THE CYCLIC HOPF H mod K THEOREM

ADRIAN C. MURZA

Abstract. The H mod K theorem gives all possible periodic solutions in a $\Gamma-$equivariant dynamical system, based on the group-theoretical aspects. In addition, it classifies the spatio temporal symmetries that are possible. By the contrary, the equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for each $\mathbf{C}-$axial subgroup of $\Gamma \times \mathbb{S}^1$. In this paper we identify which periodic solution types, whose existence is guaranteed by the H mod K theorem, are obtainable by Hopf bifurcation, when the group Γ is finite cyclic.

1. Introduction

In the formalism of equivariant differential equations [1], [2] and [3] have been described two methods for obtaining periodic solutions: the H mod K theorem and the equivariant Hopf theorem. While the H mod K theorem offers the complete set of possible periodic solutions based exclusively on the structure of the group Γ acting on the differential equation, the equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all $\mathbf{C}-$axial subgroups of $\Gamma \times \mathbb{S}^1$.

Not always all solutions predicted by the H mod K theorem can be obtained by the generic Hopf bifurcation [3]. In [4] there are described which periodic solutions, whose existence is guaranteed by the H mod K theorem are obtainable by the Hopf bifurcation when the group Γ is finite abelian. In this article, we pose a more specific question: what periodic solutions predicted by the H mod K theorem are obtainable by the Hopf bifurcation when the group Γ is finite cyclic. We will answer this question by finding which additional constraints have to be added to the Abelian Hopf H mod K theorem [4] so that the periodic solutions predicted by the H mod K theorem coincide with the ones obtained by the equivariant Hopf theorem when the group Γ is finite cyclic.

2. The H mod K theorem

We call $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ a spatio-temporal symmetry of the solution $x(t)$. A spatio-temporal symmetry of $x(t)$ for which $\theta = 0$ is called a spatial symmetry, since it fixes the point $x(t)$ at every moment of time. The group of all spatio-temporal symmetries of $x(t)$ is denoted

$$\Sigma_{x(t)} \subseteq \Gamma \times \mathbf{S}^1.$$

As shown in [3], the symmetry group $\Sigma_{x(t)}$ can be identified with a pair of subgroups H and K of Γ and a homomorphism $\Theta : H \to \mathbf{S}^1$ with kernel K. Define

$$\begin{aligned} K &= \{\gamma \in \Gamma : \gamma x(t) = x(t)\ \forall t\} \\ H &= \{\gamma \in \Gamma : \gamma x(t) = \{x(t)\}\ \forall t\}. \end{aligned} \tag{1}$$

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The subgroup $K \subseteq \Sigma_{x(t)}$ is the group of spatial symmetries of $x(t)$ and the subgroup H consists of those symmetries that preserve the trajectory of $x(t)$, ie. the spatial parts of the spatio-temporal symmetries of $x(t)$. The groups $H \subseteq \Gamma$ and $\Sigma_{x(t)} \subseteq \Gamma \times \mathbf{S}^1$ are isomorphic; the isomorphism is in fact just the restriction to $\Sigma_{x(t)}$ of the projection of $\Gamma \times \mathbf{S}^1$ onto Γ. Therefore the group $\Sigma_{x(t)}$ can be written as

$$\Sigma^{\Theta} = \{(h, \Theta(h)) : h \in H, \Theta(h) \in \mathbf{S}^1\}.$$

Moreover, we call Σ^{Θ} a twisted subgroup of $\Gamma \times \mathbf{S}^1$. In our case Γ is a finite cyclic group and the H mod K theorem states necessary and sufficient conditions for the existence of a periodic solution to a $\Gamma-$ equivariant system of ODEs with specified spatio-temporal symmetries $K \subset H \subset \Gamma$. Recall that the isotropy subgroup Σ_x of a point $x \in \mathbb{R}^n$ consists of group elements that fix x, that is they satisfy

$$\Sigma_x = \sigma \in \Gamma : \sigma x = x.$$

Let $N(H)$ be the normalizer of H in Γ, satisfying $N(H) = \{\gamma \in \Gamma : \gamma H = H\gamma\}$. Let also $\mathrm{Fix}(K) = \{x \in \mathbb{R}^n : kx = x \ \forall k \in K\}$.

Definition 1. *Let $K \subset \Gamma$ be an isotropy subgroup. The variety L_K is defined by*

$$L_K = \bigcup_{\gamma \notin K} \mathrm{Fix}(\gamma) \cap \mathrm{Fix}(K).$$

Theorem 1. *(H mod K Theorem [3]) Let Γ be a finite group acting on $\mathbb{R}^n$. There is a periodic solution to some $\Gamma-$equivariant system of ODEs on $\mathbb{R}^n$ with spatial symmetries K and spatio-temporal symmetries H if and only if the following conditions hold:*

(a) *H/K is cyclic;*
(b) *K is an isotropy subgroup;*
(c) *dim $Fix(K) \geqslant 2$. If dim $Fix(K) = 2$, then either $H = K$ or $H = N(K)$;*
(d) *H fixes a connected component of $\mathrm{Fix(K)} \backslash \mathrm{L_K}$, where L_K appears as in Definition 1 above;*

Moreover, if (a) – (d) hold, the system can be chosen so that the periodic solution is stable.

Definition 2. *The pair of subgroups (H, K) is called admissible if the pair satisfies hypotheses (a) – (d) of Theorem 1, that is, if there exist periodic solutions to some $\Gamma-$equivariant system with (H, K) symmetry.*

3. Hopf bifurcation with cyclic symmetries

In the following we recall two results from [4] needed later for the proof of the Theorem 2. Let $x_0 \in \mathbb{R}^n$. Suppose that V is an $\Sigma_{x_0}-$invariant subspace of $\mathbb{R}^n$. Let $\hat{V} = x_0 + V$, and observe that $\hat{V}$ is also $\Sigma_{x_0}-$invariant.

Lemma 1. *Let g be an $\Sigma_{x_0}-$equivariant map on $\hat{V}$ such that $g(x_0) = 0$. Then g extends to a $\Gamma-$equivariant mapping f on $\mathbb{R}^n$ so that the center subspace of $(df)_{x_0}$ equals the center subspace of $(dg)_{x_0}$.*

Proof. See [4]. □

Lemma 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $\Gamma-$equivariant and let $f(x_0) = 0$. Let V be the center subspace of $(df)_{x_0}$. Then there exists a $\Gamma-$equivariant diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi(x_0) = x_0$ and the center manifold of the transformed vector field*

$$\psi_* f(x) \equiv (d\psi)^{-1}_{\psi(x)} f(\psi(x))$$

is $\hat{V}$.

Proof. See [4]. □

In order to state the Cyclic Hopf theorem, we need first the following lemma.

Lemma 3. *The group Γ is cyclic if and only it is a homomorphic image of $\mathbb{Z}$.*

Proof. To show that Γ is cyclic if and only if it is a homomorphic image of $\mathbb{Z}$, let $\Gamma = \langle a \rangle$ then the map

$$\mathbb{Z} \to \Gamma,\ n \to a^n$$

is a homomorphism (since $a^{n+m} = a^n a^m$ for all $n, m \in \mathbb{Z}$) whose image is Γ.
Conversely, if $f : \mathbb{Z} \to \Gamma$ is an epimorphism then let $a = f(1)$. Every $\gamma \in \Gamma$ takes the form $\gamma = f(n)$ for some $n \in \mathbb{Z}$. If $n \geqslant 0$ then

$$\gamma = f(1 + \ldots + 1) = f(1) \circ_\Gamma \cdots \circ_\Gamma f(1) = (f(1))^n = a^n.$$

The same formula holds if $n < 0$. Thus $\Gamma = \langle a \rangle$. □

Theorem 2. *(cyclic Hopf theorem). In systems with finite cyclic symmetry, generically, Hopf bifurcation at a point x_0 occurs with simple eigenvalues, and there exists a unique branch of small-amplitude periodic solutions emanating from x_0. Moreover the spatio-temporal symmetries of the bifurcating periodic solutions are*

$$H = \Sigma_{x_0},\ H \textit{ is cyclic} \tag{2}$$

and

$$K = \ker_V(H),\ K \textit{ is cyclic}, \tag{3}$$

and H acts $H-$simply on V. In addition let $\mathbb{Z}_k$ act on $\mathbb{R}^k$ by a cyclic permutation of coordinates. Let $\mathbb{Z}_q \subseteq \mathbb{Z}_n \subseteq \mathbb{Z}_k$. Then there is a $\mathbb{Z}_n-$simple representation with kernel $\mathbb{Z}_q$ with the single exception when $n = k$ is even and $q = \frac{k}{2}$.

Proof. The proof relies on the proof of the homologous Theorem in [4], with changes concerning the form of the subgroups H and K. However, we will prefer to give the proof entirely, including the parts that coincide with the proof in [4], to easy the lecture of the paper. We begin as in [4], by showing that the equivariant Hopf bifurcation leads to a unique branch of small-amplitude periodic solutions emanating from x_0. From Lemma 1 it follows that the bifurcation point $x_0 = 0$ and therefore $\Gamma = \Sigma_{x_0}$. Moreover, from Lemma 2 it follows that if reducing to the center manifold, we may assume that $\mathbb{R}^n = V$ and therefore from [2] it follows that the center subspace V at the Hopf bifurcation point is $\Gamma-$simple. This means that V is either a direct sum of two absolutely irreducible representations or it is itself irreducible but not absolutely irreducible. Since the irreducible representations of abelian groups (and subsequently cyclic groups) are one-dimensional and absolutely irreducible or two-dimensional and non-absolutely irreducible, it follows that V is two-dimensional and therefore the eigenvalues obtained at the linearization about the bifurcation point x_0 are simple. Now the standard Hopf bifurcation theorem applies to obtain a unique branch of periodic solutions.

Let $x(t, \lambda)$ be the unique branch of small-amplitude periodic solutions that emanate at the Hopf bifurcation point x_0. For each t,

$$x_0 = \lim_{\lambda \to 0} x(t, \lambda).$$

Let H be the spatio-temporal symmetry subgroup of $x(\cdot,\lambda)$, and let $\Phi: H \to \mathbb{S}^1$ be the homomorphism that associates a symmetry $h \in H$ with a phase shift $\Phi(h) \in \mathbb{S}^1$. To prove that $H \subset \Sigma_{x_0}$ we have

$$\begin{aligned} hx_0 &= \lim_{\lambda\to 0} hx(0,\lambda) && \text{by continuity of } h \\ &= \lim_{\lambda\to 0} x(\Phi(h),\lambda) && \text{by definition of spatio} - \text{temporal symmetries} \\ &= x_0 \end{aligned}$$

and therefore $h \in \Sigma_{x_0}$. In the following we proof that $\Sigma_{x_0} \subset H$. Let $\gamma \in \Sigma_{x_0} \subseteq \Gamma$; therefore $\gamma x(t,\lambda)$ is also a periodic solution. Since the periodic is unique (as shown above), we have

$$\gamma\{x(t,\lambda)\} = \{x(t,\lambda)\},$$

so $\gamma \in H$. Lemma 2 allows us to assume that the center manifold at x_0 is $\hat{V} = v + x_0$, which may be identified with V, and therefore V is $H-$invariant. Therefore V is $H-$simple since γ is cyclic (and subsequently abelian). Since Γ is cyclic, all its subgroups are cyclic, in particular H and K.
The proof of the last condition is the proof of Proposition 6.2 in [4]. □

4. Constructing systems with cyclic symmetry near Hopf points

This section consists in recalling the results corresponding section 4 in [4] where the construction of systems with abelian symmetry near Hopf points has been carried out. When Γ is finite cyclic, a key step in constructing H mod K periodic solutions from Hopf bifurcation at x_0 is the construction of a locally $\Sigma_{x_0}-$equivariant vector field. We first construct, for finite symmetry groups, a $\Gamma-$equivariant vector field that has a stable equilibrium, $x_0 \in \mathbb{R}^n$, with the desired isotropy. We will use

Lemma 4. *For any finite set of distinct points $y_1,\ldots,y_l$, vectors $v_1,\ldots,v_l$ in $\mathbb{R}^n$ and matrices $A_1,\ldots,A_l \in \mathrm{GL}(n)$, there exists a polynomial map $g:\mathbb{R}^n\to\mathbb{R}^n$ such that $g(y_j)=v_j$ and $(dg)_{y_j} = A_j$.*

Proof. See [5]. □

Theorem 3. *Let Γ be a finite cyclic group acting on $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. Then there exists a $\Gamma-$equivariant system of ODEs on $\mathbb{R}^n$ with a stable equilibrium x_0.*

Proof. See [4]. □

In conclusion any point $x_0 \in \mathbb{R}^n$ can be a stable equilibrium for a $\Gamma-$equivariant vector field $f:\mathbb{R}^n\to\mathbb{R}^n$. It is clear that $(df)_{x_0}$ must commute with the isotropy subgroup Σ_{x_0} of x_0 [3]. The following result states that the linearization about the equilibrium x_0 can be any linear map that commutes with the isotropy subgroup.

Theorem 4. *Let $x_0 \in \mathbb{R}^n$ and $A:\mathbb{R}^n\to\mathbb{R}^n$ be a linear map that commutes with the isotropy subgroup Σ_{x_0} of x_0. Then there exists a polynomial $\Gamma-$equivariant vector field $f:\mathbb{R}^n\to\mathbb{R}^n$ such that $f(x_0)=0$ and $(df)_{x_0}=A$.*

Proof. See [4]. □

When constructing a Hopf bifurcation at points $x_0 \in \mathbb{R}^n$ we do not necessarily assume full isotropy. Genericity of $\Sigma_{x_0}-$simple subspaces at points of Hopf bifurcation is given by $\Gamma-$equivariant mappings as follows.

Lemma 5. *Let Γ act on $\mathbb{R}^n$ and fix $x_0 \in \mathbb{R}^n$. Let V be a $\Sigma_{x_0}-$invariant neigborhood of x_0 such that $\gamma \bar{V} \cap \bar{V} = \emptyset$ for any $\gamma \in \Gamma \backslash \Sigma_{x_0}$. Let $g : \bar{V} \times \mathbb{R} \to \mathbb{R}^n$ be a smooth $\Sigma_{x_0}-$equivariant vector field. Then there exists an extension of g to a smooth $\Gamma-$equivariant vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$.*

Proof. See [4]. □

5. The cyclic Hopf H mod K theorem

Theorem 5. *(cyclic Hopf H mod K theorem). Let Γ be a finite cyclic group acting on $\mathbb{R}^n$. There is an H mod K periodic solution that arises by a generic Hopf bifurcation if and only if the following seven conditions hold: Theorem 1 $(a)-(d)$, H is a cyclic isotropy subgroup, there exists an $H-$simple subspace V such that $K = \ker_V(H)$, K is cyclic and let $\mathbb{Z}_k$ act on $\mathbb{R}^k$ by a cyclic permutation of coordinates. Let $\mathbb{Z}_q \subseteq \mathbb{Z}_n \subseteq \mathbb{Z}_k$. Then there is a $\mathbb{Z}_n-$simple representation with kernel $\mathbb{Z}_q$ with the single exception when $n = k$ is even and $q = \frac{k}{2}$.*

Proof. Necessity follows from the H mod K theorem (Theorem 1) and the cyclic Hopf theorem (Theorem 2). We'll prove the sufficiency next. The idea of the proof will again, rely heavily on the proof of Abelian Hopf H mod K theorem in [4]. Let $x_0 \in \mathbb{R}^n$ and let H be the isotropy subgroup of the point x_0, ie. $H = \Sigma_{x_0}$. Moreover, let W be a $H-$simple representation. Since Γ is cyclic (in particular, abelian), W is two-dimensional. Now we can define the linear maps $A(\lambda) : W \to W$ by

$$A(\lambda) = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}.$$

Since W is two-dimensional it is easy to prove the commutativity with A. We have

$$A(\lambda) \cdot W = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} \cdot \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \lambda a - b & -\lambda b - a \\ a + \lambda b & a\lambda - b \end{bmatrix} = W \cdot A(\lambda).$$

Next we can extend Theorem 4 to a bifurcation problem as in Lemma 5. Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a $\Gamma-$equivariant polynomial such that for all $\gamma \in \Gamma$, $f(\gamma x_0, \lambda) = 0$ and $(df)_{\gamma x_0, \lambda}|_{\gamma W} = \gamma A(\lambda) \gamma^{-1}$. Moreover, let $g = f|_{W+x_0}$. From the way f has been constructed, g is $H-$equivariant on $W + x_0$ and $g(x_0) = 0$, hence from Lemma 2 we have that W is the center subspace of $(dg)_{x_0,0}$.

Next consider $(dg)_{x_0,\lambda}|_W$; its eigenvalues are $\sigma(\lambda) \pm i\rho(\lambda)$ with $\sigma(0) = 0$, $\rho(0) = 1$ and $\sigma'(0) \neq 0$. Then the equivariant Hopf theorem extended to a point $x_0 \in \mathbb{R}^n$ implies the existence of small-amplitude periodic solutions emanating from x_0 with spatio-temporal symmetries H and spatial symmetries K. □

6. General considerations between the differences of the results in this article and [4]

In the first place it must be highlighted that one can start we the methodology used in [4] and add the restrictions presented in this paper to obtain the Cyclic Hopf H mod K Theorem, but not vice-versa. This is obvious, because any cyclic group is abelian but not any abelian group is cyclic.

In this section we use the Cyclic Hopf H mod K Theorem to exhibit symmetry pairs (H, K) that are admissible by the Abelian Hopf H mod K Theorem but not admissible by the Cyclic Hopf H mod K Theorem. Let $\mathbb{Z}_l$ act on $\mathbb{R}^l$ by cyclic permutation of coordinates and $\Gamma = \mathbb{Z}_l \times \mathbb{Z}_k$ act on $\mathbb{R}^l \times \mathbb{R}^k$ by the diagonal action, where $l, k > 1$. We show Abelian

Hopf H mod K admissible but not Cyclic Hopf H mod K admissible pairs for this action of Γ by classifying in Theorem (6) all Cyclic Hopf H mod K admissible pairs $K \subset H \subset \Gamma$ and showing that there are admissible pairs that are not on the list.

Theorem 6. *By applying the Cyclic Hopf H mod K Theorem, the (H, K) Hopf-admissible pairs in Γ are $(\mathbb{Z}_m \times \mathbb{Z}_n,\ \mathbb{Z}_m \times \mathbb{Z}_q)$ where q divides n except when $q = \frac{k}{2}$ and $n = k$, and $(\mathbb{Z}_m \times \mathbb{Z}_n,\ \mathbb{Z}_p \times \mathbb{Z}_n)$ where p divides m except when $p = \frac{l}{2}$ and $m = l$. Moreover, m and n are coprimes, m and q are coprimes with $m \neq q$, and p and n are coprimes, with $p \neq n$.*

Proof. The proof is a restriction to the cases m and n are coprimes with $m \neq n$, m and q are coprimes with $m \neq q$, and p and n are coprimes, with $p \neq n$, of the proof of Theorem 6.1 in [4]. $\square$

To find an example of a pair (H, K) that is admissible by the Abelian Hopf H mod K Theorem but not by the Cyclic Hopf H mod K Theorem, let's take $(H = \mathbb{Z}_m \times \mathbb{Z}_n,\ K = \mathbb{Z}_m \times \mathbb{Z}_q)$ where q divides n except when $q = \frac{k}{2}$ and $n = k$, and m and n are not coprimes, m and q are not coprimes. They are admissible by the Abelian Hopf H mod K by applying Theorem 6.1 in [4]. However, they are not admissible by the Cyclic Hopf H mod K Theorem because of the application of the the Fundamental Theorem of finitely generated abelian groups. Indeed, if, for example m and n are not coprimes then they have a common divisor integer $a \in \mathbb{R}_+$ that is prime, and in this case $m = ab,\ n = ac$ for some integers $b \in \mathbb{R}_+,\ c \in \mathbb{R}_+$ and the group $\mathbb{Z}_{ab} \times \mathbb{Z}_{ac}$ is not cyclic. A similar case applies for the group $K = \mathbb{Z}_m \times \mathbb{Z}_q$ if m and q are not coprimes with $m \neq q$, or the group $K = \mathbb{Z}_p \times \mathbb{Z}_n$ if n and p are not coprimes with $n \neq p$.

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Centro de Matemática da Universidade do Porto., Rua do Campo Alegre 687, 4169-007 Porto, Portugal