CONTRIBUTIONS TO THE GEOMETRIC AND ERGODIC THEORY OF CONSERVATIVE FLOWS

MÁRIO BESSA AND JORGE ROCHA

Abstract. We prove the following dichotomy for vector fields in a C^1-residual subset of volume-preserving flows: for Lebesgue almost every point all Lyapunov exponents equal to zero or its orbit has a dominated splitting. Moreover, we prove that a volume-preserving and C^1-stably ergodic flow can be C^1-approximated by another volume-preserving flow which is non-uniformly hyperbolic.

MSC 2000: primary 37D30, 37D25; secondary 37A99, 37C10.
keywords: Volume-preserving flows; Lyapunov exponents; Dominated splitting; Stable ergodicity.

1. Introduction and statement of the results

Let M be a d-dimensional, $d \geq 3$, compact, connected and boundaryless Riemaniann manifold endowed with a volume-form ω and let μ denote the Lebesgue measure associated to it. We denote by $\mathfrak{X}^1_\mu(M)$ the space of C^1 vector fields X over M such that X is *divergence-free*, that is its associated flow X^t preserves the measure μ. We consider $\mathfrak{X}^1_\mu(M)$ endowed with the usual Whitney C^1-topology.

Given a flow X^t one usually deduces properties of it by studying its linear approximation. One way to do that is by considering the Lyapunov exponents which, in broad terms, detect if there are any exponential behavior of the linear tangent map along orbits. Given $X \in \mathfrak{X}^1_\mu(M)$ the existence of Lyapunov exponents for almost every point is guaranteed by Oseledets' theorem ([31]). Positive (or negative) exponents assure, in average, exponential rate of divergence (or convergence) of two neighboring trajectories, whereas zero exponents give us the lack of any kind of average exponential behavior. A flow is said to be *nonuniformly hyperbolic* if its Lyapunov exponents are all different from zero. In [25] Hu, Pesin and Talitskaya gave examples of nonuniformly hyperbolic flows in any compact manifold. Non-zero exponents plus some smoothness assumptions on the flow allows us to obtain invariant manifolds dynamically defined (see [32]). Since this stable/unstable manifold theory is the base of capital results on dynamical systems nowadays it is of extreme importance to detect when we have nonzero Lyapunov exponents.

In the beginning of the 1980s Ricardo Mañé, in [28], announced a dichotomy for C^1-generic discrete-time conservative systems which in broad terms says that for Lebesgue almost every point its Lyapunov exponents are

Date: March 13, 2017.

all equal to zero or else there exists a weak form of uniform hyperbolicity along its orbit.

It is well-known that hyperbolicity plays a crucial role if one wants to obtain stability. Briefly speaking, hyperbolicity means uniform expansion (or contraction) by the tangent map along the orbits and when restricted to particular invariant subbundles. A quintessential example is an Anosov flow [1].

By a weak form of hyperbolicity we mean uniform contraction of the ratio between the dynamical behavior of the tangent map when computed in an invariant subbundle and the dynamical behavior of the tangent map restricted to another invariant subbundle which is most contracting (or less expanding) than the first mentioned.

Later, in [29], Mañé presented the guidelines for the proof of the aforementioned dichotomy in the surfaces case. However, more ingredients and new tools were necessary to obtain a complete proof (see the work of Bochi [14]). Then, in a remarkable paper [17], Bochi and Viana extended the Bochi-Mañé theorem to any dimensional manifolds and recently Bochi (see [15]) was able to obtain the full statement announced in [28] for the symplectomorphisms setting.

For the flow setting the first author proved in [7] the three-dimensional version for vectors fields without equilibrium points and also a weak version for general divergence-free vector fields. Later, in [2, Theorem A], a global version for vector fields with equilibrium points was obtained. In [8] was proved a version for linear differential systems with conservativeness properties and in [9] was obtained a similar result in the Hamiltonians setting.

After the perturbation techniques developed in [7] and in [8] we expected to obtain Bochi-Viana's theorem for a C^1-dense subset of $\mathfrak{X}^1_\mu(M)$, however by an upgrade refinement on the perturbation framework we were able to obtain this result for a C^1-residual subset of $\mathfrak{X}^1_\mu(M)$, thus achieve the full counterpart of [17, Theorem 2].

More precisely we prove the following result.

Theorem 1. *There exists a C^1-residual set $\mathcal{E} \subset \mathfrak{X}^1_\mu(M)$ such that if $X \in \mathcal{E}$ then there exist two X^t-invariant subsets of M, $\mathcal{Z}$ and $\mathcal{D}$, whose union has full measure and such that:*

- *if $p \in \mathcal{Z}$ then all the Lyapunov exponents associated to p are zero;*
- *if $p \in \mathcal{D}$ then its orbit admits a dominated splitting for the linear Poincaré flow.*

We point out that the abundance of zero exponents from the generic point of view seems to be strongly related to the topology used, namely to the C^1-topology. On the other hand recent results obtained by Viana ([36]) show that, in a prevalent way, Hölder continuous linear cocycles based on a uniformly hyperbolic system with local product structure have nonzero Lyapunov exponents.

We recall that a closed orbit p with period ℓ is called *elliptic* if the linear Poincaré flow on the period ℓ has non-real eigenvalues of norm one. In the three-dimensional setting of divergence-free vector fields we know, by [10], that far from vector fields exhibiting (dense) elliptic points we have hyperbolicity, say an Anosov flow. It is natural to ask if, for dimensions ≥ 4, *global* hyperbolicity prevails when we are far from *local* non-hyperbolicity (e.g. elliptic points). Somehow related with this question Ferreira [22] proved, using Theorem 1, that, if a divergence-free vector field X has all its singularities and closed orbits of hyperbolic type and, moreover, this property prevails for any C^1-neighborhood of X, within the divergence-free class, then X is an Anosov vector field. We denote the set of divergence-free vector fields with this property by $\mathcal{G}^1_\mu(M)$. Moreover, she proved that any $X \in \mathfrak{X}^1_\mu(M)$ can be C^1 approximated by another field $Y \in \mathfrak{X}^1_\mu(M)$ such that Y is Anosov or else Y exhibits a heterodimensional cycle. Related to this subject we also mention the remarkable paper by Gan and Wen ([24]).

Let $\mathcal{KS} \subset \mathfrak{X}^1_\mu(M)$ be the set of Kupka-Smale divergence-free vector fields ([34]).

Corollary 1.1. *Let $X \in \mathcal{KS}$. If $Sing(X) \neq \emptyset$, then X can be C^1-approximated by a divergence-free vector field exhibiting a non-hyperbolic closed orbit.*

In fact, if $X \in \mathcal{KS}$ and $Sing(X) \neq \emptyset$, then X is not Anosov. So, by [22], $X \notin \mathcal{G}^1_\mu(M)$; therefore X is C^1-arbitrarily approximated by a vector field $Y \in \mathfrak{X}^1_\mu(M)$ exhibiting a non-hyperbolic critical point (closed orbit or singularity). Using the fact that $X \in \mathcal{KS}$, it is easy to show that, if Y is sufficiently C^1-close to X, then the non-hyperbolic critical point is a closed orbit.

Recall that $X \in \mathfrak{X}^1_\mu(M)$ is *ergodic* if any measurable X^t-invariant set is a zero measure set or is a full measure set. Let $\alpha > 0$; we say that $X \in \mathfrak{X}^{1+\alpha}_\mu(M)$ is a C^1-*stably ergodic* flow if there exists a C^1-neighborhood of X such that any $Y \in \mathfrak{X}^{1+\alpha}_\mu(M)$ is ergodic.

In [13] we considered the open class of C^1 partially hyperbolic volume-preserving flows with one dimensional central direction endowed with the C^1-Whitney topology. Then, we proved that, within this class, any flow can be approximated by an ergodic C^2 volume-preserving flow. Therefore, we obtained that, within this class, ergodicity is dense. Our next result deals with a somehow counterpart statement, i.e., ergodic assumptions assure some geometric and dynamical consequences.

Let us denote by $\mathcal{SE}$ the space of the C^1-stably ergodic flows in $\mathfrak{X}^{1+\alpha}_\mu(M)$. We refer the reader to the survey of Pugh and Shub ([33]) on properties of these systems. It follows from [12] that if X is stably ergodic then it does not have any singularities. From Theorem 1 it follows that if $X \in \mathcal{E} \cap \mathcal{SE}$ then either the set $\mathcal{Z}$ has full measure or else $\mathcal{D}$ is a full measure set. In the next result we prove that, for an open and dense subset of $\mathcal{SE}$, actually $\mathcal{Z}$ has zero measure and the pointwise domination given by Theorem 1 for points in the full measure set $\mathcal{D}$ is in fact uniform. This result is the continuous-time version of Bochi-Fayad-Pujals Theorem for conservative diffeomorphisms ([16]).

Theorem 2. *There exists a C^1-open and dense set $\mathcal{U} \subset \mathcal{SE}$ such that if $X \in \mathcal{U}$ then X^t is a non-uniformly hyperbolic flow and X admits a dominated splitting (for* *the linear Poincaré flow) that separates the spaces corresponding to positive and negative Lyapunov exponents.*

2. Notation, definitions and basic results

In this section we introduce some notation, fundamental definitions and basic results needed to prove our theorems.

2.1. **The setting.** Let M be a d-dimensional compact, connected and boundaryless Riemaniann manifold endowed with a volume-form ω. The measure μ associated to ω is called the Lebesgue measure. As mentioned before we denote by $\mathfrak{X}^r_\mu(M)$ the set of all *divergence-free* vector fields $X\colon M \to TM$ of class C^r ($r \geq 1$), endowed with the usual Whitney C^1-topology. Let $X \in \mathfrak{X}^1_\mu(M)$; X is the infinitesimal generator of X^t, that is, $\frac{dX^t}{dt}|_{t=s}(p) = X(X^s(p))$. We are interested in the study of the tangent map $DX^t_p\colon T_pM \to T_{X^t(p)}M$. Notice that DX^t_p is a solution of the linear variational equation $\dot{u}(t) = DX_{X^t(p)} \cdot u(t)$. Is is easy to see that $|\det(DX^t)| = 1$ for any $t \in \mathbb{R}$, that is the flow X^t is volume-preserving.

Let $Sing(X) := \{x \in M\colon X(x) = \vec{0}\}$ denote the set of *singularities* of X and let $\mathcal{R}(X) := M \setminus Sing(X)$ denote the set of *regular* points.

2.2. **Linear Poincaré flow.** Fix $X \in \mathfrak{X}^1_\mu(M)$, $p \in \mathcal{R}(X)$ and let $N_p \subset T_pM$ denote the *normal fiber* at $X(p)$, that is, the subfiber spanned by the orthogonal complement of $X(p)$. We denote by $N \subset TM$ the *normal bundle* which, of course, is only defined on $\mathcal{R}(X)$. Now, let $\mathcal{N}_p$ and $\mathcal{N}_{X^t(p)}$ be two $(d-1)$-dimensional manifolds contained in M whose tangent spaces at p and $X^t(p)$, respectively, are N_p and $N_{X^t(p)}$. We can be careful and chose these $(d-1)$-dimensional submanifolds to vary continuously along the orbit. Let also $\mathcal{V}_p$ be a small neighborhood of p in $\mathcal{N}_p$. If $\mathcal{V}_p$ can be taken small enough, then the usual *Poincaré map* $\mathcal{P}^t_X(p) : \mathcal{V}_p \subset \mathcal{N}_p \to \mathcal{N}_{X^t(p)}$ is well defined.

The *linear Poincaré flow* was first used by Liao (e.g. [27]) and formally introduced in [21]. It is the differential of the Poincaré map. To define it properly for each $t \in \mathbb{R}$ we consider the tangent map $DX^t : T_{\mathcal{R}}M \longrightarrow T_{\mathcal{R}}M$ which is defined by $DX^t(p, v) = (X^t(p), DX^t(p) \cdot v)$ and let $\Pi_{X^t(p)}$ be the canonical projection on $N_{X^t(p)}$. The linear map $P^t_X(p) : N_p \longrightarrow N_{X^t(p)}$ defined by $P^t_X(p) = \Pi_{X^t(p)} DX^t(p)$ is called the linear Poincaré flow at p associated to the vector field X.

2.3. **Local coordinates.** Given a linear map A we define its norm in the usual way, i.e.,

$$\sup_{v \neq \vec{0}} \frac{\|A \cdot v\|}{\|v\|}.$$

By Lemma 2 of [30], given a volume form ω in M there exists an atlas $\mathcal{A}^* = \{(\alpha_i, U^*_i)_i\}$ of M, such that $(\alpha_i)_*\omega = dx_1 \wedge dx_2 \wedge ... \wedge dx_d$. The fact that M is compact guarantees that $\mathcal{A}^*$ can be taken finite, say $\mathcal{A}^* = \{(\alpha_i, U^*_i)\}^k_{i=1}$. Moreover, we can always choose U^*_i containing the closure of an open set U_i, $i = 1, .., k$, such that $\{(\alpha_i, U_i)\}^k_{i=1}$ is an atlas.

Given any $x \in M$ we define $i(x) := \min\{i \in \{1, ..., k\} : x \in U_i\}$. The Riemannian norm *a priori* fixed at TM will not be used, instead we use the equivalent norm $\|v\|_x := \|(D\alpha_{i(x)})_x \cdot v\|$.

Let p and q be points in the same domain U_i and t be such that $X^t(p)$ and $X^t(q)$ are in the same domain U_j. Given linear maps $A^t(p) : N_p \to N_{X^t(p)}$ and $B^t(q) : N_q \to N_{X^t(q)}$ we define the distance $\|A^t(p) - B^t(q)\|$ in the following way. Let

- $a^t_{i,j} = (D\alpha_{j(X^t(p))})_{X^t(p)}|_{N_{X^t(p)}} \circ A^t(p) \circ (D\alpha_{i(p)})^{-1}_p|_{D\alpha_i(N_p)}$, and
- $b^t_{i,j} = (D\alpha_{j(X^t(q))})_{X^t(q)}|_{N_{X^t(q)}} \circ B^t(q) \circ (D\alpha_{i(q)})^{-1}_q|_{D\alpha_i(N_q)}$.

Now we define

$$\|A^t(p) - B^t(q)\| = \|a^t_{i,j} - b^t_{i,j}\|. \tag{2.1}$$

2.4. **Flowboxes and the modified volume-preserving property.** Given the Poincaré map associated to X and to a non-periodic point p, $\mathcal{P}^t_X(p) : \mathcal{V}_p \subseteq \mathcal{N}_p \to \mathcal{N}_{X^t(p)}$, where $\mathcal{V}_p$ is chosen sufficiently small and given $B \subseteq \mathcal{V}_p$ the set

$$\mathcal{F}^n_X(p)(B) := \{\mathcal{P}^t_X(p)(q) : q \in B, t \in [0, n]\},$$

is called the time-n length *flowbox* at p associated to the vector field X and to the sections $\mathcal{N}_p$ and $\mathcal{N}_{X^t(p)}$. We observe that, although the flowboxes depend on the initial choice of the traversal sections, in fact this is not relevant since we will take $\mathcal{V}_p$ arbitrarily small and this gives a uniform control of the angles.

Given $v_1, v_2, ..., v_{d-1} \in N_p$ we can define a pair of $(d-1)$-forms by

$$\hat{\omega}_p(v_1, v_2, ..., v_{d-1}) := \omega_p(X(p), v_1, v_2, ..., v_{d-1}),$$

and

$$\overline{\omega}_p(v_1, v_2, ..., v_{d-1}) = \omega_p(\|X(p)\|^{-1}X(p), v_1, v_2, ..., v_{d-1}),$$

both induced by the volume form ω. It turns out that $(\mathcal{P}^t_X(p))^*\hat{\omega}_p = \hat{\omega}_{X^t(p)}$. The measure $\overline{\mu}$ induced by the $(d-1)$-form $\overline{\omega}$ is not necessarily $\mathcal{P}^t_X$-invariant, however both the associated measures $\hat{\mu}$ and $\overline{\mu}$ are equivalent. We call $\overline{\mu}$ the Lebesgue measure at normal sections or *modified section volume*. In fact, given $v_1, v_2, ..., v_{d-1} \in N_p$ we have that

$$(\mathcal{P}^t_X(p))^*\overline{\omega}_p(v_1, ..., v_{d-1}) = x(t)^{-1}\overline{\omega}_{X^t(p)}(P^t_X(p) \cdot v_1, ..., P^t_X(p) \cdot v_{d-1}),$$

where $x(t) = \|X(X^t(p))\| \, \|X(p)\|^{-1}$. Since the flow is volume-preserving we have $|\det P^t_X(p)| = x(t)^{-1}$. Therefore it follows that we can give an explicit expression for the infinitesimal distortion volume factor of the linear Poincaré flow, which is expressed by the following simple adaptation of ([7, Lemma 2.2]).

Lemma 2.1. *Fix a non-periodic point $p \in M$ and let $\mathcal{N}_p$ be a $(d-1)$-dimensional manifold contained in M whose tangent spaces at p is N_p. Given $\nu > 0$ and $T > 0$, there exists $r > 0$ such that for any measurable set $K \subseteq B(p, r) \subseteq \mathcal{N}_p$ we have*

$$\left| \frac{\overline{\mu}(K)}{x(t).\overline{\mu}(\mathcal{P}^t_X(p)(K))} - 1 \right| < \nu, \textit{ for all } t \in [0, T].$$

2.5. **Multiplicative ergodic theorem for the linear Poincaré flow.** Let

$$\mathbb{R}X(p) := \{v \in T_pM : v = \eta X(p), \eta \in \mathbb{R}\},$$

be the line field direction at p. Recalling that $DX^t_p(X(p)) = X(X^t(p))$ we conclude that the vector field direction is DX^t-invariant. The existence of other DX^t-invariant fibers is guaranteed, at least for Lebesgue almost every point, by a theorem due to Oseledets (see [31]) that we re-write for the linear Poincaré flow.

Theorem 2.2. *(Oseledets' Theorem for the linear Poincaré flow) Given $X \in \mathfrak{X}^1_\mu(M)$ such that $\mu(Sing(X)) = 0$, then for μ-a.e. $p \in M$ there exist*

- *a P^t_X-invariant splitting of the fiber $N_p = N^1_p \oplus ... \oplus N^{k(p)}_p$ along the orbit of p (*Oseledets' splitting*) and*
- *real numbers $\hat{\lambda}_1(p) > ... > \hat{\lambda}_{k(p)}(p)$ (*Lyapunov exponents*),*

with $1 \leq k(p) \leq d-1$, such that:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|P^t_X(p) \cdot n^i\| = \hat{\lambda}_i(p), \tag{2.2}$$

for any $n^i \in N^i_p \setminus \{\vec{0}\}$ and $i = 1, ..., k(p)$.

We observe that the hypothesis $\mu(Sing(X)) = 0$ holds if all the singularities of X are hyperbolic, thus, it is satisfied for a C^1-open and dense subset of $\mathfrak{X}^1_\mu(M)$.

Let $\mathcal{O}(X)$ denote the set of μ-generic points given by this theorem. We note that if we take into account the multiplicities of the $\hat{\lambda}_i(p)$, then we have $d-1$ Lyapunov exponents: $\lambda_1(p) \geq \lambda_2(p) \geq ... \geq \lambda_{d-1}(p)$.

In [26] is presented a proof of Theorem 2.2 in the context of linear differential systems.

Remark 2.1. *Actually, the Oseledets Theorem gives us a splitting of $T_pM = E^1_p \oplus ... \oplus E^{k(p)}_p \oplus \mathbb{R}X(p)$ and Lyapunov exponents associated to these DX^t-invariant directions for μ-a.e. point p. Due to the fact that for any of these subspaces $E^i_p \subset T_pM$, with non-zero Lyapunov exponent, the angle between this space and $\mathbb{R}X(p)$ along the orbit has sub-exponential growth, that is*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \sin(\measuredangle(E^i_{X^t(p)}, \mathbb{R}X(X^t(p)))) = 0. \tag{2.3}$$

Therefore we conclude that the Lyapunov exponent $\hat{\lambda}_i(p) \neq 0$ for DX^t with associated subspace E^i_p is also a Lyapunov exponent for P^t_X associated to subspace $N^i_p = \Pi_p(E^i_p)$, $i \in \{1, ..., k(p)\}$, where Π_p is the projection into N_p. When the Lyapunov exponent associated to the direction E^i_p is equal to zero then, since the flow direction has also a zero exponent, we conclude that any direction of the plane defined by E^i_p and the flow direction has zero Lyapunov exponent; in particular the direction N^i_p.

2.6. **Multilinear algebra for the linear Poincaré flow.** The k^{th} exterior power of N, denoted by $\wedge^k(N)$, is a $\binom{d-1}{k}$-dimensional vector space. Let $\{e_j\}_{j\in J}$ be an orthonormal basis of N, then the family of exterior powers $e_{j_1}\wedge e_{j_2}\wedge ... \wedge e_{j_k}$ for $j_1 < ... < j_k$ with $j_\alpha \in J$ forms an orthonormal basis of $\wedge^k(N)$. Given $P^t_X(p)\colon N_p \to N_{X^t(p)}$ we define

$$\begin{array}{cccc} \wedge^k(P^t_X(p))\colon & \wedge^k(N_p) & \longrightarrow & \wedge^k(N_{X^t(p)}) \\ & \psi_1\wedge ... \wedge \psi_k & \longrightarrow & P^t_X(p)\cdot\psi_1\wedge ... \wedge P^t_X(p)\cdot\psi_k. \end{array}$$

This formalism of multilinear algebra reveals to be the adequate to prove our results. This is because we can recover the spectrum and the splitting information of the dynamics of $\wedge^k(P^t_X(p))$ from the one obtained by applying Oseledets' Theorem to $P^t_X(p)$. This is precisely the meaning of the next theorem ([5, Theorem 5.3.1]). For a deeper discussion of exterior power algebra see [5, §3.2.3].

Theorem 2.3. *(Oseledets' Theorem for the exterior power of the linear Poincaré flow) The Lyapunov exponents $\lambda^{\wedge k}_i(p)$ for $i \in \{1, ..., \binom{d-1}{k}\}$ (repeated with multiplicity) of the k^{th} exterior power operator $\wedge^k(P^t_X(p))$ are the numbers of the form:*

$$\sum_{j=1}^{k}\lambda_{i_j}(p), \text{ where } 1 \le i_1 < ... < i_k \le d-1.$$

This nondecreasing sequence starts with
- $\lambda^{\wedge k}_1(p) = \lambda_1(p) + \lambda_2(p) + ... + \lambda_k(p)$ *and ends with*
- $\lambda^{\wedge k}_{q(k)}(p) = \lambda_{d-k}(p) + \lambda_{d+1-k}(p) + ... + \lambda_{d-1}(p)$.

Take an Oseledets basis $\{e_1(p), ..., e_{d-1}(p)\}$ of N_p such that $e_i(p) \in N^\ell_p$ for

$$\dim(N^1_p) + ... + \dim(N^{\ell-1}_p) < i \le \dim(N^1_p) + ... + \dim(N^\ell_p).$$

Then the Oseledets space $\wedge^k(N^j_p)$ of $\wedge^k(P^t_X(p))$ is the subspace of $\wedge^k(N_p)$ which is generated by the k-vectors:

$$e_{i_1}\wedge ... \wedge e_{i_k} \text{ such that } 1 \le i_1 < ... < i_k \le d-1 \text{ and } \sum_{j=1}^{k}\lambda_{i_j}(p) = \lambda_j(p).$$

2.7. **Dominated splitting for the linear Poincaré flow.** Let $\mathfrak{m}(A) = \|A^{-1}\|^{-1}$ denotes the co-norm of a linear map A.

Take a X^t-invariant set Λ, without singularities, and fix $m \in \mathbb{N}$. A non-trivial P^t_X-invariant and continuous splitting $N_\Lambda = U_\Lambda \oplus S_\Lambda$ is said to have an *m-dominated splitting* for the linear Poincaré flow of X over Λ if the following inequality holds for every $p \in \Lambda$:

$$\frac{\|P^m_X(p)|_{S_p}\|}{\mathfrak{m}(P^m_X(p)|_{U_p})} \le \frac{1}{2}. \tag{2.4}$$

The *index* of the splitting is the dimension of the bundle U_Λ. The dominated splitting structure is a "weak" form of uniform hyperbolicity, in fact behaves like a uniform hyperbolic structure in the projective space $\mathbf{RP}^{d-2}$. We enumerate some basic properties of an m-dominated splitting on a set Λ, for the detailed proofs of these properties see [18] Section B.1.

(H) (Higher Domination) There exists $m_0 > m$ such that, for all $\ell \geq m_0$, $U_\Lambda \oplus S_\Lambda$ is an ℓ-dominated splitting.

(E) (Extension) It can always be extended to an m-dominated splitting over $\overline{\Lambda} \setminus Sing(X)$.

(T) (Transversality) The angles between the U_p and S_p are uniformly bounded away from zero, for $p \in \overline{\Lambda}$.

(U) (Uniqueness) For a fixed index the dominated splitting is unique.

(P) (Persistence) The dominated splitting persists under C^1-perturbations of X.

Fix $X \in \mathfrak{X}^1_\mu(M)$, $k \in \{1, ..., d-2\}$ and $m \in \mathbb{N}$. The subset of M formed by the points $p \in M \setminus Sing(X)$ such that there exists an m-dominated splitting of index k along the orbit of p is denoted by $\Lambda_k(X, m)$. The set $\Gamma_k(X, m) = M \setminus \overline{\Lambda_k(X, m)}$ is open and each element of it has an iterate where inequality (2.4) does not hold (for index k) or else it is a singularity of X. We also know, by [34], that generic divergence-free vector fields have only a finite number of (hyperbolic) singularities, thus the singularities form a zero measure set.

Along this paper we will be mainly interested in dominated splittings related to the natural P^t_X-invariant splitting given by Oseledets' Theorem (obtained in Subsection 2.5) and over the orbit of some $p \in \mathcal{O}(X)$, namely,

$$U^j_p = N^1_p \oplus ... \oplus N^j_p \text{ and } S^j_p = N^{j+1}_p \oplus ... \oplus N^\ell_p,$$

where $\ell \leq d-1$, $N^i_p \subset N_p$ for $i = \{1, ..., \ell\}$ and j is some fixed index of the splitting, $j \in \{1, ..., d-2\}$.

Now, we define some sets which will be used in the sequel:

- $\Gamma^\sharp_k(X, m) := \{p \in \Gamma_k(X, m) \cap \mathcal{O}(X) \colon \lambda_k(X, p) > \lambda_{k+1}(X, p)\}$;
- $\Gamma^*_k(X, m) := \Gamma^\sharp_k(X, m) \setminus Per(X)$, where $Per(X)$ denotes the periodic points of the flow X^t including the fixed points;
- $\Gamma_k(X, \infty) := \bigcap\limits_{m \in \mathbb{N}} \Gamma_k(X, m)$ and
- $\Gamma^\sharp_k(X, \infty) := \bigcap\limits_{m \in \mathbb{N}} \Gamma^\sharp_k(X, m)$.

The next lemma (see [17, Lemma 4.1]) allows us to focus our attention only on the non-periodic points.

Lemma 2.4. *For every $\delta > 0$ and $k \in \{1, ..., d-2\}$, there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ we have that*

$$\mu(\Gamma^\sharp_k(X, m) \setminus \Gamma^*_k(X, m)) < \delta.$$

We consider a measurable function $\rho_{X,m} \colon \Gamma^*_k(X, m) \longrightarrow \mathbb{R}$ defined by

$$\rho_{X,m}(p) = \frac{\|P^m_X(p)|_{S_p}\|}{\mathfrak{m}(P^m_X(p)|_{U_p})}.$$

It is clear that if $p \in \Gamma^*_k(X, m)$, then there exists $t \in \mathbb{R}$ satisfying the inequality $\rho_{X,m}(X^t(p)) > \frac{1}{2}$. We define the set $\Delta^*_k(X, m)$ by those points in $\Gamma^*_k(X, m)$ such that $\rho_{X,m}(p) > \frac{1}{2}$. Clearly, $\Gamma^*_k(X, m)$ is the superset of $\Delta^*_k(X, m)$ saturated by the flow, i.e., $\Gamma^*_k(X, m) = \bigcup_{t \in \mathbb{R}} X^t(\Delta^*_k(X, m))$.

In [8, Lemma 2.2] it is proved the following result relating the measures of these two sets.

Lemma 2.5. *Given $\Delta_k^*(X,m)$ and $\Gamma_k^*(X,m)$ as above, if $\mu(\Gamma_k^*(X,m)) > 0$, then $\mu(\Delta_k^*(X,m)) > 0$.*

2.8. **The integrated upper Lyapunov exponent of exterior power of the linear Poincaré flow.** We consider the following function:

$$\begin{array}{rccc} LE_k: & \mathfrak{X}^1_\mu & \longrightarrow & [0,+\infty) \\ & X & \longmapsto & \int_M \lambda_1(\wedge^k(X),p)d\mu(p), \end{array} \tag{2.5}$$

where $\lambda_1(\wedge^k(X),p) = \lambda_1^{\wedge^k}(X,p)$. In the same way we define the function $LE_k(X,\Gamma)$, where $\Gamma \subseteq M$ is an X^t-invariant set, defined by:

$$LE_k(X,\Gamma) = \int_\Gamma \lambda_1(\wedge^k(X),p)d\mu(p).$$

Let $\Sigma_k(X,p)$ denote the sum of the first k Lyapunov exponents of X, that is $\Sigma_k(X,p) = \lambda_1(X,p) + ... + \lambda_k(X,p)$. It is an easy consequence of Theorem 2.3 that for $k = 1,...,d-2$ we have $\Sigma_k(X,p) = \lambda_1(\wedge^k(X),p)$ and so $LE_k(X,\Gamma) = LE_1(\wedge^k(X),\Gamma)$, for any X^t-invariant set Γ. By using [17, Proposition 2.2] we get immediately that:

$$LE_k(X,\Gamma) = \inf_{j\in\mathbb{N}} \frac{1}{j}\int_\Gamma \log\| \wedge^k (P_X^j(p))\| d\mu(p), \tag{2.6}$$

concluding that, for all $k \in \{1,...,d-2\}$, the function (2.5) is an upper semi-continuous function.

3. Proof of Theorem 2

Our arguments to prove Theorem 2 are borrowed from the ones used by Bochi, Fayad and Pujals in [16].

Let us now discuss the three fundamental steps of the proof of Theorem 2 and at the end of the section we complete the proof.

We recall that $X \in \mathfrak{X}^{1+\alpha}_\mu(M)$ is C^1-*robustly transitive* if X^t has a dense orbit and any $Y \in \mathfrak{X}^{1+\alpha}_\mu(M)$ sufficiently C^1-close to X has also a flow with a dense orbit. It is easy to see that if $X \in \mathcal{SE}$, then X must be C^1-robustly transitive.

Let us first observe that, by [12, Theorem 1.1], a C^1-stably ergodic vector field X does not have singularities. As C^s, $s \geq 2$, divergence-free vector fields are C^1-dense in $\mathfrak{X}^1_\mu(M)$ ([37]), by Theorem 1.2 of [12] there exists a C^1-dense subset of $\mathcal{SE}$, $\mathcal{DSE}$ whose vector fields admit a dominated splitting over M.

We say that a dominated splitting $N = N^1 \oplus ... \oplus N^j$ of X is the *finest dominated splitting* if there is no dominated splitting with more that j sub-bundles. As is pointed out in [16] it is possible that the continuation of the finest dominated splitting is not the finest dominated splitting of the perturbed vector field. Hence, we say that a dominated splitting of $X \in \mathfrak{X}^1_\mu(M)$ is *stably finest* if, for every Y sufficiently C^1-close vector field in $\mathfrak{X}^1_\mu(M)$, it has a continuation which is the finest dominated splitting of Y. Flows with stably finest splittings are open and dense in the class of flows with dominated splitting. Given $X \in \mathcal{DSE}$ we take $X_1 \in \mathfrak{X}^{1+\alpha}_\mu(M)$ C^1-close to X and having a stably finest dominated splitting $N = N^1(X_1) \oplus ... \oplus N^k(X_1)$. We denote by

$$\Sigma^i(X_1) = \int_M \log|\det P^1_{X_1}(p)_{|N^i(X_1)}| d\mu(p)$$

the sum of the Lyapunov exponents of the subbundle $N^i(X_1)$. Clearly, by domination, there is at most one index i such that $\Sigma^i(X_1) = 0$. In [11] we proved the following result:

Theorem 3.1. *Let $X_1 \in \mathfrak{X}^{1+\alpha}_\mu(M)$ be a stably ergodic vector field having a (stably finest) dominated splitting. Then, either $\Sigma^i(X_1) \neq 0$, or else X_1 may be approximated, in the C^1-topology, by $X_2 \in \mathfrak{X}^2_\mu(M)$ for which $\Sigma^i(X_2) \neq 0$.*

We observe that the vector field X_2 given by the previous result is stably ergodic and has a finest dominated splitting.

We are ready to give the proof of Theorem 2:

Openness; Let $\mathcal{U}$ be the set of vector fields $X \in \mathcal{SE}$ such that X has a dominated splitting $N^u \oplus N^s$ where $\dim(N^u) = j$, $\lambda_j(X) > 0$ is the lowest exponent in N^u and $\lambda_{j+1}(X) < 0$ is the largest exponent in N^s. By § 2.7 property (P) any $Y \in \mathfrak{X}^1_\mu(M)$, arbitrarily close to X, has a dominated splitting $N^u(Y) \oplus N^s(Y)$. Moreover, since the function that gives the largest exponent $\lambda_{j+1}(\cdot)$ (in N^s) defined by

$$Y \mapsto \inf_{n \in \mathbb{N}} \int_M \log\Big(\|P^n_Y(x)|_{N^s(Y,x)}\|^{1/n}\Big) d\mu(x)$$

is upper semicontinuous, we obtain that $\lambda_{j+1}(Y)$ cannot increase abruptly. Therefore, if Y is close enough to X, we have that $\lambda_{j+1}(Y) < 0$. In the same way the function that gives the lowest exponent $\lambda_j(\cdot)$ in N^u and is defined by

$$Y \mapsto \sup_{n \in \mathbb{N}} \int_M \log\Big(\mathfrak{m}(P^n_Y(x)|_{N^u(Y,x)})^{1/n}\Big) d\mu(x)$$

is lower semicontinuous. Hence, as $\lambda_j(Y)$ cannot decrease abruptly, we have $\lambda_j(Y) > 0$ and we obtain that $\mathcal{U}$ is open.

Density; Let $X \in \mathcal{SE}$. We want to prove that X can be C^1-approximated by a vector field in $\mathcal{U}$. First we choose $X_1 \in \mathcal{DSE}$ arbitrarily close to X. The vector field X_1 has a dominated splitting and by a small C^1-perturbation we obtain a vector field $X_2 \in \mathfrak{X}^{1+\alpha}_\mu(M)$ having a dominated splitting which is stably finest. Fix some $i \in \{1, ..., k\}$; by Theorem 3.1, X_2 may be approximated, in the C^1-topology, by $X_3 \in \mathfrak{X}^2_\mu(M)$ for which $\Sigma^i(X_3) \neq 0$. For each pair of consecutive Oseledets fibers in N^i we do not have domination, thus, arguments developed in the proof of Theorem 1 allow us to C^1-perturb to obtain that the associated Lyapunov exponents are equal. Applying this argument a finite number of times we conclude that the Lyapunov exponents in N^i are all equal and, since their sum is nonzero, we conclude that they are all different from zero and have the same sign. Finally, Y is a nonuniformly hyperbolic vector field and $Y \in \mathcal{U}$.

4. Proof of Theorem 1

Next we consider an abstract object called *a realizable sequence* which will play a central role in the proofs of the two main theorems. Briefly, it consists of the following: we want to change the action of the linear Poincaré flow along the orbit of a given Oseledets point with lack of hyperbolic behavior, in order to decay its exponential asymptotic behavior. However, one single point is meaningless since we consider the Lebesgue measure and, moreover, the chosen point may not be an Oseledets point for the perturbed vector field. So, in broad terms, we consider time-t modified volume-preserving linear maps acting in the normal fiber at p, $L_t(p)\colon N_p \to N_{X^t(p)}$ which perform exactly the action that we want. Then, we build a divergence-free vector field, C^1-close to the original one such that the time-t linear Poincaré map at $q \in K$ of this new vector field has *almost* the same behavior as the map $L_t(p)$, where K is a measurable set contained in a pre-assigned open set inside a small transversal section of p such that both these sets have almost the same measure.

With this definition in mind we are able to "realize dynamically" the perturbations performed in [8] concerning the skew-product flows version of Theorem 1.

This may be seen as a Franks' Lemma (see [23] or [3, 19, 12] for the flows version) of a measure theoretical flavor type.

Definition 4.1. *Given $X \in \mathfrak{X}^1_\mu(M)$, $\epsilon > 0$, $\kappa \in (0,1)$, $\ell \in \mathbb{N}$, and a non-periodic point p, we say that the modified volume-preserving sequence of linear maps $L_j : N_{X^j(p)} \to N_{X^{j+1}(p)}$ for $j = 0, ..., \ell - 1$ is an* (ϵ, κ)-realizable sequence of length ℓ at p *if the following occurs.*

For all $\gamma > 0$, there is $r > 0$ such that for any $(d-1)$-manifold $\mathcal{N}_p$ (tangent to N_p) and any non-empty open set $U \subseteq B(p, r) \subseteq \mathcal{N}_p$, there exist a measurable set $K \subseteq U$ and a divergence-free vector field Y satisfying:

(a) $\overline{\mu}(K) > (1-\kappa)\overline{\mu}(U)$;
(b) $\|Y - X\|_{C^1} < \epsilon$;
(c) *$Y = X$ outside $\mathcal{F}^\ell_X(p)(U)$; and*
(d) *if $q \in K$, then $\|P^1_Y(Y^j(q)) - L_j\| < \gamma$ for all $j = 0, 1, ..., \ell - 1$.*

Let us consider some observations about this definition.

Remark 4.1. *We note that this definition only requires C^1-closeness of vector fields. Also observe that realizable sequences of length $\ell \in \mathbb{R}$ are also allowed and are defined in the obvious way by considering non-integers cut-points.*

Remark 4.2. *It is easy to see that, taking $r > 0$ very small, it is sufficient to prove the property of realizability for a single submanifold $\mathcal{N}_p$.*

Remark 4.3. *By basic Vitali's covering arguments we only have to prove the realizability of the linear maps for open sets $U = B(p', r')$ where $U \subseteq B(p, r)$.*

Remark 4.4. *It is obvious that the time-t linear Poincaré flow is itself (ϵ, κ)-realizable of length t for every ϵ and κ.*

Remark 4.5. *Condition (c) in the Definition 4.1 enables one to concatenate an (ϵ, κ_1)-realizable sequence of length ℓ_1 at p with an (ϵ, κ_2)-realizable sequence of* *length ℓ_2 at $X^{\ell_1}(p)$, obtaining an $(\epsilon, \kappa_1 + \kappa_2)$-realizable sequence of length $\ell_1 + \ell_2$ at p. Notice that if $\kappa_1 + \kappa_2 \geq 1$ then we do not have any useful estimate for the measure of the set $K = K_1 \cap X^{-\ell_1}(K_2)$. Actually we are just interested in the concatenation of b realizable sequences in the cases where $\sum_{j=0}^{b} \kappa_j < 1$ (in fact close to zero) obtaining a measurable set K such that the measure of $U \setminus K$ is less than $\sum_{j=0}^{b} \kappa_j$.*

The next proposition, which is the flow-version of [17, Proposition 3.1] is a key result that allows us to mix the Oseledets directions in the absence of domination. Once we get this result the two lemmas of this section and the proof of Theorem 1 are obtained borrowing [17].

Proposition 4.1. *Given $X \in \mathfrak{X}^1_\mu(M)$, $\epsilon > 0$ and $\kappa \in (0,1)$ there exists $m_0 \in \mathbb{N}$ such that for every real number $m \geq m_0$ the following property holds.*

For any non-periodic point p with a splitting $S^j_{X^t(p)} \oplus U^j_{X^t(p)}$, $t \in \mathbb{R}$ and $j \in \{1, ..., d-2\}$ fixed, satisfying

$$\frac{\|P^m_X(p)|_{S^j_p}\|}{\mathfrak{m}(P^m_X(p)|_{U^j_p})} \geq \frac{1}{2}, \tag{4.1}$$

there exist (ϵ_i, κ_i)-realizable sequences of length $\ell_i \geq 1$ at $X^{\tau_i}(p)$, denoted by $\{L_i\}_{i=1}^{b}$, where $b \in \mathbb{N}$ is the number of cut-points ($b \leq [m]$), with $\sum_{i=1}^{b} \ell_i = m$ and $\tau_i = \sum_{j=1}^{i-1} \ell_j$, and there are vectors $\mathfrak{u} \in U^j_p \setminus \{\vec{0}\}$ and $\mathfrak{s} \in S^j_{X^m(p)} \setminus \{\vec{0}\}$ such that:

(a) *The concatenation of the b realizable sequences is an (ϵ, κ)-realizable sequence of length m at p and*
(b) $L_b \circ ... \circ L_1(\mathfrak{u}) = \mathfrak{s}$

This proposition will be proved in sections 5.1, 5.2 and 5.3. In Section 5.1 we consider the easiest case, that is when we only need to do, at most, two perturbations to achieve our goal. Section 5.2 is technically harder, because we need to do many perturbations along the orbit and each time we concatenate two realizable sequences the relative measure in U of the associated set K decreases. In Section 5.3 we complete the proof.

Once we are able to realize dynamically the action which mixes the Oseledets directions, to prove Theorem 1 we need to use the following two results.

Lemma 4.2. *(Local) Let $X \in \mathfrak{X}^1_\mu(M)$. Then, given any $\epsilon, \delta > 0$, a small $\kappa > 0$ and $k \in \{1, \ldots, d-2\}$, there exist $m_0 \in \mathbb{N}$ and, for each $m \geq m_0$, a measurable function $\tilde{T}\colon \Gamma^*_k(X, m) \to \mathbb{R}$ satisfying the following properties: for μ-almost every point $q \in \Gamma^*_k(X, m)$ and every $t > \tilde{T}(q)$ there exists a modified volume-preserving sequence of linear maps $L_j : N_{X^j(q)} \to N_{X^{j+1}(q)}$ for $j = 0, ..., t-1$ which is an (ϵ, κ)-realizable sequence of length t at q satisfying*

$$\frac{1}{t} \log \| \wedge^k (L_{t-1} \circ \cdots L_1 \circ L_0)\| < \delta + \frac{1}{2} \left(\Sigma_{k-1}(X, q) + \Sigma_{k+1}(X, q)\right).$$

This lemma corresponds to Proposition 4.2 of [17] adapted to the flow setting and, in view of Proposition 4.1, its proof follows exactly as the proof of Lemma 4.2 of [8]. In rough terms it uses the lack of domination and also the different Lyapunov exponents to cause a decay, via a small C^1-perturbation, of the top Lyapunov exponent of the k^{th} exterior power of the linear Poincaré flow.

The next lemma is a global version of the previous one. Once we have the local version (Lemma 4.2) its proof follows directly the proof of Proposition 4.17 of [17] and uses a Kakutani's tower argument. We also refer to [7] for the ingredients used in the flow framework.

Lemma 4.3. *(Global) Let $X \in \mathfrak{X}^1_\mu(M)$. Then, given any $\epsilon, \delta > 0$, and $k \in \{1, \ldots, d-2\}$, there exists $Y \in \mathfrak{X}^1_\mu(M)$, ϵ-C^1-close, such that*

$$\int_M \Sigma_k(Y, p)\, d\mu(p) < \int_M \Sigma_k(X, p)\, d\mu(p) - 2J_k(X) + \delta,$$

where $J_k(X) = \int_{\Gamma_k(X,\infty)} \Big(\lambda_k(X, p) - \lambda_{k+1}(X, p)\Big)\, d\mu(p)$.

Now to prove Theorem 1 we argue exactly as in [17, pp 1467].

For $k \in \{1, ..., d-2\}$ let $\mathcal{E}_k$ be the subset of $\mathfrak{X}^1_\mu(M)$ corresponding to the points of continuity of the map LE_k, see (2.5), and define $\mathcal{E} = \cap_1^{d-2}\mathcal{E}_k$. If follows from semicontinuity of LE_k that the sets $\mathcal{E}_k$ are residual and so is $\mathcal{E}$. If $X \in \mathcal{E}_k$ then, by the definition of this set and by Lemma 4.3, $J_k(X) = 0$. Therefore $\lambda_k(X, p) = \lambda_{k+1}(X, p)$ for a.e $p \in \Gamma_k(X, \infty)$. For $X \in \mathcal{E}$ let:

- $\mathcal{Z} = \mathcal{O}(X) \cap (\cap_{k=1}^{d-2}\Gamma_k(X, \infty))$ and
- $\mathcal{D} = \mathcal{O}(X) \setminus (\cap_{k=1}^{d-2}\Gamma_k(X, \infty))$

If $p \in \mathcal{Z}$ then all the Lyapunov exponents of p are equal to zero. On the other hand if $p \in \mathcal{D}$ then $p \notin \Gamma_k(X, \infty)$ for some $1 \leq k \leq d-2$, therefore, by the definition of these sets, there exists $m \in \mathbb{N}$ such $p \in \Lambda_k(X, m)$, meaning that there exists an m-dominated splitting of index k along the orbit of p. This ends the proof of Theorem 1.

5. Main perturbation lemmas

5.1. **Local perturbations.** We start by proving a key tool that, in broad terms, assures that a time-one linear map, obtained by composing a small rotation with the linear Poincaré flow of a vector field, is realizable.

The next lemma, proved in [11, Lemma 2.1], together with an improvement of the well known Pasting Lemma ([4]), see Lemma 5.2, and Zuppa's Theorem ([37]), will be crucial to perform this construction. We recall that the result of Zuppa says that C^∞ divergence-free vectors fields are C^1-dense in the class of C^1 divergence-free vectors fields.

Lemma 5.1. *Given a vector field $X \in \mathfrak{X}^2_\mu(M)$ a non-periodic point $p \in M$ and $t_0 \in \mathbb{R}^+$, there exists a volume-preserving C^2 diffeomorphism Ψ, defined in a neighborhood of the arc $\{X^t(p);\, t \in [0, t_0]\}$, into $\mathbb{R}^d$ and such that $T = \Psi_* X$, where $T = \frac{\partial}{\partial x_1}$.*

The proof of the next lemma was suggested to us by Carlos Matheus.

Lemma 5.2. *(Smooth C^1-Pasting lemma) Let M be a compact and boundaryless Riemannian manifold of dimension ≥ 2. Given $\epsilon > 0$, $X \in \mathfrak{X}^1_\mu(M)$, a compact $\mathcal{K} \subset M$ and an open neighborhood $\mathcal{U}$ of $\mathcal{K}$, there are $\delta > 0$ and an open set*

$\mathcal{K} \subset \mathcal{V} \subset \mathcal{U}$ such that, if $Y \in \mathfrak{X}^2_\mu(M)$ is δ-C^1-close to X in $\mathcal{U}$, then there exists $Z \in \mathfrak{X}^1_\mu(M)$ satisfying

a) *Z is ϵ-C^1-close to Y in $\mathcal{V}$,*
b) *Z is of class C^2 in $\mathcal{V}$,*
c) *Z is ϵ-C^1-close to X and*
d) *$Z = X$ outside $\mathcal{U}$.*

Proof. We consider $\mathcal{V} \supset \mathcal{K}$ such that $\partial\mathcal{V}$ is C^∞, $\overline{\mathcal{V}} \subset \mathcal{U}$ and $\{\mathcal{U}, \mathrm{int}(M \setminus \overline{\mathcal{V}})\}$ is an open covering of M.

Let $\alpha\colon M \to [0,1]$ be a C^∞ function such that $\alpha = 1$ in $\mathcal{V}$, $\alpha = 0$ outside $\mathcal{U}$ and $|\nabla\alpha| \leq K$, where K is a positive constant depending only on $\mathcal{U}$ and $\mathcal{V}$. Define

$$Z_0(\cdot) := \alpha(\cdot)Y(\cdot) + (1-\alpha(\cdot))X(\cdot) \tag{5.1}$$

where $Y \in \mathfrak{X}^2_\mu(M)$ is δ-C^1-close to X on a small open neighborhood $\mathcal{U}$ of $\mathcal{K}$. We observe that $Z_0 = Y$ inside $\mathcal{K}$ and $Z_0 = X$ outside $\mathcal{U}$. However, although the divergence $\nabla \cdot Z_0$ is close to zero, in general $\nabla \cdot Z_0 \neq 0$. Actually,

$$\begin{aligned} \nabla \cdot Z_0 &= (\nabla\alpha)\cdot Y + \alpha(\nabla \cdot Y) - (\nabla\alpha)\cdot X + (1-\alpha)\nabla \cdot Y \\ &= (\nabla\alpha)\cdot Y - (\nabla\alpha)\cdot X \\ &= (\nabla\alpha)\cdot(Y - X), \end{aligned}$$

and we have $|\nabla \cdot Z_0| < K\delta$.

Now we will make use of [20, Theorem 2] in order to obtain $Z_1 \in \mathfrak{X}^2_\mu(M)$ (supported in $\mathcal{U}$) such that $\nabla \cdot Z_1 = -\nabla \cdot Z_0$.

Finally, we define

$$Z := Z_0 + Z_1. \tag{5.2}$$

Clearly $\nabla \cdot Z = 0$ and b) and d) hold.

In $\mathcal{U}$ we have

$$\begin{aligned} \|Z - Y\|_{C^1} &= \|Z_0 + Z_1 - Y\|_{C^1} \\ &\leq \|Z_1\|_{C^1} + \|Z_0 - Y\|_{C^1} \\ &\leq C\|\nabla \cdot Z_1\|_{C^0} + \|Z_0 - Y\|_{C^1}, \end{aligned}$$

where $C > 0$ is a constant given in Dacorogna-Moser theorem (see [4, Theorem 2.3]) and depending on $\mathcal{U}$ satisfying $\|Z_1\|_{C^1} \leq C\|\nabla \cdot Z_1\|_{C^0}$.

Going back to the beginning of the proof we take $\delta < \min\left\{\epsilon, \frac{\epsilon}{2CK}\right\}$.

Now, using $|\nabla \cdot Z_0| < K\delta$ we get,

$$\begin{aligned} \|Z - Y\|_{C^1} &\leq C\|\nabla \cdot Z_1\|_{C^0} + \|Z_0 - Y\|_{C^1} \\ &\leq CK\delta + \|\alpha Y + (1-\alpha)X - Y\|_{C^1} \\ &\leq \frac{\epsilon}{2} + \|(1-\alpha)X - (1-\alpha)Y\|_{C^1} \\ &\leq \frac{\epsilon}{2} + |1-\alpha|\|X - Y\|_{C^1} \\ &\leq \frac{\epsilon}{2} + \|X - Y\|_{C^1} \leq \frac{\epsilon}{2} + \delta < \epsilon. \end{aligned}$$

$\square$

We fix $X \in \mathfrak{X}^1_\mu(M)$ and we choose a non-periodic point p. Let V_p be a two-dimensional subspace of N_p; given $\xi \in \mathbb{R}$ let $\mathfrak{R}_\xi \colon V_p \to V_p$ denote the rotation of angle ξ. Let $V_p^\perp$ denote the orthogonal complement of V_p in N_p and define $R_\xi \colon N_p \to N_p$ as $R_\xi = \mathfrak{R}_\xi \oplus Id$, that is $R_\xi(v + v^\perp) = \mathfrak{R}_\xi(v) + v^\perp$, where $v \in V_p$ and $v^\perp \in V_p^\perp$.

Lemma 5.3. *Given $X \in \mathfrak{X}^1_\mu(M)$, $\epsilon > 0$ and $\kappa \in (0,1)$, there exists $\xi_0 > 0$ such that for any $\xi \in (0, \xi_0)$, any $p \in M$ (non-periodic or with period larger than one) and any two-dimensional vector space $V_p \subset N_p$ one has that the time-one map $L_1 = P^1_X(p) \circ R_\xi$ is an (ϵ,κ)-realizable sequence of length 1 at p, where $R_\xi = \mathfrak{R}_\xi \oplus Id$ and $\mathfrak{R}_\xi$ is the rotation of angle ξ in V_p.*

Proof. Fix ϵ, κ, X, and take ξ_0 to be fixed later. Let $p \in M$ be a non-periodic point or a periodic point with period larger than one, and let V_p be a two-dimensional subspace of N_p.

Once fixed $\gamma > 0$ of the definition of realizable sequence, the first step is to get $r > 0$ as a function of $X, \epsilon, \kappa, \gamma$ and p and then, for each $U \subset B(p,r) \subset \mathcal{N}_p$, perturb X in $\mathcal{V}$, a flowbox to be defined below, in order to obtain $K \subset U$ and Y as in Definition 4.1. The way we get r is by shrinking step by step its value along the proof. According to Remark 4.3 we can assume that $U = B(p', r') \subset B(p,r) \subset \mathcal{N}_p$, for some $p' \in B(p,r)$ and $r' < r$. We observe that if we prove the lemma for this particular normal section then the general case follows just by shrinking r.

Now, for fixed U and κ, we will choose K and the vector field Y such that $Y = X$ outside the flowbox $\mathcal{U} := \mathcal{F}^1_X(p')(U)$. Let $K = B(p', r'')$, where $r'' < r'$ is chosen such that $\overline{\mu}(K) > (1-\kappa)\overline{\mu}(U)$, and fix $V = B(p', s)$, where $r'' < s < r'$. Let us define $\mathcal{K} := \mathcal{F}^1_X(p')(K)$ and $\mathcal{V} := \mathcal{F}^1_X(p')(V)$. Let Y_1 be a C^2-divergence-free vector field, arbitrarily close to X, given by Zuppa's theorem ([37]); in particular their C^1-distance is less then δ. Now we apply Lemma 5.2 in order to get a vector field Z as in the conclusions of that lemma.

As Z is of class C^2 in $\mathcal{V}$ we use Lemma 5.1 to obtain a C^2 local change of coordinates Ψ which trivialize Z, that is $T = \Psi_* Z$ in $\mathcal{V}$.

Let $\mathcal{K} := \Psi(K)$, $\mathcal{V} := \Psi(V)$ and $\mathcal{U} := \Psi(U)$. If the initial r is small enough there are two balls B_1 and B_2, centered at $0 \in \mathbb{R}^d$ and contained in $\{0\} \times \mathbb{R}^{d-1}$, such that $\mathcal{K} \subset B_1 \subset B_2 \subset \mathcal{V}$.

The perturbation will be done in $B_2 \times [\tau, 1-\tau]$, for a small $\tau > 0$. Consider $\overline{p} := \Psi^{-1}(T^\tau(0)) = \Psi^{-1}((\tau, 0, \cdots, 0))$, and let $V_\tau := D\Psi_{\overline{p}}(V_{\overline{p}})$, where $V_{\overline{p}}$ is the parallel transport of V_p to $\overline{p}$. In Figure 1 we illustrate the sets we are considering.

Now to obtain the conclusions of the lemma, we are going to induce a rotation in V_τ and obtain the desired perturbed vector field Y by making a pullback via Ψ.

Actually, following the proof of [11, Lemma 2.2], we obtain a constant $\tilde{\xi}_0 > 0$ and, for any $0 < \xi < \tilde{\xi}_0$, a divergence-free vector field $T + P$ satisfying the following conditions:

(i) P is C^1-arbitrarily close to the null vector field and is supported in $B_2 \times [\tau, 1-\tau]$;

(ii) $P^{1-2\tau}_{T+P}(\overline{q})_{|W_{\overline{q}}}$ is the identity, where $W_{\overline{q}}$ is the orthogonal complement of $\overline{q}+\overline{V}_{\overline{p}}$ in $N_{\overline{q}}$, for all $\overline{q}\in B_1$, where $\overline{V}_{\overline{p}} := D\Psi_{\overline{p}}(V_{\overline{p}})$;

(iii) $P^{1-2\tau}_{T+P}(\overline{q})\cdot v = P^{1-2\tau}_{T}(\overline{p})\circ R_\xi \cdot v$, $\forall v\in \overline{q}+\overline{V}_{\overline{p}}$, where R_ξ is the rotation of angle ξ on $\overline{q}+\overline{V}_{\overline{p}}$, for all $\overline{q}\in B_1$.

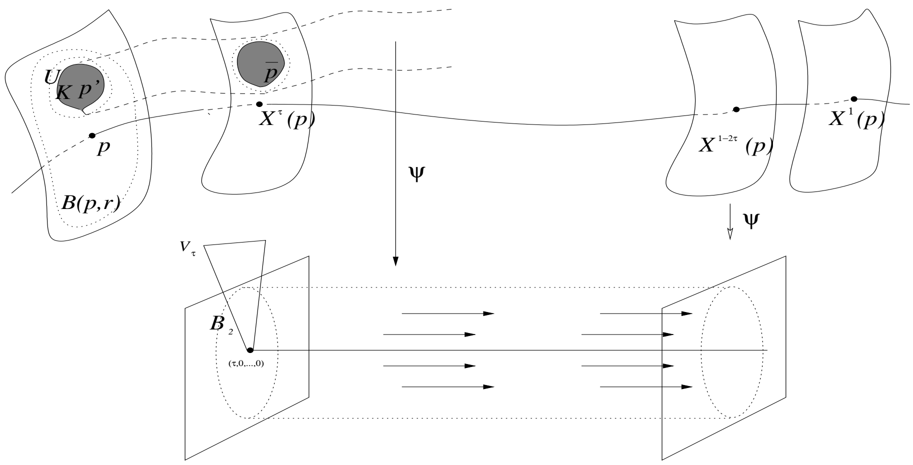

FIGURE 1

Notice that condition (i) completely determines ξ_0.

As we mention before we take $Y := \Psi_*(T+P)$. It straightforward to verify that we can obtain such an Y satisfying the properties:

(i) Y is C^1-arbitrarily close to X and equal to X outside $\Psi^{-1}(B_2\times[\tau,1-\tau])$;

(ii) $P^1_Y(q)_{|W_q}$ is the identity, where W_q is the orthogonal complement of V_q in N_q, for all $q\in K$;

(iii) $P^1_Y(q)\cdot v$ is close to $P^1_X(p')\circ R_\xi\cdot v$, $\forall v\in V_q$, where, for all $q\in K$, R_ξ is the rotation of angle ξ on V_q ,

as long as we consider r and $\tilde{\xi}_0$ small enough.

Now it is easy to check that $L_1 = P^1_X(p)\circ R_\xi$ is an (ϵ,κ)-realizable sequence of length 1 at p.

□

Remark 5.1. *A completely analog proof of the previous lemma, this time considering the vector field* $(-X)$ *and a rotation of angle* $(-\xi)$*, guarantees that the time-*1 *map* $R_\xi\circ P^1_X(X^{-1}(p))$ *is also* (ϵ,κ)*-realizable sequence of length* 1 *at* $X^{-1}(p)$*, for small* ξ*.*

If we want to prove that certain time-2 map is (ϵ,κ)-realizable sequence, which is a rotation in a fixed two-dimensional subspace V of the normal space, we can try to do it by concatenating two (ϵ,κ_i)- realizable sequences being the first one a rotation on V and the other one being an elliptical rotation on the image of V by the time-one linear Poincaré flow. For that we need to adapt Lemma 5.3 for elliptical rotations. Note that, according to Remark 4.5, the resulting linear flow is $(\epsilon,\kappa_1+\kappa_2)$-realizable, and so the measure of the set K (see Definition 4.1) decreases. Therefore there is no hope to use directly this concatenation argument to prove that a rotation of a given angle is (ϵ, κ)-realizable sequence of length ℓ, for large ℓ, although we need large time to get the desired rotation by a composition of rotations of small angle. Lemma 5.5 below solves this problem.

Fix a regular point $p \in M$ and a two-dimensional subspace V_p of N_p. A *right cylinder* centered at p is a subset of N_p of the form $\mathcal{B} \oplus \mathcal{A}$, where $\mathcal{B}$ is called the basis of the cylinder and it is an ellipse contained in V_p, and $\mathcal{A}$ is called the axis of the cylinder and it is a $(d-3)$-dimensional ellipsoid contained in the $V_p^{\perp}$, the orthogonal complement of V_p in N_p (see Figure 2).

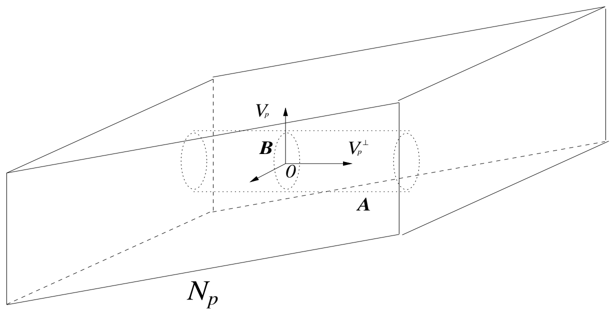

FIGURE 2

Let $\mathcal{B} \oplus \mathcal{A}$ be a right cylinder centered at p. Let $h\colon V_p \to \mathbb{R}^2$ be an area-preserving map such that $h(\mathcal{B})$ is a disk. An elliptical rotation of angle θ of the cylinder is a map $R_\theta\colon \mathcal{B} \oplus \mathcal{A} \to \mathcal{B} \oplus \mathcal{A}$ of the form $\mathcal{E}_\theta \oplus Id$, where $h \circ \mathcal{E}_\theta \circ h^{-1}$ is a rotation of angle θ. We point out that in the sequel, and since we are in an almost conformal case (see (5.10)), we will deal with deformations h close to the identity.

Concerning the case of elliptical rotations a direct adaptation of the proof of the previous lemma jointly with the strategy followed in [17, Lemma 3.4] give the following result.

Lemma 5.4. *Given $X \in \mathfrak{X}^1_\mu(M)$, $\epsilon > 0$ and $\sigma \in]0,1[$. There exists $\xi > 0$ such that for any $p \in M$ a non-periodic point or with period larger than one, the following holds: Let $\mathcal{B} \oplus \mathcal{A}$ be a right cylinder centered at p and consider an elliptical rotation $R_\xi = \mathcal{E}_\xi \oplus Id$ close to the identity. For $a, b > 0$, consider the cylinder $\mathcal{C} = b\mathcal{B} \oplus a\mathcal{A}$. There exists $\tau > 1$ such that if $a \geq \tau b$ and diam$(\mathcal{C}) < \epsilon$, then there exists $Y \in \mathfrak{X}^1_\mu(M)$ satisfying*

- *$Y = X$ outside $\mathcal{F}^1_X(p)(\hat{\mathcal{C}})$, where $\hat{\mathcal{C}} = \alpha(\mathcal{C})$ for some volume-preserving chart α (according to Section 2.3);*
- *$\|P^1_Y(q) - P^1_X(q) \circ R_\xi\| \ll \xi$, for all $q \in \sigma\mathcal{C}$, and*
- *Y and X are ϵ-C^1-close.*

5.2. **Large length perturbations.** Next lemma is the continuous-time version of [17, Lemma 3.3] that allows us to realize a large concatenation of time-one perturbations under certain conditions. The proof of the lemma follows closely the arguments of Bochi-Viana's aforementioned lemma with some extra care when dealing with the involved measure. We present the guidelines of the proof emphasizing these aspects.

Lemma 5.5. *Given $X \in \mathfrak{X}^1_\mu(M)$, $\epsilon > 0$ and $\kappa > 0$, there exists $\xi > 0$ with the following properties: assume that $p \in M$ is a non-periodic point and that for some $n \in \mathbb{N}$, and for each $j \in \{0, 1, ..., n-1\}$ we have*

- *co-dimension two spaces $H_j \subset N_{X^j(p)}$ such that $H_j = P^j_X(p)(H_0)$;*
- *ellipses $\mathcal{B}_j \subset (N_{X^j(p)})/H_j$ centered at zero[1] such that $\mathcal{B}_j = (P^j_X(p)/H_0)(\mathcal{B}_0)$ and*
- *linear maps $\mathcal{E}_j \colon (N_{X^j(p)})/H_j \to (N_{X^j(p)})/H_j$ such that $\mathcal{E}_j(\mathcal{B}_j) = \mathcal{B}_j$ and $\|\mathcal{E}_j - Id\| < \xi$.*

For each $j \in \{0, 1, ..., n-1\}$ we define $L_j = P^1_X(X^j(p)) \circ R_j$, where $R_j = \mathcal{E}_j \oplus Id$. Then $\{L_0, L_1, ..., L_{n-1}\}$ is an (ϵ, κ)-realizable sequence of length n at p.

Proof. Let us fix a small $\gamma > 0$ according to the definition of realizable sequence. We have to choose a sufficiently small $r > 0$ such that $\mathcal{F}^n_X(p)(B(p, r))$ is a flowbox, and

$$\|P^1_X(q) - P^1_X(X^j(p))\| \cdot \|R_j\| < \frac{\gamma}{2},$$

for every $q \in \mathcal{P}^j_X(p)(B(p, r))$ and $j \in \{0, 1, ..., n-1\}$. This r will be shrunk along the proof.

We start with a ball $\mathcal{A}_0 \subset H_0$ and, for $j \in \{1, ..., n-1\}$, let $\mathcal{A}_j = P^j_X(p)(\mathcal{A}_0)$. By [17, Lemma 3.6], for each j, there exist $\hat{\tau}_j > 1$ such that if $a > \hat{\tau}_j b$ then,

$$P^1_X(X^j(p))(b\mathcal{B}_j \oplus a\mathcal{A}_j) \supset b\mathcal{B}_{j+1} \oplus \lambda a\mathcal{A}_{j+1}, \tag{5.3}$$

for some $\lambda \in (0, 1)$ chosen sufficiently close to 1 and depending on κ. This allows us to "rightify", at each step, the basis of the iterated cylinder and keeping almost the same $\overline{\mu}$-measure.

We feed Lemma 5.4 with ϵ, σ and we get $\xi_j > 0$ and, for each j, $\tau_j > 1$ such that if $a \geq \tau_j b$ and the diameter of each cylinder C_j is sufficiently small, then we can realize a rotation on the basis $\mathcal{B}_j$ for ϵ-close vector fields Y_j.

Let $\tau = \max_{j=0,\cdots,n}\{\tau_j, \hat{\tau}_j\}$ and $\xi = \min_{j=0,\cdots,n}\{\xi_j\}$. We have to consider cylinders with the axis much larger than the basis. Actually, we fix $a_0, b_0 > 0$ satisfying $a_0 > b_0\lambda^{-n}\tau$.

We define, for each j, $C_j = \lambda^j b_0 \mathcal{B}_j \oplus \lambda^{2j} a_0 \mathcal{A}_j$.

By linear approximation properties (see [17, Lemma 3.5]), there exists $\{r_j\}_{j=0}^n$ such that for $\rho > 0$ and $q_j \in B(X^j(p), r_j) \subset \mathcal{N}_{X^j(p)}$, if[2] $\rho C_j + q_j \subset B(X^j(p), r_j)$, then

$$X^1(\rho C_j + q_j) \supset \lambda P^1_X(X^j(p))(\rho C_j) + X^1(q_j).$$

Now, if necessary, we decrease r in order to have, for each j, $X^j(B(p, r)) \subset B(X^j(p), r_j)$.

Let Y be the divergence-free vector field and K the set defined by:

- $Y|_{\mathcal{F}^1_X(X^j(p))(C_j)} = Y_j$;
- $K = \mathcal{P}^{-n}_Y(Y^n(p))(\sigma\rho C_n + q_n)$.

---

[1]We can identify each $\mathcal{B}_j \subset (N_{X^j(p)})/H_j$ as a subset of a 2-dimensional space $H_j^\perp$ using an inner product that makes de projection $H_j^\perp \to (N_{X^j(p)})/H_j$ an isometry.

[2]In order to simplify the presentation, instead of using the Moser atlas (cf. §2.3) we perform the computations assuming that we are working in the Euclidian space.

Note that

$$\sigma\rho C_n + q_n = \sigma\rho(\lambda^n b\mathcal{B}_n \oplus \lambda^{2n} a\mathcal{A}_n) + q_n.$$

Let μ^* be the Lebesgue measure with a density given by the pull-back of $\overline{\mu}$ by the volume-preserving charts. Given a right cylinder C defined by ellipses $\mathcal{B}$ and $\mathcal{A}$, applying the Rokhlin Theorem ([35]), locally, using the right cylinder structure we can decompose this measure as $\mu^* = \mu_{\mathcal{B}} \times \mu_{\mathcal{A}}$.

As a consequence of Lemma 2.1 we get that, for $S \in N_p$, $\overline{\mu}(S)$ is close to $x(t)\overline{\mu}(P^t_X(p)(S))$. For a sufficiently small set S the disintegration gives

$$\mu^*(S) = \int_S d\mu_{\mathcal{B}} d\mu_{\mathcal{A}} = \int_{P^t_X(p)(S)} \delta(\cdot) d\mu_{\mathcal{B}_t} d\mu_{\mathcal{A}_t},$$

where

- $\delta(\cdot)$ is the density with respect to the $(n-1)$-volume and depends on t and on a space variable of dimension $n-1$;
- $\mathcal{B}_t = (P^t_X(p)/H_0)(\mathcal{B})$ and
- $\mathcal{A}_t = P^t_X(p)(\mathcal{A})$.

Since S is taken arbitrarily small, the density δ can be replaced by a density φ depending only on t such that $\mu^*(S)$ is close to

$$\int_{P^t_X(p)(S)} \varphi(\cdot) d\mu_{\mathcal{B}_t} d\mu_{\mathcal{A}_t}$$

Note that the same holds if one considers the restriction to $\mathcal{A}$ or to $\mathcal{B}$. Now

$$\begin{aligned}
\frac{\overline{\mu}(K)}{\overline{\mu}(U)} &= \frac{\overline{\mu}(\mathcal{P}^{-n}_Y(Y^n(p))(\sigma\rho C_n + q_n))}{\mu^*(\rho b\mathcal{B}_0 \oplus \rho a\mathcal{A}_0)} \\
&= \frac{\|X(p)\|\,\overline{\mu}(\mathcal{P}^{-n}_Y(Y^n(p))(\sigma\rho C_n + q_n))}{\|X(p)\|\,\mu^*(\rho b\mathcal{B}_0 \oplus \rho a\mathcal{A}_0)} \\
&\approx \frac{\|X(X^n(p))\|\,\mu^*(\sigma\rho\lambda^n b\mathcal{B}_n \oplus \sigma\rho\lambda^{2n} a\mathcal{A}_n + q_n)}{\|X(p)\|\,\mu^*(\rho b\mathcal{B}_0 \oplus \rho a\mathcal{A}_0)} \\
&= x(n)\frac{\mu^*\Big(\sigma\lambda^n b(P^n_X(p)/H_0)(\mathcal{B}_0) \oplus \sigma\lambda^{2n} a P^n_X(p)(\mathcal{A}_0) + q_n\Big)}{\mu^*(b\mathcal{B}_0 \oplus a\mathcal{A}_0)} \\
&= x(n)\frac{\mu^*\Big(\sigma\lambda^n (P^n_X(p)/H_0)(\mathcal{B}_0) \oplus \sigma\lambda^{2n} P^n_X(p)(\mathcal{A}_0) + q_n\Big)}{\mu^*(\mathcal{B}_0 \oplus \mathcal{A}_0)} \\
&= x(n)\frac{(\lambda^n\sigma)^3 \mu_{\mathcal{B}_n}((P^n_X(p)/H_0)(\mathcal{B}_0))(\lambda^{2n}\sigma)^{d-3}\mu_{\mathcal{A}_n}(P^n_X(p)(\mathcal{A}_0))}{\mu_{\mathcal{B}}(\mathcal{B}_0)\mu_{\mathcal{A}}(\mathcal{A}_0)} \\
&\approx x(n)\frac{(\lambda^n\sigma)^3(\lambda^{2n}\sigma)^{d-3}x(n)^{-1}\mu_{\mathcal{B}}(\mathcal{B}_0)\mu_{\mathcal{A}}(\mathcal{A}_0)}{\mu_{\mathcal{B}}(\mathcal{B}_0)\mu_{\mathcal{A}}(\mathcal{A}_0)},
\end{aligned}$$

and we obtain that $\frac{\overline{\mu}(K)}{\overline{\mu}(U)}$ is of order of $\lambda^{2nd-3n}\sigma^d$. Now it is clear that λ and σ can be chosen such that condition (a) of Definition 4.1 is satisfied.

We are left to prove that if $q \in K$, then $\|P^1_Y(Y^j(q)) - L_j\| < \gamma$ for $j = 0, 1, ..., \ell - 1$. Since $L_j = P^1_X(X^j(p)) \circ R_j$ we get

$$\begin{aligned} \|P^1_Y(Y^j(q)) - L_j\| &= \|P^1_Y(Y^j(q)) - P^1_X(X^j(p)) \circ R_j\| \\ &= \|P^1_Y(Y^j(q)) - P^1_X(X^j(q)) \\ &\quad + P^1_X(X^j(q)) - P^1_X(X^j(p)) \circ R_j\| \\ &\leq \|P^1_Y(Y^j(q)) - P^1_X(X^j(q))\| + \frac{\gamma}{2} < \gamma, \end{aligned}$$

where the last inequality is assured if we take r small enough. □

5.3. **Proof of Proposition 4.1.** In order to prove Proposition 4.1 we follow the strategy in [17, Proposition 3.1]. This proposition has an easy proof when the lack of dominated splitting comes from a small angle between the two fibers U and S or else comes from the fact that S "expands" much more than U.

Let $X \in \mathfrak{X}^1_\mu(M)$, $\epsilon > 0$ and $0 < \kappa < 1$ be given as in Proposition 4.1. Take $\kappa' \in \left(0, \frac{1}{2}\kappa\right)$ and let $\xi_0 = \xi_0(X, \epsilon, \kappa')$ be given by Lemma 5.3.

Finally, take:

$$c \geq \frac{1}{\sin^2(\xi_0)} \text{ and } c \geq \sup_{t \in [0,2]} \left(\sup_{x \in \mathcal{R}(X)} \frac{\|P^t_X(x)\|}{\mathfrak{m}(P^t_X(x))} \right) \tag{5.4}$$

Let $\theta > 0$ be such that $8\sqrt{2}c \sin\theta < \epsilon \sin^6(\xi_0)$. Take $m \geq 2\pi/\theta$.

Given a non-periodic point p and a splitting of the normal bundle at p, $N_p = U_p \oplus S_p$ such that

$$\frac{\|P^m_X(p)|_{S_p}\|}{\mathfrak{m}(P^m_X(p)|_{U_p})} \geq \frac{1}{2},$$

and we assume that

$$\text{there exists } t \in [0, m] \text{ such that } \measuredangle(U_t, S_t) = \xi \leq \xi_0, \tag{5.5}$$

where we use the notation $U_t = P^t_X(p)(U_p)$ and $S_t = P^t_X(p)(S_p)$ for $t \in [0, m]$.

Then we take unit vectors $s_t \in S_t$ and $u_t \in U_t$ with $\measuredangle(s_t, u_t) < \xi_0$. If $t \in [0, m-1]$, then we use Lemma 5.3 with $V_{X^t(p)} = \langle s_t, u_t \rangle$, where $\langle e_1, e_2 \rangle$ denotes the vector space spanned by e_1 and e_2, and we define the sequence:

$$L_1 = P^t_X(p)\,,\, L_2 = P^1_X(X^t(p)) \circ R_\xi \text{ and } L_3 = P^{m-t-1}_X(X^{t+1}(p)) \tag{5.6}$$

On the other hand, if $t \in (m-1, m]$, then we use Remark 5.1 and we define:

$$L_1 = P^{t-1}_X(p)\,,\, L_2 = R_\xi \circ P^1_X(X^{t-1}(p)) \text{ and } L_3 = P^{m-t}_X(X^t(p)) \tag{5.7}$$

It is clear, using Remark 4.4, that the concatenation of three realizable sequence (5.6) is an (ϵ, κ)-realizable sequence of length m at p. The same works for (5.7).

In both cases we obtain vectors $\mathfrak{u} \in U_p \setminus \{\vec{0}\}$ and $\mathfrak{s} \in P^m_X(S_p) \setminus \{\vec{0}\}$ such that $L_3 \circ L_2 \circ L_1(\mathfrak{u}) = \mathfrak{s}$. Therefore, under the hypothesis (5.5), the proof of Proposition 4.1 is completed.

Now we assume that there exist $r > 0$ and $t \in \mathbb{R}$ with $0 \le t + r \le m$ such that:

$$\frac{\|P_X^r(X^t(p))|_{S_t}\|}{\mathfrak{m}(P_X^r(X^t(p))|_{U_t})} \ge c. \tag{5.8}$$

Observe that, by the choice of c, we have that $r \ge 2$.

We consider the unit vectors:

- $s_t \in S_t$ such that $\|P_X^r(X^t(p)) \cdot s_t\| = \|P_X^r(X^t(p))|_{S_t}\|$;
- $u_t \in U_t$ such that $\|P_X^r(X^t(p)) \cdot u_t\| = \mathfrak{m}(P_X^r(X^t(p))|_{U_t})$;
- $u_{t+r} = \frac{P_X^r(X^t(p)) \cdot u_t}{\|P_X^r(X^t(p)) \cdot u_t\|} \in U_{t+r}$ and
- $s_{t+r} = \frac{P_X^r(X^t(p)) \cdot s_t}{\|P_X^r(X^t(p)) \cdot s_t\|} \in S_{t+r}$.

The vector $\hat{u}_t = u_t + \sin(\xi_0) s_t$ is such that $\measuredangle(\hat{u}_t, u_t) < \xi_0$ so we consider $L_2 = P_X^1(X^t(p)) \circ R_2$, where R_2 is the rotation on $V_{X^t(p)} = \langle s_t, u_t \rangle$, which sends u_t into $\frac{\hat{u}_t}{\|\hat{u}_t\|}$.

Let

$$\varrho = \frac{\|P_X^r(X^t(p)) \cdot u_t\|}{\sin(\xi_0)\, \|P_X^r(X^t(p)) \cdot s_t\|}.$$

Let us define a vector in $N_{X^{t+r}(p)}$ by $\hat{s}_{t+r} = \varrho u_{t+r} + s_{t+r}$. We have that,

$$\begin{aligned}
P_X^r(X^t(p)) \cdot \hat{u}_t &= P_X^r(X^t(p)) \cdot u_t + \sin(\xi_0) P_X^r(X^t(p)) \cdot s_t \\
&= P_X^r(X^t(p)) \cdot u_t + \frac{\|P_X^r(X^t(p)) \cdot u_t\|}{\varrho \|P_X^r(X^t(p)) \cdot s_t\|} P_X^r(X^t(p)) \cdot s_t \\
&= \frac{1}{\varrho} \|P_X^r(X^t(p)) \cdot u_t\| \cdot (\varrho u_{t+r} + s_{t+r}) \\
&= \frac{1}{\varrho} \|P_X^r(X^t(p)) \cdot u_t\| \cdot \hat{s}_{t+r}.
\end{aligned}$$

It follows from (5.4), (5.8) and definition of ϱ, u_t and s_t, that

$$\varrho = \frac{\mathfrak{m}(P_X^r(X^t(p))|_{U_t})}{\|P_X^r(X^t(p))|_{S_t}\| \sin \xi_0} \le \frac{1}{c \sin \xi_0} < \sin \xi_0.$$

So, $\measuredangle(s_{t+r}, \hat{s}_{t+r}) < \xi_0$. Let $L_4 = R_4 \circ P_X^1(X^{t+r-1}(p))$, where R_4 acts in $V_{X^{t+r}(p)} = \langle s_{t+r}, \hat{s}_{t+r} \rangle$ and sends $\frac{\hat{s}_{t+r}}{\|\hat{s}_{t+r}\|}$ into s_{t+r}. By Remark 5.1 we obtain that L_4 is a realizable sequence of length 1 at $X^{t+r-1}(p)$. Now we concatenate as follows:

$$N_p \xrightarrow{L_1} N_{X^t(p)} \xrightarrow{L_2} N_{X^{t+1}(p)} \xrightarrow{L_3} N_{X^{t+r-1}(p)} \xrightarrow{L_4} N_{X^{t+r}(p)} \xrightarrow{L_5} N_{X^m(p)},$$

where $L_1 = P_X^t(p)$, $L_3 = P_X^{r-2}(X^{t+1}(p))$ and $L_5 = P_X^{m-t-r}(X^{t+r}(p))$, and $L_3 = Id$ if $r = 2$. In this way, applying Lemma 5.3 twice and recalling that L_2 and L_4 are (ϵ, κ')-realizable sequence of length 1, we obtain an (ϵ, κ)-realizable sequence of length m at p such that $L_5 \circ L_4 \circ L_3 \circ L_2 \circ L_1(\mathfrak{u}) = \mathfrak{s}$, where $\mathfrak{u} = P_X^{-t}(X^t(p)) \cdot u_t$ and $\mathfrak{s}$ is a vector co-linear with $P_X^{m-t}(X^t(p)) \cdot s_t$.

So, assuming (5.8), the proof of Proposition 4.1 is done.

Now we shall finish the proof of Proposition 4.1 by considering the last case, that is, when we have that neither (5.5) nor (5.8) is satisfied, that is:

$$\text{for all } t \in [0, m] \text{ we have that } \measuredangle(U_t, S_t) > \xi_0, \tag{5.9}$$

and for all $r > 0$ and $t \in \mathbb{R}$ with $0 \le r + t \le m$ we have:

$$\frac{\|P^r_X(X^t(p))|_{S_t}\|}{\mathfrak{m}(P^r_X(X^t(p))|_{U_t})} < c. \tag{5.10}$$

We define unit vectors $u \in U_0$ and $s \in S_0$ such that

- $\|P^m_X(p) \cdot u\| = \mathfrak{m}(P^m_X(p)|_{U_0})$ and
- $\|P^m_X(p) \cdot s\| = \|P^m_X(p)|_{S_0})\|$.

Now define $s' = \frac{P^m_X(p)\cdot s}{\|P^m_X(p)\cdot s\|} \in S_m$. Like in [17, Lemma 3.8] we consider $G_0 = U_0 \cap u^{\perp}$, $G_t = P^t_X(p)(G_0) \subseteq U_t$ for $t \in [0, m]$, $F_m = S_m \cap (s')^{\perp}$ and $F_t = P^{t-m}_X(p)(F_m) \subseteq S_t$ for $t \in [0, m]$. We consider unit vectors $v_t \in U_t \cap G_t^{\perp}$ and $w_t \in S_t \cap F_t^{\perp}$ for $t \in [0, m]$.

We continue defining some useful objects; let, for $t \in [0, m]$, $H_t = G_t \oplus F_t$ and $I_t = v_t \cdot \mathbb{R} \oplus w_t \cdot \mathbb{R}$. Let $\mathcal{B}_0 \subset N_p/H_0$ be a ball and, for $t \in [0, m]$, $\mathcal{B}_t = (P^t_X(p)/H_0)(\mathcal{B}_0)$.

The conditions (5.9) and (5.10) together with [17, Lemma 3.8] allows us to conclude that for every $t \in [0, m]$ we have

$$\frac{\|P^t_X(q)/H_0\|}{\mathfrak{m}(P^t_X(q)/H_0)} \le \frac{8c}{\sin^6(\xi_0)}, \tag{5.11}$$

where the norm in the quotient space is defined in a standard way (see [17, pp. 1441]).

Since $m \ge 2\pi/\theta$ we take $\{\theta_j\}_{j=0}^{m-1}$ such that $|\theta_j| \le \theta$ for all j and $\sum_{j=0}^{m-1} \theta_j = \measuredangle(v_0 + H_0, w_0 + H_0)$.

Let $\mathcal{E}_{\theta_j}$ be the rotation of angle θ_j in N_p/H_0 and define:

$$R_{\theta_j} = (P^j_X(p)/H_0) \circ \mathcal{E}_{\theta_j} \circ (P^j_X(p)/H_0)^{-1}.$$

Clearly we have $R_{\theta_j}(\mathcal{B}_j) = \mathcal{B}_j$ for all j. Moreover, by (5.11) we obtain,

$$\|R_{\theta_j} - Id\| \le \frac{\|P^j_X(p)/H_0\|}{\mathfrak{m}(P^j_X(p)/H_0)} \|\mathcal{E}_{\theta_j} - Id\| \le \frac{8c}{\sin^6(\xi_0)} \sqrt{2} \sin\theta < \epsilon.$$

For each $j \in \{0, 1, ..., n-1\}$ we define $L_j = P^1_X(X^j(p)) \circ R_j$. Then, by Lemma 5.5, $\{L_0, L_1, ..., L_{n-1}\}$ is an (ϵ, κ)-realizable sequence of length n at p.

Notice that $L_j|_{H_j} = P^1_X(X^j(p))|_{H_j}$ and

$$L_j/H_j = (P^1_X(X^j(p))/H_j) \circ R_{\theta_j} = (P^{j+1}_X(p)/H_0) \circ \mathcal{E}_{\theta_j} \circ (P^j_X(p)/H_0)^{-1}.$$

By composing the sequences we obtain,

$$(L_{m-1} \circ ... \circ L_0)/H_0 = (P^m_X(p)/H_0) \circ \mathcal{E}_{\theta_j} \circ ... \circ \mathcal{E}_{\theta_1}.$$

Applying to $v_0 + H_0$ we get

$$(L_{m-1} \circ ... \circ L_0)(v_0 + H_0) = (P^m_X(p)/H_0)(w_0 + H_0) = P^m_X(p)(w_0) + H_m.$$

Note that $H_m = G_m \oplus I_m$. Then,

$$(L_{m-1} \circ ... \circ L_0)(v_0) = P^m_X(p)(w_0) + g_m + i_m,$$

where $g_m \in G_m$ and $i_m \in I_m$.

Let $g_0 = (P^m_X(p))^{-1}(g_m) \in G_0 \subset H_0 \cap U_0$.

Clearly, $(L_{m-1} \circ ... \circ L_0)(g_0) = g_m$, therefore the vector $v_0 - g_0 \in U_0$ is such that

$$(L_{m-1} \circ ... \circ L_0)(v_0 - g_0) = P_X^m(p)(w_0) + i_m \in S_m,$$

and Proposition 4.1 is proved.

Acknowledgements

We would like to thank Carlos Matheus for the suggestion given for the proof of Lemma 5.2. We would like to thank also the referees for their suggestions, corrections and careful reading of the manuscript.

Research funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT – Fundação para a Ciência e a Tecnologia under the project PEst – C/MAT/UI0144/2011. The authors were partially supported through the project PTDC/MAT/099493/2008.

References

[1] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Trudy Mat. Inst. Steklov., 90, (1967), 209 pp.
[2] V. Araújo and M. Bessa, *Dominated splitting and zero volume for incompressible three-flows*, Nonlinearity, 21, 7, (2008), 1637–1653.
[3] V. Araújo and M. J. Pacifico., *Three-Dimensional Flows*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol 53 - Springer 2010.
[4] A. Arbieto and C. Matheus, *A pasting lemma and some applications for conservative systems*, Ergod. Th. & Dynam. Sys., 27, 5 (2007), 1399–1417.
[5] L. Arnold, *Random Dynamical Systems* Springer Verlag, (1998).
[6] A. Avila, *On the regularization of conservative maps*, Acta Mathematica, 205 (2010), 5–18.
[7] M. Bessa, *The Lyapunov exponents of zero divergence three-dimensional vector fields*, Ergod. Th. & Dynam. Sys. 27, 5 (2007), 1445-1472.
[8] M. Bessa, *Dynamics of generic multidimensional linear differential systems* Adv. Nonlinear Stud., 8 (2008), 191–211.
[9] M. Bessa and J.L. Dias, *Generic dynamics of 4-dimensional C^2 Hamiltonian systems*, Comm. Math. Phys., 281, (2008), 597–619.
[10] M. Bessa and P. Duarte, *Abundance of elliptic dynamics on conservative 3-flows*, Dynamical Systems, 23, 4, (2008), 409–424.
[11] M. Bessa and J. Rocha, *Removing zero Lyapunov exponents in volume-preserving flows*, Nonlinearity 20 (2007), 1007–1016.
[12] M. Bessa and J. Rocha, *On C^1-robust transitivity of volume-preserving flows*, Journal Diff. Eq. 245, 11, (2008), 3127–3143.
[13] M. Bessa and J. Rocha, *Denseness of ergodicity for a class of volume-preserving flows*, Portugaliae Mathematica, 68, 1, (2011) 1–17.
[14] J. Bochi, *Genericity of zero Lyapunov exponents* Ergod. Th. & Dynam. Sys., 22: (2002) 1667–1696.
[15] J. Bochi, *C^1-generic symplectic diffeomorphisms: partial hyperbolicity and zero center Lyapunov exponents*, Journal of the Institute of Mathematics of Jussieu, 9, 1 (2010), 49–93.
[16] J. Bochi, B. Fayad and E. Pujals, *A remark on conservative diffeomorphisms*, C.R. Acad. Sci. Paris, Ser. I, 342, (2006), 763–766.
[17] J. Bochi and M. Viana, *The Lyapunov exponents of generic volume-preserving and symplectic maps*. Ann. of Math. (2) 161, no. 3 (2005), 1423–1485.
[18] C. Bonatti, L. J. Díaz and M. Viana, *Dynamics beyond uniform hyperbolicity*, EMS 102, Springer 2005.

[19] C. Bonatti, N. Gourmelon and T. Vivier, *Perturbations of the derivative along periodic orbits.* Ergod. Th. & Dynam. Sys. 26, 5 (2006), 1307–1337.

[20] Dacorogna, B., Moser, J. On a partial differential equation involving the Jacobian determinant. Ann. Inst. Henri Poincaré, 7, 1, (1990) 1–26.

[21] C. Doering, *Persistently transitive vector fields on three-dimensional manifolds*. Proceedings on Dynamical Systems and Bifurcation Theory, Vol. 160 (1987), 59–89, Pitman.

[22] C. Ferreira, *Stability properties of divergence-free vector fields*, Dynamical Systems: An International Journal, Vol. 27, 2 (2012), 223–238.

[23] J. Franks, *Necessary conditions for the stability of diffeomorphisms*, Trans. Amer. Math. Soc., 158 (1971), 301-308.

[24] S. Gan and Lan Wen, *Nonsingular star flows satisfy Axiom A and the no-cycle condition*, Invent. Math. 164, 2 (2006), 279–315.

[25] H. Hu, Y.Pesin and A. Talitskaya *Every compact manifold carries a hyperbolic Bernoulli flow*, Modern dynamical systems and applications, 347–358, Cambridge Univ. Press, Cambridge, 2004.

[26] R. Johnson, K. Palmer and G. Sell, *Ergodic properties of linear dynamical systems*, SIAM J. Math. Anal. **18** (1987), 1–33.

[27] S. T. Liao *Obstruction sets (I)* (in chinese), Acta Math. Sin. 23, (1980), 411–453.

[28] R. Mañé, *Oseledec's theorem from the generic viewpoint*. Proc. Int. Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 1269–1276, PWN, Warsaw, 1984.

[29] R. Mañé, *The Lyapunov exponents of generic area preserving diffeomorphisms* International Conference on Dynamical Systems (Montevideo, 1995), Pitman Res. Notes Math. Ser., 362, pp. 110-119, 1996

[30] J. Moser, *On the volume elements on a manifold*. Trans. Amer. Math. Soc., 120 (1965), 286–294.

[31] V. Oseledets, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, *Trans. Moscow Math. Soc.*, **19** (1968), 197–231.

[32] Y. Pesin, Families of invariant manifolds that correspond to nonzero characteristic exponents, *Izv. Akad. Nauk SSSR Ser. Mat.*, **40** (1976), 6, 1332–1379.

[33] C. Pugh and M. Shub, *Stable ergodicity*, With an appendix by Alexander Starkov, Bull. Amer. Math. Soc. (N.S.) 41 (2004), no. 1, 1–41.

[34] C. Robinson, *Generic properties of conservative systems*, Amer. J. Math. 92 (1970), 562–603.

[35] V. A. Rokhlin, *On the fundamental ideas of measure theory*, Transl. Amer. Math. Soc., Series 1, 10 (1962), 1–52.

[36] M. Viana, *Almost all cocycles over any hyperbolic system have non-vanishing Lyapunov exponents*, Annals of Math. 167 (2008), 643–680.

[37] C. Zuppa, *Regularisation C^∞ des champs vectoriels qui préservent l'elément de volume*, Bol. Soc. Bras. Mat., 10, 2 (1979), 51–56.

Departamento de Matemática, Universidade da Beira Interior, Rua Marquês d'Ávila e Bolama, 6201-001 Covilhã Portugal.

E-mail address: bessa@fc.up.pt

Departamento de Matemática, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

E-mail address: jrocha@fc.up.pt