

Is there switching for replicator dynamics and bimatrix games?

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Abstract

We consider heteroclinic networks for replicator dynamics and bimatrix games; that is, on a simplex or product of simplices; with equilibria at the vertices and connections at the edges – edge networks. Switching dynamics near a heteroclinic network occurs whenever every (infinite) sequence of connections in the network is shadowed by at least one trajectory in its neighbourhood. Aguiar and Castro (Chaotic switching in a two-person game, *Physica D* **239** (16), 1598–1609) prove switching near an edge network for the dynamics of the Rock-Scissors-Paper game. Here we give conditions for switching dynamics in general bimatrix games and show that switching near an edge network can never occur for replicator dynamics.

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1 Introduction

Let ξ_i and ξ_j be saddle equilibria for the flow of a smooth vector field. By *saddle equilibria* we mean that the equilibria ξ_k , $k = i, j$, have non-trivial stable and unstable manifolds, $W^s(\xi_k) \neq \{\xi_k\}$ and $W^u(\xi_k) \neq \{\xi_k\}$. A *heteroclinic connection* from ξ_i to ξ_j , denoted by $[\xi_i \rightarrow \xi_j]$, is a trajectory in $W^u(\xi_i) \cap W^s(\xi_j)$. A *heteroclinic network* corresponds to an

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assembly of saddle equilibria and heteroclinic connections between them that is strongly connected. Thus, every equilibria in a heteroclinic network has at least one incoming and one outgoing connection; and given any two equilibria in the network, there is a sequence of connections taking one to the other.

A particular example of a heteroclinic network occurs when the equilibria are cyclically connected. That is, up to a reordering of the equilibria, there are heteroclinic connections $[\xi_i \rightarrow \xi_{i+1}]$, for $i = 1, \dots, r - 1$, and $[\xi_r \rightarrow \xi_1]$. This is called a *heteroclinic cycle*. Note that a heteroclinic network can be defined as a connected union of heteroclinic cycles. A subset of a heteroclinic network Σ that is a network is called a *heteroclinic subnetwork* of Σ .

The existence of heteroclinic networks is a common phenomenon in the presence of invariant spaces. This can be a consequence of symmetry, see Krupa [27] and Field [12], of the coupling structure in coupled cell systems, see Aguiar *et al.* [1], or of the formulation of the problem itself, as is the case of games or population dynamics, see Hofbauer [17] and Hofbauer and Sigmund [19].

Here we are interested in heteroclinic networks in games and replicator dynamics and thus contained in a simplex or product of simplices.

In particular, we consider the replicator equation

$$\dot{x}_j = x_j [(A\mathbf{x})_j - \mathbf{x}^T A\mathbf{x}] \tag{1}$$

with $\mathbf{x} = (x_1, \dots, x_{n+1})$ in the n -dimensional simplex

$$S_n = \left\{ (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 1, x_i \geq 0, i = 1, \dots, n + 1 \right\},$$

and $A = [a_{ij}]$ an $(n + 1) \times (n + 1)$ real matrix, and we consider the equations for the dynamics of bimatrix games

$$\begin{aligned} \dot{x}_i &= x_i [(A\mathbf{y})_i - \mathbf{x}^T A\mathbf{y}] \\ \dot{y}_j &= y_j [(B\mathbf{x})_j - \mathbf{y}^T B\mathbf{x}] \end{aligned} \tag{2}$$

with $(\mathbf{x}, \mathbf{y}) = (x_1, \dots, x_{n+1}, y_1, \dots, y_{m+1})$ in the product of simplices $S_n \times S_m$, and $A = [a_{ij}]$ an $(n + 1) \times (m + 1)$ real matrix and $B = [b_{ji}]$ an $(m + 1) \times (n + 1)$ real matrix.

Replicator equation and bimatrix games play a fundamental role in describing evolutionary dynamics in a broad variety of modelling situations in different and seemingly unrelated fields ranging from biology to economics, see, for example, Weibull [41], Fudenberg and Levine [13], Hofbauer and Sigmund [19] and Nowak [31]. For more details on replicator and evolutionary game dynamics see also Hofbauer and Sigmund [20] and Szabó and Fáth [37].

In both settings all the vertices of the simplex or product of simplices are equilibria with eigendirections at the edges. Here, we consider that there are no more equilibria at the edges, besides the ones at the vertices, and we assume that they are all hyperbolic saddles. Thus, the edges of the simplex or product of simplices correspond to heteroclinic connections and form a heteroclinic network between the equilibria at the vertices. This

type of network, that we will call *edge network*, is common in replicator dynamics and bimatrix games.

Eventually, as we will see in Proposition 7, there can be other heteroclinic connections between the equilibria at the vertices and we extend the definition of edge heteroclinic network to those cases. But, we are excluding the situations where there are heteroclinic connections involving the equilibria at the vertices and other equilibria at the boundary of the simplex or product of simplices.

In [8], Brannath observes that for the case of an edge network in the simplex S_3 there is no orbit whose ω -limit contains more than one heteroclinic cycle in the network. He claims that ‘there should be a quite general (but non-trivial) argument which excludes such a ‘switching’ for networks on S_3 consisting of corners and edges.’ Moreover, he argues that ‘it would be interesting to find (more complicated or higher dimensional) networks with orbits ‘jumping’ between the cycles because these orbits could form an invariant set with quite unusual dynamical behaviour.’

At the same time, Kirk and Silber [26] study the competition between two heteroclinic cycles that have a heteroclinic connection in common. More specifically, let ξ_i and ξ_j be two equilibria such that there is a connection $[\xi_i \rightarrow \xi_j]$. Let ξ_k and ξ_l be other two equilibria such that there are the connections $[\xi_k \rightarrow \xi_i]$ and $[\xi_l \rightarrow \xi_i]$ and the connections $[\xi_j \rightarrow \xi_k]$ and $[\xi_j \rightarrow \xi_l]$. The four equilibria and the heteroclinic connections between them form a heteroclinic network with two heteroclinic cycles with one heteroclinic connection in common (see Figure 1). We shall call such a network a *Kirk&Silber network*. The conclusion in [26] is that if one of the cycles is unstable and the other attracts nearly all trajectories lying in an open neighbourhood of the heteroclinic network, then switching between the cycles can occur in only one direction; a trajectory passing near the common heteroclinic connection may make excursions around the unstable cycle and then switch to the attracting one but thereafter cannot switch back to the unstable one. The dynamics does not allow for more complicated sequences of visits to the equilibria on the network with a *random switching* between the cycles comprising the network.

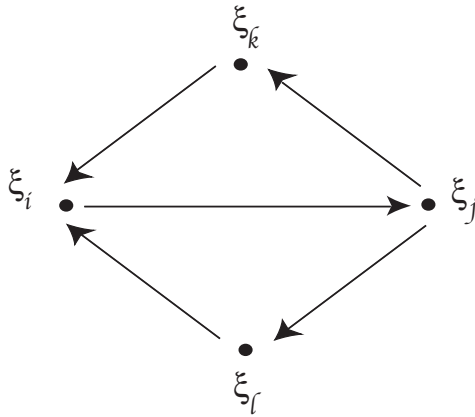


Figure 1: A Kirk&Silber network.

We now make the definition of switching more precise.

A (finite) *path* on a network Σ is a sequence of connections (c_i) , $i = 1, \dots, s$ in Σ such that $c_i = [\xi_i \rightarrow \xi_j]$ and $c_{i+1} = [\xi_j \rightarrow \xi_k]$, with ξ_i, ξ_j and ξ_k equilibria in Σ . An *infinite path* corresponds to an infinite sequence of connections (c_i) , $i \in \mathcal{Z}$.

Given a path on a network Σ , we say that there is a trajectory for the flow of a vector field f that *follows* or *shadows* that path, if for every neighbourhood V of the sequence of connections in Σ defining that path, there is a trajectory for the flow of f contained in V . That is to say, there is a trajectory for the flow of f as close as required to the sequence of connections in Σ defining the path.

We say there is finite (infinite) *switching* near a network if for every finite (infinite) path on the network there is a trajectory, near the network, for the flow of f that follows that path. Here we will prove switching only for $t \geq 0$. This is sometimes called as *forward switching*, as in Homburg and Knobloch [22]. Here we will simplify and drop the word *forward*.

In Section 2 we prove that the reason for the exclusion of switching near an edge network in S_3 , as reported by Brannath [8], is the existence of at least a Kirk&Silber subnetwork. In fact, we show that, this is the reason for the absence of switching near an edge network in S_n , for all $n \geq 2$.

In [11], Fatás-Villafranca *et al.* derive a differential equation for the evolution of a dynamical model for discretionary consumption activities with n social groups that reduces to a linear replicator equation. They conclude from the numerical simulations that for $n = 4$ social groups, the interior orbits always approach one of the cycles of the heteroclinic network. The ‘attracting’ cycle depending on the values of the parameters. For $n \geq 5$, the behaviour also looks like the attraction to a heteroclinic cycle. However, it seems that there is irregularity in the order and duration of the visits to the equilibria. As observed, the orbits seem to switch ‘randomly’ between the cycles of the network in each new return. However, from our results in Section 2, not all sequences of connections in the network can be realized by an interior trajectory.

In [35, 36], Sato *et al.* present numerical evidence for the existence of complicated behaviour arising from the existence of a heteroclinic network for the dynamics of the Rock-Scissors-Paper game with two players. Aguiar and Castro [2] provide an analytical proof for the numerical results obtained in [35, 36], showing that in the Rock-Scissors-Paper game with two players there is infinite switching, leading to behaviour ranging from almost deterministic actions to chaotic-like dynamics. In Section 3 we consider general bimatrix games finding conditions for the existence of switching near edge networks for the case of two players and more than three actions.

Switching dynamics near homoclinic and heteroclinic networks has been a subject of increasing interest lately, see, for example, [5], [3], [4], [2], [25] and [22]. It is important to identify the different general properties that entail switching; that is, the different mechanisms for switching. According to our knowledge, besides the noise-induced switching in [5], in all the studies so far, the mechanisms for switching are rotating nodes (either equilibria with complex eigenvalues or periodic trajectories) and/or transverse intersections of invariant manifolds.

This paper contributes with a new mechanism for switching: heteroclinic networks in conservative dynamics such that any two cycles in the network only share equilibria and have no heteroclinic connection in common. We believe that the results in the paper

should apply to any such network for a volume preserving flow defined on a compact manifold.

2 No switching in replicator dynamics

We consider the replicator equation

$$\dot{x}_j = x_j \left[\sum_{i=1}^{n+1} a_{ji} x_i - \mathbf{x}^T A \mathbf{x} \right] \quad (3)$$

with $\mathbf{x}^T A \mathbf{x} = \sum_{j=1}^{n+1} x_j \sum_{i=1}^{n+1} a_{ji} x_i$, defined on the n -dimensional simplex S_n .

All coordinate hyperplanes $x_j = 0$, $j = 1, \dots, n+1$, and all their intersections are invariant by the flow of (3). The intersection of those invariant subspaces with the simplex S_n makes all the boundary n -faces, $n \geq 0$, of the simplex invariant by the flow. Thus, the vertices ξ_i , $i = 1, \dots, n+1$, are equilibria, which we assume to be saddles. Moreover, if we assume that there are no other equilibria at the edges then they correspond to heteroclinic connections.

We remark that, for each vertex there is at least one incoming and one outgoing connection. Thus, with our assumptions, we conclude that there is an edge heteroclinic network at the boundary of S_n . Moreover, for every pair of equilibria ξ_i and ξ_j at the vertices there is always one of the connections $[\xi_i \rightarrow \xi_j]$ or $[\xi_j \rightarrow \xi_i]$. The number of connections to and from a vertex add up to n .

The main result in this section is Theorem 6 which states that there can be no switching near an edge network in S_n , for all $n \in \mathcal{N}$. The idea of the proof is to show that every edge network in S_n has a Kirk&Silber network as a subnetwork. As it is shown in Theorem 1, there can be no switching near such a network. The proof of Theorem 1 consists in showing that there is no switching along the common connection in the Kirk&Silber subnetwork.

We say there is *switching along a connection* $[\xi_k \rightarrow \xi_j]$ if, for any neighbourhood of a point in a connection leading to node ξ_k , there exist trajectories starting in that neighbourhood that follow along the connection $[\xi_k \rightarrow \xi_j]$ and then along all the possible connections forward from ξ_j .

Note that, switching along a connection is a necessary condition for switching near a network.

Theorem 1 *There is no switching near a heteroclinic network with a Kirk&Silber subnetwork.*

Proof We can assume, without loss of generality, that the Kirk&Silber subnetwork is formed by the equilibria ξ_n , ξ_{n+1} , ξ_1 and ξ_2 and the connections $[\xi_n \rightarrow \xi_{n+1}]$, $[\xi_1 \rightarrow \xi_n]$, $[\xi_2 \rightarrow \xi_n]$, $[\xi_{n+1} \rightarrow \xi_1]$ and $[\xi_{n+1} \rightarrow \xi_2]$. See Figure 2.

Let $-c_1$ and $-c_2$, with $c_1, c_2 > 0$, be the eigenvalues of ξ_n in the direction of ξ_1 and ξ_2 , respectively, and $e > 0$ the eigenvalue in the direction of ξ_{n+1} . Let e_1 and e_2 , with

$e_1, e_2 > 0$, be the eigenvalues of ξ_{n+1} in the direction of ξ_1 and ξ_2 , respectively, and $-c < 0$ the eigenvalue in the direction of ξ_n .

We follow the computations in [2]. By the C^1 extension by Ruelle [34] of Hartman's results [14], the vector field may be linearised around each equilibrium ξ_i , up to a set of measure zero. So, generically, there is a C^1 linearization for all the equilibria in the network.

We can choose local coordinates $(w_1, \dots, w_n) \in S_n$ such that the linearization of the flow near ξ_n is given by

$$\begin{aligned}\dot{w}_1 &= -c_1 w_1 \\ \dot{w}_2 &= -c_2 w_2 \\ \dot{w}_i &= \lambda_i^n w_i \quad i = 3, \dots, n-1 \\ \dot{w}_n &= e w_n\end{aligned}$$

with λ_i^n , $i = 3, \dots, n-1$ the remaining eigenvalues of ξ_n and the linearization of the flow near ξ_{n+1} is given by

$$\begin{aligned}\dot{w}_1 &= e_1 w_1 \\ \dot{w}_2 &= e_2 w_2 \\ \dot{w}_i &= \lambda_i^{n+1} w_i \quad i = 3, \dots, n-1 \\ \dot{w}_n &= -c w_n\end{aligned}$$

with λ_i^{n+1} , $i = 3, \dots, n-1$ the remaining eigenvalues of ξ_{n+1} .

The flow near ξ_n is then given by

$$\mathcal{F}_t^n(w_1, w_2, \dots, w_{n-1}, w_n) = (w_1 e^{-c_1 t}, w_2 e^{-c_2 t}, \dots, w_n e^{e t}), \quad (4)$$

and the flow near ξ_{n+1} is given by

$$\mathcal{F}_t^{n+1}(w_1, w_2, \dots, w_{n-1}, w_n) = (w_1 e^{e_1 t}, w_2 e^{e_2 t}, \dots, w_n e^{-c t}). \quad (5)$$

Near a neighbourhood of each node ξ_n and ξ_{n+1} where the flow can be linearized, we define cross-sections for the connection $[\xi_n \rightarrow \xi_{n+1}]$ and for each of the connections involving those nodes and the nodes ξ_1 and ξ_2 as follows

$$\begin{aligned}\Sigma_{n,1}^{\text{in}} &= \{(h, w_2, w_3, \dots, w_n) : 0 < w_2, w_3, \dots, w_n < h\} \\ \Sigma_{n,2}^{\text{in}} &= \{(w_1, h, w_3, \dots, w_n) : 0 < w_1, w_3, \dots, w_n < h\} \\ \Sigma_{n,n+1}^{\text{out}} &= \{(z_1, z_2, z_3, \dots, z_{n-1}, h) : 0 < z_1, z_2, \dots, z_{n-1} < h\} \\ \\ \Sigma_{n+1,n}^{\text{in}} &= \{(w_1, w_2, w_3, \dots, w_{n-1}, h) : 0 < w_2, w_3, \dots, w_{n-1} < h\} \\ \Sigma_{n+1,1}^{\text{out}} &= \{(h, z_2, z_3, \dots, z_n) : 0 < z_2, z_3, \dots, z_n < h\} \\ \Sigma_{n+1,2}^{\text{out}} &= \{(z_1, h, z_3, \dots, z_n) : 0 < z_1, z_3, \dots, z_n < h\}\end{aligned}$$

where $0 < h < 1$ is a positive number, small enough to guarantee transversality of the flow near each saddle. A two-dimensional representation of these sections is given in Figure 2.

For easy of computations, and without loss of generality, we will consider $h = 1$.

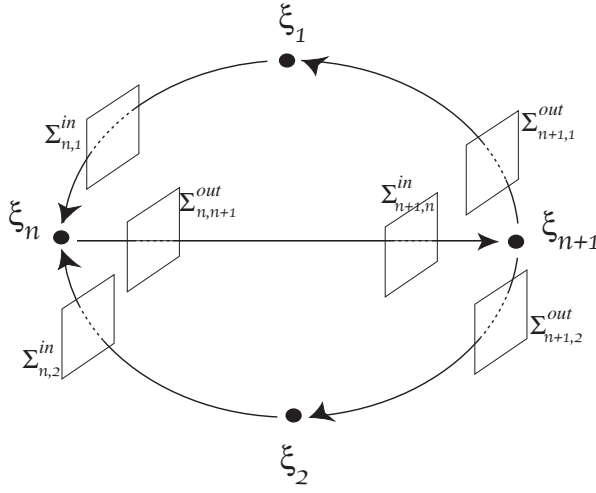


Figure 2: Two-dimensional representation of the cross-sections.

The points in $\Sigma_{k,j}^{\text{out}}$ follow the connection from saddle ξ_k to saddle ξ_j . Analogously, the points in $\Sigma_{k,j}^{\text{in}}$ come from a neighbourhood of saddle ξ_j and are taken close to saddle ξ_k .

Using the linear flow near ξ_n we define the maps $\Psi_{i,n,n+1} : \Sigma_{n,i}^{\text{in}} \rightarrow \Sigma_{n,n+1}^{\text{out}}$, $i = 1, 2$, by the following rules,

$$\Psi_{1,n,n+1}(w_2, w_3, \dots, w_{n-1}, w_n) = \left(w_n^{\frac{c_1}{e}}, w_2 w_n^{\frac{c_2}{e}}, w_3 w_n^{-\frac{\lambda_3^n}{e}}, \dots, w_{n-1} w_n^{-\frac{\lambda_{n-1}^n}{e}} \right) = (z_1, \dots, z_n),$$

$$\Psi_{2,n,n+1}(w_1, w_3, \dots, w_{n-1}, w_n) = \left(w_1 w_n^{\frac{c_1}{e}}, w_n^{\frac{c_2}{e}}, w_3 w_n^{-\frac{\lambda_3^n}{e}}, \dots, w_{n-1} w_n^{-\frac{\lambda_{n-1}^n}{e}} \right) = (z_1, \dots, z_n),$$

Thus, the points in $\Sigma_{n,n+1}^{\text{out}}$ with $w_2 < w_1^{\frac{c_2}{e_1}}$ come from $\Sigma_{n,1}^{\text{in}}$, and the ones satisfying $w_2 > w_1^{\frac{c_2}{e_1}}$ come from $\Sigma_{n,2}^{\text{in}}$.

In the neighbourhood of ξ_{n+1} , using the linearized flow, we define the maps $\Psi_{n,n+1,i} : \Sigma_{n+1,n}^{\text{in}} \rightarrow \Sigma_{n+1,i}^{\text{out}}$, $i = 1, 2$ by

$$\Psi_{n,n+1,1}(w_1, w_2, \dots, w_{n-1}) = \left(w_2 w_1^{-\frac{e_2}{e_1}}, w_3 w_1^{-\frac{\lambda_3^{n+1}}{e_1}}, \dots, w_{n-1} w_1^{-\frac{\lambda_{n-1}^{n+1}}{e_1}}, w_1^{\frac{c}{e_1}} \right) = (z_1, \dots, z_n),$$

$$\Psi_{n,n+1,2}(w_1, w_2, \dots, w_{n-1}) = \left(w_1 w_2^{-\frac{e_1}{e_2}}, w_3 w_2^{-\frac{\lambda_3^{n+1}}{e_2}}, \dots, w_{n-1} w_2^{-\frac{\lambda_{n-1}^{n+1}}{e_2}}, w_2^{\frac{c}{e_2}} \right) = (z_1, \dots, z_n),$$

Thus, the points in $\Sigma_{n+1,n}^{\text{in}}$ that go to $\Sigma_{n+1,1}^{\text{out}}$ satisfy $w_2 < w_1^{\frac{e_2}{e_1}}$ and the points that go to $\Sigma_{n+1,2}^{\text{out}}$ satisfy $w_2 > w_1^{\frac{e_2}{e_1}}$.

We can consider the transition map $\Phi_{n,n+1}$ from $\Sigma_{n,n+1}^{\text{out}}$ to $\Sigma_{n+1,n}^{\text{in}}$ as

$$\Phi_{n,n+1}(z_1, \dots, z_{n-1}) = (a_1 z_1, \dots, a_{n-1} z_{n-1})$$

with $a_i, i = 1, \dots, n - 1$ constants.

As a simplification, and without loss of generality, for the proposes of the proof we can approximate the transition map $\Phi_{n,n+1}$ by the identity map.

So, in $\Sigma_{n+1,n}^{\text{in}}$, the surface $w_2 = w_1^{\frac{e_2}{e_1}}$ divides the region of points that come from ξ_1 from the region of points that come from ξ_2 , and the surface $w_2 = w_1^{\frac{e_2}{e_1}}$ divides the region of points that go to ξ_1 from the region of points that go to ξ_2 .

If $\frac{e_2}{e_1} \neq \frac{e_2}{c_1}$, the two curves divide $\Sigma_{n+1,n}^{\text{in}}$ in three regions of points and if $\frac{e_2}{e_1} = \frac{e_2}{c_1}$, in two regions.

Thus, independently of the magnitude of the eigenvalues $e_i, -c_i, i = 1, 2$, there is always one transition $\xi_p \rightarrow \xi_n \rightarrow \xi_{n+1} \rightarrow \xi_q, p, q = 1, 2$ that is not followed by points near the heteroclinic connection $[\xi_n \rightarrow \xi_{n+1}]$.

Then there is no switching along the connection $[\xi_i \rightarrow \xi_j]$ and thus, no switching near the network. \square

The following three lemmas allow to prove the result in Theorem 5 that every edge network in S_n has a Kirk&Silber subnetwork.

The *type of a saddle equilibria* ξ_i is the tuple (s, u) , where $s = \dim W^s(\xi_i)$ and $u = \dim W^u(\xi_i)$.

Lemma 2 *Consider an edge network Σ in S_n with two equilibria ξ_i and ξ_j of type (s^i, u^i) and (s^j, u^j) , respectively, such that there is a connection $[\xi_i \rightarrow \xi_j]$. If $s^i + u^j - (n - 1) \geq 2$ then Σ has a Kirk&Silber subnetwork.*

Proof Let $E = \{\xi_1, \dots, \xi_{n+1}\} \setminus \{\xi_i, \xi_j\}$, then $\#E = n - 1$. We have that ξ_i receives a connection from s^i equilibria in E and ξ_j sends a connection to u^j equilibria in E .

If $s^i + u^j - (n - 1) \geq 2$ then there are at least two equilibria, say ξ_k and ξ_l , such that there is a connection from ξ_k and from ξ_l to ξ_i and there is a connection from ξ_j to ξ_k and to ξ_l .

The equilibria ξ_i, ξ_j, ξ_k and ξ_l with the heteroclinic connections between them form a Kirk&Silber subnetwork of Σ . \square

Lemma 3 *Let Σ be an edge network in S_n . For each equilibrium ξ_k in Σ , let (s^k, u^k) be the type of ξ_k . We have that, $s^i + u^j - (n - 1) < 2$ for every connection $[\xi_i \rightarrow \xi_j]$ if and only if n is even and all saddles are of type $(\frac{n}{2}, \frac{n}{2})$.*

Proof If all saddles are of type $(\frac{n}{2}, \frac{n}{2})$, then $s^i + u^j - (n - 1) = 1 < 2$, for all connections $[\xi_i \rightarrow \xi_j]$.

Now, suppose that for every connection $[\xi_i \rightarrow \xi_j]$ in Σ we have $s^i + u^j - (n - 1) < 2$. We start by observing that the network Σ cannot have connections $[\xi_i \rightarrow \xi_j]$ such that ξ_i is of type $(s^i, n - s^i)$ and ξ_j is of type $(s^j, n - s^j) = (s^i - m, n - s^i + m)$, with $m \geq 1$. Otherwise, $s^i + u^j - (n - 1) = m + 1 \geq 2$.

Thus, saddles of type $(1, n - 1)$, if any, can only receive a connection from a saddle of type $(1, n - 1)$. So, there can be no saddle of type $(1, n - 1)$, otherwise there would be a subnetwork of Σ with only saddles of type $(1, n - 1)$, which is impossible. The same

argument applies to saddles of type $(n - 1, 1)$, since saddles of this type can only send a connection to a saddle of type $(n - 1, 1)$.

With no saddles of types $(1, n - 1)$ and $(n - 1, 1)$ in the network Σ , the same reasoning applies to saddles of types $(2, n - 2)$ and $(n - 2, 2)$. That is, saddles of type $(2, n - 2)$ can only receive connections from two saddles of type $(2, n - 2)$ and saddles of type $(n - 2, 2)$ can only send connections to two saddles of type $(n - 2, 2)$. So, there can be no saddles of types $(2, n - 2)$ and $(n - 2, 2)$ in Σ .

This reasoning applies recursively until we end up with no saddles if n is odd and only with saddles of type $(\frac{n}{2}, \frac{n}{2})$ in Σ if n is even. \square

Lemma 4 *Every edge network Σ in S_n , with $n > 2$ even, with all the saddles of the type $(\frac{n}{2}, \frac{n}{2})$ has a Kirk&Silber subnetwork.*

Proof Let ξ_i and ξ_j be any two equilibria in Σ . Then, there exists one of the connections $[\xi_i \rightarrow \xi_j]$ or $[\xi_j \rightarrow \xi_i]$ in Σ . Without loss of generality, assume that there exists the connection $[\xi_i \rightarrow \xi_j]$. There are $n - 1$ more equilibria in Σ . We have that ξ_i receives a connection from $\frac{n}{2}$ equilibria $\eta_1, \dots, \eta_{\frac{n}{2}}$ and ξ_j sends a connection to $\frac{n}{2}$ equilibria $\zeta_1, \dots, \zeta_{\frac{n}{2}}$. See Figure 3 a).

Thus, there is at least one equality $\eta_r = \zeta_s$, $r, s \in \{1, \dots, \frac{n}{2}\}$, since there are only $n - 1$ equilibria in Σ , besides ξ_i and ξ_j . If there are more than one such equality then we get a Kirk&Silber subnetwork. Let's assume then that there is only one such equality, which we can assume, without loss of generality, to be $\eta_1 = \zeta_1 = \xi_k$. And so, $\eta_p \neq \zeta_q$, for all $p, q \in \{2, \dots, \frac{n}{2}\}$. Moreover, ξ_j receives a connection from $\xi_i, \eta_2, \dots, \eta_{\frac{n}{2}}$ and ξ_i sends a connection to $\xi_j, \zeta_2, \dots, \zeta_{\frac{n}{2}}$.

In order for a Kirk&Silber subnetwork not to exist we must have that ξ_k receives a connection from $\xi_j, \eta_2, \dots, \eta_{\frac{n}{2}}$ and sends a connection to $\xi_i, \zeta_2, \dots, \zeta_{\frac{n}{2}}$. See Figure 3 b) and c).

We have then that, for example, the equilibria ζ_2 receives a connection from ξ_i, ξ_j and ξ_k and thus must send a connection to two of the η 's, let's say, η_r and η_s . See Figure 3 d).

And so, the equilibria ξ_i, ζ_2, η_r and η_s and the connections between them form a Kirk&Silber subnetwork. See Figure 4. \square

Theorem 5 *Every edge network in S_n , for all $n > 2$ has a Kirk&Silber subnetwork.*

Proof Let Σ be an edge network in S_n , with $n > 2$. Each saddle equilibria ξ_i of Σ is of the type $(s^i, n - s^i)$, with $s^i = 1, \dots, n - 1$.

According to Lemma 2, Σ has a Kirk&Silber subnetwork if there are connections $[\xi_i \rightarrow \xi_j]$ such that $s^i + (n - s^j) - (n - 1) \geq 2$.

By Lemma 3, for such subnetwork not to exist, n must be even and the network Σ can only have saddles of type $(\frac{n}{2}, \frac{n}{2})$.

By Lemma 4, an edge network with all saddles of the type $(\frac{n}{2}, \frac{n}{2})$ contains a Kirk&Silber subnetwork. \square

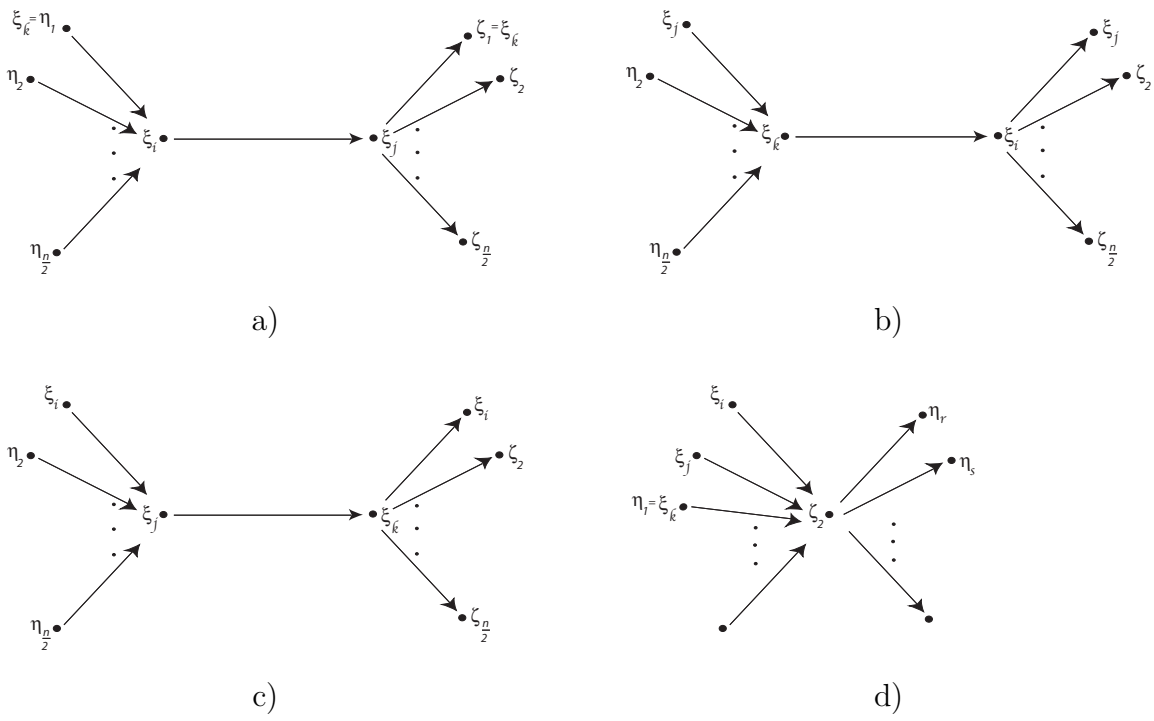


Figure 3: Illustration of heteroclinic connections mentioned in the proof of Lemma 4.

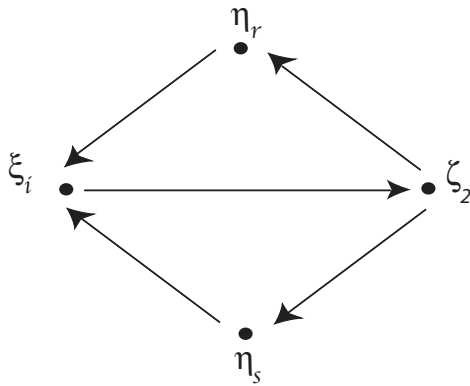


Figure 4: Kirk&Silber subnetwork obtained in the proof of Lemma 4.

Theorem 6 *There is no switching near an edge network in S_n , for all n .*

Proof If $n = 2$ then the edge heteroclinic network is a heteroclinic cycle and so switching makes no sense. For $n > 2$ the result follows from Theorem 1 and Theorem 5. \square

3 Switching in bimatrix games

We consider the analog of the replicator equation for bimatrix games,

$$\begin{aligned}\dot{x}_i &= x_i \left[\sum_{j=1}^{m+1} a_{ij} y_j - \mathbf{x}^T \mathbf{A} \mathbf{y} \right] \\ \dot{y}_j &= y_j \left[\sum_{i=1}^{n+1} b_{ji} x_i - \mathbf{y}^T \mathbf{B} \mathbf{x} \right]\end{aligned}\tag{6}$$

with $\mathbf{x}^T \mathbf{A} \mathbf{y} = \sum_{j=1}^{m+1} y_j \sum_{i=1}^{n+1} a_{ji} x_i$ and $\mathbf{y}^T \mathbf{B} \mathbf{x} = \sum_{i=1}^{n+1} x_i \sum_{j=1}^{m+1} b_{ij} y_j$, for $(\mathbf{x}, \mathbf{y}) \in S_n \times S_m$.

For more on this evolutionary dynamics for bimatrix games, see Hofbauer [18] and Hofbauer and Sigmund [19] and [20]. In particular, this bimatrix dynamics preserves volume inside $S_n \times S_m$. Conservative systems tend to be complicated because they do not contract. In fact, there can be no compact set contained in the interior of the product of simplices $S_n \times S_m$ which attracts all nearby trajectories, but they can occur in the boundary, Eshel and Akin [10].

We start by characterizing in Proposition 7 the edge heteroclinic networks in a product of simplices and then show in Proposition 8 that this networks can never have a Kirk&Silber subnetwork. On account of the absence of a Kirk&Silber subnetwork, that is, of the absence of heteroclinic cycles with common connections, by the results in Section 2, there is the possibility of switching in the neighbourhood of these networks. First, we find the conditions for the existence of switching along the connections of an edge network, as this is a necessary condition for switching near the network, Theorem 9 and Corollaries 10 and 11. In Theorem 12 the Poincaré recurrence theorem allow us to show switching dynamics near edge heteroclinic networks in $S_n \times S_m$ with the condition that $n + m$ is even and under the assumption that the flow is topologically mixing.

The boundary of the product of simplices $S_n \times S_m$ is invariant by the flow of (6) as it corresponds to the intersection of $S_n \times S_m$ with all the coordinate hyperplanes $x_i = 0 \wedge y_j = 0$, which are all flow-invariant. In particular, the vertices and the edges of $S_n \times S_m$ are flow-invariant. If there are no more equilibria at the edges, we can assume an edge network in the boundary of $S_n \times S_m$ connecting the equilibria ξ_{ij} at the vertices.

Proposition 7 *Let Σ be an edge heteroclinic network in $S_n \times S_m$. Then*

1. every equilibria ξ_{ij} in Σ is given by $\xi_{ij} = (\xi_i^n, \xi_j^m)$ with ξ_i^n a vertex in S_n and ξ_j^m a vertex in S_m ;
2. each equilibrium ξ_{ij} has an edge heteroclinic connection to or from the equilibria ξ_{kj} and ξ_{il} with $k \in \{1, \dots, n+1\}$ and $l \in \{1, \dots, m+1\}$;
3. at some square 2-faces of $S_n \times S_m$ there can be a continua of heteroclinic connections connecting two equilibria at the vertices.

Proof Statement 1. is trivial. Statement 2. follows immediatly from the computation of the Jacobian matrix at the equilibria ξ_{ij} .

The partial derivatives of (6) are given by

$$\begin{aligned}\frac{\partial \dot{x}_i}{\partial x_i} &= (1 - x_i) \sum_{j=1}^{m+1} a_{ij} y_j - \mathbf{x}^T \mathbf{A} \mathbf{y} & \frac{\partial \dot{y}_j}{\partial y_j} &= (1 - y_j) \sum_{i=1}^{n+1} b_{ji} x_i - \mathbf{y}^T \mathbf{B} \mathbf{x} \\ \frac{\partial \dot{x}_i}{\partial x_k} &= -x_i \sum_{j=1}^{m+1} a_{kj} y_j, \text{ for } k \neq i & \frac{\partial \dot{y}_j}{\partial y_k} &= -y_j \sum_{i=1}^{n+1} b_{ki} x_i, \text{ for } k \neq j \\ \frac{\partial \dot{x}_i}{\partial y_j} &= x_i [a_{ij} - \sum_{l=1}^{n+1} a_{lj} x_l] & \frac{\partial \dot{y}_j}{\partial x_i} &= y_j [b_{ji} - \sum_{l=1}^{m+1} b_{li} y_l]\end{aligned}$$

The Jacobian matrix at a vertex equilibria ξ_{ij} is a block matrix of the form

$$\begin{bmatrix} J_A & 0_{(n+1) \times (m+1)} \\ 0_{(m+1) \times (n+1)} & J_B \end{bmatrix}$$

with

$$J_A = \begin{bmatrix} a_{1j} - a_{ij} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & a_{2j} - a_{ij} & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{i-1j} - a_{ij} & 0 & 0 & \dots & 0 \\ -a_{1j} & -a_{2j} & \dots & -a_{i-1j} & -a_{ij} & -a_{i+1j} & \dots & -a_{(n+1)j} \\ 0 & 0 & \dots & 0 & 0 & a_{i+1j} - a_{ij} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_{(n+1)j} - a_{ij} \end{bmatrix}$$

and

$$J_B = \begin{bmatrix} b_{1i} - b_{ji} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & b_{2i} - b_{ji} & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{j-1i} - b_{ji} & 0 & 0 & \dots & 0 \\ -b_{1i} & -b_{2i} & \dots & -b_{j-1i} & -b_{ji} & -b_{j+1i} & \dots & -b_{(m+1)i} \\ 0 & 0 & \dots & 0 & 0 & b_{j+1i} - b_{ji} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & b_{(m+1)i} - b_{ji} \end{bmatrix}$$

In Figure 5 we list all the possible dynamics, up to conjugacy, at the square 2-faces of $S_n \times S_m$. Statement 3. corresponds to case b) in Figure 5.

□

Note that all the eigenvalues of the equilibria at the vertices are real.

Remark 1 *In our study we are excluding the existence of heteroclinic connections between the equilibria at the vertices and other equilibria at the boundary of $S_n \times S_m$, and thus we are not considering the case c) in Figure 5.*

Next we prove that no edge heteroclinic network in a product of simplices has a Kirk&Silber subnetwork. This leaves to believe that there can be switching along the connections of an edge network. We identify those connections in Theorem 9.

Proposition 8 *An edge heteroclinic network Σ in $S_n \times S_m$ never has a Kirk&Silber subnetwork.*

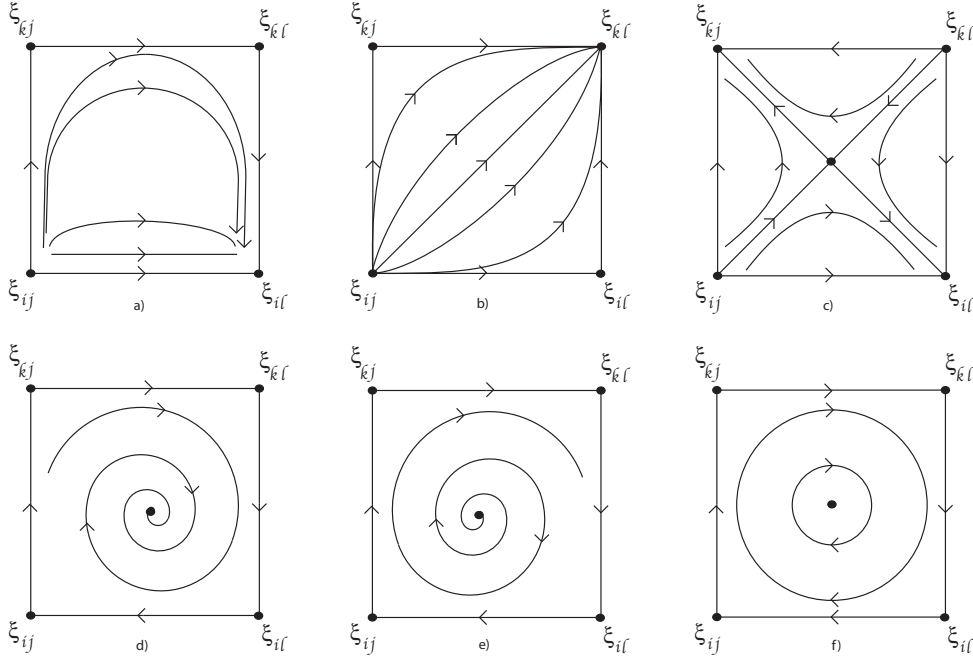


Figure 5: Possible dynamics, up to conjugacy, at the square 2-faces of $S_n \times S_m$.

Proof Suppose that an edge heteroclinic network Σ has a Kirk&Silber subnetwork, then there are four equilibria $e_{i_k j_k}$, $k = 1, \dots, 4$, in Σ such that there are the following connections in Σ

$$[e_{i_1 j_1} \rightarrow e_{i_2 j_2}] \quad [e_{i_2 j_2} \rightarrow e_{i_3 j_3}] \quad [e_{i_2 j_2} \rightarrow e_{i_4 j_4}] \quad [e_{i_3 j_3} \rightarrow e_{i_1 j_1}] \quad [e_{i_4 j_4} \rightarrow e_{i_1 j_1}]$$

In the connection $[e_{i_1 j_1} \rightarrow e_{i_2 j_2}]$, by Proposition 7, we must have $i_1 = i_2$ or $j_1 = j_2$. Let us assume, without loss of generality, that $j_1 = j_2$ and thus $i_1 \neq i_2$. Then, we must have $j_3 = j_1 = j_2$ and $i_3 \neq i_1$ and $i_3 \neq i_2$. Otherwise, if $j_3 \neq j_1 = j_2$ then, by Proposition 7, we should have $i_1 = i_2 = i_3$, which is impossible. By the same argument, we must have $j_4 = j_1 = j_2$ and $i_4 \neq i_1$.

We would then have the following connections in Σ

$$[e_{i_1 j_1} \rightarrow e_{i_2 j_1}], \quad [e_{i_2 j_1} \rightarrow e_{i_3 j_1}], \quad [e_{i_2 j_1} \rightarrow e_{i_4 j_1}], \quad [e_{i_3 j_1} \rightarrow e_{i_1 j_1}] \text{ and } [e_{i_4 j_1} \rightarrow e_{i_1 j_1}],$$

and thus we would have

$$a_{i_2 j_1} - a_{i_1 j_1} > 0, \quad a_{i_3 j_1} - a_{i_2 j_1} > 0, \quad a_{i_4 j_1} - a_{i_2 j_1} > 0, \quad a_{i_1 j_1} - a_{i_3 j_1} > 0 \text{ and } a_{i_1 j_1} - a_{i_4 j_1} > 0. \quad (7)$$

But,

$$(a_{i_2 j_1} - a_{i_1 j_1}) + (a_{i_3 j_1} - a_{i_2 j_1}) + (a_{i_1 j_1} - a_{i_3 j_1}) = 0$$

and

$$(a_{i_2 j_1} - a_{i_1 j_1}) + (a_{i_4 j_1} - a_{i_2 j_1}) + (a_{i_1 j_1} - a_{i_4 j_1}) = 0.$$

So, (7) is impossible. Thus, there cannot be any Kirk&Silber subnetwork in Σ . \square

Theorem 9 *Let Σ be an edge network in $S_n \times S_m$. There is switching along an edge heteroclinic connection $[\xi_i \rightarrow \xi_j]$ if and only if $s^i \leq s^j$, with (s^k, u^k) the type of ξ_k , for $k = i, j$.*

Proof The network Σ contains $p = (n + 1) \times (m + 1)$ equilibria and each equilibrium ξ_k has $r = m + n = s^k + u^k$ eigendirections at the product of simplices $S_n \times S_m$.

In the first part of the proof we ignore the existence of continua of face heteroclinic connections in Σ , if any, and consider only the edge heteroclinic connections.

We can consider, without loss of generality, the heteroclinic connection $[\xi_{p-1} \rightarrow \xi_p]$. Let $s = s^{p-1}$, $u = u^{p-1}$, $\bar{s} = s^p$ and $\bar{u} = u^p$.

Let $-c_1, \dots, -c_s < 0$ be the eigenvalues of ξ_{p-1} in the direction of $\xi_{i_1}, \dots, \xi_{i_s}$, respectively, with $i_k \in \{1, \dots, p-2\}$, for $k = 1, \dots, s$, and $e > 0$ the eigenvalue in the direction of ξ_p . Let $\bar{e}_1, \dots, \bar{e}_{\bar{u}} > 0$ be the eigenvalues of ξ_p in the direction of $\xi_{j_1}, \dots, \xi_{j_{\bar{u}}}$, respectively, with $j_l \in \{1, \dots, p-2\}$, for $l = 1, \dots, \bar{u}$, and $-\bar{c} < 0$ the eigenvalue in the direction of ξ_{p-1} .

We follow the computations in [2] and in the proof of Theorem 1. By the C^1 extension by Ruelle [34] of Hartman's results [14], the vector field may be linearised around each equilibrium ξ_i , up to a set of measure zero.

We can choose local coordinates $(w_1, \dots, w_n) \in S_n$ such that the linearization of the flow near ξ_{p-1} is given by

$$\begin{aligned} \dot{w}_k &= -c_k w_k & k &= 1, \dots, s \\ \dot{w}_{s+l} &= e_l w_{s+l} & l &= 1, \dots, u-1 \\ \dot{w}_r &= e w_r \end{aligned}$$

with $e_l > 0$, $l = 1, \dots, u-1$ the remaining eigenvalues of ξ_{p-1} and the linearization of the flow near ξ_p is given by

$$\begin{aligned} \dot{w}_k &= -\bar{c}_k w_k & k &= 1, \dots, \bar{s}-1 \\ \dot{w}_{\bar{s}-1+l} &= \bar{e}_l w_{\bar{s}-1+l} & l &= 1, \dots, \bar{u} \\ \dot{w}_r &= -\bar{c} w_r \end{aligned}$$

with $-\bar{c}_k < 0$, $k = 1, \dots, \bar{s}-1$ the remaining eigenvalues of ξ_p .

The flow near ξ_{p-1} is then given by

$$\mathcal{F}_t^{p-1}(w_1, w_2, \dots, w_{r-1}, w_r) = (w_1 e^{-c_1 t}, \dots, w_s e^{-c_s t}, w_{s+1} e^{e_1 t}, \dots, w_{r-1} e^{e_{u-1} t}, w_r e^{e t}), \quad (8)$$

and the flow near ξ_p is then given by

$$\mathcal{F}_t^p(w_1, w_2, \dots, w_{r-1}, w_r) = (w_1 e^{-\bar{c}_1 t}, \dots, w_{\bar{s}-1} e^{-\bar{c}_{\bar{s}-1} t}, w_{\bar{s}} e^{\bar{e}_1 t}, \dots, w_{r-1} e^{\bar{e}_{\bar{u}} t}, w_r e^{-\bar{c} t}). \quad (9)$$

Near a neighbourhood of the nodes ξ_{p-1} and ξ_p where the flow can be linearized, we define cross-sections for the connection $[\xi_{p-1} \rightarrow \xi_p]$ and for each of the connections involving those nodes and the nodes $\xi_{i_1}, \dots, \xi_{i_s}$ and the nodes $\xi_{j_1}, \dots, \xi_{j_{\bar{u}}}$ as follows

$$\begin{aligned} \Sigma_{p-1, i_k}^{\text{in}} &= \{(w_1, \dots, w_{k-1}, h, w_{k+1}, \dots, w_r) : 0 < w_1, w_2, \dots, w_r < h\} & k &= 1, \dots, s \\ \Sigma_{p-1, p}^{\text{out}} &= \{(z_1, z_2, z_3, \dots, z_{r-1}, h) : 0 < z_1, z_2, \dots, z_{r-1} < h\} \end{aligned}$$

$$\begin{aligned}
\Sigma_{p,p-1}^{\text{in}} &= \{(w_1, w_2, w_3, \dots, w_{r-1}, h) : 0 < w_2, w_3, \dots, w_{r-1} < h\} \\
\Sigma_{p,j_1}^{\text{out}} &= \{(z_1, z_2, \dots, z_{\bar{s}-1}, h, z_{\bar{s}+1}, \dots, z_r) : 0 < z_1, z_3, \dots, z_r < h\} \\
&\dots \\
\Sigma_{p,j_{\bar{u}}}^{\text{out}} &= \{(z_1, z_2, z_3, \dots, h, z_r) : 0 < z_1, z_2, \dots, z_r < h\}
\end{aligned}$$

where $0 < h < 1$ is a positive number, small enough to guarantee transversality of the flow near each saddle. For easy of computations, and without loss of generality, we will consider $h = 1$.

Note that, for each equilibrium ξ_{p-1} and ξ_p , the union of the cross sections, as defined above, is the boundary of a neighbourhood of the equilibrium and thus it intersects all the trajectories approaching or leaving that neighbourhood of the equilibrium.

Using the linear flow near ξ_{p-1} we define the maps $\Psi_{i_k,p-1,p} : \Sigma_{p-1,i_k}^{\text{in}} \rightarrow \Sigma_{p-1,p}^{\text{out}}$, $k = 1, \dots, s$, by the following rule,

$$\begin{aligned}
\Psi_{i_k,p-1,p}(w_1, w_2, \dots, w_{k-1}, w_{k+1}, \dots, w_s, \dots, w_r) &= \\
&= \left(w_1 w_r^{\frac{c_1}{e}}, \dots, w_{k-1} w_r^{\frac{c_{k-1}}{e}}, w_r^{\frac{c_k}{e}}, w_{k+1} w_r^{\frac{c_{k+1}}{e}}, \dots, w_s w_r^{\frac{c_s}{e}}, w_{s+1} w_r^{-\frac{e_1}{e}}, \dots, w_{r-1} w_r^{-\frac{e_{u-1}}{e}} \right),
\end{aligned}$$

Thus, the points in $\Sigma_{p-1,p}^{\text{out}}$ that come from $\Sigma_{p-1,i_k}^{\text{in}}$, $k = 1, \dots, s$, satisfy

$$\left\{ \begin{array}{l} z_1 < z_k^{\frac{c_1}{c_k}} \\ \dots \\ z_{k-1} < z_k^{\frac{c_{k-1}}{c_k}} \\ z_{k+1} < z_k^{\frac{c_{k+1}}{c_k}} \\ \dots \\ z_s < z_k^{\frac{c_s}{c_k}} \end{array} \right.$$

In the neighbourhood of ξ_p , using the linearized flow, we define the maps $\Psi_{p-1,p,j_l} : \Sigma_{p,p-1}^{\text{in}} \rightarrow \Sigma_{p,j_l}^{\text{out}}$, $l = 1, \dots, \bar{u}$ by

$$\begin{aligned}
\Psi_{p-1,p,j_1}(w_1, w_2, \dots, w_{r-1}) &= \left(w_1 w_{\bar{s}}^{\frac{\bar{c}_1}{\bar{e}_1}}, \dots, w_{\bar{s}-1} w_{\bar{s}}^{\frac{\bar{c}_{\bar{s}-1}}{\bar{e}_1}}, w_{\bar{s}+1} w_{\bar{s}}^{-\frac{\bar{e}_2}{\bar{e}_1}}, \dots, w_{r-1} w_{\bar{s}}^{-\frac{\bar{c}_{\bar{u}}}{\bar{e}_1}}, w_{\bar{s}}^{\frac{c}{\bar{e}_1}} \right), \\
&\dots\dots\dots
\end{aligned}$$

$$\Psi_{p-1,p,j_{\bar{u}}}(w_1, w_2, \dots, w_{r-1}) = \left(w_1 w_{r-1}^{\frac{\bar{c}_1}{\bar{e}_{\bar{u}}}}, \dots, w_{\bar{s}-1} w_{r-1}^{\frac{\bar{c}_{\bar{s}-1}}{\bar{e}_{\bar{u}}}}, w_{\bar{s}} w_{r-1}^{-\frac{\bar{e}_1}{\bar{e}_{\bar{u}}}}, \dots, w_{r-2} w_{r-1}^{-\frac{\bar{c}_{\bar{u}}}{\bar{e}_{\bar{u}}}}, w_{r-1}^{\frac{c}{\bar{e}_{\bar{u}}}} \right),$$

Thus, the points in $\Sigma_{p,p-1}^{\text{in}}$ that go to $\Sigma_{p,j_1}^{\text{out}}$ satisfy

$$\left\{ \begin{array}{l} w_{\bar{s}+1} < w_{\bar{s}}^{\frac{\bar{e}_2}{\bar{e}_1}} \\ \dots \\ w_{r-1} < w_{\bar{s}}^{\frac{\bar{c}_{\bar{u}}}{\bar{e}_1}} \end{array} \right.$$

and the points that go to $\Sigma_{p,\bar{p}}^{\text{out}}$ satisfy

$$\begin{cases} w_{\bar{s}} < w_{r-1}^{\frac{\bar{e}_1}{\bar{e}_u}} \\ \dots \\ w_{r-2} < w_{r-1}^{\frac{\bar{e}_{u-1}}{\bar{e}_u}} \end{cases}$$

We can consider the transition map $\Phi_{p-1,p}$ from $\Sigma_{p-1,p}^{\text{out}}$ to $\Sigma_{p,p-1}^{\text{in}}$ as

$$\Phi_{p-1,p}(z_1, \dots, z_{r-1}) = (a_1 z_1, \dots, a_{r-1} z_{r-1})$$

with $a_i, i = 1, \dots, n-1$ constants.

As a simplification, and without loss of generality, for the proposes of the proof we can approximate the transition map $\Phi_{p-1,p}$ by the identity map.

So, in $\Sigma_{p,p-1}^{\text{in}}$, we get the conditions

$$\begin{cases} w_2 < w_1^{\frac{c_2}{c_1}} \\ w_3 < w_1^{\frac{c_3}{c_1}} \\ \dots \\ w_s < w_1^{\frac{c_s}{c_1}} \end{cases} \quad \begin{cases} w_1 < w_2^{\frac{c_1}{c_2}} \\ w_3 < w_2^{\frac{c_3}{c_2}} \\ \dots \\ w_s < w_2^{\frac{c_s}{c_2}} \end{cases} \quad \dots \quad \begin{cases} w_1 < w_s^{\frac{c_1}{c_s}} \\ w_2 < w_s^{\frac{c_2}{c_s}} \\ \dots \\ w_{s-1} < w_s^{\frac{c_{s-1}}{c_s}} \end{cases}$$

and

$$\begin{cases} w_{\bar{s}+1} < w_{\bar{s}}^{\frac{\bar{e}_1}{\bar{e}_2}} \\ w_{\bar{s}+2} < w_{\bar{s}}^{\frac{\bar{e}_3}{\bar{e}_2}} \\ \dots \\ w_{r-1} < w_{\bar{s}}^{\frac{\bar{e}_u}{\bar{e}_2}} \end{cases} \quad \begin{cases} w_{\bar{s}} < w_{\bar{s}+1}^{\frac{\bar{e}_1}{\bar{e}_3}} \\ w_{\bar{s}+2} < w_{\bar{s}+1}^{\frac{\bar{e}_3}{\bar{e}_4}} \\ \dots \\ w_{r-1} < w_{\bar{s}+1}^{\frac{\bar{e}_u}{\bar{e}_4}} \end{cases} \quad \dots \quad \begin{cases} w_{\bar{s}} < w_{r-1}^{\frac{\bar{e}_1}{\bar{e}_u}} \\ w_{\bar{s}+1} < w_{r-1}^{\frac{\bar{e}_2}{\bar{e}_u}} \\ \dots \\ w_{r-2} < w_{r-1}^{\frac{\bar{e}_{u-1}}{\bar{e}_u}} \end{cases}$$

It follows from the proof of Theorem 1 that a necessary and sufficient condition for switching along the connection $[\xi_{p-1} \rightarrow \xi_p]$ is that there are no two conditions of the form, for example, $w_k < w_l^{\frac{\alpha}{\beta}}$ and $w_l < w_k^{\frac{\gamma}{\delta}} \Leftrightarrow w_k > w_l^{\frac{\gamma}{\delta}}$, with $\frac{\alpha}{\beta} \neq \frac{\gamma}{\delta}$, involving each pair of variables w_k and w_l , for $k, l \in 1, \dots, r$ and $k \neq l$. As we have seen in the proof of Theorem 1, in the plane (w_k, w_l) , the curves $w_k = w_l^{\frac{\alpha}{\beta}}$ and $w_k = w_l^{\frac{\gamma}{\delta}}$, intersect at the origin, and thus divide $\Sigma_{p,p-1}^{\text{in}}$ in at most three regions, when we would need four region to get switching.

Thus, a necessary and sufficient condition for switching along the connection $[\xi_{p-1} \rightarrow \xi_p]$ is that $s \leq \bar{s}$. For example, if $s = \bar{s}+1$, then we would have $w_{\bar{s}} < w_{\bar{s}+1}^{\frac{c_{\bar{s}}}{c_{\bar{s}+1}}}$ and $w_{\bar{s}} > w_{\bar{s}+1}^{\frac{\bar{e}_1}{\bar{e}_2}}$.

Until now we have ignored the existence of continua of face connections. Now, we are going to analyse what differs in the presence of continua of connections.

So, assume that there is a continua of connections, for example, in the two-dimensional square face containing the connections $[\xi_{i_1} \rightarrow \xi_{p-1}]$ and $[\xi_{i_2} \rightarrow \xi_{p-1}]$. Let ξ_{i_k} be the other equilibrium vertex at that face. Thus, the other edge heteroclinic connections in the face are $[\xi_{i_k} \rightarrow \xi_{i_1}]$ and $[\xi_{i_k} \rightarrow \xi_{i_2}]$ and there is a continua of connections from ξ_{i_k} to ξ_{p-1} .

The difference, between this situation and the situation analysed above without continua of connections, is that not all points in Σ_{i_k, i_j}^{out} , for $j = 1, 2$, will follow the connection $[\xi_{i_k} \rightarrow \xi_{i_j}]$. Some will follow the continua of connections $[\xi_{i_k} \rightarrow \xi_{p-1}]$.

So roughly speaking, is such that like from the points in $\Sigma_{p-1, p}^{out}$ that come from Σ_{p-1, i_1}^{in} , some follow the connection $[\xi_{i_1} \rightarrow \xi_{p-1}]$, while the others follow a connection $[\xi_{i_k} \rightarrow \xi_{p-1}]$ and from the points in $\Sigma_{p-1, p}^{out}$ that come from Σ_{p-1, i_2}^{in} , some follow the connection $[\xi_{i_2} \rightarrow \xi_{p-1}]$, while the others follow a connection $[\xi_{i_k} \rightarrow \xi_{p-1}]$. See Figure 6.

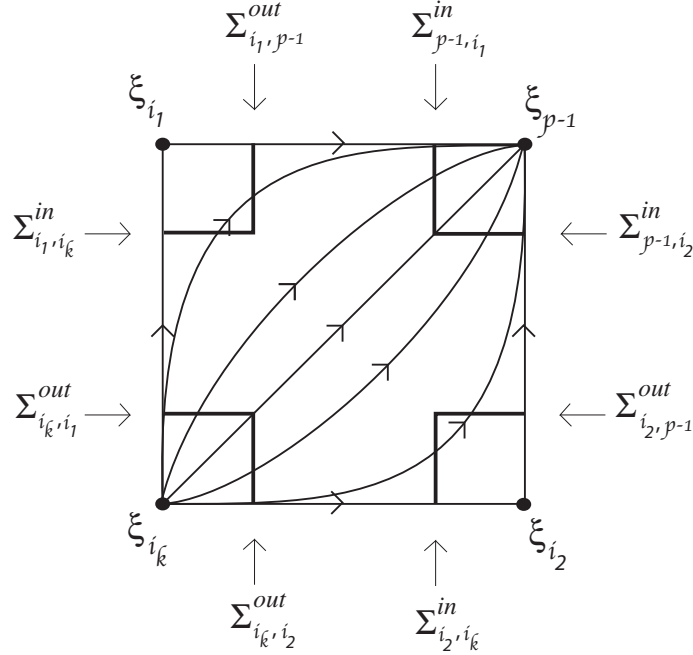


Figure 6: .

More specifically, there are $\alpha > \frac{c_{i_2}}{c_{i_1}}$ and $\beta < \frac{c_{i_2}}{c_{i_1}}$ such that from the points in $\Sigma_{p-1, p}^{out}$ that come from Σ_{p-1, i_1}^{in} , the ones satisfying $z_{i_2} < z_{i_1}^\alpha$, follow the connection $[\xi_{i_1} \rightarrow \xi_{p-1}]$, while the ones satisfying $z_{i_1}^\alpha < z_{i_2} < z_{i_1}^{\frac{c_{i_2}}{c_{i_1}}}$ are the points that follow the connection $[\xi_{i_k} \rightarrow \xi_{p-1}]$ and the points in $\Sigma_{p-1, p}^{out}$ that come from Σ_{p-1, i_2}^{in} , the ones satisfying $z_{i_2} > z_{i_1}^\beta$, follow the connection $[\xi_{i_2} \rightarrow \xi_{p-1}]$, while the ones satisfying $z_{i_1}^{\frac{c_{i_2}}{c_{i_1}}} < z_{i_2} < z_{i_1}^\beta$ are the points that follow the connection $[\xi_{i_k} \rightarrow \xi_{p-1}]$. See Figure 7.

Analogously, if there is a continua of connections in a two-dimensional square face containing any two connections forward from ξ_p .

So, the necessary and sufficient condition for switching, that there are no two conditions of the form $w_k < w_l^{\frac{\alpha}{\delta}}$ and $w_k > w_l^{\frac{\gamma}{\delta}}$, is still valid. \square

It follows from the proof of the above theorem that,

Corollary 10 *If for all edge heteroclinic connections $[\xi_i \rightarrow \xi_j]$, we have $s^i \leq s^j$, then there is switching along every continua of connections.*

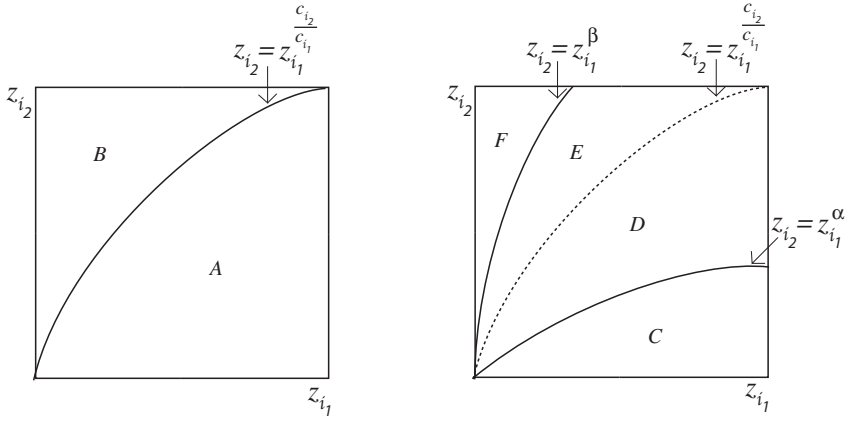


Figure 7: .

Corollary 11 *There is switching along all the connections of an edge network Σ in $S_n \times S_m$ if and only if $n + m$ is even and all saddles are of type $(\frac{r}{2}, \frac{r}{2})$, with $r = n + m$.*

Proof By Theorem 9 and Corollary 10, there is switching along a connection $[\xi_i \rightarrow \xi_j]$ of Σ if and only if $s^i \leq s^j$. This happens for all connections in the network if and only if $n + m$ is even and all saddles are of type $(\frac{r}{2}, \frac{r}{2})$, with $r = n + m$. The proof is analogous to the proof of Lemma 3. \square

Denote by $\lambda_{\bar{l}j}$ the eigenvalue of the equilibria ξ_{ij} in the direction of ξ_{lj} and by $\lambda_{i\bar{j}k}$ the eigenvalue of the equilibria ξ_{ij} in the direction of ξ_{ik} . Analysing the Jacobian matrix of (6) at ξ_{ij} (see the proof of Proposition 7), we conclude that the eigenvalues at ξ_{ij} satisfy $\lambda_{\bar{l}j} = -\lambda_{\bar{l}i}$, for $i \neq l$ and $\lambda_{i\bar{j}k} = -\lambda_{i\bar{k}j}$, for $k \neq j$ for all $i, l \in \{1, \dots, n\}$ and $j, k \in \{1, \dots, m\}$. It follows from this resonance that, for each hyperplane the sum of the expanding eigenvalues of the equilibria of the network in that hyperplane equals the sum of the expanding eigenvalues. This means that, on average the contraction around the network equals the expansion, and so, we can expect some *neutral stability* of the network in the sense that trajectories orbit around the network in closed invariant sets which contain no equilibria without leading towards neither away from the network in average. See Appendix A for more considerations on this.

In fact, since the flow of (6) in $\text{int}S_n \times S_m$ preserves volume (see Hofbauer and Sigmund [19]), by the Poincaré recurrence theorem (see Walters [40]), we can conclude the existence of a tubular neighbourhood of the edge network such that almost all orbits will stay infinitely inside that neighbourhood. That is, the distribution of almost all solutions traveling around the network form a ‘tube’ centered on the network.

This, together with the existence of switching along every connection of the edge network, suggests the existence of infinite switching near the network. We prove in Theorem 12 that, provided there is switching along all the connections in the network, switching near the network is equivalent to the flow being *topologically mixing*: a continuous flow $X^t : M \rightarrow M$ is topologically mixing, if given any two nonempty open subsets A and B of M , there exists $\tau \in \mathcal{R}$ such that $X^t(A) \cap B \neq \emptyset$ for any $t \geq \tau$.

Theorem 12 *Let Σ be an edge network in $S_n \times S_m$ with $n + m$ even and all saddles of type $(\frac{r}{2}, \frac{r}{2})$, for $r = n + m$. There is (infinite) switching near Σ if and only if the flow is topologically mixing within a neighbourhood of Σ .*

Proof

That switching near the heteroclinic network Σ implies the flow to be topologically mixing in its neighbourhood, follows immediatly from the definition. The reverse implication requires, besides topological mixing, switching along all the connections of Σ and we give next a detailed proof of it.

Let X^t denote the flow associated to the vector field (6) defined in $S_n \times S_m$. Since in $\text{int}S_n \times S_m$ volume is conserved by the flow X^t , we have, following the proof of the Poincaré recurrence theorem (see Walters [40]), that $U = X^t(V)$ is a neighbourhood of the edge network such that almost every orbit starting in U will stay infinitely in U . Recall that, by the Poincaré recurrence theorem (see Walters [40] and Barreira [6]), if a flow preserves volume and has only bounded orbits then for each open set V there exist orbits that intersect the set infinitely often. That is, the existence of a finite invariant measure guarantees that almost every orbit starting in a set V returns infinitely often to this set.

Near each equilibrium ξ_k consider for each connection $[\xi_i \rightarrow \xi_k]$ the intersection Σ_{ki}^{in} of a cross-section, transversal to the connection and to the flow, with the neighbourhood U . For each connection $[\xi_k \rightarrow \xi_j]$, let $C^{i,k,j}$ denote the open subset of points in $\Sigma_{k,i}^{\text{in}}$ that come from a neighbourhood of saddle ξ_i are taken close to saddle ξ_k and then follow the connection from saddle ξ_k to saddle ξ_j .

As follows from Theorem 9 and Corollary 11, all the sets $C^{i,k,j}$ are nonempty if and only if $n + m$ is even and all saddles are of type $(\frac{r}{2}, \frac{r}{2})$, with $r = n + m$, which proves that, under these conditions, there is switching along all the connections of an edge network Σ in $S_n \times S_m$. Switching along every connection of a network is a necessary condition for the existence of switching near the network, but it is not enough.

Let $\xi_{i_1} \rightarrow \xi_{i_2} \rightarrow \xi_{i_3} \rightarrow \xi_{i_4} \rightarrow \dots \rightarrow \xi_{i_n}$ be any finite sequence of connections in the edge network. We shall prove the existence of an open set of initial conditions near the edge network whose trajectories shadow the given sequence.

Since we are assuming the vector field (6) to be topologically mixing in $\text{int}S_n \times S_m$, we have that

$$\exists \tau_1 > 0 \text{ such that } \forall t > \tau_1, X^t(C^{i_1, i_2, i_3}) \cap C^{i_2, i_3, i_4} \neq \emptyset.$$

Moreover, $\cup_{t > \tau_1} X^t(C^{i_1, i_2, i_3}) \cap C^{i_2, i_3, i_4}$ is an open set (as X^t is a C^∞ flow and the intersection of a finite number of open sets is open).

Now, consider the open set $C^{i_1, i_2, i_3, i_4} = \cup_{t > \tau_1} X^{-t}(X^t(C^{i_1, i_2, i_3}) \cap C^{i_2, i_3, i_4}) \cap C^{i_1, i_2, i_3}$ which corresponds to the points in $\Sigma_{i_2, i_1}^{\text{in}}$ that come from a neighbourhood of saddle ξ_{i_1} are taken close to saddle ξ_{i_2} , follow the connection from saddle ξ_{i_2} to saddle ξ_{i_3} and then follow the connection from ξ_{i_3} to ξ_{i_4} .

Since the vector field (6) is topologically mixing, we have that

$$\exists \tau_2 > \tau_1 > 0 \text{ such that } \forall t > \tau_2, X^t(C^{i_1, i_2, i_3, i_4}) \cap C^{i_3, i_4, i_5} \neq \emptyset.$$

Let $C^{i_1, i_2, i_3, i_4, i_5} = \cup_{t > \tau_2} X^{-t}(X^t(C^{i_1, i_2, i_3, i_4}) \cap C^{i_3, i_4, i_5}) \cap C^{i_1, i_2, i_3, i_4}$ which corresponds to the points in $\Sigma_{i_2, i_1}^{\text{in}}$ that come from a neighbourhood of saddle ξ_{i_1} are taken close to saddle

ξ_{i_2} , follow the connection from saddle ξ_{i_2} to saddle ξ_{i_3} , then follow the connection from ξ_{i_3} to ξ_{i_4} and then follow the connection from ξ_{i_4} to ξ_{i_5} .

Continuing iteratively we end up with an open subset $C^{i_1, i_2, i_3, \dots, i_n} \subset C^{i_1, i_2, i_3}$ of points whose trajectory follows the given sequence of connections $\xi_{i_1} \rightarrow \xi_{i_2} \rightarrow \xi_{i_3} \rightarrow \xi_{i_4} \rightarrow \dots \rightarrow \xi_{i_n}$.

If we consider an infinite sequence of connections, that is, if we take $n \rightarrow +\infty$, then we get infinite intersections of open sets which need not be open. Nevertheless, by the topological mixing property, we know that it is nonempty and thus contains at least one point. \square

We remark that the assumption of topological mixing seems reasonable as it seems likely to happen. For example, in [7], Bessa proves that, generically the flow of a divergence-free vector field defined on a n -dimensional compact connected, boundaryless C^∞ Riemannian manifold is topologically mixing.

Theorem 12 gives conditions for switching dynamics in games with two players and three or more strategies per player. A particular example is the RSP-game studied in [36] and [2]. Note that in the case of bimatrix games with two strategies there can be no switching. This is the case, for example, of the Matching Pennies (see Section 4.1 in [36]).

In [2], Aguiar and Castro prove switching near an edge network in $S_3 \times S_3$ for the dynamics of the RSP-game. The results proved here add to the dynamical conclusions in [2], showing that there is switching dynamics near a bigger heteroclinic network than the one described there. More concretely, by statement 3. in Proposition 7, we get the heteroclinic connections $(R, R) \rightarrow (S, S)$, $(S, S) \rightarrow (P, P)$ and $(S, S) \rightarrow (R, R)$. The author thanks this observation to Peter Ashwin.

As in [2], depending on the sign of the eigenvalues of the equilibria, besides switching near an edge network for the dynamics of a bimatrix game, one or more heteroclinic cycles in the network can be relatively asymptotically stable.

4 Conclusions

It seems that the minimal dimension for the existence of switching, without rotating nodes (i.e., equilibria with complex eigenvalues or periodic trajectories) and/or transversal intersections of invariant manifolds, is four; consider an edge network in $S_3 \times S_3$ satisfying the hypotheses in Theorem 12.

For edge heteroclinic networks in bimatrix games, the mechanism for switching seems to be the absence in the network of heteroclinic cycles with heteroclinic connections in common, and the conservation of volume by the flow.

As shown here, the absence of heteroclinic cycles with common connections is a necessary condition for switching dynamics. This condition is not verified in the Kirk and Silber network [26] where switching was prove not to exist. However, in the examples in Chawanya [9] and Sato *et al.* [36], the condition is verified and, in both cases, switching is observed.

A consequence of volume preservation is the resonance on the eigenvalues, $\lambda_{\bar{l}j} = -\lambda_{\bar{l}i}$, for $i \neq l$ and $\lambda_{\bar{i}k} = -\lambda_{\bar{i}j}$, for $k \neq j$ for all $i, j, k, l \in \{1, \dots, n\}$. This allowed us to prove the existence of a trapping region containing the heteroclinic network. An analogous

resonance occurs for homoclinic orbits to saddle points in conservative flows, $\lambda^s = -\lambda^u$, see Homburg and Knobloch [21] and Vanderbauwhede and Fiedler [39]. We can ask if the same can occur for the replicator dynamics. More specifically, we ask if there can be a resonance on the eigenvalues and, if so, if that happens only in the conservative case. By Theorem 13, in the Appendix B, the answer to both questions is positive. We conjecture that every heteroclinic connection in a volume preserving flow satisfies the resonance relation $\lambda^s = -\lambda^u$, with λ^s and λ^u the stable and unstable eigenvalues associated with the connection.

Here we have analysed the existence of complicated behaviour, more specifically switching, in the neighbourhood of an edge network in $S_n \times S_m$, for bimatrix game dynamics, which is conservative. In [21], Homburg and Knobloch show the existence of hyperbolic suspended Smale horseshoes near homoclinic belts (union of two homoclinic orbits to the same saddle point) in conservative systems. We can ask about special behaviour near an edge network in S_n when the flow is conservative.

We have performed some numerical simulations for replicator dynamics in the 4-dimensional simplex S_4 in \mathcal{R}^5 . In the case of no resonance on the eigenvalues, the trajectories tend to follow one of the heteroclinic cycles, as in the Kirk and Silber case [26]. In the resonance case, the observed behaviour resembles the situation in Postlethwaite and Dawes [32] with the trajectories following the heteroclinic network in a ‘random’ way. We can identify the edge heteroclinic network as the union of 5 heteroclinic cycles. It seems that trajectories tend to loop around those cycles. For some parameter values there are initial conditions such that trajectories loop around the cycles in a sequence and it seems that the number of loops around each cycle is always the same. Other trajectories seem to visit the cycles in a more irregular way.

In [32], Postlethwaite and Dawes analyse the behaviour near a heteroclinic network in a topological globally attracting invariant five-dimensional sphere in \mathcal{R}^6 . The network has 6 equilibria, each with two incoming and two outgoing connections, as in the RSP-game. The difference is that in the heteroclinic network in the RSP-game there are 9 equilibria, and so, no Kirk&Silber subnetwork. In [32], for some parameter values they observe what they call ‘regular cycling’ and ‘irregular cycling’. In both cases, a trajectory follows three cycles in the network in a sequential way. But, in the case of regular cycling, there is the same number n of turns around each cycle, whereas, in the irregular cycling, that number of turns varies in an irregular way. We note that, each two cycles in the network that are visited sequentially have an heteroclinic connection in common.

The key ingredient for (infinite) switching in bimatrix games seems to be the existence of an edge heteroclinic network without a Kirk&Silber subnetwork in an incompressible flow. We conjecture that this should apply to more general settings.

Acknowledgments

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A Neutral stability of edge heteroclinic networks in bimatrix games

Let $\lambda_{\bar{l}j}$ be the eigenvalue of the equilibria ξ_{ij} in the direction of ξ_{lj} and $\lambda_{i\bar{j}k}$ the eigenvalue of the equilibria ξ_{ij} in the direction of ξ_{ik} . By the proof of Proposition 7, the eigenvalues at ξ_{ij} satisfy $\lambda_{\bar{l}j} = -\lambda_{\bar{l}i}$, for $i \neq l$ and $\lambda_{i\bar{j}k} = -\lambda_{i\bar{k}j}$, for $k \neq j$ for all $i, l \in \{1, \dots, n\}$ and $j, k \in \{1, \dots, m\}$. We have then that, for each hyperplane the sum of the expanding eigenvalues of the equilibria in the network contained in it equals the sum of the expanding eigenvalues. Thus, on average the contraction around the network equals the expansion, and so, we conjecture the heteroclinic network to be neutrally stable.

To support this conjecture, we rely on the work of Hofbauer in [17] making use of the notion of an average Lyapunov function and of the characteristic matrix C of an edge heteroclinic network. Roughly speaking, an average Lyapunov function is a weaker version of a Lyapunov function in which the time average behaves as a Lyapunov function. For an account of average Lyapunov functions, see Hofbauer [16], Hutson [23], Jansen [24], Law and Blackford [30], and Vance [38].

Following [17], we use the average ‘Lyapunov’ function

$$P(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n+1} (x_i)^{p_i} \prod_{j=1}^{m+1} (y_j)^{p_{n+1+j}},$$

with $p_k \in \mathcal{R}$, $k = 1, \dots, n + m + 2$.

The function P satisfies

$$P(\mathbf{x}, \mathbf{y}) = 0 \text{ if } (\mathbf{x}, \mathbf{y}) \in \text{bd}S_n \times S_m$$

$$P(\mathbf{x}, \mathbf{y}) > 0 \text{ if } (\mathbf{x}, \mathbf{y}) \in \text{int}S_n \times S_m.$$

We want to analyse the sign of \dot{P} close to the boundary of $S_n \times S_m$. We have

$$\frac{\dot{P}}{P} = \sum_{i=1}^{n+1} p_i \frac{\dot{x}_i}{x_i} + \sum_{j=1}^{m+1} p_{n+1+j} \frac{\dot{y}_j}{y_j}$$

thus, $\dot{P}(\mathbf{x}, \mathbf{y}) = P(\mathbf{x}, \mathbf{y})\Psi(\mathbf{x}, \mathbf{y})$ with Ψ a continuous function, and so, it is enough to analyse the sign of Ψ at $\text{bd}S_n \times S_m$. Moreover, we reduce the analysis to the equilibria in the heteroclinic network, as trajectories near the network spend most of the time near the equilibria.

We now make use of the characteristic matrix C of the edge network which is a matrix with rows indexed by the equilibria in the network and the columns indexed by the boundary hyperplanes of $S_n \times S_m$. The entry in row k column j is the eigenvalue of the equilibria corresponding to row k in the eigendirection corresponding to column j . For each equilibrium the ‘radial’ eigenvalues are zero and so there are two zero entries in each row. Moreover, due to the resonance on the eigenvalues of the equilibria at the edge network, the elements of each column can be associated in pairs such that the two elements of each pair have symmetric value.

We have that \dot{P}/P takes the (average) value $(Cp)_k$ at each equilibrium ξ_k , $k = (n+1) \times (m+1)$. Since the sum of the elements of each column of the characteristic matrix C is zero, it follows that $\sum_{k=1}^{(n+1) \times (m+1)} (Cp)_k = 0$ and so, intuitively it seems plausible to expect P to be constant (in the average) near the heteroclinic network and so the network to be neutrally stable.

B Conservative Replicator Dynamics

Theorem 13 *Up to a change in velocity, the flow of (1) in $\text{int}S_n$ is incompressible if and only if $\lambda_{ij} = -\lambda_{ji}$ for $i \neq j$.*

The proof of the theorem is given by the following two lemmas.

Lemma 14 *Up to a change in velocity, the flow of (1) in $\text{int}S_n$ is incompressible if and only if $a_{ij} + a_{ji} = a_{ii} + a_{jj}$ for all $i, j \in \{1, \dots, n\}$.*

Proof We follow the proof of Theorem 11.3.1 in [19]. The idea of the proof is to use Liouville’s formula to determine the conditions for $\dot{V}(t)$ to be zero in the interior of the simplex S_n . This is equivalent to find the conditions for the divergence of the vector field (1) in the interior of the simplex S_n to be zero.

We start by dividing the vector field (1) by the positive function $P = \prod_{i=1}^n x_i$ getting the modified vector field

$$\dot{x}_j = \frac{1}{\prod_{i=1 \wedge i \neq j}^n x_i} \left[\sum_{i=1}^n a_{ji} x_i - \mathbf{x}^T A \mathbf{x} \right]. \quad (10)$$

The divergence of (10) in $\text{int}\mathcal{R}_n$ is given by

$$\begin{aligned}
div &= \sum_{j=1}^n \frac{\partial \dot{x}_j}{\partial x_j} = \\
&= \frac{1}{\prod_{i=1}^n x_i} \left[\sum_{j=1}^n a_{jj} x_j - \sum_{j=1}^n x_j \sum_{i=1}^n (a_{ji} + a_{ij}) x_i \right] = \\
&= \frac{1}{\prod_{i=1}^n x_i} \left[\sum_{j=1}^n a_{jj} x_j - \sum_{j=1}^n x_j \sum_{i=1}^n a_{ji} x_i - \sum_{i=1}^n x_i \sum_{j=1}^n a_{ij} x_j \right] = \\
&= \frac{1}{\prod_{i=1}^n x_i} \left[\sum_{j=1}^n a_{jj} x_j - 2\mathbf{x}^T \mathbf{A} \mathbf{x} \right]
\end{aligned}$$

In order to obtain the divergence div_0 within the state space $\text{int}S_n$, we have to subtract the eigenvalues of the Jacobian which are orthogonal to S_n . Since

$$\left(\sum x_j \right)' = \left(1 - \sum x_j \right) \mathbf{x}^T \mathbf{A} \mathbf{x},$$

the superfluous eigenvalues are $-\mathbf{x}^T \mathbf{A} \mathbf{x}$.

Taking into account the factor $1/P$, we have

$$\begin{aligned}
div_0 &= \frac{1}{P} \left[\sum_{j=1}^n a_{jj} x_j - \sum_{j=1}^n x_j \sum_{i=1}^n a_{ji} x_i \right] = \\
&= \frac{1}{P} \left[\sum_{j=1}^{n-1} (a_{jj} - a_{nn}) x_j + a_{nn} - \sum_{j=1}^{n-1} \left[\sum_{i=1}^{n-1} (a_{ji} - a_{jn}) x_i + a_{jn} \right] x_j - \right. \\
&\quad \left. - \left[\sum_{j=1}^{n-1} (a_{nj} - a_{nn}) x_j + a_{nn} \right] \left(1 - \sum_{i=1}^{n-1} x_i \right) \right]
\end{aligned}$$

since $x_n = 1 - \sum_{i=1}^{n-1} x_i$.

Rewriting the expression of div_0 , we have

$$\begin{aligned}
div_0 &= \frac{1}{P} \left[\sum_{j=1}^{n-1} (a_{jj} - a_{jn} - a_{nj} + a_{nn}) x_j + \sum_{j=1}^{n-1} \left[\sum_{i=1}^{n-1} (-a_{ji} + a_{jn} + a_{ni} - a_{nn}) x_i \right] x_j \right] = \\
&= \frac{1}{P} \left[\sum_{j=1}^{n-1} ((a_{jj} - a_{jn} - a_{nj} + a_{nn})(1 - x_j)) x_j + \right. \\
&\quad \left. + \sum_{j=1}^{n-1} \left[\sum_{i=j+1}^{n-1} (-a_{ji} + a_{jn} + a_{ni} - a_{nn} - a_{ij} + a_{in} + a_{nj} - a_{nn}) \right] x_i x_j \right]
\end{aligned}$$

Thus, $div_0 \equiv 0$ if and only if

$$\begin{cases} (a_{jj} - a_{jn} - a_{nj} + a_{nn}) = 0 \\ -a_{ji} + a_{jn} + a_{ni} - a_{nn} - a_{ij} + a_{in} + a_{nj} - a_{nn} = 0 \end{cases}$$

for all $j = 1, \dots, n$ and $i = 1, \dots, n-1$

That is, if and only if $a_{ij} + a_{ji} = a_{ii} + a_{jj}$ for all $i, j \in \{1, \dots, n\}$. \square

Lemma 15 *The eigenvalues of the equilibria at the vertices satisfy $\lambda_{ij} = -\lambda_{ji}$, for $i \neq j$, if and only if $a_{ij} + a_{ji} = a_{ii} + a_{jj}$ for all $i, j \in \{1, \dots, n\}$.*

Proof The partial derivatives of (1) are given by

$$\begin{aligned}
\frac{\partial \dot{x}_j}{\partial x_j} &= \sum_{i=1}^n a_{ji} x_i - \mathbf{x}^T \mathbf{A} \mathbf{x} + x_j \left(a_{jj} - \frac{\partial}{\partial x_j} \mathbf{x}^T \mathbf{A} \mathbf{x} \right) \\
\frac{\partial \dot{x}_j}{\partial x_k} &= a_{jk} x_j - x_j \frac{\partial}{\partial x_j} \mathbf{x}^T \mathbf{A} \mathbf{x}, \text{ for } k \neq j
\end{aligned}$$

with $\frac{\partial}{\partial x_j} \mathbf{x}^T \mathbf{A} \mathbf{x} = \sum_{i=1}^n (a_{ji} + a_{ij}) x_i$.

The Jacobian matrix at a vertice equilibria e_i is given by

$$\begin{bmatrix} a_{1i} - a_{ii} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & a_{2i} - a_{ii} & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{i-1i} - a_{ii} & 0 & 0 & \dots & 0 \\ -a_{1i} & -a_{2i} & \dots & -a_{i-1i} & -a_{ii} & -a_{i+1i} & \dots & -a_{ni} \\ 0 & 0 & \dots & 0 & 0 & a_{i+1i} - a_{ii} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_{ni} - a_{ii} \end{bmatrix}$$

Thus, $\lambda_{ij} = a_{ji} - a_{ii}$ for all $i \neq j$ and so $\lambda_{ij} = -\lambda_{ji} \Leftrightarrow a_{ij} + a_{ji} = a_{ii} + a_{jj}$ □