

# THE CYCLIC HOPF $H \bmod K$ THEOREM

ADRIAN C. MURZA

ABSTRACT. The  $H \bmod K$  theorem gives all possible periodic solutions in a  $\Gamma$ -equivariant dynamical system, based on the group-theoretical aspects. In addition, it classifies the spatio-temporal symmetries that are possible. By the contrary, the equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for each  $\mathbf{C}$ -axial subgroup of  $\Gamma \times \mathbb{S}^1$ . In this paper we identify which periodic solution types, whose existence is guaranteed by the  $H \bmod K$  theorem, are obtainable by Hopf bifurcation, when the group  $\Gamma$  is finite cyclic.

## 1. INTRODUCTION

In the formalism of equivariant differential equations [1], [2] and [3] have been described two methods for obtaining periodic solutions: the  $H \bmod K$  theorem and the equivariant Hopf theorem. While the  $H \bmod K$  theorem offers the complete set of possible periodic solutions based exclusively on the structure of the group  $\Gamma$  acting on the differential equation, the equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all  $\mathbf{C}$ -axial subgroups of  $\Gamma \times \mathbb{S}^1$ .

Not always all solutions predicted by the  $H \bmod K$  theorem can be obtained by the generic Hopf bifurcation [3]. In [4] there are described which periodic solutions, whose existence is guaranteed by the  $H \bmod K$  theorem are obtainable by the Hopf bifurcation when the group  $\Gamma$  is finite abelian. In this article, we pose a more specific question: what periodic solutions predicted by the  $H \bmod K$  theorem are obtainable by the Hopf bifurcation when the group  $\Gamma$  is finite cyclic. We will answer this question by finding which additional constraints have to be added to the Abelian Hopf  $H \bmod K$  theorem [4] so that the periodic solutions predicted by the  $H \bmod K$  theorem coincide with the ones obtained by the equivariant Hopf theorem when the group  $\Gamma$  is finite cyclic.

## 2. THE $H \bmod K$ THEOREM

We call  $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$  a spatio-temporal symmetry of the solution  $x(t)$ . A spatio-temporal symmetry of  $x(t)$  for which  $\theta = 0$  is called a spatial symmetry, since it fixes the point  $x(t)$  at every moment of time. The group of all spatio-temporal symmetries of  $x(t)$  is denoted

$$\Sigma_{x(t)} \subseteq \Gamma \times \mathbf{S}^1.$$

As shown in [3], the symmetry group  $\Sigma_{x(t)}$  can be identified with a pair of subgroups  $H$  and  $K$  of  $\Gamma$  and a homomorphism  $\Theta : H \rightarrow \mathbf{S}^1$  with kernel  $K$ . Define

$$(1) \quad \begin{aligned} K &= \{\gamma \in \Gamma : \gamma x(t) = x(t) \ \forall t\} \\ H &= \{\gamma \in \Gamma : \gamma x(t) = \{x(t)\} \ \forall t\}. \end{aligned}$$

---

*Date:* March 12, 2015.

*2000 Mathematics Subject Classification.* 34C23, 34C25, 37G40.

*Key words and phrases.* Hopf bifurcation, spatio-temporal symmetry, finite cyclic group, periodic solutions.

This work was supported by FCT grant *SFRH/BD/64374/2009*.

The subgroup  $K \subseteq \Sigma_{x(t)}$  is the group of spatial symmetries of  $x(t)$  and the subgroup  $H$  consists of those symmetries that preserve the trajectory of  $x(t)$ , ie. the spatial parts of the spatio-temporal symmetries of  $x(t)$ . The groups  $H \subseteq \Gamma$  and  $\Sigma_{x(t)} \subseteq \Gamma \times \mathbf{S}^1$  are isomorphic; the isomorphism is in fact just the restriction to  $\Sigma_{x(t)}$  of the projection of  $\Gamma \times \mathbf{S}^1$  onto  $\Gamma$ . Therefore the group  $\Sigma_{x(t)}$  can be written as

$$\Sigma^\Theta = \{(h, \Theta(h)) : h \in H, \Theta(h) \in \mathbf{S}^1\}.$$

Moreover, we call  $\Sigma^\Theta$  a twisted subgroup of  $\Gamma \times \mathbf{S}^1$ . In our case  $\Gamma$  is a finite cyclic group and the  $H$  mod  $K$  theorem states necessary and sufficient conditions for the existence of a periodic solution to a  $\Gamma$ -equivariant system of ODEs with specified spatio-temporal symmetries  $K \subset H \subset \Gamma$ . Recall that the isotropy subgroup  $\Sigma_x$  of a point  $x \in \mathbb{R}^n$  consists of group elements that fix  $x$ , that is they satisfy

$$\Sigma_x = \{\sigma \in \Gamma : \sigma x = x\}.$$

Let  $N(H)$  be the normalizer of  $H$  in  $\Gamma$ , satisfying  $N(H) = \{\gamma \in \Gamma : \gamma H = H\gamma\}$ . Let also  $\text{Fix}(K) = \{x \in \mathbb{R}^n : kx = x \forall k \in K\}$ .

**Definition 1.** Let  $K \subset \Gamma$  be an isotropy subgroup. The variety  $L_K$  is defined by

$$L_K = \bigcup_{\gamma \notin K} \text{Fix}(\gamma) \cap \text{Fix}(K).$$

**Theorem 1.** (*H mod K Theorem [3]*) Let  $\Gamma$  be a finite group acting on  $\mathbb{R}^n$ . There is a periodic solution to some  $\Gamma$ -equivariant system of ODEs on  $\mathbb{R}^n$  with spatial symmetries  $K$  and spatio-temporal symmetries  $H$  if and only if the following conditions hold:

- (a)  $H/K$  is cyclic;
- (b)  $K$  is an isotropy subgroup;
- (c)  $\dim \text{Fix}(K) \geq 2$ . If  $\dim \text{Fix}(K) = 2$ , then either  $H = K$  or  $H = N(K)$ ;
- (d)  $H$  fixes a connected component of  $\text{Fix}(K) \setminus L_K$ , where  $L_K$  appears as in Definition 1 above;

Moreover, if (a) – (d) hold, the system can be chosen so that the periodic solution is stable.

**Definition 2.** The pair of subgroups  $(H, K)$  is called admissible if the pair satisfies hypotheses (a) – (d) of Theorem 1, that is, if there exist periodic solutions to some  $\Gamma$ -equivariant system with  $(H, K)$  symmetry.

### 3. HOPF BIFURCATION WITH CYCLIC SYMMETRIES

In the following we recall two results from [4] needed later for the proof of the Theorem 2. Let  $x_0 \in \mathbb{R}^n$ . Suppose that  $V$  is an  $\Sigma_{x_0}$ -invariant subspace of  $\mathbb{R}^n$ . Let  $\hat{V} = x_0 + V$ , and observe that  $\hat{V}$  is also  $\Sigma_{x_0}$ -invariant.

**Lemma 1.** Let  $g$  be an  $\Sigma_{x_0}$ -equivariant map on  $\hat{V}$  such that  $g(x_0) = 0$ . Then  $g$  extends to a  $\Gamma$ -equivariant mapping  $f$  on  $\mathbb{R}^n$  so that the center subspace of  $(df)_{x_0}$  equals the center subspace of  $(dg)_{x_0}$ .

*Proof.* See [4]. □

**Lemma 2.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be  $\Gamma$ -equivariant and let  $f(x_0) = 0$ . Let  $V$  be the center subspace of  $(df)_{x_0}$ . Then there exists a  $\Gamma$ -equivariant diffeomorphism  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\psi(x_0) = x_0$  and the center manifold of the transformed vector field

$$\psi_* f(x) \equiv (d\psi)_{\psi(x)}^{-1} f(\psi(x))$$

is  $\hat{V}$ .

*Proof.* See [4]. □

In order to state the Cyclic Hopf theorem, we need first the following lemma.

**Lemma 3.** *The group  $\Gamma$  is cyclic if and only if it is a homomorphic image of  $\mathbb{Z}$ .*

*Proof.* To show that  $\Gamma$  is cyclic if and only if it is a homomorphic image of  $\mathbb{Z}$ , let  $\Gamma = \langle a \rangle$  then the map

$$\mathbb{Z} \rightarrow \Gamma, \quad n \rightarrow a^n$$

is a homomorphism (since  $a^{n+m} = a^n a^m$  for all  $n, m \in \mathbb{Z}$ ) whose image is  $\Gamma$ .

Conversely, if  $f : \mathbb{Z} \rightarrow \Gamma$  is an epimorphism then let  $a = f(1)$ . Every  $\gamma \in \Gamma$  takes the form  $\gamma = f(n)$  for some  $n \in \mathbb{Z}$ . If  $n \geq 0$  then

$$\gamma = f(1 + \dots + 1) = f(1) \circ_{\Gamma} \dots \circ_{\Gamma} f(1) = (f(1))^n = a^n.$$

The same formula holds if  $n < 0$ . Thus  $\Gamma = \langle a \rangle$ . □

**Theorem 2.** (*cyclic Hopf theorem*). *In systems with finite cyclic symmetry, generically, Hopf bifurcation at a point  $x_0$  occurs with simple eigenvalues, and there exists a unique branch of small-amplitude periodic solutions emanating from  $x_0$ . Moreover the spatio-temporal symmetries of the bifurcating periodic solutions are*

$$(2) \quad H = \Sigma_{x_0}, \quad H \text{ is cyclic}$$

and

$$(3) \quad K = \ker_V(H), \quad K \text{ is cyclic,}$$

and  $H$  acts  $H$ -simply on  $V$ . In addition let  $\mathbb{Z}_k$  act on  $\mathbb{R}^k$  by a cyclic permutation of coordinates. Let  $\mathbb{Z}_q \subseteq \mathbb{Z}_n \subseteq \mathbb{Z}_k$ . Then there is a  $\mathbb{Z}_n$ -simple representation with kernel  $\mathbb{Z}_q$  with the single exception when  $n = k$  is even and  $q = \frac{k}{2}$ .

*Proof.* The proof relies on the proof of the homologous Theorem in [4], with changes concerning the form of the subgroups  $H$  and  $K$ . However, we will prefer to give the proof entirely, including the parts that coincide with the proof in [4], to ease the lecture of the paper. We begin as in [4], by showing that the equivariant Hopf bifurcation leads to a unique branch of small-amplitude periodic solutions emanating from  $x_0$ . From Lemma 1 it follows that the bifurcation point  $x_0 = 0$  and therefore  $\Gamma = \Sigma_{x_0}$ . Moreover, from Lemma 2 it follows that if reducing to the center manifold, we may assume that  $\mathbb{R}^n = V$  and therefore from [2] it follows that the center subspace  $V$  at the Hopf bifurcation point is  $\Gamma$ -simple. This means that  $V$  is either a direct sum of two absolutely irreducible representations or it is itself irreducible but not absolutely irreducible. Since the irreducible representations of abelian groups (and subsequently cyclic groups) are one-dimensional and absolutely irreducible or two-dimensional and non-absolutely irreducible, it follows that  $V$  is two-dimensional and therefore the eigenvalues obtained at the linearization about the bifurcation point  $x_0$  are simple. Now the standard Hopf bifurcation theorem applies to obtain a unique branch of periodic solutions.

Let  $x(t, \lambda)$  be the unique branch of small-amplitude periodic solutions that emanate at the Hopf bifurcation point  $x_0$ . For each  $t$ ,

$$x_0 = \lim_{\lambda \rightarrow 0} x(t, \lambda).$$

Let  $H$  be the spatio-temporal symmetry subgroup of  $x(\cdot, \lambda)$ , and let  $\Phi : H \rightarrow \mathbb{S}^1$  be the homomorphism that associates a symmetry  $h \in H$  with a phase shift  $\Phi(h) \in \mathbb{S}^1$ . To prove that  $H \subset \Sigma_{x_0}$  we have

$$\begin{aligned} hx_0 &= \lim_{\lambda \rightarrow 0} hx(0, \lambda) && \text{by continuity of } h \\ &= \lim_{\lambda \rightarrow 0} x(\Phi(h), \lambda) && \text{by definition of spatio-temporal symmetries} \\ &= x_0 \end{aligned}$$

and therefore  $h \in \Sigma_{x_0}$ . In the following we proof that  $\Sigma_{x_0} \subset H$ . Let  $\gamma \in \Sigma_{x_0} \subseteq \Gamma$ ; therefore  $\gamma x(t, \lambda)$  is also a periodic solution. Since the periodic is unique (as shown above), we have

$$\gamma\{x(t, \lambda)\} = \{x(t, \lambda)\},$$

so  $\gamma \in H$ . Lemma 2 allows us to assume that the center manifold at  $x_0$  is  $\hat{V} = v + x_0$ , which may be identified with  $V$ , and therefore  $V$  is  $H$ -invariant. Therefore  $V$  is  $H$ -simple since  $\gamma$  is cyclic (and subsequently abelian). Since  $\Gamma$  is cyclic, all its subgroups are cyclic, in particular  $H$  and  $K$ .

The proof of the last condition is the proof of Proposition 6.2 in [4].  $\square$

#### 4. CONSTRUCTING SYSTEMS WITH CYCLIC SYMMETRY NEAR HOPF POINTS

This section consists in recalling the results corresponding section 4 in [4] where the construction of systems with abelian symmetry near Hopf points has been carried out. When  $\Gamma$  is finite cyclic, a key step in constructing  $H$  mod  $K$  periodic solutions from Hopf bifurcation at  $x_0$  is the construction of a locally  $\Sigma_{x_0}$ -equivariant vector field. We first construct, for finite symmetry groups, a  $\Gamma$ -equivariant vector field that has a stable equilibrium,  $x_0 \in \mathbb{R}^n$ , with the desired isotropy. We will use

**Lemma 4.** *For any finite set of distinct points  $y_1, \dots, y_l$ , vectors  $v_1, \dots, v_l$  in  $\mathbb{R}^n$  and matrices  $A_1, \dots, A_l \in \text{GL}(n)$ , there exists a polynomial map  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $g(y_j) = v_j$  and  $(dg)_{y_j} = A_j$ .*

*Proof.* See [5].  $\square$

**Theorem 3.** *Let  $\Gamma$  be a finite cyclic group acting on  $\mathbb{R}^n$  and  $x_0 \in \mathbb{R}^n$ . Then there exists a  $\Gamma$ -equivariant system of ODEs on  $\mathbb{R}^n$  with a stable equilibrium  $x_0$ .*

*Proof.* See [4].  $\square$

In conclusion any point  $x_0 \in \mathbb{R}^n$  can be a stable equilibrium for a  $\Gamma$ -equivariant vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . It is clear that  $(df)_{x_0}$  must commute with the isotropy subgroup  $\Sigma_{x_0}$  of  $x_0$  [3]. The following result states that the linearization about the equilibrium  $x_0$  can be any linear map that commutes with the isotropy subgroup.

**Theorem 4.** *Let  $x_0 \in \mathbb{R}^n$  and  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map that commutes with the isotropy subgroup  $\Sigma_{x_0}$  of  $x_0$ . Then there exists a polynomial  $\Gamma$ -equivariant vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $f(x_0) = 0$  and  $(df)_{x_0} = A$ .*

*Proof.* See [4].  $\square$

When constructing a Hopf bifurcation at points  $x_0 \in \mathbb{R}^n$  we do not necessarily assume full isotropy. Genericity of  $\Sigma_{x_0}$ -simple subspaces at points of Hopf bifurcation is given by  $\Gamma$ -equivariant mappings as follows.

**Lemma 5.** *Let  $\Gamma$  act on  $\mathbb{R}^n$  and fix  $x_0 \in \mathbb{R}^n$ . Let  $V$  be a  $\Sigma_{x_0}$ -invariant neighborhood of  $x_0$  such that  $\gamma\bar{V} \cap \bar{V} = \emptyset$  for any  $\gamma \in \Gamma \setminus \Sigma_{x_0}$ . Let  $g : \bar{V} \times \mathbb{R} \rightarrow \mathbb{R}^n$  be a smooth  $\Sigma_{x_0}$ -equivariant vector field. Then there exists an extension of  $g$  to a smooth  $\Gamma$ -equivariant vector field  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ .*

*Proof.* See [4]. □

## 5. THE CYCLIC HOPF $H \bmod K$ THEOREM

**Theorem 5.** *(cyclic Hopf  $H \bmod K$  theorem). Let  $\Gamma$  be a finite cyclic group acting on  $\mathbb{R}^n$ . There is an  $H \bmod K$  periodic solution that arises by a generic Hopf bifurcation if and only if the following seven conditions hold: Theorem 1 (a) – (d),  $H$  is a cyclic isotropy subgroup, there exists an  $H$ -simple subspace  $V$  such that  $K = \ker_V(H)$ ,  $K$  is cyclic and let  $\mathbb{Z}_k$  act on  $\mathbb{R}^k$  by a cyclic permutation of coordinates. Let  $\mathbb{Z}_q \subseteq \mathbb{Z}_n \subseteq \mathbb{Z}_k$ . Then there is a  $\mathbb{Z}_n$ -simple representation with kernel  $\mathbb{Z}_q$  with the single exception when  $n = k$  is even and  $q = \frac{k}{2}$ .*

*Proof.* Necessity follows from the  $H \bmod K$  theorem (Theorem 1) and the cyclic Hopf theorem (Theorem 2). We'll prove the sufficiency next. The idea of the proof will again, rely heavily on the proof of Abelian Hopf  $H \bmod K$  theorem in [4]. Let  $x_0 \in \mathbb{R}^n$  and let  $H$  be the isotropy subgroup of the point  $x_0$ , ie.  $H = \Sigma_{x_0}$ . Moreover, let  $W$  be a  $H$ -simple representation. Since  $\Gamma$  is cyclic (in particular, abelian),  $W$  is two-dimensional. Now we can define the linear maps  $A(\lambda) : W \rightarrow W$  by

$$A(\lambda) = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}.$$

Since  $W$  is two-dimensional it is easy to prove the commutativity with  $A$ . We have

$$A(\lambda) \cdot W = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} \cdot \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \lambda a - b & -\lambda b - a \\ a + \lambda b & a\lambda - b \end{bmatrix} = W \cdot A(\lambda).$$

Next we can extend Theorem 4 to a bifurcation problem as in Lemma 5. Let  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  be a  $\Gamma$ -equivariant polynomial such that for all  $\gamma \in \Gamma$ ,  $f(\gamma x_0, \lambda) = 0$  and  $(df)_{\gamma x_0, \lambda}|_{\gamma W} = \gamma A(\lambda) \gamma^{-1}$ . Moreover, let  $g = f|_{W+x_0}$ . From the way  $f$  has been constructed,  $g$  is  $H$ -equivariant on  $W + x_0$  and  $g(x_0) = 0$ , hence from Lemma 2 we have that  $W$  is the center subspace of  $(dg)_{x_0, 0}$ .

Next consider  $(dg)_{x_0, \lambda}|_W$ ; its eigenvalues are  $\sigma(\lambda) \pm i\rho(\lambda)$  with  $\sigma(0) = 0$ ,  $\rho(0) = 1$  and  $\sigma'(0) \neq 0$ . Then the equivariant Hopf theorem extended to a point  $x_0 \in \mathbb{R}^n$  implies the existence of small-amplitude periodic solutions emanating from  $x_0$  with spatio-temporal symmetries  $H$  and spatial symmetries  $K$ . □

## 6. GENERAL CONSIDERATIONS BETWEEN THE DIFFERENCES OF THE RESULTS IN THIS ARTICLE AND [4]

In the first place it must be highlighted that one can start with the methodology used in [4] and add the restrictions presented in this paper to obtain the Cyclic Hopf  $H \bmod K$  Theorem, but not vice-versa. This is obvious, because any cyclic group is abelian but not any abelian group is cyclic.

In this section we use the Cyclic Hopf  $H \bmod K$  Theorem to exhibit symmetry pairs  $(H, K)$  that are admissible by the Abelian Hopf  $H \bmod K$  Theorem but not admissible by the Cyclic Hopf  $H \bmod K$  Theorem. Let  $\mathbb{Z}_l$  act on  $\mathbb{R}^l$  by cyclic permutation of coordinates and  $\Gamma = \mathbb{Z}_l \times \mathbb{Z}_k$  act on  $\mathbb{R}^l \times \mathbb{R}^k$  by the diagonal action, where  $l, k > 1$ . We show Abelian

Hopf  $H \bmod K$  admissible but not Cyclic Hopf  $H \bmod K$  admissible pairs for this action of  $\Gamma$  by classifying in Theorem (6) all Cyclic Hopf  $H \bmod K$  admissible pairs  $K \subset H \subset \Gamma$  and showing that there are admissible pairs that are not on the list.

**Theorem 6.** *By applying the Cyclic Hopf  $H \bmod K$  Theorem, the  $(H, K)$  Hopf-admissible pairs in  $\Gamma$  are  $(\mathbb{Z}_m \times \mathbb{Z}_n, \mathbb{Z}_m \times \mathbb{Z}_q)$  where  $q$  divides  $n$  except when  $q = \frac{k}{2}$  and  $n = k$ , and  $(\mathbb{Z}_m \times \mathbb{Z}_n, \mathbb{Z}_p \times \mathbb{Z}_n)$  where  $p$  divides  $m$  except when  $p = \frac{l}{2}$  and  $m = l$ . Moreover,  $m$  and  $n$  are coprimes,  $m$  and  $q$  are coprimes with  $m \neq q$ , and  $p$  and  $n$  are coprimes, with  $p \neq n$ .*

*Proof.* The proof is a restriction to the cases  $m$  and  $n$  are coprimes with  $m \neq n$ ,  $m$  and  $q$  are coprimes with  $m \neq q$ , and  $p$  and  $n$  are coprimes, with  $p \neq n$ , of the proof of Theorem 6.1 in [4].  $\square$

To find an example of a pair  $(H, K)$  that is admissible by the Abelian Hopf  $H \bmod K$  Theorem but not by the Cyclic Hopf  $H \bmod K$  Theorem, let's take  $(H = \mathbb{Z}_m \times \mathbb{Z}_n, K = \mathbb{Z}_m \times \mathbb{Z}_q)$  where  $q$  divides  $n$  except when  $q = \frac{k}{2}$  and  $n = k$ , and  $m$  and  $n$  are not coprimes,  $m$  and  $q$  are not coprimes. They are admissible by the Abelian Hopf  $H \bmod K$  by applying Theorem 6.1 in [4]. However, they are not admissible by the Cyclic Hopf  $H \bmod K$  Theorem because of the application of the the Fundamental Theorem of finitely generated abelian groups. Indeed, if, for example  $m$  and  $n$  are not coprimes then they have a common divisor integer  $a \in \mathbb{R}_+$  that is prime, and in this case  $m = ab$ ,  $n = ac$  for some integers  $b \in \mathbb{R}_+$ ,  $c \in \mathbb{R}_+$  and the group  $\mathbb{Z}_{ab} \times \mathbb{Z}_{ac}$  is not cyclic. A similar case applies for the group  $K = \mathbb{Z}_m \times \mathbb{Z}_q$  if  $m$  and  $q$  are not coprimes with  $m \neq q$ , or the group  $K = \mathbb{Z}_p \times \mathbb{Z}_n$  if  $n$  and  $p$  are not coprimes with  $n \neq p$ .

**Acknowledgements.** The author would like to thank the helpful suggestions received from the referee, which improved the presentation of this paper. He also acknowledges economical support form FCT grant SFRH/ BD/ 64374/ 2009.

#### REFERENCES

- [1] M. GOLUBITSKY, D.G. SCHAEFFER, *Singularities and groups in bifurcation theory I*, Applied mathematical sciences **51**, Springer-Verlag, (1985).
- [2] M. GOLUBITSKY, I. STEWART, D. G. SCHAEFFER, *Singularities and groups in bifurcation theory II*, Applied mathematical sciences **69**, Springer-Verlag, (1988), 388–399.
- [3] M. GOLUBITSKY, J. STEWART, *The symmetry perspective*, Birkhauser Verlag, (2003), 63–68. 487–509.
- [4] N. FILIPSKI, M. GOLUBITSKY, *The Abelian Hopf  $H \bmod K$  Theorem*, SIAM J. Appl. Dynam. Sys. **9**, (2010), 283–291.
- [5] P. LANCASTER, M. TISMENETSKY, *The theory of matrices, 2-nd edition*, Academic Press New-York, (1985).

CENTRO DE MATEMÁTICA DA UNIVERSIDADE DO PORTO., RUA DO CAMPO ALEGRE 687, 4169-007 PORTO, PORTUGAL