ON THE GENERALIZED FENG-RAO NUMBERS OF NUMERICAL SEMIGROUPS GENERATED BY INTERVALS

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Abstract. We give some general results concerning the computation of the generalized Feng-Rao numbers of numerical semigroups. In the case of a numerical semigroup generated by an interval, a formula for the r^{th} Feng-Rao number is obtained.

1. Introduction

The Feng-Rao distance for a numerical semigroup was introduced in coding theory as a lower bound for the minimum distance of a one-point algebraic geometry (error-correcting) code (see [8]). This *order bound*, computed from Weierstrass semigroups, improves the lower bound for the minimum distance given by Goppa with the aid of the Riemann-Roch theorem. Moreover, the Feng-Rao distance is essential in a majority voting decoding procedure, that is the most efficient one for such kind of codes (see [11]).

Even though the Feng-Rao distance was introduced for Weierstrass semigroups and for decoding purposes, it is just a combinatorial concept that makes sense for arbitrary numerical semigroups. This problem has been broadly studied in the literature for different types of semigroups (see [2], [3] or [12]). In numerical terms, the above mentioned improvement of the Goppa distance in coding theory means the following: For a semigroup S with genus g and $m \in S$ the *Feng-Rao distance* satisfies

$$\delta_{FR}(m+1) \geq m+2-2g$$

if $m > 2g-2$, and equality holds for $m >> 0$.

On the other hand, the concept of minimum distance for an error-correcting code has been generalized to the so-called *generalized Hamming weights*. They were introduced independently by Helleseth et al. in [10] and Wei in [14] for applications in coding theory and cryptography, respectively.

The natural generalization of the Feng-Rao distance to higher weights was introduced in [9]. The computation of these generalized Feng-Rao distances turns out to

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be a very hard problem. Actually, very few results are known about this subject, and they are completely scattered in the literature (see for example [1], [9] or [7]).

This paper studies the asymptotical behaviour of the generalized Feng-Rao distances, that is, $\delta^r_{FR}(m)$ for $r \geq 2$ and $m >> 0$. In fact, it was proven in [7] that

$$\delta^r_{FR}(m) = m + 1 - 2g + E_r \tag{1}$$

for $m >> 0$ (details in the next section). The number $E_r \equiv \mathrm{E}(S, r)$ is called the r-th Feng-Rao number of the semigroup S, and they are unknown but for very few semigroups and concrete r's. For example, it was proven in [6] that

$$\mathrm{E}(S, r) = \rho_r$$

for hyper-elliptic semigroups $S = \langle 2, 2g+1\rangle$, with multiplicity 2 and genus g, and for Hermitian-like semigroups $S = \langle a, a+1\rangle$, where $S = \{\rho_1 = 0 < \rho_2 < \cdots\}$. In fact, it is not even known yet if this formula holds for arbitrary numerical semigroups generated by two elements $S = \langle a, b\rangle$. Nevertheless, our experimental results point in this direction.

The main purpose of this paper is precisely to compute $\mathrm{E}(S, r)$ for semigroups generated by intervals, as a certain generalization of the Hermitian-like case. As a byproduct, we provide some general algorithms, implemented in GAP [5], to compute Feng-Rao numbers.

The paper is written as follows. Section 2 presents the general definitions concerning numerical semigroups, Feng-Rao distances and Feng-Rao numbers, and some convenient visualizations of integers for a given semigroup. The reader may find useful to see some images in Subsection 2.3.

The concept of amenable subset of a numerical semigroup is introduced in section 3. It consists of a set that is closed for taking divisors. It implies that distances between elements are somehow controlled. Amenable sets play a fundamental role in some general results on Feng-Rao numbers of numerical semigroups. These results allowed the implementation of a function to compute the Feng-Rao numbers of a numerical semigroup which works quite well. It uses some of the functionalities of the GAP package numericalsgps [5] and will hopefully be part of a future release of that package. We give in this way some general results. Among them, an important lemma shows that the divisors of a configuration are the divisors of the shadow plus the elements above the ground.

Many examples computed with the referred function helped us to gain the necessary intuition to obtain a formula for the r^{th} Feng-Rao number of a numerical semigroup generated by an interval, which is presented in Section 4. This is the last and main result of this paper, and we briefly explain it in the sequel.

Recall that we are aiming to find a formula for $\delta^r(m)$, when S is a semigroup generated by an interval of integers. The strategy will be as follows: Suppose that there is an amenable set M which is an optimal configuration whose shadow $L_M = [m, m+a+b) \cap M$ does not contain the ground (that is, $L_M \neq [m, m+a+b) \cap \mathbb{N}$). Then, using the results of Subsection 4.2, we can construct an r-amenable set N (said to be ordered amenable) whose shadow L_N is an interval starting in m and has no more elements than $\sharp L_M$. Furthermore, by Lemma 30, $\sharp\mathrm{D}(L_N) \leq \sharp\mathrm{D}(L_M)$ which implies that the number of divisors of N is no bigger than the number of divisors of M and therefore N is also an optimal configuration. It follows that ordered amenable sets are optimal configurations. Thus, the problem of computing the generalized Feng-Rao numbers is reduced to counting the divisors of intervals of the form $[m, m+\ell] \cap \mathbb{N}$, with $\ell \leq a+b-1$. This is done by Corollary 26. The main result, which gives a formula, then follows.

2. Definitions and basic results

This section is divided into several subsections. We start with several basic definitions and we introduce some notation. The reader is referred to the book [13] for details. Then we give the definition of generalized Feng-Rao numbers and end the section by giving a way to visualize the integers which is convenient for our purposes.

2.1. **Basic definitions and notation.** Let S be a numerical semigroup, that is, a submonoid of $\mathbb{N}$ such that $\sharp(\mathbb{N} \setminus S) < \infty$ and $0 \in S$. Denote respectively by $g := \sharp(\mathbb{N} \setminus S)$ and $c \in S$ the *genus* and the *conductor* of S, being c by definition the (unique) element in S such that $c-1 \notin S$ and $c+l \in S$ for all $l \in \mathbb{N}$. Note that if S is the Weierstrass semigroup of a curve χ at a point P, g equals to the geometric genus of χ, and the elements of $G(S) := \mathbb{N} \setminus S$ are called the Weierstrass gaps at P. For an arbitrary semigroup, these elements are simply called gaps.

It is well known (see for instance [13, Lemma 2.14]) that $c \leq 2g$, and thus the "largest gap" of S is $c-1 \leq 2g-1$. The number $c-1$ is precisely the *Frobenius number* of S. The *multiplicity* of a numerical semigroup is the least positive integer belonging to it.

We say that a numerical semigroup S is generated by a set of elements $G \subseteq S$ if every element $x \in S$ can be written as a linear combination

$$x = \sum_{g \in G} \lambda_g g,$$

where finitely many $\lambda_g \in \mathbb{N}$ are non-zero. In fact, it is classically known that every numerical semigroup is finitely generated, that is, we can find a finite set G generating S. Furthermore, every generator set contains the set of irreducible elements, $x \in S$ being irreducible if $x = u+v$ and $u, v \in S$ implies $u \cdot v = 0$, and this set actually generates S, so that it is usually called "the" generator set of S, whose cardinality is called *embedding dimension* of S (more details in [13]). Most of the times, we will suppose S is minimally generated by $\{n_1 < \cdots < n_e\}$. Its embedding dimension is e. Note that if a and b are integers, with $b < a$, and S is minimally generated by the interval $[a, a+b] \cap \mathbb{N}$, then n_1 is a, $n_e - n_1$ is b and the embedding dimension is $b+1$.

Finally, if we enumerate the elements of S in increasing order

$$S = \{\rho_1 = 0 < \rho_2 < \cdots\},$$

we note that every $x \geq c$ is the $(x+1-g)$-th element of S, that is $x = \rho_{x+1-g}$.

The last part of this paper will be devoted to semigroups generated by intervals.

Let a be a positive integer and b an integer with $0 < b < a$. Let $S = \langle a, a+1, \ldots, a+b \rangle$. Then S is a numerical semigroup with multiplicity a and embedding dimension $b+1$. As usual, let c denote the conductor of S and $m \geq 2c-1$.

2.2. **Feng-Rao numbers.** Next we introduce the definitions for generalized Feng-Rao distances. Although there is a subsection dedicated to the concept of divisor, we already need the definition.

Definition 1. *Given $x \in S$, we say that $\alpha \in S$* divides *x if $x - \alpha \in S$. We denote by $\mathrm{D}(x) = \{\alpha \in S \mid x - \alpha \in S\}$ the set of* divisors *of x.*

Definition 2. *Let S be a numerical semigroup. For $m_1 \in S$, let $\nu(m_1) := \sharp\mathrm{D}(m_1)$. The (classical)* Feng-Rao distance *of S is defined by the function*

$$\begin{array}{rcccl} \delta_{FR} & : & S & \longrightarrow & \mathbb{N} \\ & & m & \mapsto & \delta_{FR}(m) := \min\{\nu(m_1) \mid m_1 \geq m, \ \ m_1 \in S\}. \end{array}$$

There are some well-known facts about the functions ν and δ_{FR} for an arbitrary semigroup S (see [11], [12] or [2] for further details). An important one is that $\delta_{FR}(m) \geq m+1-2g$ for all $m \in S$ with $m \geq c$, and that equality holds if moreover $m \geq 2c-1$ (see also Proposition 9).

The classical Feng-Rao distance corresponds to $r = 1$ in the following definition.

Definition 3. *Let S be a numerical semigroup. For any set of distinct $m_1, \ldots, m_r \in S$, let $\nu(m_1, \ldots, m_r) := \sharp\mathrm{D}(m_1, \ldots, m_r)$, where $\mathrm{D}(m_1, \ldots, m_r) := \mathrm{D}(m_1) \cup \cdots \cup \mathrm{D}(m_r)$.*

For any integer $r \geq 1$, the r-th Feng-Rao distance *of S is defined by the function*

$$\begin{array}{rcccl} \delta^r_{FR} & : & S & \longrightarrow & \mathbb{N} \\ & & m & \mapsto & \delta^r_{FR}(m) \end{array}$$

where $\delta^r_{FR}(m) = \min\{\nu(m_1, \ldots, m_r) \mid m \leq m_1 < \cdots < m_r, \ \ m_i \in S\}$.

Very few results are known for the numbers δ^r_{FR}, and their computation is very hard from both a theoretical and computational point of view. The main result we need describes the asymptotical behavior for $m >> 0$, and was proven in [7]. This result tells us that there exists a certain constant $E_r = \mathrm{E}(S, r)$, depending on r and S, such that

$$\delta^r_{FR}(m) = m + 1 - 2g + E_r$$

for $m \geq 2c - 1$.

Definition 4. *This constant $E(S, r)$ is called the r-th* Feng-Rao number *of the semigroup S.*

Furthermore, it is also true that $\delta^r_{FR}(m) \geq m + 1 - 2g + \mathrm{E}(S, r)$ for $m \geq c$ (see [7]).

Note that, for any non-negative integer k and $m \geq 2c - 1$, $\delta^r_{FR}(m + k) = k + \delta^r_{FR}(m)$.

We will simplify the notation by writting $\delta^r(m)$ for $\delta^r_{FR}(m)$.

Definition 5. *Let S be a numerical semigroup and let $m \in S$. A finite subset of $S \cap [m, \infty)$ is called a (S, m)*-configuration*, or simply a configuration. A configuration M of cardinality r is said to be* optimal *if $\delta^r(m) = \sharp\mathrm{D}(M)$, where $\mathrm{D}(M) := \cup_{x \in M}\mathrm{D}(x)$.*

2.3. **A convenient visualisation of the integers.** We can think of the integers as points disposed regularly on a cylindrical helix (Figure 1).

As using the sketch of Figure 1 some of the integers would be hidden, we will consider planifications of the cylinder instead. They are usually obtained by cutting the cylinder through a vertical line passing through a point previously chosen. Note that a planification corresponds to taking a partition of the integers. The reader may think on the letters m, a, b as being $2c - 1$, n_1 and $n_e - n_1$ for a semigroup

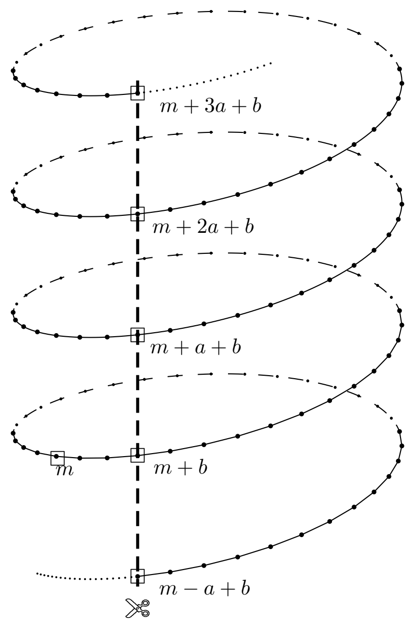

FIGURE 1. The integers on an helix

generated by $\{n_1 < \cdots < n_e\}$ whose conductor is c. This will be the case when dealing with semigroups generated by intervals.

We shall use this drawings to depict the most relevant parts of the sets considered. For instance, if we want to highlight the elements of a numerical semigroup, we do not add any information by depicting the points below 0 and those above the conductor.

The parallelograms in Figure 2 highlight the elements of the semigroup $S = \langle 9, 13, 15\rangle$, and the elements of $60 - S$, respectively.

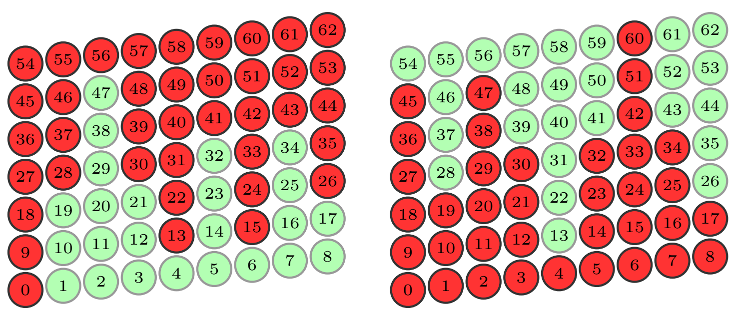

FIGURE 2. The semigroup $S = \langle 9, 13, 15\rangle$ and $60 - S$, respectively

Most times we are interested in finite sets of integers which are non smaller than a given integer m. In this case we prefer to draw all the points from m to $m + a + b$ at the same level. See Figure 3 for an example. Its caption will soon become clear.

For convenience, the columns are numbered. Having such a picture in mind, we can think on a partition of the set of integers greater than m whose classes are the columns (the ith column of a set is the set its elements congruent with i modulo a).

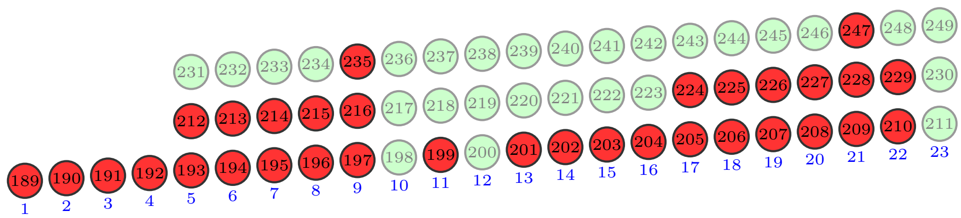

FIGURE 3. An amenable set

3. A generic algorithm

We shall start the section by giving a quite efficient algorithm to compute the divisors of an element of a numerical semigroup. The aim is then to find an optimal configuration. Note that if M is wanted to be an optimal configuration, we just have to control the cardinality of the difference $\mathrm{D}(M) \setminus \mathrm{D}(m)$, for all possible $m \in M$.

Among the optimal configurations there is an amenable set (Proposition 12). Thus, one can search for an optimal configuration among the amenable sets, which can be constructed using Algorithm 2. Due to the results in Section 3.3 (Corollary 16, to be more specific), one only needs to consider one amenable set for each shadow.

3.1. **Divisors.** Recall that given $x \in S$, we say that $\alpha \in S$ divides x if $x - \alpha \in S$. We denote by $\mathrm{D}(x)$ the set of *divisors* of x.

Note that $\mathrm{D}(x) \subseteq [0, x]$ and $s \in \mathrm{D}(x)$ implies $\mathrm{D}(s) \subseteq \mathrm{D}(x)$.

Lemma 6. $\mathrm{D}(x) = S \cap (x - S)$.

Proof. Let $\alpha \in \mathrm{D}(x)$. By definition, $\alpha \in S$ and $x-\alpha \in S$. But then $\alpha = x-(x-\alpha) \in x - S$.

Conversely, let $\alpha \in S$ be such that there exists $\beta \in S$ for which $x - \beta = \alpha$. But then $x - \alpha = \beta \in S$, proving that α divides x. □

We observe that elements greater than x need not to be used to compute the divisors of x. Denoting $S_x = \{n \in S \mid n \leq x\}$, we get the following:

Corollary 7. $\mathrm{D}(x) = S_x \cap (x - S_x)$.

The computation of the divisors of an element can be easily implemented (Algorithm 1) due to this consequence of Lemma 6. Note also that, once we compute the elements of S smaller than x (which can easily be done if the conductor is known), the computation of the divisors is immediate.

The highlighted elements in Figure 4 represent the divisors of $60 \in \langle 9, 13, 15\rangle$. They are obtained intersecting the highlighted elements of the pictures in Figure 2.

Another immediate consequence of Lemma 6, which has interest in concrete implementations, is the following corollary:

Algorithm 1: Divisors

Input : A numerical semigroup S, $x \in S$
Output: The divisors of x

$S_x := \{s \in S \mid s \leq x\}$`/* Compute the elements of` S `smaller than` x `*/`
return $\{s \in S_x \mid x - s \in S_x\}$

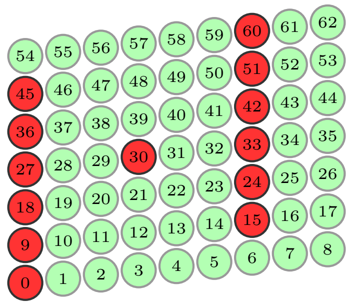

FIGURE 4. The divisors of 60 in the semigroup $S = \langle 9, 13, 15\rangle$

Corollary 8. *If $c \leq x \leq y$, then* $\mathrm{D}(y) \cap [x, \infty) = (y - S) \cap [x, \infty)$.

We remember that

$$\mathrm{D}(m_1, \ldots, m_r) = \mathrm{D}(m_1) \cup \cdots \cup \mathrm{D}(m_r) = \{p \in S \mid m_i - p \in S \text{ for some } i \in \{1, \ldots, r\}\}$$

The highlighted elements in Figure 3 are the elements of $\mathrm{D}(235, 199, 247, 229)$ which are greater than 189, when S is the semigroup $\langle 19, 20, 21, 22, 23\rangle$.

Observe that $x - S$ contains all the integers not greater than $x - c$ and that the number of integers smaller than x not belonging to $x - S$ is precisely the genus of S. As the number of non-negative integers not greater than x is $x + 1$, one gets immediately the well known fact (see [11], [12] or [2]):

Proposition 9. *If $x \geq 2c - 1$, then* $\sharp\mathrm{D}(x) = \sharp S \cap (x - S) = x + 1 - 2g$.

3.2. Amenable sets.

Definition 10. *Let S be a numerical semigroup with conductor c. Let $M = \{m_1, \ldots, m_r\} \subseteq S$ with $2c - 1 \leq m = m_1 < \cdots < m_r$. We say that the set M is* (S, m, r)-amenable *if:*

$$\text{for all } i \in \{1, \ldots, r\}, \mathrm{D}(m_i) \cap [m, \infty) \subseteq M. \tag{2}$$

We will refer a set satisfying (2) as being *m-closed under division*. So, a subset of $S \cap [m, \infty)$ with cardinality r is (S, m, r)-amenable if and only if it contains m and is m-closed under division.

As a convention, the empty set is considered an $(S, m, 0)$-amenable set, for any m. When no confusion arises or only the concept is important, we say (m, r)-*amenable* set or simply *amenable* set.

Example 11. (1) Let $S = \langle 19, 20, 21, 22, 23\rangle$. Its conductor is $c = 95$. Take $m = 2c - 1 = 189$. The set M consisting of the highlighted elements in Figure 3 is an amenable subset of S.

(2) Let S be a numerical semigroup with conductor c. Let $m \geq 2c-1$, and r a non negative integer. Then the interval $[m, m+r-1] \cap \mathbb{N}$ is a (S, m, r)-amenable set.

The importance of amenable sets comes from the following result, which states that among the optimal configurations of cardinality r there is at least one (S, m, r)-amenable set.

Proposition 12. *Let S be a numerical semigroup with conductor c and let $m \geq 2c-1$. Let r be a positive integer. Among the optimal configurations of cardinality r there is one (S, m, r)-amenable set.*

Proof. Let $M = \{m_1, \ldots, m_r\}$ be an optimal configuration. As $m \geq 2c-1$, $\delta^r(m)$ is strictly increasing in m, and thus m cannot be less than m_1, which implies that $m_1 = m$.

If M is not m-closed under division, we may assume that for some $i \in \{1, \ldots, r\}$ there exists $t \in S$ such that $m_i - t > m$ and $m_i - t \notin \{m_1, \ldots, m_r\}$. Clearly $\mathrm{D}(m_i - t) \subset \mathrm{D}(m_i)$, and thus $\mathrm{D}(m_1, \ldots, m_{i-1}, m_i - t, m_{i+1}, \ldots, m_r) \subseteq \mathrm{D}(m_1, \ldots, m_r)$. In other words, we can change m_i by $m_i - t$ and the number of divisors does not increase. Now we can repeat the process with the set obtained until we reach a m-closed under division set. Note that this must happen in a finite number of steps ($\mathbb{N}^r$ has no infinite descending chains). □

The definition of amenable set, which seems to be suitable for proofs, does not seem to help very much to do computations unless we can prove some consequences. The following one, showing that the distances between elements is somehow controlled, guarantees that the search of the amenable sets can be done in a bounded subset of S, and therefore amenable sets can be effectively computed. An algorithm will be presented (Algorithm 2).

Proposition 13. *Let S be a numerical semigroup with conductor c and let $m \geq 2c-1$. Let $M = \{m_1, \ldots, m_r\} \subseteq S$ be an (S, m, r)-amenable set and suppose that $S = \{0 = \rho_1 < \rho_2 < \cdots\}$. Then*
(a) $m_i \leq m + \rho_i$, for all $i \in \{1, \ldots, r\}$,
(b) $m_{i+1} - m_i \leq \rho_2$, for all $i \in \{1, \ldots, r-1\}$.

Proof. (a) Suppose that there exists $i_0 \in \{1, \ldots, r\}$ such that $m_{i_0} - \rho_{i_0} > m$. Let $D = \{m_{i_0} - \rho_j \mid j \in \{1, \ldots, i_0\}\}$. All the elements of D are bigger than m, that is, $D \subseteq (m, \infty)$. On the other hand, by using Lemma 6, $D \subseteq \mathrm{D}(m_{i_0})$. Thus $D \subseteq \mathrm{D}(m_{i_0}) \cap (m, \infty) \subsetneqq \{m_1, \ldots, m_{i_0}\}$. The containment is strict since $m_1 = m$. But this is absurd, since the two ends of the chain have the same cardinality.

(b) Note that $m_{i+1} - \rho_2$ is a divisor of m_{i+1}. This implies that, if $m_{i+1} - \rho_2 \geq m$, then $m_{i+1} - \rho_2 \in M$. As $m_{i+1} - \rho_2 < m_{i+1}$ and there is no element in M strictly between m_i and m_{i+1}, $m_{i+1} - \rho_2$ must be non greater than m_i. □

For efficiency reasons, the following result is important. It shows that we do not have to consider all divisors.

Proposition 14. *A subset $M = \{m = m_1, \ldots, m_r\}$ of a numerical semigroup S is (S, m, r)-amenable if and only if*

$$\begin{array}{l} \textit{for all } i \in \{1, \ldots, r\} \textit{ and } g \textit{ minimal generator of } S, \\ \textit{if } m_i - g \geq m, \textit{ then } m_i - g \in \{m_1, \ldots, m_r\}. \end{array} \tag{3}$$

Proof. Let $m_i \in M$ and $u \in \mathrm{D}(m_i) \cap [m, \infty)$, with $u \neq m_i$. We shall prove that if (3) holds, then $u \in M$, thus concluding that M is (S, m, r)-amenable. We can write $u = m_i - \gamma$, with $\gamma \in S \setminus \{0\}$. Assume as induction hypothesis that $m_i - \alpha \in \mathrm{D}(m_i) \cap [m, \infty)$ implies $m_i - \alpha \in M$, for all α less than γ. Let g be a minimal generator that divides γ. As $\gamma - g < \gamma$, and $m_i - (\gamma - g) = m_i - \gamma + g \in \mathrm{D}(m_i) \cap [m, \infty)$, we have, by hypothesis, that $m_i - \gamma + g \in M$. But then, by (3), $m_i - \gamma = (m_i - \gamma + g) - g \in M$. □

Propositions 13 and 14 led to an algorithm to compute the set of (S, m, r)-amenable sets. Pseudo-code is presented in Algorithm 2 .

Algorithm 2: (S, m, r)-amenable sets

```
Input  : A numerical semigroup S, m ≥ 2c − 1 and r an integer
Output: The set of (S, m, r)-amenable sets
SM := [[m]]/* the set of amenable sets                              */
Compute the generators gens = {n_1 < ... < n_e} and the elements
{0 = ρ_1 < ρ_2 < ...} of S
for i in [2..r] do
    newM := [ ]
    for x in SM do
        min := Minimum(x[Length(x)] + ρ_2, m + ρ_i)/* the consequences
            in Proposition 13 should be satisfied:  the next element
            to be added must not be greater than the last + rho2
            neither m+el[i]                                         */
        for m_j in [x[Length(x)] + 1..min] do
            divs := {d ∈ m_j − gens | d > m} /* strict divisors of m_j
                greater than m                                      */
            if divs ⊆ x then
                /* in order to get condition (3) of Proposition 14
                    satisfied                                       */
                Append(newM, [Union(x, [mj])])
    SM := newM;
return SM
```

As we will see in the next subsection, we do not need all the amenable sets.

3.3. **The ground.** We continue considering S a numerical semigroup minimally generated by $\{n_1 < \cdots < n_e\}$ with conductor c. Let $m \geq 2c - 1$. The set $\{m, \ldots, m + n_e - 1\}$ is called the (S, m)-*ground*, or simply *ground*.

The intersection of an (S, m, r)-amenable set M with the (S, m)-ground is called the *shadow* of M.

Note that the shadow of an amenable set is amenable.

Lemma 15. *Let S be a numerical semigroup minimally generated by $\{n_1 < \cdots < n_e\}$ with conductor c. Let $m \geq 2c - 1$ and let $M = \{m = m_1 < \cdots < m_r\}$ be an amenable set. Let $L = M \cap [m, m + n_e)$ be the shadow of M. Then*

$$\mathrm{D}(M) = (M \setminus L) \cup \mathrm{D}(L),$$

and furthermore $\sharp\mathrm{D}(M) = \sharp(M \setminus L) + \sharp\mathrm{D}(L)$.

Proof. The inclusion $(M \setminus L) \cup \mathrm{D}(L) \subseteq \mathrm{D}(M)$ is clear. For the other inclusion, let $x \in \mathrm{D}(M) \setminus (M \setminus L) = (\mathrm{D}(M) \setminus M) \cup L$. We want to prove that $x \in \mathrm{D}(L)$. Since $L \subseteq \mathrm{D}(L)$, we can assume that $x \in \mathrm{D}(M) \setminus M$. Then $x \in \mathrm{D}(m_i)$ for some $i \in \{1, \ldots, r\}$ and $m_i \geq m + n_e$. As $m_i - x \in S \setminus \{0\}$, there exists $j \in \{1, \ldots, e\}$ such that $m_i - x - n_j \in S$. Hence $x \in \mathrm{D}(m_i - n_j)$. By hypothesis M is amenable and thus $m_i - n_j \in M$, since $m_i - n_j \in \mathrm{D}(m_i) \cap [m, \infty)$. If needed, we can repeat the process until $m_i - n_j \in L$, that is, $x \in \mathrm{D}(L)$.

The second assertion follows easily since the above union is disjoint. □

As an easy but useful consequence, we get the following corollary.

Corollary 16. *Let M and N be (m, r)-amenable sets with shadows L_M and L_N respectively.* $L_M \subseteq L_N \implies \sharp\mathrm{D}(M) \leq \sharp\mathrm{D}(N)$.

Proof. Suppose that L_N is the disjoint union of L_M and a set K of cardinality k. Observe that $\sharp(M \setminus L_M) = \sharp(N \setminus L_N) + k$.

As $\mathrm{D}(L_N) = \mathrm{D}(L_M) \cup \mathrm{D}(K) \supseteq \mathrm{D}(L_M) \cup K$, it follows that $\sharp\mathrm{D}(L_N) \geq \sharp\mathrm{D}(L_M) + k$, that is, $\sharp\mathrm{D}(L_M) \leq \sharp\mathrm{D}(L_N) - k$.

$\sharp\mathrm{D}(M) = \sharp(M \setminus L_M) + \sharp\mathrm{D}(L_M) \leq \sharp(N \setminus L_N) + k + \sharp\mathrm{D}(L_N) - k$. □

Corollary 17. *Let S be a numerical semigroup minimally generated by $\{n_1 < \cdots < n_e\}$ with conductor c. Let $m \geq 2c - 1$ and let $M \subset [m, \infty)$ be an amenable set which is an optimal configuration of cardinality r. Let $L = M \cap [m, m + n_e)$ be the shadow of M. Then $\delta^r(m) = \sharp\mathrm{D}(L) + \sharp(M \setminus L)$.*

Corollary 18. *In particular, if there exists an optimal configuration M of cardinality r such that $[m, m + n_e) \cap \mathbb{N} \subseteq M$, then $[m, m + r - 1 + k] \cap \mathbb{N}$ is also an optimal configuration of cardinality $r + k$.*

3.4. **An algorithm to compute generalized Feng-Rao numbers.** In the cases where computing divisors is "easy", finding optimal configurations is as difficult as computing generalized Feng-Rao numbers. This problem is referred to as "hard" in the literature, even from the computational point of view.

Algorithm 3 can be used to compute generalized Feng-Rao numbers of any numerical semigroup. Note that its efficiency depends on the number of amenable sets. Due to Corollary 16, it can be sharpened, since we only need to consider one amenable set for each possible shadow.

This algorithm (even preliminary versions of it) has been extensively used by the authors to perform computations which gave the intuition that ultimately led to the main results of this paper.

4. Numerical semigroups generated by intervals

From now on we assume that $S = \langle a, \ldots, a + b \rangle$ with a and b positive integers, and $b < a$.

4.1. **Some counting lemmas.** As we have seen above, it is crucial to know the number of divisors of subsets of the ground (this is obtained in Remark 24). In this section we prove some technical lemmas on counting the divisors of elements, and then apply them for elements in the ground. The main result (Lemma 30) shows that the minimum is obtained when the elements form an interval starting in m.

Algorithm 3: Generalized Feng-Rao numbers

Input : A numerical semigroup S, $m \in S$, $r \in \mathbb{N}$
Output: $\delta^r_{FR}(m)$

$SM := \emptyset$

1 $AM := \{M \subset S \mid M$ is a (S, m, r)-amenable set$\}$`/* Compute the` (m, r)`-amenable sets, by making a call to Algorithm 2 */`

2 For each possible shadow s, add to SM an element of AM with shadow s, if it exists

$\nu := m + r$`/* an obvious upper bound */`

3 **for** M *in* SM **do**

  $D := \bigcup\{Divisors(x) \mid x \in M\}$`/* Compute the divisors of` M`, by using Algorithm 1 */`

  $\nu := minimum(\sharp D, \nu)$

4 `return` ν

Membership problem for semigroups generated by intervals is trivial as the following known result (and with many different formulations) shows.

Lemma 19. [4, Lemma 10, for $d = 1$] *Let k and r be integers such that $0 \le r \le a-1$. Then $ka + r \in S$ if and only if $r \le kb$.*

Lemma 20. *Let $m \ge 2c-1$. Let q be a nonnegative integer and $j \in \{0, \dots, a-1\}$.*

$$\begin{aligned}\mathrm{D}(m, m+qa+j) = \mathrm{D}(m)& \\ \cup &\left\{m-(ka+r) \mid 0 \le r \le a-j-1,\ \frac{r+j}{b} - q \le k < \frac{r}{b}\right\} \\ \cup &\left\{m-(ka+r) \mid a-j \le r \le a-1,\ \frac{r+j-(a+b)}{b} - q \le k < \frac{r}{b}\right\},\end{aligned}$$

and this union is disjoint.

Proof. We describe the set $\mathrm{D}(m+qa+j) \setminus \mathrm{D}(m)$. Let x be an integer such that $m+qa+j-x \in S$ and $m-x \notin S$. In particular, as $m \ge 2c-1$, $m-x \notin S$ implies that $m-x < c$, and thus $c-1 \le m-c < x$, which leads to $x \in S$. Thus $x \in \mathrm{D}(m+qa+j) \setminus \mathrm{D}(m)$. Set $n = m-x$, and let k and r be integers such that $n = ka+r$ ($x = m-(ka+r)$). Then $n = ka+r \notin S$ and $n+qa+j = (q+k)a+(j+r) \in S$. In view Lemma 19, this implies that $kb < r < a$ and

- if $r+j \le a-1$, then $0 \le r+j \le (q+k)b$,
- if $r+j \ge a$, by writing $(q+k)a+(j+r) = (q+k+1)a+(j+r-a)$, we obtain $0 \le r+j-a \le (q+k+1)b$.

□

Figure 5 shows how are the divisors of $\mathrm{D}(m, m+\lambda)$ with $\lambda \in \{42, 59\}$ and $S =< 9, 10, 11, 12, 13 >$.

Remark 21. For the particular case $q = 0$ and $0 < j < a$, we get

$$\mathrm{D}(m, m+j) = \mathrm{D}(m) \cup \{m-(ka+r) \mid a-j \le r \le a-1,\ 0 < r-kb \le (a+b)-j\}.$$

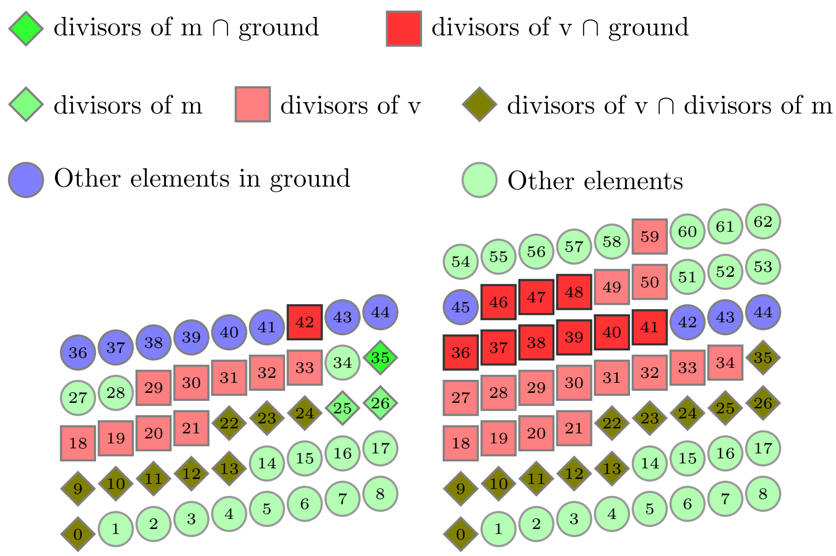

FIGURE 5. $\mathrm{D}(m, m+\lambda)$

For $q = 1$ and $j = 0$,

$$\mathrm{D}(m, m+a) = \mathrm{D}(m) \cup \{m - (ka + r) \mid 0 \leq r \leq a-1,\ 0 < r - kb \leq b\},$$

which is the same as above by taking $j = a$.

For the case $q = 1$, we get

$$\begin{aligned}\mathrm{D}(m, m+a+j) = \mathrm{D}(m) & \\ &\cup \{m - (ka + r) \mid 0 \leq r \leq a-j-1,\ 0 < r - kb \leq b - j\} \\ &\cup \{m - (ka + r) \mid a - j \leq r \leq a-1,\ 0 < r - kb \leq (a+b) + b - j\}.\end{aligned}$$

This describes all elements $\mathrm{D}(m, m+\ell)$, with $\ell \in \{1, \ldots, a+b-1\}$ (i.e., $m+\ell$ in the ground).

Lemma 22. *Let $0 = i_0 < i_1 < \cdots < i_t < i_{t+1} < a+b$ be such that $\{m, m+i_1, \ldots, m+i_{t+1}\}$ is amenable. Then*

$$\begin{aligned}\mathrm{D}(m, m+i_1, \ldots, m+i_{t+1}) = \mathrm{D}(m, m+i_1, \ldots, m+i_t) \cup & \\ \{m - (ka + r) \mid a - i_{t+1} \leq r \leq a - 1 - i_t,\ 0 < r - kb \leq (a+b) - i_{t+1}\}.\end{aligned}$$

Proof. Assume first that $i_{t+1} \leq a$. Note that $\mathrm{D}(m + i_{t+1}) \setminus (\mathrm{D}(m, m+i_1, \ldots, m+i_t)) = \bigcap_{j=0}^{t} \mathrm{D}(m+i_{t+1}) \setminus \mathrm{D}(m+i_j)$, and this equals

$$\bigcap_{j=0}^{t} \{m + i_j - (ka + r') \mid a - (i_{t+1} - i_j) \leq r' \leq a-1,\ 0 < r' - kb \leq (a+b) - (i_{t+1} - i_j)\}$$

(Remark 21). If we make the change of variables $r = r' - i_j$ for each j, we obtain

$$\bigcap_{j=0}^{t} \{m - (ka + r) \mid a - i_{t+1} \leq r \leq a - 1 - i_j,\ -i_j < r - kb \leq (a+b) - i_{t+1}\}.$$

Intersecting means choosing the least intervals for r and $r-kb$, and we get the desired result.

Now assume that $a < i_{t+1} < a+b$. By hypothesis there exists s such that $i_{t+1} - i_s < a$ and $i_{t+1} - i_{s-1} \geq a$ (by amenability). For $i_{t+1} - i_j \geq a$, write $i_{t+1} - i_j = a + h_j$. Hence $\bigcap_{j=0}^{t} \mathrm{D}(m+i_{t+1}) \setminus \mathrm{D}(m+i_j)$ equals

$$\bigcap_{j=s}^{t} \{m+i_j-(ka+r') \mid a-(i_{t+1}-i_j) \leq r' \leq a-1,\ 0 < r'-kb \leq (a+b)-(i_{t+1}-i_j)\}$$

$$\bigcap \Big(\bigcap_{j=0}^{s-1} \big(\{m+i_j-(ka+r') \mid 0 \leq r' \leq a-(i_{t+1}-i_j-a)-1,\ 0 < r'-kb \leq b-(i_{t+1}-i_j-a)\}$$

$$\cup \{m+i_j-(ka+r') \mid a-(i_{t+1}-i_j-a) \leq r' \leq a-1,\ 0 < r'-kb \leq (a+b)+b-(i_{t+1}-i_j-a)\}\big)\Big).$$

If we perform again the change of variables $r = r' - i_j$, we obtain that $C = \bigcap_{j=s}^{t} \{m+i_j-(ka+r') \mid a-(i_{t+1}-i_j) \leq r' \leq a-1,\ 0 < r'-kb \leq (a+b)-(i_{t+1}-i_j)\} = \{m-(ka+r) \mid a-i_{t+1} \leq r \leq a-i_t-1,\ -i_s < r-kb \leq (a+b)-i_{t+1}\}$. Analogously, for every $j \in \{0, \ldots, s-1\}$,

$$\{m+i_j-(ka+r') \mid 0 \leq r' \leq a-(i_{t+1}-i_j-a)-1,\ 0 < r'-kb \leq b-(i_{t+1}-i_j-a)\}$$
$$\cup \{m+i_j-(ka+r') \mid a-(i_{t+1}-i_j-a) \leq r' \leq a-1,\ 0 < r'-kb \leq (a+b)+b-(i_{t+1}-i_j-a)\}$$

equals

$$\{m-(ka+r) \mid -i_j \leq r \leq 2a-i_{t+1}-1,\ -i_j < r-kb \leq a+b-i_{t+1}\}$$
$$\cup \{m-(ka+r) \mid 2a-i_{t+1} \leq r \leq a-i_j-1,\ -i_j < r-kb \leq 2(a+b)-i_{t+1}\}.$$

Observe that $a - i_t - 1 \leq 2a - i_{t+1} - 1$ if and only if $i_{t+1} - i_t \leq a$, which is the case since we are using an amenable set. Hence C does not cut the second set in the above union, and the whole intersection is as in the case $i_{t+1} < a$. □

Corollary 23. *Let* $0 = i_0 < i_1 < \cdots < i_t < i_{t+1} < a+b$. *Then*

$$\sharp\mathrm{D}(m, m+i_1, \ldots, m+i_{t+1}) = \sharp\mathrm{D}(m, m+i_1, \ldots, m+i_t) + \sum_{j=i_t+1}^{i_{t+1}} \left\lceil \frac{a+b-j}{b} \right\rceil - \left\lceil \frac{i_{t+1}-j}{b} \right\rceil.$$

Proof. We compute the cardinality of $\{m-(ka+r) \mid a-i_{t+1} \leq r \leq a-1-i_t,\ 0 < r-kb \leq (a+b)-i_{t+1}\}$. Note that $m-(ka+r) = m-(k'a+r')$ with $0 \leq r, r' < a$ implies that $k = k'$ and $r = r'$. Thus we must calculate $\sharp\{(k,r) \mid a-i_{t+1} \leq r \leq a-1-i_t,\ 0 < r-kb \leq (a+b)-i_{t+1}\}$, which equals $\sharp\{(k,r) \mid i_t+1 \leq a-r \leq i_{t+1},\ \frac{i_{t+1}+r-(a+b)}{b} \leq k < \frac{r}{b}\}$. By taking $j = a-r$, we get

$$\sum_{j=i_t+1}^{i_{t+1}} \sharp\Big\{k \mid \frac{i_{t+1}-j}{b} - 1 \leq k < \frac{a-j}{b}\Big\} = \sum_{j=i_t+1}^{i_{t+1}} \Big(\big(\left\lceil \frac{a-j}{b} \right\rceil - 1 \big) - \big(\left\lceil \frac{i_{t+1}-j}{b} \right\rceil - 1 \big) + 1 \Big),$$

and the proof follows easily. □

Remark 24. Recall that

$$\sharp\mathrm{D}(m, m+i_1, \ldots, m+i_t) = \sharp\mathrm{D}(m, m+i_1, \ldots, m+i_{t-1}) + \sum_{j=i_{t-1}+1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil - \left\lceil \frac{i_t-j}{b} \right\rceil,$$

and by applying several times this process we obtain

$$\begin{aligned}\sharp\mathrm{D}(m, m+i_1, \ldots, m+i_t) &= \sharp\mathrm{D}(m) + \sum_{k=1}^{t} \sum_{j=i_{k-1}+1}^{i_k} \left\lceil \frac{a+b-j}{b} \right\rceil - \left\lceil \frac{i_k - j}{b} \right\rceil \\ &= \sharp\mathrm{D}(m) + \sum_{j=1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil - \sum_{k=1}^{t} \sum_{j=i_{k-1}+1}^{i_k} \left\lceil \frac{i_k - j}{b} \right\rceil .\end{aligned}$$

And thus

$$\sharp\mathrm{D}(m, m+i_1, \ldots, m+i_t) = \sharp\mathrm{D}(m) + \sum_{j=1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil - \sum_{k=1}^{t} \sum_{j=1}^{i_k - i_{k-1}} \left\lceil \frac{(i_k - i_{k-1}) - j}{b} \right\rceil .$$

If we write $d_k = i_k - i_{k-1}$, this rewrites as

$$\sharp\mathrm{D}(m, m+i_1, \ldots, m+i_t) = \sharp\mathrm{D}(m) + \sum_{j=1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil - \sum_{k=1}^{t} \sum_{j=1}^{d_k} \left\lceil \frac{d_k - j}{b} \right\rceil .$$

Hence the value of $\sharp\mathrm{D}(m, m+i_1, \ldots, m+i_t)$ depends on $d_1, \ldots, d_t$, subject to $\sum_{k=1}^{t} d_k = i_t$.

Corollary 25. *Let $t \in \{1, \ldots, a+b-1\}$. Then*

$$\sharp\mathrm{D}(m, m+1, \ldots, m+t) = \sharp\mathrm{D}(m) + \sum_{j=1}^{t} \left\lceil \frac{a+b-j}{b} \right\rceil .$$

As a consequence of this, when the shadow of a configuration is an interval containing m, then we can compute the number of divisors of its elements.

Corollary 26. *Let m be an integer greater than or equal to $2c-1$. Let $m = m_1 < m_2 < \cdots < m_t$ be integers such that $\{m_1, m_2, \ldots, m_t\}$ is amenable. Assume that $l = \sharp\{m_1, \ldots, m_t\} \cap [m, m+a+b) = \{m, m+1, \ldots, m+l-1\}$. Then*

$$\sharp\mathrm{D}(m_1, \ldots, m_t) = m - 2g + t + \sum_{j=1}^{l-1} \left\lceil \frac{a-j}{b} \right\rceil .$$

Proof. This is a direct consequence of Proposition 9, Lemma 15, and Corollary 25. □

Indeed, we will show that among these configurations there is an optimal one. To do this, we first prove that the best shadows are those of the form $\{m, m+1, \ldots, m+l-1\}$, and later (in the next section) we will have to compute the smallest possible value of l.

Let us see how to compute sums of the form $\sum_{j=1}^{t} \left\lceil \frac{a-j}{b} \right\rceil$.

Lemma 27. *Let x and y be postive integers. Assume that $y = cb + r$ with c an integer and $0 \leq r < b$, and that k is an integer such that $kb \leq x - r < (k+1)b$. Then*

(1) if $r \neq 0$, $\sum_{j=1}^{x} \left\lceil \frac{y-j}{b} \right\rceil = (c+1)(r-1) + b\sum_{i=0}^{k-1}(c-i) + (x-(kb+r)+1)(c-k) = x(c-k) + (k+1)(r-1) + b\frac{k(k+1)}{2}$,

(2) if $r = 0$, $\sum_{j=1}^{x} \left\lceil \frac{y-j}{b} \right\rceil = -c + b\sum_{i=0}^{k-1}(c-i) + (x-kb+1)(c-k) = (x+1)(c-k) + b\frac{k(k+1)}{2} - c$.

Proof. Observe that

$$\sum_{j=1}^{x}\left\lceil\frac{y-j}{b}\right\rceil=\sum_{j=1}^{r-1}\left\lceil\frac{y-j}{b}\right\rceil+\sum_{j=r}^{r+b-1}\left\lceil\frac{y-j}{b}\right\rceil+\cdots+\sum_{j=(k-1)b+r}^{kb+r-1}\left\lceil\frac{y-j}{b}\right\rceil+\sum_{j=kb+r}^{x}\left\lceil\frac{y-j}{b}\right\rceil,$$

and

$$\sum_{j=r+lb}^{(l+1)b+r-1}\left\lceil\frac{y-j}{b}\right\rceil=\sum_{j=0}^{b-1}\left\lceil\frac{y-(r+lb)-j}{b}\right\rceil=\sum_{j=0}^{b-1}\left\lceil\frac{(c-l)b-j}{b}\right\rceil=b(c-l).$$

In the same way the first and last summand are computed. If $r=0$, the first summand does not appear, and the second sum starts on 0, and so we have to decrease the total amount by $\left\lceil\frac{cb}{b}\right\rceil=c$. □

Actually, as we see next it suffices to consider the following type of sums.

Remark 28. For the case $x=y$, we get $k=c$ and

(1) if $r\neq 0$, $\sum_{j=1}^{x}\left\lceil\frac{x-j}{b}\right\rceil=(c+1)(r-1)+b\frac{c(c+1)}{2}=\frac{c+1}{2}(x+r)-c-1$,
(2) if $r=0$, $\sum_{j=1}^{x}\left\lceil\frac{x-j}{b}\right\rceil=\frac{c+1}{2}x-c$.

Observe also that $\sum_{j=1}^{x}\left\lceil\frac{x-j}{b}\right\rceil=\sum_{j=1}^{x-1}\left\lceil\frac{j}{b}\right\rceil$.

The following trick will allow us to prove that the best possible shadows are those that are intervals starting in m.

Remark 29. Let $d_k=i_k-i_{k-1}$, $i\in\{1,\ldots,t\}$. Then $\sum_{k=1}^{t}d_k=i_t$. If we replace $\{d_1,d_2\}$ with $\{1,d_1+d_2-1\}$, the total sum of the d_k's remains the same (we are thus assuming that both d_1 and d_2 are greater than one). Let us see what happens to

$$\sum_{k=1}^{t}\sum_{j=1}^{i_k-i_{k-1}}\left\lceil\frac{(i_k-i_{k-1})-j}{b}\right\rceil=\sum_{k=1}^{t}\sum_{j=1}^{d_k}\left\lceil\frac{d_k-j}{b}\right\rceil.$$

Write $d_k=c_kb+r_k$, with c_k and integer and $0\le r_k<b$. Set $s_k=\sum_{j=1}^{d_k}\left\lceil\frac{d_k-j}{b}\right\rceil$. Then $s_k=\frac{c_k+1}{2}(d_k+r_k)-1-c_k$, if $r_k\neq 0$, and $s_k=\frac{c_k+1}{2}d_k-c_k$, otherwise. Let c and r be the quotient and remainder of the division of d_1+d_2-1 by b. Let $\Delta=\sum_{j=1}^{d_1+d_2-1}\left\lceil\frac{d_1+d_2-1-j}{b}\right\rceil-s_1-s_2$. If $b=1$, then $r_1=r_2=r=0$, $c_i=d_i$, $c=d_1+d_2-1$, and $\Delta=d_1d_2$, which is a nonnegative integer. For $b>1$ we distinguish three cases depending on the value of r_1+r_2.

- If $r_1+r_2=0$ (this means $r_1=r_2=0$), then $d_1=c_1b$, $d_2=c_2b$, $c=c_1+c_2-1$, and $r=b-1$. Then $\Delta=bc_1c_2-(c_1+c_2)$. Since we are assuming that $b\geq 2$, and c_1 and c_2 are positive integers, this amount is nonnegative.
- If $0<r_1+r_2\leq b$, then $c=c_1+c_2$ and $r=r_1+r_2-1$. Thus if $rr_1r_2\neq 0$, $\Delta=bc_1c_2+c_1(r_2-1)+c_2(r_1-1)$, which is greater than or equal to zero. For $r=0$, either $r_1=0$ (and $r_2=1$) or $r_2=0$ (and $r_1=1$). Assume without loss of generality that $r_1=0$. We obtain $\Delta=(bc_1-1)c_2$, which is again nonnegative.
- Finally if $r_1+r_2\geq b+1$, then $c=c_1+c_2+1$ and $r=r_1+r_2-b-1$. In this setting $r_1\neq 0\neq r_2$. If $r\neq 0$, then $\Delta=bc_1c_2+c_1(r_2-1)+c_2(r_1-1)+r_1+r_2-(b+2)$, which is nonnegative since $r_1+r_2\geq b+2$. For $r_1+r_2=b+1$ ($r=0$), we obtain a nonnegative $\Delta=bc_1c_2+c_1(r_2-1)+c_2(r_1-1)$.

With all this in mind, we are able to prove the main result of this section, that is, if the elements in the ground form an interval containing m, then we get the least possible number of divisors.

Lemma 30. *Let $0 = i_0 < i_1 < \cdots < i_t < a+b$. Then*

$$\sharp\mathrm{D}(m, m+i_1, \ldots, m+i_t) \geq \sharp\mathrm{D}(m, m+1, \ldots, m+t).$$

Proof. Let $d_k = i_k - i_{k-1}$ for $k \in \{1, \ldots, t\}$ ($i_0 = 0$). We know that (see Remark 24)

$$\sharp\mathrm{D}(m, m+i_1, \ldots, m+i_t) = \sharp\mathrm{D}(m) + \sum_{j=1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil - \sum_{k=1}^{t} \sum_{j=1}^{d_k} \left\lceil \frac{d_k - j}{b} \right\rceil.$$

By applying several times the above remark, we obtain that $\sharp\mathrm{D}(m, m+i_1, \ldots, m+i_t) \geq \sharp\mathrm{D}(m, m+i'_1, m+i'_2, \ldots, m+i'_t)$, with $i_t = i'_t$ and $d'_k = i'_k - i'_{k-1} = 1$ for $k \in \{2, \ldots, t\}$. By using again the above expression, but now for d'_k instead of d_k, we get

$$\sharp\mathrm{D}(m, m+i'_1, \ldots, m+i'_t) = \sharp\mathrm{D}(m) + \sum_{j=1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil - \sum_{j=1}^{i'_1} \left\lceil \frac{i'_1 - j}{b} \right\rceil.$$

Hence by Corollary 25, in order to prove the inequality of the statement, it suffices to show that

$$\sum_{j=1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil - \sum_{j=1}^{i'_1} \left\lceil \frac{i'_1 - j}{b} \right\rceil \geq \sum_{j=1}^{t} \left\lceil \frac{a+b-j}{b} \right\rceil,$$

or equivalently,

$$\sum_{j=t+1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil - \sum_{j=1}^{i'_1} \left\lceil \frac{i'_1 - j}{b} \right\rceil \geq 0.$$

Now, if we take into account that $\sum_{j=t+1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil = \sum_{j=1}^{i_t-t} \left\lceil \frac{a+b-t-j}{b} \right\rceil$, that $\sum_{j=1}^{i'_1} \left\lceil \frac{i'_1-j}{b} \right\rceil = \sum_{j=1}^{i'_1-1} \left\lceil \frac{i'_1-j}{b} \right\rceil$, and that $i'_1 + (t-1) = i'_t = i_t$, we get $\sum_{j=t+1}^{i_t} \left\lceil \frac{a+b-j}{b} \right\rceil = \sum_{j=1}^{i'_1-1} \left\lceil \frac{a+b-t-j}{b} \right\rceil$. Since $a+b-t \geq i_t + 1 - t = i'_1$, we obtain the desired inequality. □

4.2. **Ordered amenable sets.** As we have seen above, the minimum number of divisors of elements in the ground is reached when these elements form an interval starting in m. In this section we study configurations fulfilling this condition.

Let M be a configuration and let

$$j_0 = max\{j \in \{0, \ldots, a-1\} \mid x - (m+b) = qa + j, \text{for some } q \in \mathbb{Z} \text{ and } x \in M\}.$$

Let us call *wagon* of M the set $\{x \in M \mid x - (m+b) = qa + j_0, \text{for some integer } q\}$.

An element P of a configuration M is said to be the *pivot* of M if either ($P < m+b$ and P is the maximum of M) or, P is the maximum of the wagon.

Note that the wagon so as the pivot element of a configuration can be determined in an algorithmic way.

Checking Figure 6 may be useful. The wagon is column 22 and the pivot is 305.

The wagon consists of the rightmost elements of M. The highest of these is the pivot element. Note that the index of the column containing the wagon is $b + j_0$.

An (S, m, r)-amenable set M with pivot element P is said to be *ordered amenable* if its shadow is of the form $\{m, m+1, \ldots, m+t\}$, for some integer t, $0 \leq t < a+b-1$, and the only element that can possibly be added to obtain an $(S, m, r+1)$-amenable set without increasing the shadow is $P + \rho_2$.

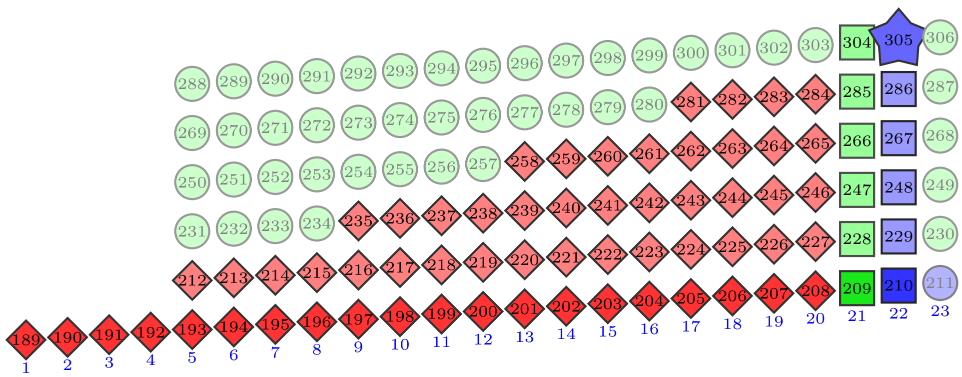

FIGURE 6. The biggest ordered amenable.

We now show how to construct (S, m, r)-amenable sets.

Remark 31. In view of Lemma 20, for every positive integer q with $qb < a$,

$$\mathrm{D}(m+qa+qb) = \mathrm{D}(m) \cup \{m - (ka+r) \mid a - qb \le r \le a-1, \frac{r-(a+b)}{b} \le k < \frac{r}{b}\}.$$

By performing a change of variables (change $a - r$ to r and $-k-1$ to k), we obtain that

$$\mathrm{D}(m+qa+qb) = \mathrm{D}(m) \cup \{m + ka + r \mid 1 \le r \le qb, -\frac{a-r}{b} - 1 < k \le \frac{r}{b}\}.$$

Hence

$$\mathrm{D}(m+qa+qb) \cap [m, \infty) = \{m\} \cup \{m + ka + r \mid 1 \le r \le qb, 0 \le k \le \frac{r}{b}\}$$

and

$$\mathrm{D}(m+qa+qb) \cap [m, m+a+b) = \{m, m+1, \dots, m+qb\}$$

(observe that for k to be one, r must be at least b, and in this case we obtain $m + a + b$ which is not in $[m, m + a + b)$).

Moreover,

$$\sharp\mathrm{D}(m+qa+qb) \cap [m, \infty) = 1 + q + \frac{b}{2}q(q+1),$$

since the cardinality of the set $\{m+ka+r \mid 1 \le r \le qb, 0 \le k \le \frac{r}{b}\}$ is $\sum_{r=1}^{qb}(\lfloor\frac{r}{b}\rfloor+1)$, which can be rewritten as $qb + \sum_{i=0}^{q-1}\sum_{j=0}^{b-1}\lfloor\frac{ib+j}{b}\rfloor + \lfloor\frac{qb}{b}\rfloor$, and this equals $qb + q + \sum_{i=0}^{q-1} bi = qb + q + b\frac{q(q-1)}{2} = q + \frac{b}{2}q(q+1)$. By adding now the cardinality of $\{m\}$, we obtain the desired equality.

Clearly the sets $\mathrm{D}(m + \lambda) \cap [m, \infty)$ are amenable sets. The following lemma shows that some of these are indeed ordered amenable sets. The problem is that their cardinalities do not cover all possible r's.

Lemma 32. *Let q be a positive integer such that $qb < a$. Then $\mathrm{D}(m + qa + qb) \cap [m, \infty)$ is an ordered amenable set.*

Proof. We already know that its shadow is an interval containing m (the condition $qb < a$, ensures that the shadow is not the whole ground), and as pointed out above, it is amenable. In order to conclude the proof, we show that for all $s > m$, $s \notin \mathrm{D}(m + qa + qb)$, if the set $\mathrm{D}(m + qa + qb) \cup \{s\}$ is amenable, then its shadow is larger than $\{m, m + 1, \ldots, m + qb\}$. Write $s = m + ua + v$, with $0 \le v < a$. If u is zero, as $s \notin \mathrm{D}(m + qa + qb)$, we obtain that $v > qb$, obtaining in this way a new element in the shadow. So u must be positive. We distinguish to cases.

- If $v \le qb$, then as $s \notin \mathrm{D}(m + qa + qb)$, by the preceding remark, we deduce that u must be greater than $\frac{v}{b}$. But then $m + ua + v - (m + a + b - 1) = (u-1)a+(v-b)+1$, and this element is in S if and only if $v-b+1 \le (u-1)b$ (Lemma 19), or equivalently, $v < ub$, which holds since $u > \frac{v}{b}$. This proves that $m + a + b - 1$ is in the shadow of $\mathrm{D}(m + qa + qb) \cup \{s\}$ (under the assumption that this set is amenable), and it is not in $\{m, m+1, \ldots, m+qb\}$, a contradiction.
- Now assume that $v > qb$. Then the element $m + v$ is in the shadow of $\mathrm{D}(m + qa + qb) \cup \{s\}$, obtaining again a contradiction.

□

From an (S, m, r)-ordered amenable set, we can construct another $(S, m, r - 1)$-ordered amenable set, just by removing its pivot.

Lemma 33. *If M is an ordered amenable set, which shadow is not the whole ground, and P is its pivot, then $M \setminus \{P\}$ is ordered amenable.*

Proof. Observe that P does not belong to $\mathrm{D}(M \setminus \{P\})$, and thus $M \setminus \{P\}$ is still amenable. From the definition of pivot, it follows easily that this set is also ordered amenable. □

Let r be a positive integer, there exists $q \in \mathbb{Z}$ such that

$$q + \frac{1}{2}bq(q-1) \le r < 1 + q + \frac{1}{2}bq(q+1).$$

Define $\mathrm{h}(r) = q$. Thus we can write $r = \mathrm{h}(r) + \frac{1}{2}b\mathrm{h}(r)(\mathrm{h}(r) - 1) + s$, with $0 \le s \le \mathrm{h}(r)b$. Hence

$$r = \mathrm{h}(r) + \frac{1}{2}b\mathrm{h}(r)(\mathrm{h}(r) - 1) + k\mathrm{h}(r) + j, \tag{4}$$

with $-1 \le k \le b - 1$ and $0 < j \le \mathrm{h}(r)$ ($k = -1$ only in the case $r = \mathrm{h}(r) + \frac{1}{2}b\mathrm{h}(r)(\mathrm{h}(r) - 1)$, and then $j = \mathrm{h}(r)$). Note that $\mathrm{h}(r) = 0$ leads to $r = 0$, so we may assume that $\mathrm{h}(r) > 0$. Observe also that $j + k = 0$ only when $\mathrm{h}(r) = 1 = j$ and $k = -1$.

Proposition 34. *Let r be a positive integer. Let k and j be as above. If $b(\mathrm{h}(r) - 1) + k + 1 < a + b - 1$, then the set*

$$\begin{aligned}(\mathrm{D}(m + (\mathrm{h}(r) - 1)(a + b)) &\cap [m, \infty)) \\ &\cup \{m + ua + v \mid (\mathrm{h}(r) - 1)b + 1 \le v \le (\mathrm{h}(r) - 1)b + k, 0 \le u \le \mathrm{h}(r) - 1\} \\ &\qquad \cup \{m + ((\mathrm{h}(r) - 1)b + k + 1)a + v \mid 0 \le v < j\}\end{aligned}$$

is an r-ordered amenable set.

Proof. This set is obtained from $\mathrm{D}(m + \mathrm{h}(r)(a + b))$ by repeating the Lemma 33 $1 + \mathrm{h}(r) + \frac{b}{2}\mathrm{h}(r)(\mathrm{h}(r) + 1) - r$ times. □

Next we prove that ordered (S,m,r)-amenable sets have minimal shadow in the set of all (S,m,r)-amenable sets with shadow an interval containing m. As a consequence any two (S,m,r)-ordered amenable sets have the same shadow.

Proposition 35. *Let M be an ordered (S,m,r)-amenable subset of S whose shadow has t elements and let N be another (S,m,r)-amenable subset of S whose shadow is an interval containing m. Then, the shadow of N has at least also t elements.*

Proof. Suppose that the shadow of N has less than t elements. This implies that the set $N \setminus M$ is non empty (since both sets have cardinality r) and therefore it has a minimum z. Furthermore, N has no elements in the wagon of M. It is straightforward to observe that $M \cup \{z\}$ is amenable.
In fact, as N is amenable, we have that $\mathrm{D}(z) \cap [m,\infty) \subset N$; $(\mathrm{D}(z) \setminus \{z\}) \cap [m,\infty) \subset M$ because z is minimum. So $\mathrm{D}(z) \cap [m,\infty) \subset M \cup \{z\}$. As z is not in the column containing the wagon of M we conclude that $z \neq P + \rho_2$, which contradicts the assumption that M is ordered. □

Considering M and N ordered amenable sets in the above proposition and applying it in both directions, we get the following consequence.

Corollary 36. *The shadows of ordered (S,m,r)-amenable sets coincide.*

With all these ingredients we can effectively compute the cardinality of the shadow of an ordered (S,m,r)-amenable set.

Corollary 37. *Let M be an ordered (S,m,r)-amenable set, and let k and j be as in (4), then $\#(M \cap [m, m+a+b)) = (\mathrm{h}(r)-1)b + k + 2$.*

Proof. As any two ordered amenable sets with the same cardinality have the same elements in the ground, we can use the ordered ameneable set of the preceding proposition. Observe that the ground for this set is $\{m, m+1, \ldots, m+(\mathrm{h}(r)-1)b, m+(\mathrm{h}(r)-1)b+1, \ldots, m+(\mathrm{h}(r)-1)+k+1\}$. □

Observe that this result gives a bound for integers r such that there exists an ordered (S,m,r)-amenable set.

Now we prove that if M is an (S,m,r)-amenable set whose shadow is not an interval containing m, then we can remove the trailing spaces in the shadow without increasing the number of divisors, that is we can find N, an (S,m,r)-amenable with shadow an interval containing m and such that $\sharp\mathrm{D}(N) \leq \sharp\mathrm{D}(M)$. By using what we already know for ordered (S,m,r)-amenable set, as a consequence we will obtain that they are optimal configurations. To this end we need several tools.

The first one enables us to push an (S,m,r)-amenable set to the right, obtaining an $(S,m+1,r)$-amenable set.

Lemma 38. *Let M be an (S,m,r)-amenable set. Then $N = M+1 = \{x+1 \mid x \in M\}$ is an $(S,m+1,r)$-amenable set.*

Proof. Let $x = y+1 \in N$, with $y \in M$, and suppose that $h \in S$ is such that the divisor $x-h$ of x is greater than m. We have to prove that $x-h \in N$. As $y-h$ is greater than or equal to m, we have that $y-h \in M$. It follows that $x-h = (y+1)-h = (y-h)+1 \in M+1 = N$. □

If we shift an (S,m,r)-amenable set to the left, we get an $(S,m-1,r)$-amenable set (provided $m-1 \geq 2c-1$).

Lemma 39. *Let M be an (S,m,r)-amenable set with $m \geq 2c$. Then $M-1 = \{x-1 \mid x \in M\}$ is an $(S,m-1,r)$-amenable set.*

Proof. Let $y \in M-1$, say $y = x-1$, with $x \in M$. Note that $y \geq m$. Now we use Corollary 8. By hypothesis $\mathrm{D}(x) \cap [m,\infty) = (x-S) \cap [m,\infty) \subseteq M$, but then $\mathrm{D}(y) \cap [m,\infty) = ((x-1)-S) \cap [m,\infty) \subseteq (M-1)$. □

If we add the element m to an $(S,m+1,r)$-amenable set, we get an $(S,m,r+1)$-amenable set.

Lemma 40. *Let N be an $(S,m+1,r)$-amenable set. The set $M = \{m\} \cup N$ is $(S,m,r+1)$-amenable.*

Proof. If $N = \{m+1 = m_2 < \ldots < m_{r+1}\}$ then $M = \{m < m_2 < \ldots < m_{r+1}\}$. Write $m_1 = m$.

We have to check that M is m-closed under division. It clearly holds for $i=1$. Let $i \geq 2$. The divisors of m_i non smaller than $m+1$ belong to N and thus to M. Therefore, divisors of m_i non smaller than m belong to M. □

It is immediate that if we remove the biggest element of an (m,r)-amenable set, then we get an $(m,r-1)$-amenable set (provided $r>1$).

Lemma 41. *Let M be an (m,r)-amenable set and suppose that $r>1$. Let u be the maximum of M. Then $M \setminus \{u\}$ is an $(m,r-1)$-amenable set.*

Before removing the trailing spaces of an (S,m,r)-amenable set, we need it to no contain the last element in the ground, that is $m+a+b-1$. If this is the case, next we give a procedure to obtain another (S,m,r)-amenable set whose shadow is at most as large as the original set, but not containing $m+a+b-1$.

Proposition 42. *Given an (S,m,r)-amenable set M with shadow L_M not coinciding with the ground, we can construct an (S,m,r)-amenable set N with shadow L_N not containing $m+a+b-1$ and such that $\sharp L_N \leq \sharp L_M$.*

Proof. Assume that M is an (S,m,r)-amenable set containing $m+a+b-1$, and with a shadow different to the ground. Then, there is at least an element x in the ground, such that x is not in M, and thus $x < m+a+b-1$. Let $N = \{m\} \cup (M+1) \setminus \{\max\{\{m\} \cup (M+1)\}\}$. Thus N is a shifting to the right, and then its maximum is replaced by m. So N is by the preceding lemmas an (S,m,r)-amenable set. Observe also that $x+1 \notin N$. Hence we repeat this procedure until $x+k$ becomes $m+a+b-1$. □

Suppose we have a configuration not containing $m+a+b-1$. We can shrink it so that the shadow of the configuration obtained is an interval containing m. It can be done using the following results.

Lemma 43. *Let M be an (S,m,r)-amenable set not containing $m+a+b-1$. Assume there exists a column c, such that $c \cap M = \emptyset$, and if M_1 are the elements in M in the columns to the left of c, and $M_2 = M \setminus M_1$, then $M_2 \neq \emptyset$ (c is a splitting column for M). Let $r_1 = \sharp M_1$, $r_2 = \sharp M_2$ and $m_2 = \min M_2$. Then*

1. *M_1 is an (S,m,r_1) amenable set,*
2. *M_2 is an (S,m_2,r_2) amenable set,*
3. *$M_1 \cup (M_2 - 1)$ is an (S,m,r) amenable set.*

Proof. It suffices to show that no element in M_1 divides an element in M_2, and vice-versa. Assume that there is $x \in M_1$ and $y \in M_2$ such that $y - x \in S$, that is, $y - x = ka + r$ for some r, k nonnegative integers with $r \leq \min\{a-1, kb\}$ (Lemma 19). Hence $y = x + ka + r$, and $y - (ka + i) \in \mathrm{D}(M) \cap [m, \infty) = M$ for all $i \in \{0, \dots, r\}$. Assume that c corresponds with the elements s in $[m, \infty) \cap \mathbb{N}$ such that $s - (m+b) \bmod a = j$. Then, by hypothesis

$$\begin{aligned} y - r - (m+b) \bmod a &= y - (ka + r) - (m+b) \bmod a \\ &= x - (m+b) \bmod a < y - (m+b) \bmod a, \end{aligned}$$

and thus there is $i \in \{0, \dots, r\}$ such that $y - (ka + i) - (m+b) \bmod a = y - i - (m+b) \bmod a = j$, contradicting that c was an empty column of M.

Now assume that $x \in M_1$ and $y \in M_2$ are such that $x - y = ka + r$ for some r, k as above. In this setting, $x - (ka + i) \in M$ for all $i \in \{0, \dots, r\}$. By hypothesis $x - (m+b) \bmod a < y - (m+b) \bmod a$. And

$$\begin{aligned} y + r - (m+b) \bmod a &= y + ka + r - (m+b) \bmod a \\ &= x - (m+b) \bmod a < y - (m+b) \bmod a. \end{aligned}$$

It follows that for some $i \in \{0, \dots, r\}$, $y + i - (m+b) \bmod a = a - 1$, but this is impossible, since as $m+a+b-1 \notin M$, the column $\{s \in M \mid s-(m+b) \bmod a = a-1\}$ is empty. □

Proposition 44. *Let M be an (S, m, r)-amenable set whose shadow L_M has t elements. There exists an (S, m, r)-amenable set T whose shadow L_T is an interval containing m and has no more than t elements, i.e., $\sharp L_T \leq \sharp L_M$.*

Proof. Every time you find a splitting column as in the statement of Lemma 43, change M with $M_1 \cup (M_2 - 1)$. This procedure does not increase the number of elements in the shadow of M. □

Lemma 45. *Let M be an (S, m, r)-amenable set whose shadow L_M has t elements. There exists an (S, m, r)-amenable set T whose shadow is an interval containing m, and $\sharp\mathrm{D}(T) \leq \sharp\mathrm{D}(M)$.*

Proof. Let T be as in Proposition 44, and assume that $\sharp L_T = t - k$. In view of Lemma 30, $\sharp\mathrm{D}(L_M) \geq \sharp\mathrm{D}(m, m+1, \dots, m+t-1)$. By Corollary 25, $\sharp\mathrm{D}(m, m+1, \dots, m+t-1) = \sharp\mathrm{D}(m) + \sum_{j=1}^{t-1} \left\lceil \frac{a+b-j}{b} \right\rceil$, and as $\left\lceil \frac{a+b-j}{b} \right\rceil \geq 1$, this amount is greater than or equal to $\sharp\mathrm{D}(m) + \sum_{j=1}^{t-k-1} \left\lceil \frac{a+b-j}{b} \right\rceil + k$, which according to Corollary 25 equals $\sharp\mathrm{D}(m, m+1, \dots, m+t-k-1) + k = \sharp\mathrm{D}(L_T) + k$. Now we use Lemma 15, having $\sharp\mathrm{D}(M) = \sharp\mathrm{D}(L_M) + \sharp M \setminus L_M = \sharp\mathrm{D}(L_M) + r - t \geq \sharp\mathrm{D}(L_T) + k + r - t = \sharp\mathrm{D}(L_T) + \sharp T \setminus L_T = \sharp\mathrm{D}(T)$. □

Theorem 46. *Let S be a numerical semigroup with conductor c, and let $m \geq 2c-1$. Then every ordered (S, m, r)-amenable set is an optimal configuration.*

Proof. Let M be an ordered (S, m, r)-amenable set. By Proposition 12, among the optimal configurations, there is always an (S, m, r)-amenable set. Let N be an (S, m, r)-amenable set that is an optimal configuration. In light of Lemma 45, we can assume that its shadow is an interval containing m. By Proposition 35, the shadow of M is contained in that of N, and by Corollary 16, we get that $\#\mathrm{D}(M) \leq$

$\#\mathrm{D}(N)$. As N is an optimal configuration we deduce that $\#\mathrm{D}(M) = \#\mathrm{D}(N)$, and thus M is also an optimal configuration. $\square$

Corollary 47. *Let $S = \langle a, a+1, \ldots, a+b\rangle$ with integers a, b such that $0 < b < a$. Write r as in formula (4), that is, $r = \mathrm{h}(r) + \frac{1}{2}b\mathrm{h}(r)(\mathrm{h}(r)-1) + k\mathrm{h}(r) + j$, with $-1 \le k \le b-1$ and $0 < j \le \mathrm{h}(r)$. Then $\mathrm{E}(r, \langle a, a+1, \ldots, a+b\rangle)$ equals*

$$r - 1 + \begin{cases} \sum_{i=1}^{b(\mathrm{h}(r)-1)+k+1} \left\lceil \frac{a-i}{b} \right\rceil, & \text{if } b(\mathrm{h}(r)-1)+k+2 < a+b, \\ \sum_{i=1}^{a+b-1} \left\lceil \frac{a-i}{b} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. Assume that $b(\mathrm{h}(r)-1)+k+1 < a+b-1$. Then by Proposition 34, there exists an ordered (S, m, r)-amenable set. In this setting the proof follows from Corollaries 26 and 37 and Theorem 46.

Observe also that if $b(\mathrm{h}(r)-1)+k+1 = a+b-2$, by Proposition 34, the set

$$\begin{aligned} M = {} & (\mathrm{D}(m + (\mathrm{h}(r)-1)(a+b)) \cap [m, \infty)) \\ & \cup \{m + ua + v \mid (\mathrm{h}(r)-1)b + 1 \le v \le (\mathrm{h}(r)-1)b + k + 1, 0 \le u \le \mathrm{h}(r) - 1\} \end{aligned}$$

is an ordered amenable set, and thus by Theorem 46 an optimal configuration for $r = \#M$. As $\mathrm{D}(M \cup \{m+a+b-1\}) = \mathrm{D}(M) \cup \{m+a+b-1\}$, the set $M \cup \{m+a+b-1\}$ is an optimal configuration of cardinality $r+1$, whose shadow fills the whole ground. By using now Corollary 18, we get optimal configurations for cardinalies greater than $r+1$. And the proof follows easily by Corollary 26. $\square$

Needless to say that, by using this formula for numerical semigroups generated by intervals, we have no need of the general Algorithm 3, speeding-up the computation of Feng-Rao distances for such semigroups.

Remark 48. The reader can check that we have $\mathrm{E}(r, S) = \rho_r$ exactly in the following cases:

(A) If either $r = b\sigma(p)+1, \ldots, b\sigma(p)+p+1$ and the ground is not completely filled, or

(B) $r \ge r_0$, where r_0 is the first r filling the ground,

being $\sigma(p) := 1 + \cdots + p = \frac{1}{2}p(p+1)$ and $p \ge 1$.

Besides, since both sequences $\mathrm{E}(r, S)$ and ρ_r are strictly increasing, the largest difference between them is for $r = 2$ and for the first r after each element in the first case, that is, $r = b\sigma(p)+p+2$ where ρ_r jumps from one interval to the next.

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