
- [33] K. Yokogawa, Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves, *J. Math. Kyoto Univ.* **33** (1993) 451–504.

- [34] K. Yokogawa, Infinitesimal deformation of parabolic Higgs sheaves, *Internat. J. Math.* **6** (1995) 125–148.

INSTITUTO DE MATEMÁTICAS Y FÍSICA FUNDAMENTAL, CSIC, SERRANO 113 BIS, 28006 MADRID,  
SPAIN

*E-mail address:* oscar.garcia-prada@uam.es

DEPARTAMENTO DE MATEMÁTICA PURA, FACULDADE DE CIÊNCIAS DA UNIVERSIDADE DO PORTO,  
RUA DO CAMPO ALEGRE 687, 4169-007 PORTO, PORTUGAL

*E-mail address:* pbgothen@fc.up.pt

INSTITUTO DE MATEMÁTICAS Y FÍSICA FUNDAMENTAL, CSIC, SERRANO 113 BIS, 28006 MADRID,  
SPAIN

*E-mail address:* vicente.munoz@imaff.cfmac.csic.es