Simple Vector Fields with Complex Behaviour

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Abstract. We construct examples of vector fields on a three-sphere, amenable to analytic proof of properties that guarantee the existence of complex behaviour.

The examples are restrictions of symmetric polynomial vector fields in $\mathbf{R}^4$ and possess heteroclinic networks producing switching and nearby suspended horseshoes.

The heteroclinic networks in our examples are persistent under symmetry preserving perturbations.

We prove that some of the connections in the networks are the transverse intersection of invariant manifolds. The remaining connections are symmetry-induced.

The networks lie in an invariant three-sphere and may involve connections exclusively between equilibria or between equilibria and periodic trajectories.

The same construction technique may be applied to obtain other examples with similar features.

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1. Introduction

Although chaotic dynamics is known to be a prevalent feature of dynamical systems, there are not many examples in the literature where a chaotic invariant set can be obtained analytically. In this paper we construct examples that, although exhibiting complicated behaviour, are sufficiently simple to be treated analytically.

Simple systems may be used as prototypes for partial behaviour in more complicated ones. For example, in dynamical systems equivariant by the action of a symmetry group, it is natural to reduce the study of the original problem to its restriction to the quotient space by part of the group action (see for instance, [Aguiar *et al.*, 2005] and also [Chossat, 2002] and references therein).

A lot of attention has been given recently to heteroclinic networks, both in their own right and as the cause of complicated nearby dynamics. In this paper we construct examples of polynomial vector fields with heteroclinic networks that originate chaotic dynamics around them. These networks have some connections arising through the transverse intersection of invariant manifolds. Analytical proof of this type of property is usually difficult but can be achieved in our examples. Our examples exhibit networks with connections between equilibria and between equilibria and a periodic trajectory. An example with a connection involving uniquely limit cycles may be found in ([Field, 1996], example 7.2).

The complicated dynamics in our examples arises in two ways [Aguiar *et al.*, 2005]. First, transverse intersection of two-dimensional manifolds, together with equilibria with complex eigenvalues, ensures the existence of switching on the network: every sequence of connections in the network can be shadowed by nearby trajectories of the flow. Second, through heteroclinic cycles and complex eigenvalues and, near these cycles, through suspended horseshoes. Dimension three is the lowest compatible with this type of dynamical behaviour. We work in $\mathbf{R}^4$, on an invariant three-sphere, obtained explicitly for each example. Hence, we can use the compactness of the three-sphere to simplify both the analytical proofs and the numerical study.

We use a construction technique that relies heavily on symmetry and may be used to obtain examples with different features, thus providing a set of tools for the construction of symmetric vector fields with prescribed properties. The use of symmetry is, by no means, a handicap as persistence of heteroclinic phenomena is natural in a symmetric setting and not in the absence of symmetry. Furthermore, the dynamics near the heteroclinic network will persist under small symmetry-breaking perturbations, even if the network itself disappears. However, proving the existence of complex behaviour in cases where too much symmetry is broken may require the use of aditional tools such as, for instance, the computation of Lyapunov exponents. We conclude this section with some preliminary definitions. Our construction is described in generic terms in Sec. 2 and applied to the construction of specific examples in Secs. 4 and 5. Section 3 contains several results concerning how properties of a $\mathbf{Z}_2$ symmetric vector field in $\mathbf{R}^3$ are reflected in properties of the lifted vector field in $\mathbf{R}^4$. These results will be essential for proving that the examples in the following sections possess the desired features.

Preliminaries. Let Γ be a compact Lie group acting linearly on $\mathbf{R}^n$ and $\mathbf{X}$ a Γ-equivariant vector field on $\mathbf{R}^n$. A *relative equilibrium* is a Γ-orbit, $\Gamma(x_0) = \{\gamma.x_0 \in \mathbf{R}^n : \ \gamma \in \Gamma\}$, that is invariant by the flow of $\mathbf{X}$. If the group Γ is finite then relative equilibria are finite sets of equilibria.

Let A be a compact invariant set for the flow of $\mathbf{X}$. Following Field [1996] we say that A is an *invariant saddle* if both $\overline{W^s(A) \setminus A}$ and $\overline{W^u(A) \setminus A}$ contain A. Notice that invariant saddles do not have to be hyperbolic. In our examples they are hyperbolic (relative) equilibria. We distinguish saddles which have complex eigenvalues and call them *saddle-foci*.

Given two invariant saddles A and B, a k-dimensional connection from A to B, $[A \to B]$, is a k-dimensional $\mathbf{X}$-invariant connected manifold contained in $W^u(A) \cap W^s(B)$. The connection is *heteroclinic* if $A \neq B$.

Let $\{A_i, i = 0, \ldots, n-1\}$ be a finite ordered set of mutually disjoint invariant saddles for the vector field $\mathbf{X}$. If there is a connection $[A_i \to A_{i+1}]$ for each $i = 0, \ldots, n-1 \pmod n$ then we say that

$$\bigcup_{i=0}^{n-1} A_i \cup [A_i \to A_{i+1}]$$

is a *heteroclinic cycle* with invariant saddles $\{A_i\}$.

We think of a *heteroclinic network* as a finite union of heteroclinic cycles. The saddles defining the heteroclinic cycles and network are called *nodes* of the network.

Denote by $\mathbf{S}^n_r = \{X \in \mathbf{R}^{n+1} : |X| = r\}$, $r \geq 0$, the n-dimensional sphere of radius r.

If $\mathbf{S}^n_r$ is flow invariant, we say it is *globally attracting* if every trajectory with nonzero initial condition is asymptotic to $\mathbf{S}^n_r$ in forward time.

2. Heuristics of the Construction

Our aim is to construct examples of polynomial vector fields $\mathbf{X}$ on $\mathbf{R}^4$ with the following properties:

- $\mathbf{X}$ is equivariant for some discrete subgroup of $O(4)$.
- There is an invariant globally attracting three-sphere preserved by the group action.

- On the invariant sphere there is a heteroclinic network whose nodes are either equilibria or closed trajectories of $\mathbf{X}$.
- The connections in the network are one-dimensional and of two types:
  - (c1) intersection of the invariant sphere with a two-dimensional fixed-point subspace,
  - (c2) transverse intersection of invariant manifolds.

Examples are constructed in three essential steps:

(1) Construction of a $\mathbf{Z}_2$-equivariant vector field $\mathbf{X}_3$ on $\mathbf{R}^3$ with an attracting two-sphere and a heteroclinic network of one-dimensional connections of type (c1) on the sphere. The $\mathbf{Z}_2$-equivariance is needed for the next step.

(2) Construction of an $SO(2)$-equivariant vector field $\mathbf{X}_4$ on $\mathbf{R}^4$ — by a rotation of $\mathbf{X}_3$ — with a globally attracting three-sphere and a heteroclinic network on this sphere. Some of the heteroclinic connections will be two-dimensional and typically non-transverse.

(3) Perturbation of $\mathbf{X}_4$ to $\mathbf{X}_4^p$, by adding terms that destroy the $SO(2)$-symmetry while preserving the invariant three-sphere. The symmetry-breaking terms are chosen so as to perturb the non-transverse two-dimensional connections into transverse intersections of invariant manifolds.

This construction is loosely inspired by [Swift, 1988] and initially suggested by Mike Field.

Step 1. Consider the $\mathbf{Z}_2$ action on $\mathbf{R}^3$ that keeps a two-dimensional vector subspace fixed. In suitable coordinates, this action is given by:

$$k \cdot (\rho, z, w) = (-\rho, z, w). \tag{1}$$

We denote this representation by $\mathbf{Z}_2(k)$.

We want the $\mathbf{Z}_2(k)$-equivariant vector field $\mathbf{X}_3$ to have an invariant two-sphere $\mathbf{S}^2_r$ and, on this sphere, heteroclinic connections between relative equilibria. This is easily achieved if $\mathbf{X}_3$ has more symmetry than the minimal $\mathbf{Z}_2(k)$-equivariance needed for lifting it to $\mathbf{R}^4$. Symmetry provides natural flow-invariant subspaces (fixed-point spaces) where connections are easy to find, especially if there is an invariant two-sphere.

Start with the vector field $\mathbf{X}_0(X) = (r^2 - |X|^2)X$ for $X = (\rho, z, w) \in \mathbf{R}^3$, $r > 0$. Then $\mathbf{X}_0$ is equivariant under the standard $O(3)$ action on $\mathbf{R}^3$ and the sphere $\mathbf{S}^2_r$ is invariant and globally attracting.

Now choose a finite subgroup Γ of $O(3)$ with the following properties:

- there is an element of Γ that acts as k in (1),
- there are at least two isotropy subgroups of Γ with two-dimensional fixed-point spaces.

Among the Γ-equivariant polynomial vector fields choose those that are tangent to $\mathbf{S}^2_r$. Perturb $\mathbf{X}_0$ by adding some of these to obtain a Γ-equivariant vector field $\mathbf{X}_3$.

At this stage the vector field $\mathbf{X}_3$ possesses an invariant sphere $\mathbf{S}^2_r$, and from the symmetry, flow-invariant planes and invariant lines where any two planes meet. These subspaces meet $\mathbf{S}^2_r$ at pairs of equilibria and arcs connecting them. The number and location of other equilibria on $\mathbf{S}^2_r$ can be controlled, by a suitable choice of the perturbation terms, to obtain a vector field with a made-to-order heteroclinic cycle or network on the two-sphere.

Step 2. The vector field $\mathbf{X}_4$ on $\mathbf{R}^4$ is obtained by adding the auxiliary equation $\dot{\varphi} = 1$ and interpreting the coordinates (ρ, φ) as polar coordinates.

The lifted vector field $\mathbf{X}_4$ is $SO(2)$-equivariant for the action given by a phase shift $\varphi \mapsto \varphi + \psi$ in the angular coordinate φ. In rectangular coordinates (x, y, z, w) on $\mathbf{R}^4$, with $x = \rho \cos \varphi$ and $y = \rho \sin \varphi$, the action is

$$\psi \cdot (x, y, z, w) = (x \cos \psi - y \sin \psi, x \sin \psi + y \cos \psi, z, w).$$

Because of the $\mathbf{Z}_2(k)$-equivariance, $\mathbf{X}_3$ has the form

$$\begin{array}{rcl} \dot{\rho} & = & \rho f_1(\rho^2, z, w), \\ \dot{z} & = & f_2(\rho^2, z, w), \\ \dot{w} & = & f_3(\rho^2, z, w), \end{array}$$

with $f_j : \mathbf{R}^3 \to \mathbf{R}$, $j = 1, 2, 3$, and it lifts by rotation to a vector field $\mathbf{X}_4$ of the form,

$$\begin{array}{rcl} \dot{x} & = & x f_1(x^2 + y^2, z, w) - y, \\ \dot{y} & = & y f_1(x^2 + y^2, z, w) + x, \\ \dot{z} & = & f_2(x^2 + y^2, z, w), \\ \dot{w} & = & f_3(x^2 + y^2, z, w). \end{array}$$

The original vector field $\mathbf{X}_3$ may be recovered from the last three equations of $\mathbf{X}_4$ by taking $x = 0$ and $y = \rho$. The $\mathbf{Z}_2(k)$ symmetry is essential to guarantee that the rotation, and therefore $\mathbf{X}_4$, are well-defined.

The result is a vector field $\mathbf{X}_4$ with $SO(2)$-symmetry coming from k, plus extra symmetries inherited from other elements of the group Γ.

The rotated vector field $\mathbf{X}_4$ will have, arising from its extra symmetries, at least one two-dimensional connection between two of its relative equilibria: these connections are the intersection of the invariant sphere with invariant hyperplanes that are fixed-point spaces, and are non-transverse intersections of the stable and unstable manifolds of the relative equilibria.

Step 3. Perturb $\mathbf{X}_4$ by adding a polynomial vector field that breaks some of the extra symmetry and is tangent to $\mathbf{S}^3_r$. The aim is to break the two-dimensional heteroclinic connections into transverse intersections while preserving the invariance of $\mathbf{S}^3_r$.

Suppose the perturbing terms do not affect the equation for φ ($\dot{\varphi} = 1$). Then the dynamics of the perturbed vector field $\mathbf{X}_4^p$ is described by a time-dependent vector field $\mathbf{X}_3^r$ on $\mathbf{R}^3$ obtained by integrating the equation for φ and replacing $\varphi(t) = t$ in the remaining equations. We call $\mathbf{X}_3^r$ the reduced vector field. Note that $\mathbf{X}_3^r$ is a time-dependent perturbation of $\mathbf{X}_3$.

We show that the perturbed heteroclinic connections correspond to transverse intersection of invariant manifolds by applying a generalization of Melnikov's method (see [Bertozzi, 1988]) to $\mathbf{X}_3^r$. The transversality of the intersection is preserved by the lift to $\mathbf{X}_4^p$.

3. Lifting $\mathbf{Z}_2$-equivariant Fields

We summarize some properties of $\mathbf{X}_3$ that, with the construction of step 2, lift to properties of $\mathbf{X}_4$. We address, in particular, the relationship between various flow-invariant sets of $\mathbf{X}_3$ and $\mathbf{X}_4$.

Given $\Sigma \subset \mathbf{R}^3$, define its *lift by rotation* $\mathcal{L}(\Sigma) \subset \mathbf{R}^4$ to be the set of points (x, y, z, w) such that either (ρ, z, w) or $(-\rho, z, w)$ lies in Σ, where $\rho = ||(x, y)||$. If Σ is $\mathbf{Z}_2(k)$-invariant then $\mathcal{L}(\Sigma)$ is the set of points (x, y, z, w) such that (ρ, z, w) lies in Σ.

Consider the inclusion map $i : \mathbf{R}^3 \to \mathbf{R}^4$, say $i(\rho, z, w) = (\rho, 0, z, w)$. Then $\mathcal{L}(\Sigma)$ is the $SO(2)$-orbit of $i(\Sigma)$.

With this notation it follows:

Proposition 1. *Let $\mathbf{X}_3$ be a $\mathbf{Z}_2(k)$-equivariant vector field on $\mathbf{R}^3$ and $\mathbf{X}_4$ its lift to $\mathbf{R}^4$ by rotation. If $\Sigma \subset \mathbf{R}^3$ is invariant by the flow of $\mathbf{X}_3$, then $\mathcal{L}(\Sigma)$ is invariant by the flow of $\mathbf{X}_4$. In particular, if p_0 is an equilibrium of $\mathbf{X}_3$ then $\mathcal{L}(\{p_0\})$ is a relative equilibrium of $\mathbf{X}_4$.*

Proof: Any $\mathbf{X}_3$-invariant set is the union of $\mathbf{X}_3$-trajectories, so we only need to prove the result for the case when Σ is a trajectory of $\mathbf{X}_3$.

For a point $p = (0, z, w)$ in $Fix(\mathbf{Z}_2(k))$, $\mathcal{L}(\{p\}) = \{i(p)\}$. Hence, if the $\mathbf{X}_3$-trajectory Σ meets $Fix(\mathbf{Z}_2(k))$ then $\Sigma \subset Fix(\mathbf{Z}_2(k))$ by equivariance, and $\mathcal{L}(\Sigma) = i(\Sigma)$ is a $\mathbf{X}_4$-trajectory. In particular, if $p_0 \in Fix(\mathbf{Z}_2(k))$ is an equilibrium then $\mathcal{L}(\{p_0\})$ is an equilibrium of $\mathbf{X}_4$.

If the trajectory Σ does not meet $Fix(\mathbf{Z}_2(k))$ and $p \in \Sigma$ then each point in $\mathcal{L}(\{p\})$ lies in the $\mathbf{X}_4$-trajectory of another point of $i(\Sigma)$. In particular, if $\Sigma = \{p_0\}$ is an equilibrium then $\mathcal{L}(\{p_0\})$ is a closed trajectory, a relative equilibrium of $\mathbf{X}_4$. □

Corollary 2. *Let $\mathbf{X}_3$ be a $\mathbf{Z}_2(k)$-equivariant vector field on $\mathbf{R}^3$ and $\mathbf{X}_4$ its lift to $\mathbf{R}^4$ by rotation. Trajectories connecting relative equilibria of $\mathbf{X}_3$ lift to invariant manifolds connecting relative equilibria of $\mathbf{X}_4$. In particular, if p_0 and p_1 are equilibria of $\mathbf{X}_3$ and connected by a trajectory ξ, then:*

(1) *If ξ lies in $Fix(\mathbf{Z}_2(k))$, then p_0 and p_1 also lie in $Fix(\mathbf{Z}_2(k))$ and ξ lifts to a trajectory connecting the two equilibria $i(p_0)$ and $i(p_1)$ of $\mathbf{X}_4$.*

(2) *If ξ lies outside $Fix(\mathbf{Z}_2(k))$, then ξ lifts to a two dimensional connection of relative equilibria of $\mathbf{X}_4$ — note that here p_0 and p_1 may either lift to equilibria or to closed trajectories.*

Proposition 3. *Let $\mathbf{X}_3$ be a $\mathbf{Z}_2(k)$-equivariant vector field on $\mathbf{R}^3$ and $\mathbf{X}_4$ its lift to $\mathbf{R}^4$ by rotation. If Σ is a compact $\mathbf{X}_3$-invariant asymptotically stable set then $\mathcal{L}(\Sigma)$ is a compact $\mathbf{X}_4$-invariant asymptotically stable set.*

Proof: Compactness of $\mathcal{L}(\Sigma)$ follows from compactness of Σ and of $SO(2)$.

We have by hypothesis that there is a neighbourhood $\mathcal{V}$ of Σ such that for $\tilde{\mathcal{V}} \subset \mathcal{V}$ the forward trajectory of $p \in \tilde{\mathcal{V}}$ by the flow of $\mathbf{X}_3$ is contained in $\mathcal{V}$ and $\omega(p) = \Sigma$. Thus, the forward trajectories of points in $\mathcal{L}(\tilde{\mathcal{V}})$ are contained in $\mathcal{L}(\mathcal{V})$ and have $\mathcal{L}(\Sigma)$ as ω-limit set, proving the asymptotic stability of $\mathcal{L}(\Sigma)$. □

Corollary 4. *Let $\mathbf{X}_3$ be a $\mathbf{Z}_2(k)$-equivariant vector field on $\mathbf{R}^3$ and $\mathbf{X}_4$ its lift to $\mathbf{R}^4$ by rotation. If $\mathbf{S}^2_r$ is an $\mathbf{X}_3$-invariant globally attracting sphere then $\mathcal{L}(\mathbf{S}^2_r) = \mathbf{S}^3_r$ is an $\mathbf{X}_4$-invariant globally attracting sphere.*

The result follows by propositions 1 and 3 and by observing that if in the proof of proposition 3 the set $\mathbf{S}^2_r$ is globally attracting then $\mathcal{V}$ and $\tilde{\mathcal{V}}$ may be chosen to be $\mathbf{R}^3 \setminus \{0\}$.

Proposition 5. *Let $\mathbf{X}_3$ be a $\mathbf{Z}_2(k)$-equivariant vector field on $\mathbf{R}^3$ and $\mathbf{X}_4$ its lift to $\mathbf{R}^4$ by rotation. Let p_0 be a hyperbolic equilibrium of $\mathbf{X}_3$. Then $\mathcal{L}(\{p_0\})$ is also hyperbolic.*

Proof: The result follows by [Krupa, 1990] where it is shown that near relative equilibria the vector field can be decomposed as the sum of two equivariant vector fields: one tangent and the other normal to the group orbit. The asymptotic dynamics of the vector field is determined by the asymptotic dynamics of the normal vector field modulo drifts along the group orbit. Hence, hyperbolicity of an equilibrium p_0 of $\mathbf{X}_3$ implies hyperbolicity of the relative equilibrium $\mathcal{L}(\{p_0\})$ of $\mathbf{X}^4$. □

If $p_0 \in Fix(\mathbf{Z}_2(k))$ then, by proposition 1, $\mathcal{L}(\{p_0\}) = \{i(p_0)\}$ is an equilibrium of $\mathbf{X}_4$. Let $d\mathbf{X}_4(i(p_0))$ and $d\mathbf{X}_3(p_0)$ be the linearizations of $\mathbf{X}_4$ and $\mathbf{X}_3$ at $i(p_0)$ and p_0, respectively. The restrictions of $d\mathbf{X}_4(i(p_0))$ and $d\mathbf{X}_3(p_0)$ to $Fix(\mathbf{Z}_2(k))$ have the same eigenvalues. In the complementary plane, $d\mathbf{X}_4(i(p_0))$ has a pair of complex eigenvalues with real part given by the remaining eigenvalue of $d\mathbf{X}_3(p_0)$.

Remark 6. (a) *The $SO(2)$-orbit of any $\mathbf{X}_4$-invariant set is always the lift of an $\mathbf{X}_3$-invariant set. In particular, any $SO(2)$-relative equilibrium of $\mathbf{X}_4$ is the lift of an equilibrium of $\mathbf{X}_3$.*

(b) *Any $\mathbf{X}_4$-heteroclinic connection of relative equilibria is the lift of an $\mathbf{X}_3$-heteroclinic connection of equilibria. This lift is the union of one-dimensional heteroclinic connections of the same relative equilibria.*

4. Heteroclinic Network Between Two Saddle-foci

In this section we apply the heuristics of Sec. 2 to obtain a vector field on $\mathbf{R}^4$ with a structurally stable heteroclinic network involving two saddle points. These points have a pair of complex eigenvalues and their invariant manifolds of dimension ≥ 2 intersect transversely.

From the results in [Aguiar, 2003], [Aguiar *et al.*, 2005] it follows that arbitrarily close to this network there is a suspended horseshoe. It also follows from [Aguiar, 2003] that there is switching on this network.

Step 1: Example on $\mathbf{R}^3$. Let $\Gamma \subset O(3)$ be the group of order 8 generated by:

$$\begin{array}{rcl} d(\rho, z, w) & = & (\rho, -z, w), \\ q(\rho, z, w) & = & (-z, \rho, -w), \end{array}$$

of orders 2 and 4, respectively, with $k = dq^2$ acting as in (1).

The subgroups $\mathbf{Z}_2(d)$ and $\mathbf{Z}_2(k) = \mathbf{Z}_2(dq^2)$ have two-dimensional fixed-point spaces,

$$Fix(\mathbf{Z}_2(d)) = \{(\rho, z, w) : z = 0\}$$

and

$$Fix(\mathbf{Z}_2(dq^2)) = \{(\rho, z, w) : \rho = 0\}.$$

The other fixed-point spaces are

$$Fix(\mathbf{Z}_4(d, q^2)) = \{(\rho, z, w) : \rho = 0,\ z = 0\},$$

$$Fix(\mathbf{Z}_2(dq^3)) = \{(\rho, z, w) : \rho = z,\ w = 0\}$$

and

$$Fix(\mathbf{Z}_2(dq)) = \{(\rho, z, w) : \rho = -z,\ w = 0\}.$$

The next theorem shows that perturbing $\mathbf{X}_0(X) = (r^2 - |X|^2)X$ with $\mathbf{S}^2_r$-preserving Γ-equivariant polynomials we obtain a family of vector fields $\mathbf{X}_3$ with phase portrait as in Fig. 1.

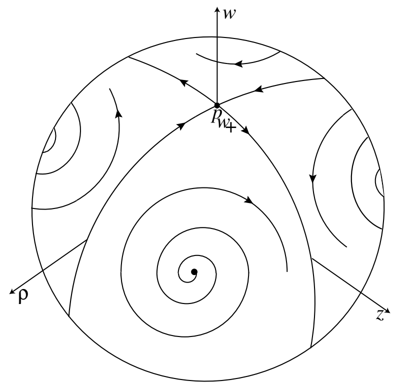

FIGURE 1. Dynamics of $\mathbf{X}_3$ on the invariant two-sphere $\mathbf{S}^2_r$.

Theorem 7. *Consider the* Γ*-equivariant vector field* $\mathbf{X}_3$ *on* $\mathbf{R}^3$ *with equations given by*

$$\begin{array}{rcl} \dot{\rho} & = & \rho(\lambda - r^2) - \alpha\rho w + \beta\rho w^2, \\ \dot{z} & = & z(\lambda - r^2) + \alpha z w + \beta z w^2, \\ \dot{w} & = & w(\lambda - r^2) - \alpha(z^2 - \rho^2) - \beta w(\rho^2 + z^2), \end{array}$$

with $r^2 = \rho^2 + z^2 + w^2$.

For $\lambda > 0$, $\beta < 0$, $\lambda\beta^2 < 8\alpha^2$, *and* $|\lambda\beta| < |\alpha|\sqrt{\lambda}$ *the following assertions hold:*

(a) *The sphere* $\mathbf{S}^2_r$, *of radius* $r = \sqrt{\lambda}$, *is invariant by the flow of* $\mathbf{X}_3$ *and globally attracting.*
(b) *The North and South poles* $p_{w_\pm} = (0, 0, \pm r)$ *are hyperbolic saddles of* $\mathbf{X}_3$.
(c) *When restricted to the invariant sphere* $\mathbf{S}^2_r$, *the invariant manifolds of* p_{w_-} *and* p_{w_+} *satisfy* $W^s(p_{w_-}) = W^u(p_{w_+})$ *and* $W^s(p_{w_+}) = W^u(p_{w_-})$, *forming an asymptotically stable heteroclinic network with four connections between the saddles* $p_{w_\pm}$.
(d) *Besides* p_{w_-}, p_{w_+} *and the origin,* $\mathbf{X}_3$ *has four equilibria which are unstable foci on the restriction to* $\mathbf{S}^2_r$.
(e) *The vector field* $\mathbf{X}_3$ *has no compact limit sets other than the ones mentioned above.*

Proof: Both the Γ-equivariance and assertion (a) follow from the construction.

The equilibria in (b) and (d) are obtained by intersecting the one-dimensional fixed-point subspaces with the sphere. A direct computation shows that these are the only equilibria in $\mathbf{S}^2_r$.

At the equilibria $p_{w_\pm} = (0, 0, \pm r)$ the non-radial eigenvalues are $\left(\lambda\beta \pm \alpha\sqrt{\lambda}\right)$ and therefore they are hyperbolic saddles for the parameter values in the hypothesis. Their invariant manifolds meet $\mathbf{S}^2_r$ on its intersection with the two-dimensional fixed-point subspaces.

At the other four equilibria the non-radial eigenvalues are $\left(-\lambda\beta \pm \sqrt{\lambda^2\beta^2 - 8\lambda\alpha^2}\right)$, and thus, for the parameter values used, they are unstable foci on $\mathbf{S}^2_r$.

The stability of the network and assertion (e) follow from lemma 8 below. □

Lemma 8. *Under the conditions of Theorem 7, all points on $\mathbf{S}^2_r$, except the unstable foci, are forward asymptotic to the heteroclinic network.*

Proof: We prove the result for the invariant sector S given by $\rho \geq 0$, $z \leq 0$ on $\mathbf{S}^2_r$. The dynamics on the other three sectors is the same, due to the symmetry.

The Lie derivative of $f(\rho, z, w) = (\rho - z)^2 + w^2$ with respect to $\mathbf{X}_3$, on the invariant sphere, is:

$$L_{\mathbf{X}_3} f|_{\mathbf{S}^2_r} = -4\beta\rho z w^2.$$

For $\beta < 0$ we have $L_{\mathbf{X}_3} f \leq 0$ in S. Let M be the largest invariant set in S contained in $\{L_{\mathbf{X}_3} f = 0\}$. By La Salle's theorem ([La Salle & Lefschetz, 1961], Th VI, Chap 2, §13), every trajectory in S tends to M as $t \to \infty$.

Given that $L_{\mathbf{X}_3} f = 0$ for $\rho = 0$, $z = 0$ or $w = 0$, and that $\{\rho = 0\} \cup \{z = 0\}$ is the heteroclinic network, it remains to study the set $\{w = 0\}$. On $\mathbf{S}^2_r \cap \{w = 0\}$ the third coordinate of $\mathbf{X}_3$ is $\dot{w} = -\alpha(z^2 - \rho^2)$ and this is zero only for $z = -\rho$, the unstable focus. Thus, in the sector S, the ω-limit set is $M = \{(\rho, z, w) : \ \rho = 0 \wedge z = 0\} \cap S$. By symmetry, on $\mathbf{S}^2_r$, the ω-limit set is the heteroclinic network. □

Remark 9. *Since all Γ-equivariant polynomials of degree 3 tangent to $\mathbf{S}^2_r$ and satisfying the properties below are used in the construction of $\mathbf{X}_3$, any G-equivariant polynomial vector field of degree 3 on $\mathbf{R}^3$ with those properties is equivalent to $\mathbf{X}_3$ for some choice of parameters.*

Step 2: Example on $\mathbf{R}^4$. We use the procedure of Sec. 2 to lift the three-dimensional vector field $\mathbf{X}_3$ to a vector field $\mathbf{X}_4$ on $\mathbf{R}^4$. The expression for $\mathbf{X}_4$ is given in the next theorem.

The action of d on $\mathbf{R}^3$ induces the following action on $\mathbf{R}^4$

$$\sigma(x, y, z, w) \quad = \quad (x, y, -z, w).$$

The symmetry group of $\mathbf{X}_4$ (below) is isomorphic to $\mathbf{Z}_2(\sigma) \times SO(2)$, with the usual action of $SO(2)$ only in the first two coordinates.

Theorem 10. *Consider the $\mathbf{Z}_2(\sigma)\times SO(2)$-equivariant vector field $\mathbf{X}_4$ on $\mathbf{R}^4$ with equations given by*

$$\begin{array}{rcl}\dot{x} &=& x(\lambda-r^2)-\alpha xw+\beta xw^2-y,\\ \dot{y} &=& y(\lambda-r^2)-\alpha yw+\beta yw^2+x,\\ \dot{z} &=& z(\lambda-r^2)+\alpha zw+\beta zw^2,\\ \dot{w} &=& w(\lambda-r^2)-\alpha(z^2-x^2-y^2)-\beta w(x^2+y^2+z^2),\end{array}$$

with $r^2=x^2+y^2+z^2+w^2$.

For the parameter values in theorem 7 the vector field $\mathbf{X}_4$ satisfies

(C1) *There is a three-dimensional sphere, $\mathbf{S}^3_r$, that is invariant by the flow and globally attracting.*

(C2) *On the invariant three-sphere, $\mathbf{X}_4$ has an asymptotically stable heteroclinic network with two saddle-foci, p_{w_-}, p_{w_+}. The invariant manifolds of the equilibria satisfy, on the invariant sphere, $W^s(p_{w_-})=W^u(p_{w_+})$ and $W^s(p_{w_+})=W^u(p_{w_-})$. One of the connections is two-dimensional, the others are one-dimensional. The two-dimensional connection coincides with $\mathbf{D}-\{p_{w_-},p_{w_+}\}$, with $\mathbf{D}$ a two-dimensional sphere.*

(C3) *The vector field has no equilibria other than the origin, p_{w_-} and p_{w_+}.*

(C4) *The vector field has two hyperbolic periodic trajectories. On the invariant sphere $\mathbf{S}^3_r$ the periodic trajectories are repelling.*

Proof: The proof relies on the results in Sec. 3.

Assertion (C1) follows directly from corollary 4 and the existence of the invariant sphere on $\mathbf{R}^3$.

Assertion (C4) follows from the existence of the unstable foci on $\mathbf{S}^2_r$, noting they do not lie in $Fix(\mathbf{Z}_2(k))$. From propositions 1 and 5 it follows that each pair of unstable foci lifts to an unstable periodic trajectory.

As an immediate consequence of propositions 1 and 5 and assertion (b) in theorem 7, $p_{w_\pm}$ on $\mathbf{S}^3_r$ are saddle-foci. By remark 6(a) and assertion (e) in theorem 7 we obtain (C3).

Corollary 2, remark 6(b) and the existence of the heteroclinic network connecting the north and south poles of $\mathbf{S}^2_r$, prove the existence of a heteroclinic network on $\mathbf{S}^3_r$ also connecting the north and south poles.

Two of the four connections of the network on $\mathbf{S}^2_r$ lie in $Fix(\mathbf{Z}_2(k))$ and the other two do not. This creates a two-dimensional connection on the lifted network.

The asymptotic stability of the network on $\mathbf{S}^3_r$ follows from the asymptotic stability of the network on $\mathbf{S}^2_r$ and proposition 3. □

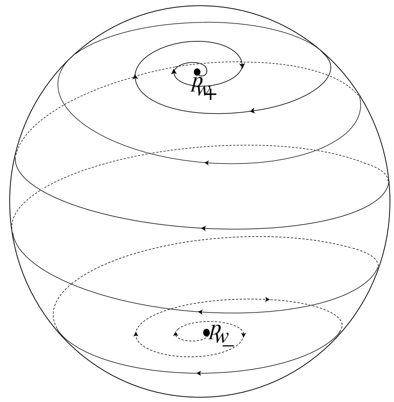

FIGURE 2. Invariant two-sphere $\mathbf{D}$ on $\mathbf{S}^3_r$ that coincides with the two-dimensional heteroclinic connection from p_{w_-} to p_{w_+}, when $\alpha > 0$. (If $\alpha < 0$ the arrows are reversed.)

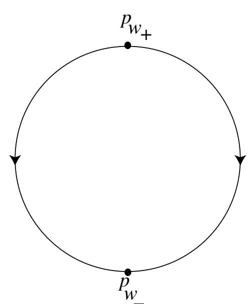

FIGURE 3. The one-dimensional connection from p_{w_+} to p_{w_-} on the intersection of $Fix(SO(2))$ and $\mathbf{S}^3_r$.

Step 3: Perturbation and transverse intersection of manifolds. We perturb $\mathbf{X}_4$ keeping $\mathbf{S}^3_r$ invariant while breaking the invariance of $\mathbf{D}$. The perturbed system $\mathbf{X}^p_4$ is:

$$\begin{array}{rcl} \dot{x} & = & x(\lambda - r^2) - \alpha xw + \beta xw^2 - y, \\ \dot{y} & = & y(\lambda - r^2) - \alpha yw + \beta yw^2 + x, \\ \dot{z} & = & z(\lambda - r^2) + \alpha zw + \beta zw^2 + \delta xw^2, \\ \dot{w} & = & w(\lambda - r^2) - \alpha(z^2 - x^2 - y^2) - \beta w(x^2 + y^2 + z^2) - \delta xzw, \end{array}$$

with $r^2 = x^2 + y^2 + z^2 + w^2$.

The perturbing term $(0, 0, xw^2, xzw)$ is tangent to $\mathbf{S}^3_r$, destroys the $SO(2)$-equivariance but still has the plane $P = \{(x, y, z, w) : \ x = y = 0\}$ as a fixed-point subspace (for the remaining action of the rotation by π). This guarantees the persistence of the one-dimensional connections between the equilibria $p_{w_\pm}$.

There are no perturbing terms in the x and y components of the vector field, the perturbing terms in the z and w components are zero when $w = 0$; this simplifies computations.

Theorem 11. *For the parameter values in theorem 7 and $|\delta| < -2\beta$, the vector field $\mathbf{X}_4^p$ satisfies (C1), (C3) and (C4) of theorem 10, and also,*

(C5) *In the restriction to the invariant sphere, the vector field $\mathbf{X}_4^p$ has a stable heteroclinic network involving the saddle-foci p_{w_+} and p_{w_-}. The two-dimensional manifolds of the equilibria intersect transversely along one-dimensional trajectories.*

Proof: Statement (C1) follows from theorem 10 and the construction of $\mathbf{X}_4^p$. For the remaining statements we rewrite $\mathbf{X}_4^p$ in spherical polar coordinates $(r, \theta, \phi, \varphi)$ to obtain,

$$\begin{aligned} \dot{r} &= r(\lambda - r^2), \\ \dot{\theta} &= \alpha r \sin\theta \cos(2\phi) + \tfrac{\beta}{2} r^2 \sin(2\theta) + \tfrac{\delta}{4} r^2 \sin(2\theta)\sin(2\phi)\cos\varphi, \\ \dot{\phi} &= -\alpha r \cos\theta \sin(2\phi) - \delta r^2 (\cos\theta)^2 (\sin\phi)^2 \cos\varphi, \\ \dot{\varphi} &= 1. \end{aligned}$$

The behaviour on $\mathbf{S}_r^2$ is governed by the time-dependent vector field $\mathbf{X}_3^r$ obtained by integrating the equation for $\dot{\varphi}$ and by taking $r = \sqrt{\lambda}$, (2)

$$\begin{aligned} \dot{\theta} &= \alpha r \sin\theta \cos(2\phi) + \beta \tfrac{r^2}{2} \sin(2\theta) + \delta \tfrac{r^2}{4} \sin(2\theta)\sin(2\phi)\cos(t), \\ \dot{\phi} &= -\alpha r \cos\theta \sin(2\phi) - \delta r^2 \cos^2\theta \sin^2\phi \cos(t). \end{aligned}$$

The vector field $\mathbf{X}_3^r$ can be seen as a non-autonomous perturbation of $\mathbf{X}_3$ — for $\delta = 0$ we recover $\mathbf{X}_3$ in spherical polar coordinates. Moreover, the equations for $\dot{\theta}$ and $\dot{\phi}$ of vector field $\mathbf{X}_3^r$ are time periodic with period 2π and thus can be lifted to $\mathbf{X}_4^p$ by considering the rotation described by the fourth coordinate, φ. We can thus use $\mathbf{X}_3^r$ and the lifting properties of the results in Sec. 3 in this proof.

The constant solutions of $\mathbf{X}_3^r$ and their stability remain unchanged for the parameter values we are using, regardless of whether δ is zero or not. Thus assertions (C3) and (C4) and the stability statement in (C5) follow as in theorem 10.

In the next proposition we prove that in the restriction to the three-sphere the two-dimensional invariant manifolds of the equilibria p_w intersect transversely. This ends the proof of (C5). □

Proposition 12. *With the hypotheses of theorem 11, for the restriction of $\mathbf{X}_4^p$ to the invariant three-sphere $\mathbf{S}_r^3$, the two-dimensional invariant manifolds of p_{w_-} and p_{w_+} intersect transversely.*

Proof: The transversality of the intersection of the two-dimensional invariant manifolds in the flow of $\mathbf{X}_4^p$ restricted to $\mathbf{S}_r^3$ follows from the transversality of the intersection of the corresponding invariant manifolds in the flow of the reduced vector field $\mathbf{X}_3^r$. The latter is proved using Melnikov's method (see [Guckenheimer & Holmes, 1983], [Bertozzi, 1988]).

The Eqs. (2) for $\mathbf{X}_3^r$ on $\mathbf{S}_r^2$ can be written as,

$$\begin{aligned}\dot{\theta} &= f_1(\theta,\phi)+\delta g_1(\theta,\phi,t),\\ \dot{\phi} &= f_2(\theta,\phi)+\delta g_2(\theta,\phi,t),\end{aligned}$$

where g_1 and g_2 are periodic in t with period 2π. We denote $f=(f_1,f_2)$ and $g=(g_1,g_2)$.

We consider (2) as a time-periodic perturbation of $\dot{\theta}=f_1(\theta,\phi)$, $\dot{\phi}=f_2(\theta,\phi)$. As the unperturbed vector field is non-Hamiltonian the Melnikov function is given by (see [Guckenheimer & Holmes, 1983], Sec. 4.5)

$$M(t_0)=\int_{-\infty}^{\infty} f(q_0(t))\wedge g(q_0(t),t+t_0)\, exp\left(-\int_0^t traceDf(q_0(s))ds\right)dt,$$

with $q_0(t)$ a parametrization of the unperturbed heteroclinic orbit.

The unperturbed $\mathbf{X}_4$-connection between p_{w_-} and p_{w_+} lies in $D=\{x^2+y^2+w^2=\frac{\lambda}{R}, z=0\}$. In spherical polar coordinates, it is given by $\phi=\frac{\pi}{2}$ and $\phi=\frac{3\pi}{2}$. Let $q_0(t)=(\theta(t),\frac{\pi}{2})$ or $q_0(t)=(\theta(t),\frac{3\pi}{2})$.

As we have

$$\begin{aligned}f_1(q_0(t)) &= -\alpha r\sin\theta(t)+\tfrac{\beta}{2}r^2\sin(2\theta(t)),\\ f_2(q_0(t)) &= 0,\\ g_1(q_0(t),t+t_0) &= 0,\\ g_2(q_0(t),t+t_0) &= -r^2\cos^2\theta(t)\cos(t+t_0),\end{aligned}$$

the Melnikov function becomes

$$M(t_0)=\textstyle\int_{-\infty}^{\infty}\cos(t+t_0)E(t)dt, \tag{3}$$

with

$$E(t)=r^2\cos^2\theta(t)\left(\alpha r\sin\theta(t)-\frac{\beta}{2}r^2\sin(2\theta(t))\right)e^{\left[-\int_0^t \alpha r\cos(\theta(s))+\beta r^2\cos(2\theta(s))ds\right]}.$$

We prove in the appendix that the integral defining $M(t_0)$ converges. In order to prove the transverse intersection of the invariant manifolds it only remains to prove that $M(t_0)$ has simple zeros. Rewrite $M(t_0)$ as

$$M(t_0) = \cos(t_0)C-\sin(t_0)S. \tag{4}$$

where $C=\int_{-\infty}^{\infty}\cos(t)E(t)dt$ and $S=\int_{-\infty}^{\infty}\sin(t)E(t)dt$ are convergent.

From (4), the Melnikov function has infinitely many zeros satisfying

$$\tan(t_0) = \tfrac{C}{S}, \qquad t_0\in\mathbf{R}. \tag{5}$$

The zeros t_0' of the Melnikov function are simple if $\frac{dM}{dt_0}(t_0') \neq 0$. We have from (4),

$$\frac{dM}{dt_0}(t_0) \quad = \quad -\sin(t_0)C - \cos(t_0)S.$$

Thus the zeros of the Melnikov function are simple, provided

$$\tan(t_0) \quad \neq \quad -\frac{S}{C}, \qquad t_0 \in \mathbf{R},$$

which is trivially verified, since the zeros of the Melnikov function satisfy (5). □

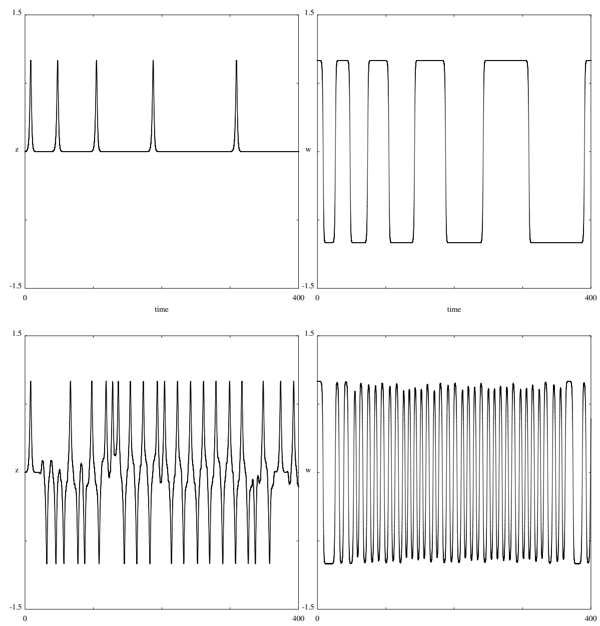

FIGURE 4. Time series for the variables z (pictures on the left) and w (pictures on the right), for the flow of the unperturbed (pictures on the top) and perturbed (pictures on the bottom) vector fields, when $\lambda = 1,\ \alpha = 1,\ \beta = -0.1,\ \delta = 0.3$.

Figures 4 and 5 (obtained using Dstool [Guckenheimer *et al.*, 1997]) provide a numerical illustration of the transverse intersection of the invariant manifolds and indicate chaotic behaviour. That this is indeed the case is discussed below. Before that we need to introduce some terminology. Let Σ be a network with a finite set of nodes. We define a *path* on Σ as a bi-infinite sequence $(c_j)_{j\in\mathbf{Z}}$ of connections in Σ such that $c_j = [n_{j-1} \to n_j]$, with n_j nodes of Σ.

Let N_Σ be any neighbourhood of a network Σ and U_n arbitrary neighbourhoods of the nodes $n \in \Sigma$. For every connection contained in Σ, let p be an arbitrary point on it and consider an arbitrary neighbourhood

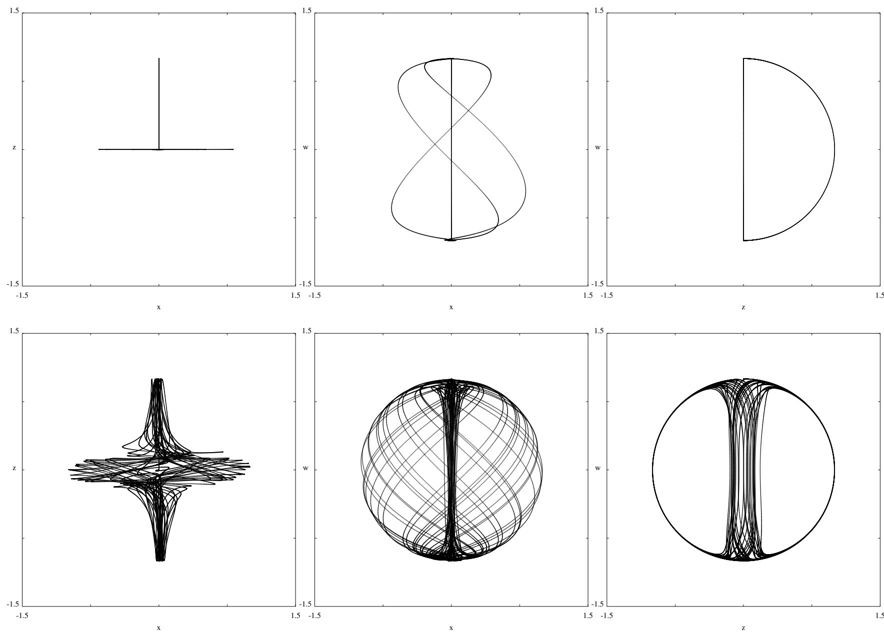

FIGURE 5. Projections on the (x, z)-plane (pictures on the left), (x, w)-plane (pictures in the middle) and on the (z, w) (pictures on the right), for the flow of the unperturbed (pictures at the top) and perturbed (pictures at the bottom) vector fields, when $\lambda = 1,\ \alpha = 1,\ \beta = -0.1,\ \delta = 0.3$.

U_p of each p. We say there is *switching on the network* if, for each path $(c_i)_{i\in\mathbf{Z}}$ contained in Σ, there is a trajectory $x(t) \subset N_\Sigma$ and sequences (t_i), (s_i) with $t_{i-1} < s_i < t_i$ such that $x(s_i) \in U_{p_i}$ and $x(t_i) \in U_{n_i}$, where $p_i \in c_i$.

Proposition 13. *Let Σ be the heteroclinic network for $\mathbf{X}_4^p$ of theorem 11. Then, for the parameter values of theorem 11:*

(1) *There is switching on the network Σ.*
(2) *There is a suspended horseshoe in any neighbourhood of each cycle in Σ.*

Proof: The proposition follows from the results in Sec. 6 of [Aguiar *et al.*, 2005]. The hypotheses either are valid by construction, or proved in theorem 11 and proposition 12. [1] □

Remark 14. *Had we chosen, in perturbing $\mathbf{X}_4$, the only perturbation tangent to $\mathbf{S}^3_r$ that preserves the $SO(2)$-symmetry, we would have seen*

[1] The existence of complex behaviour may be confirmed by calculating Lyapunov exponents. One of the referees obtained estimates indicating that there exists one positive Lyapunov exponent of about 0.9156.

bifurcation of the heteroclinic network to an invariant two-torus close to it. There is numerical evidence that the dynamics restricted to the two-torus is quasi-periodic. See [Aguiar, 2003] for more detail.

5. Heteroclinic Network Between Saddle-foci and a Periodic Trajectory

We use the same technique to construct another example — the details are similar to those in Sec. 4.

In step 1 we consider the finite group $\Gamma \subset O(3)$ generated by,

$$\begin{array}{rcl} p(\rho, z, w) & = & (z, w, \rho), \\ k(\rho, z, w) & = & (-\rho, z, w). \end{array}$$

The degree 3 normal form for the Γ-equivariant vector fields is given in [Guckenheimer & Holmes, 1988]. We consider a perturbation $\mathbf{X}_3$ of degree 5 given by,

$$\begin{array}{rcl} \dot{\rho} & = & \rho\left(\lambda + \alpha\rho^2 + \beta z^2 + w^2 + \delta(z^4 - \rho^2 w^2)\right), \\ \dot{v} & = & v\left(\lambda + \alpha z^2 + \beta w^2 + \rho^2 + \delta(w^4 - \rho^2 z^2)\right), \\ \dot{w} & = & w\left(\lambda + \alpha w^2 + \beta\rho^2 + z^2 + \delta(\rho^4 - z^2 w^2)\right). \end{array}$$

For $\lambda > 0, \beta + \gamma = 2\alpha, \beta < \alpha < \gamma < 0$ and $\delta < 0$, the sphere $\mathbf{S}^2_r$, of radius $r = \sqrt{-\frac{\lambda}{\alpha}}$, is invariant by the flow of $\mathbf{X}_3$ and globally attracting. On the invariant sphere, $\mathbf{X}_3$ has an asymptotically stable heteroclinic network connecting the equilibria, $p_\rho = (\pm r, 0, 0)$, $p_v = (0, \pm r, 0)$, and $p_w = (0, 0, \pm r)$. Besides the equilibria in (b) and the origin, system $\mathbf{X}_3$ has eight unstable foci. The proof is analogous to those in the previous section and can be found in [Aguiar, 2003].

Using step 2, the vector field $\mathbf{X}_3$ is lifted to $\mathbf{X}_4$ on $\mathbf{R}^4$. In step 3 we use a perturbation of degree 5 to obtain $\mathbf{X}_4^p$ given by

(6)

$$\begin{array}{rcl} \dot{x} & = & x\left(\lambda + \alpha\left(x^2+y^2\right) + \beta z^2 + w^2 + \delta\left(z^4 - (x^2+y^2)w^2\right)\right) - \eta y, \\ \dot{y} & = & y\left(\lambda + \alpha\left(x^2+y^2\right) + \beta z^2 + w^2 + \delta\left(z^4 - (x^2+y^2)w^2\right)\right) + \eta x, \\ \dot{z} & = & v\left(\lambda + \alpha z^2 + \beta w^2 + (x^2+y^2) + \delta\left(w^4 - (x^2+y^2)z^2\right)\right) + \xi xyw\left(\lambda + 3\alpha(x^2+y^2)\right), \\ \dot{w} & = & w\left(\lambda + \alpha w^2 + \beta(x^2+y^2) + z^2 + \delta\left((x^2+y^2)^2 - z^2w^2\right)\right) - \xi xyz\left(\lambda + 3\alpha(x^2+y^2)\right). \end{array}$$

Theorem 15. *For $\lambda > 0$, $\beta + \gamma = 2\alpha$, $\beta < \alpha < \gamma < 0$, $\delta < 0$, $\eta \in \mathbf{R}$ and for values of ξ sufficiently small and such that $|\xi| < \frac{-\alpha\beta+\alpha\gamma+\delta\lambda}{2\lambda\alpha}$ and $\xi^2 < \frac{(\gamma-\beta)(2\delta\lambda-\alpha\beta+\alpha\gamma)}{4\alpha\lambda^2}$ system (6) satisfies*

- (D1) *There is a three-dimensional sphere, $\mathbf{S}^3_r$, that is invariant by the flow and globally attracting, in the sense that every trajectory with nonzero initial condition is asymptotic to the sphere in forward time.*
- (D3) *The only equilibria are the origin and the four saddle-foci $p_{\pm v}$ and $p_{\pm w}$. On the invariant sphere, $p_{\pm v}$ are of type $2, 1$ and $p_{\pm w}$ are of type $1, 2$.*

(D5) *In the restriction to the invariant sphere, system (6) has a structurally stable heteroclinic network involving the four saddle-foci $p_{\pm v}$ and $p_{\pm w}$, and the periodic trajectory c. When $\alpha(\beta-\alpha)+\delta\lambda > 0$ the two dimensional manifolds of the periodic trajectory intersect transversely the two-dimensional manifolds of the saddle-foci $p_{\pm v}$ and $p_{\pm w}$.*

(D) *In addition to the periodic trajectory c, the system has four hyperbolic periodic trajectories each in one connected component of $\mathbf{S}^3_r \setminus (\mathbf{D}_1 \cup \mathbf{D}_2)$. On the invariant sphere, the four periodic trajectories are repelling.*

Proof: The proof is similar to that of theorem 11 and proposition 12 in Sec. 4. See [Aguiar, 2003] for details. □

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Appendix A

Lemma 16. *With the hypotheses of theorem 11, the Melnikov integral*

$$M(t_0) = \int_{-\infty}^{\infty} f(t) e^{-g(t)} dt,$$

with

$$f(t) = \left(\cos\left(t + t_0\right) r^2 \cos^2\theta(t)\right) \left(\alpha r \sin\theta(t) - \frac{\beta}{2} r^2 \sin\left(2\theta(t)\right)\right),$$

and

$$g(t) = \int_0^t \alpha r \cos(\theta(s)) + \beta r^2 \cos(2\theta(s)) ds,$$

converges.

Proof: In order to prove that $M(t_0)$ converges, and since $f(t)$ is bounded, it is sufficient to prove that $\int_{-\infty}^{\infty} e^{-g(t)} dt$ converges.

We take $\alpha > 0$. Recall that, we are studying the perturbation of the heteroclinic connection in the plane $\{(\rho, z, w) \ : \ z = 0\}$, and thus

$$\dot{\theta} \ = \ r \sin\theta \left(-\alpha + \beta r \cos\theta\right).$$

From the expression of the eigenvalues of the equilibria $p_{w_\pm}$, in the proof of theorem 7, when $\alpha > 0$ the heteroclinic connection is from p_{w_-} to p_{w_+}.

Thus, we have $\lim_{t \to +\infty} \theta(t) = 0$, $\lim_{t \to -\infty} \theta(t) = \pi$ and $\theta(t) \in [0, \pi], \forall t \in \mathbf{R}$. For the parameter values we are considering, we have $\alpha > |\beta r|$, and thus $-\alpha + \beta r \cos\theta < 0$, and $\dot{\theta} < 0$ for $\theta \in]0, \pi[$.

We change variables $u = \theta(s)$ and obtain

$$g(t) = \int_{\theta(0)}^{\theta(t)} \alpha r \cos(u) + \beta r^2 \cos(2u) \frac{du}{\dot{\theta}}.$$

Computations with Maple give

$$g(t) = -A \ln J(u) \Big]_{\theta(0)}^{\theta(t)},$$

with $A = \frac{1}{(\alpha-\beta r)(\alpha+\beta r)}$, and

$$J(u) = \frac{\left(1+\tan^2(\frac{u}{2})\right)^{2(\alpha^2-\beta^2 r^2)}\left(\tan(\frac{u}{2})\right)^{(\alpha+\beta r)^2}}{\left((\alpha-\beta r)+(\alpha+\beta r)\tan^2(\frac{u}{2})\right)^{(3\alpha^2-\beta^2 r^2)}}.$$

Since $\alpha > |\beta r|$, we have $A > 0$. Also, we have

$$e^{-g(t)} \;=\; J\left(\theta(0)\right)^{-A} J\left(\theta(t)\right)^{A}.$$

To prove that $\int_{-\infty}^{\infty} e^{-g(t)}dt$ converges, it is thus sufficient to prove the convergence of $\int_{-\infty}^{\infty} J(\theta(t))^A dt$.

For the parameter values we are considering we have (see lemma 21 in [Aguiar, 2003]),

$$0 \le J(\theta(t)) < \frac{1}{(\alpha+\beta r)^{(3\alpha^2-\beta^2 r^2)}}\left(\frac{\sin(\theta(t))}{2}\right)^{(\alpha+\beta r)^2}.$$

Thus, we conclude that

$$J(\theta(t))^A \;<\; \left(\frac{1}{(\alpha+\beta r)^{(3\alpha^2-\beta^2 r^2)}}\right)^A \left(\frac{\sin(\theta(t))}{2}\right)^B,$$

with $0 < B = \frac{\alpha+\beta r}{\alpha-\beta r} < 1$.

Thus, to prove that $\int_{-\infty}^{\infty} J(\theta(t))^A dt$ converges we only need to prove that

$$\int_{-\infty}^{\infty}\left(\frac{\sin\theta(t)}{2}\right)^B dt \tag{7}$$

converges.

By arguments above, (7) is equal to

$$\int_0^{\pi} \frac{\left(\frac{\sin\theta}{2}\right)^B}{r\sin\theta(\alpha-\beta r\cos\theta)}d\theta.$$

Since $\alpha > 0$, $\beta < 0$ and $\alpha > |\beta r|$, we have $\alpha - \beta r\cos\theta > 0$. For $\theta \in]0,\pi[$, we have,

$$\frac{\left(\frac{\sin\theta}{2}\right)^B}{r\sin\theta\,(\alpha-\beta r\cos\theta)} = \frac{1}{2^B r\,(\sin\theta)^{1-B}\,(\alpha-\beta r\cos\theta)},$$

with $0 < 1-B < 1$.

It remains to prove that,

$$\int_0^{\pi}\frac{1}{(\sin\theta)^{1-B}\,(\alpha-\beta r\cos\theta)}d\theta$$

converges, which is easily seen since $\frac{1}{\alpha-\beta r\cos\theta}$ is bounded, and

$$\int_0^{\pi}\frac{1}{(\sin\theta)^{1-B}}d\theta$$

converges, by comparison with $\int_0^{\pi}\frac{1}{\theta^{1-B}}d\theta$. $\square$