Generic Profit Singularities in Time Averaged Optimization for Phase Transitions in Polydynamical Systems

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Abstract

We consider the optimization problem that consists in maximizing the time averaged profit for a motion of a smooth polydynamical system on the circle in the presence of a smooth profit density. When the problem depends on a k-dimensional parameter the optimal averaged profit is a function of the parameter. It is known that an optimal motion can always be selected among stationary strategies and a special type of periodic cyclic motions called *level cycles*. We present the classification of all generic singularities of the optimal averaged profit when $k \leq 2$ for phase transitions between these two optimal strategies.

Keywords: singularities, averaged optimization, control systems

1 Introduction

Consider a *smooth control system* on the circle S^1:

$$\dot{x} = v(x, u)$$

where x is an angle on the circle and u is a control parameter that belongs to the control space U, which is a smooth closed manifold (or a disjoint union of smooth closed manifolds) with at least two different points.

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Each vector field $v(\,\cdot\,, u)$ that is obtained by fixing the control parameter value u is called an *admissible velocity*. Given a point x_0 on the circle, the *set of admissible velocities at* x_0 is given by

$$V(x_0) = \{v(x_0, u) : \ u \in U\}.$$

A motion $x : \mathbb{R} \to S^1$ of the control system is said to be an *admissible motion* if it is absolutely continuous and the velocity of motion at each time of differentiability, $\dot{x}(t)$, is an admissible velocity, that is, $\dot{x}(t) \in V(x(t))$.

Remark 1 The phase space is compact and so any admissible motion of the control system can be defined for all $t \in \mathbb{R}$.

In this context, the existence of a smooth *profit density* $f : S^1 \to \mathbb{R}$ on the circle leads to the following optimal control problem:

To maximize the averaged profit on the infinite time horizon

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x(t))dt$$

over all the system's admissible motions on the positive semiaxis.

We take the upper limit whenever the above limit does not exist.

This problem of control theory includes in particular the optimization of the averaged profit of periodic processes when the phase space is the circle. References to various applications of periodic control problems (e.g. economic, electrical engineering and chemical reaction engineering) as well as a case study to production planning can be found in [8]. In this paper we approach this problem from the singularity theory point of view.

Consider that this optimization problem depends on a parameter p belonging to a smooth manifold, that is, that both the control system and the profit density depend on a parameter p. Then, the optimal strategy can vary with p and the optimal averaged profit on the infinite time horizon is the function A defined by

$$A(p) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x(t), p)dt,$$

and can have singularities (points where it is not smooth). This brings us to the problem of classifying such singularities.

This problem was firstly considered in [1] and then in [3] and [4]. In these works, the determination of the optimal averaged profit on the infinite horizon of a controlled dynamical system is based on two different types of admissible motions: a *level cycle* (motion that uses the maximum, respectively minimum, velocity when the profit density is less, respectively greater, than a specific constant) and a *stationary strategy* (motion associated with an equilibrium point of the controlled dynamical system, i. e., a point where the convex hull of all admissible velocities contains the zero velocity).

In fact, the maximal averaged profit can always be provided by an admissible motion of these types (see [4]). Therefore in order to classify the singularities of the optimal averaged profit, it is enough to consider the following three situations: singularities for (a) stationary strategies, (b) level cycles and (c) transitions between stationary strategies and level cycles.

The classification of all generic singularities was already treated in the case of a one dimensional parameter ([1], [3], [4] for a control space without boundary and [9] for a control space with a regular boundary).

In this work we consider this Singularity Theory problem for a special type of control systems, namely, for *polydynamical systems*, that is, for control systems with a finite number of (at least two) admissible velocities. Note that in this case the control space U is of the form $\{1, \ldots, n\}$, $n \geq 2$ and, for simplicity, we denote by v_i the admissible velocity $v(\cdot, \cdot, i)$, $1 \leq 1 \leq n$. Hence, the set of admissible velocities at a point (x, p) takes the form

$$V(x, p) = \{v_1(x, p), \ldots, v_n(x, p)\}.$$

For polydynamical systems, the classification of all generic singularities corresponding to stationary strategies and level cycles has already been done: [12] (for stationary strategies and 1-dimensional parameter), [10] (for stationary strategies and $k \leq 3$ dimensional parameter) and [6] (for level cycles and $k \leq 2$ dimensional parameter).

In this work we complete the classification of all generic singularities of the optimal averaged profit for the case of polydynamical systems and of a k-dimensional parameter, with $k \leq 2$. Namely, we present the generic singularities of the optimal averaged profit for transitions between stationary strategies and level cycles (this classification is described in detail in [13]).

2 Preliminary concepts

We consider the fine smooth Whitney topology on the various spaces that we will deal with (families of vector fields, families of polydynamical systems, etc.). If a property holds for any object belonging to some open everywhere dense subset of such a space, we say that this property is *generic* (or *holds generically*).

In the next sections there will arise different objects defined on product spaces. Each product space $X \times P$ is regarded as a fibered space over the k-dimensional parameter space P, that is, as the union of all sets of the form $X \times \{p\}$, with $p \in P$, which are called *fibers*. A map from a fibered space to itself is *fibered* if it preserves the fibration, that is, if it sends fibers to fibers. Note that a fibered map on a fibered space $X \times P$ over P has the form $(x, p) \mapsto (\varphi(x, p), h(p))$.

We consider the following equivalence relation on a fibered space: two objects, of the same nature, are *$\mathcal{F}$-equivalent* if there exists a fibered diffeomorphism transforming one object into the other.

In this setting, the germs of two functions $f, g : P \to \mathbb{R}$ are *Γ-equivalent* when their graphs are $\mathcal{F}$-equivalent on the product space $P \times \mathbb{R}$ fibered over P. The diffeomorphism taking one graph into the other is necessarily of the form $(p, a) \mapsto (\varphi(p), h(p, a))$, with $(p, a) \in P \times \mathbb{R}$.

A particular case of Γ-equivalence is called *R^+-equivalence* and is obtained when the second component h of the diffeomorphism is of the form $a + c(p)$, for some smooth function c. For example, the germ of a smooth function at a point is R^+-equivalent to the germ of the zero function at the origin. In fact, if G is a smooth function at a point p_0 on a manifold, we consider a coordinate system φ with origin at p_0 and, finally, the germ of the fibered diffeomorphism $(p, a) \mapsto (\varphi(p), a - G(p))$ at $(p_0, G(p_0))$, to obtain the desirable form.

Remark 2 Another *R^+-equivalence* is well known in the Singularity Theory literature. It is defined for germs of families of functions as follows ([2], p.304): *two families F_1 and F_2 are said to be R^+-equivalent if one of them is mapped to the other by a suitable fibered diffeomorphism Ψ composed with the addition of a smooth function ψ of the parameter, that is,*

$$F_1(x, p) = F_2(h(x, p), \varphi(p)) + \psi(p),$$

where $\Psi(x, p) = (h(x, p), \varphi(p))$. There is an analogous definition for germs.

In order to avoid confusion with these different concepts, in this work we call *$\mathcal{F}^+$-equivalence* to the R^+-equivalence of families of functions.

3 Optimal motions

In this section we do not introduce any new results. We merely present results which are fundamental for understanding the next section.

Consider the space of families of polydynamical systems on the circle. The *stationary domain* is the set of points where the convex hull of all admissible velocities contains the zero velocity. At a point of the stationary domain either there exists an admissible velocity vanishing or none of the admissible velocities vanishes but 0 is contained in their convex hull. In the last case, for the fixed value of the parameter, there is an admissible motion converging to x_0 as time goes to infinity ([5, 11]), that is, the motion can be stabilized by a chattering control. In particular, the averaged profit on the infinite horizon provided by this motion equals the value of the profit density at this point, that is, the profit gained by staying at that point.

A *stationary strategy* is an admissible motion converging to a point of the stationary domain.

Given a value c of the profit density, the *c-level motion* is the motion that uses the minimum velocity at points where the profit density is greater than the constant c, and that uses the maximum velocity elsewhere. A value of the profit density is *cyclic* if in a neighborhood of this value all corresponding level motions provide rotations along the circle. The level motion associated with a cyclic value is called a *level cycle*. A level cycle is *optimal* if it provides the maximum averaged profit on the infinite horizon; the corresponding cyclic value is an *optimal level*.

It was proved in [4] that the optimal averaged profit on the infinite time horizon can always be provided either by a level cycle or by a stationary strategy. This result allows us to subdivide the classification of the singularities of the optimal averaged profit into three groups: singularities for (a) stationary strategies, (b) level cycles and (c) transitions between stationary strategies and level cycles. The aim of this paper is to treat the last group. The other two groups are studied in [10] and [6], and in the next two subsections we recall the results related to them that are essential for section 3.

3.1 Stationary Strategies

The *optimal averaged profit for stationary strategies*, denoted by A_s, is the function that to each parameter value assigns the maximum value of the averaged profit on the infinite horizon among all stationary strategies. A stationary strategy at a point of the stationary domain provides an averaged profit on the infinite horizon equal to the value of the family of profit densities at that point. So, A_s is given by:

$$A_s(p) = \max_{x \in S(p)} f(x,p), \tag{1}$$

where $S(p)$ is the set of points x on the phase space for which (x,p) belongs to the stationary domain. If $S(p)$ is nonempty, then it is compact and so $A_s(p)$ is well defined. So, for classifying all generic singularities of A_s, we firstly classify all generic singularities of the stationary domain and, after that, those of the solution of the problem with constraints (1).

The stationary domain around an interior point is locally $\mathcal{F}$-equivalent to $\mathbb{R} \times \mathbb{R}^k$. At a boundary point, at least one of admissible velocities vanishes. Therefore, to classify the stationary domain around its boundary points, we just have to examine the equilibria of the admissible velocities.

An equilibrium point of a family of vector fields on a 1-dimensional manifold is called an *equilibrium point of type* A_m $(m \geq 0)$ if the germ of the zero level of the family at that point is $\mathcal{F}$-equivalent to the germ at the origin of the zero level of the family A_m where

$$A_0(x,p) = x \quad \text{ and } \quad A_m(x,p) = x^{m+1} + \sum_{i=1}^{m} p_i x^{m-i}, m \geq 1$$

where x and p are local coordinates along the phase space and the parameter space, respectively.

Generically, the germ of a k-parameter family of vector fields on a 1-dimensional manifold at an equilibrium point is $\mathcal{F}$-equivalent to the germ at the origin of one of the families $A_m \cdot V$, for some smooth function V that does not vanish at the origin and $0 \leq m \leq k$. In particular, in a generic case every equilibrium point of a k-parameter family of vector fields on a 1-dimensional manifold is a point of one of the types A_m, with $0 \leq m \leq k$.

Consider the set E of equilibria of all admissible velocities of a family of polydynamical systems. Following the notation in [10], let $I_j = (i_1, \cdots, i_j)$ where $j, i_1, \cdots, i_j$ are nonnegative integers with $0 \leq i_1 \leq \cdots \leq i_j$. Then a point of $P \in E$ is said to be *of type* A_{I_j} if P is an equilibrium point of precisely j admissible velocities $v_1, \cdots, v_j$, and furthermore for each $k \in \{1, ..., j\}$, P is a point of type A_{i_k} for v_k. We also use the notation $|I_j| = j - 1 + i_1 + \cdots + i_j$.

A point of type A_{I_j} of the stationary domain is called a *point of type* ${A_{I_j}}^i$ if i is the smallest natural such that the derivative $\frac{\partial^i f}{\partial x^i}$ does not vanish at that point, where x is a coordinate along the circle. An interior point of the stationary domain is called a *point of type* I^i if i is defined in the same manner.

We say that the *profit* $A_s(p_0)$ *is attained at* (x_0, p_0) when $A_s(p_0) = f(x_0, p_0)$.

Lemma 1 ([10]) *Generically, the optimal averaged profit for stationary strategies can be attained at points of type:*

- I^2, ${A_0}^1$, ${A_1}^1$, ${A_{0,0}}^1$, ${A_0}^2$ *if* $k = 1$, *or else*
- I^4, ${A_2}^1$, ${A_{0,1}}^1$, ${A_{0,0,0}}^1$, ${A_1}^2$, ${A_{0,0}}^2$, ${A_0}^3$ *if* $k = 2$.

Notice that all singularities presented in Lemma 1 involve at most three admissible velocities. Indeed, we are considering a parameter space P with dimension at most 2 and thus a product space $S^1 \times P$ with dimension at most 3, which implies that generically A_s can not be attained at points of the stationary domain satisfying more than 3 independent conditions. For the vanishing of 4 velocities there must be satisfied 4 independent conditions which generically can not occur in a space of dimension at most 3.

Note that I^2 and I^4 correspond both to a maximum of the optimal averaged profit attained at an interior point (x_0, p_0) of the stationary domain. At such a point there is at least a pair of velocities with opposite directions, that is, $v_1(x_0, p_0) < 0 < v_2(x_0, p_0)$, for some velocities v_1 and v_2. In the case I^2, this maximum is nondegenerate, that is, $\frac{\partial^2 f}{\partial x^2}(x_0, p_0) \neq 0$. The case I^4 corresponds to a degenerate maximum, being $\frac{\partial^4 f}{\partial x^4}(x_0, p_0)$ the first nonvanishing derivative. We illustrate for $k = 1$ the other singularities in Table 1.

3.2 Level cycles

Let us consider now a c-level cycle, that is, a rotation along the circle that uses the following velocity:

$$v_c(x,p) = \begin{cases} v_{\max}(x,p) & \text{if } f(x,p) \leq c \\ v_{\min}(x,p) & \text{if } f(x,p) > c \end{cases}$$

where

$$v_{\max}(x,p) = \max\{v_1(x,p), ..., v_n(x,p)\}$$

and

$$v_{\min}(x,p) = \min\{v_1(x,p), ..., v_n(x,p)\}$$

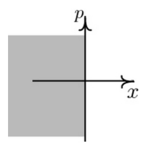
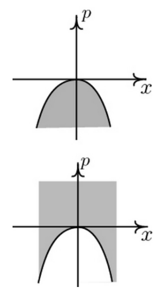
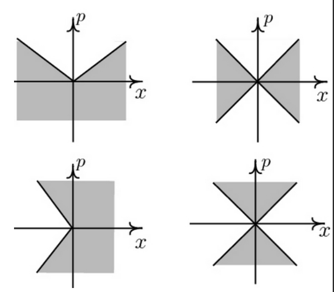
<table>
<tr><th></th><th>$A_0{}^1$</th><th>$A_0{}^2$</th><th>$A_1{}^1$</th><th>$A_{0,0}{}^1$</th></tr>
<tr><td>Stationary domain</td><td colspan="2"></td><td></td><td></td></tr>
<tr><td>Conditions on f</td><td>$\frac{\partial f}{\partial x}(0,0) \neq 0$</td><td>$\frac{\partial f}{\partial x}(0,0) = 0$
$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$</td><td>$\frac{\partial f}{\partial x}(0,0) \neq 0$</td><td>$\frac{\partial f}{\partial x}(0,0) \neq 0$</td></tr>
</table>

Table 1: Illustration of some singularities in Lemma 1, for $k = 1$.

are the *maximum* and the *minimum velocities*, respectively. These velocities are also called *extremal velocities*.

Recall that c is a cyclic value, that is, in a neighborhood of c, all corresponding level motions provide rotation along the circle. It is easy to see that, for a fixed parameter value, the set of cyclic values is either empty or is an interval.

Because every level cycle is a periodic motion we define the *Period* function, T, as the function that to each pair (p, c) of a parameter value and a corresponding cyclic value assigns the period (the smallest) of the c-level cycle. We also define the *Profit* function, P, as the function that to each pair (p, c) as before assigns the profit of a complete rotation along the circle.

Thus, $P(p, c) = \int_0^{T(p,c)} f(x(t), p)dt$, where $x(t)$ is the c-level motion. Since $v_c(\cdot, p)$ has always the same sign, we assume, without loss of generality, that it is positive. So, it is possible to rewrite both the period and the profit in terms of spatial integrals as:

$$T(p,c) = \oint \frac{1}{v_c(x,p)} dx \qquad \text{and} \qquad P(p,c) = \oint \frac{f(x,p)}{v_c(x,p)} dx.$$

The averaged profit on the infinite time horizon for the c-level cycle is given by the equality $A(p,c) = \frac{P(p,c)}{T(p,c)}$. For each parameter value whose optimal averaged profit is provided by a level cycle, we define the optimal averaged profit A_l for level cycles

$$A_l(p) = \max_c A(p,c).$$

In the following theorem we join some results presented in [3] and [4] for the general case of control systems, which are fundamental for obtaining the classification of all generic singularities of A_l.

Theorem 1 ([3], [4]) *Suppose that the differentiable profit density f has a finite number of critical points and the maximum and minimum velocities of the continuous control system are equal at isolated points only. If for the parameter value p_0, the maximum averaged profit is provided by a c_0-level cycle, then*

1. *the period T, the profit P and the averaged profit A along level cycles are continuous functions;*
2. *$A(p_0, \cdot)$ is a differentiable function near c_0;*
3. *near p_0 the optimal averaged profit is the unique solution $c(p)$ of equation $A(p,c) = c$;*
4. *if A is a C^k-function around (p_0, c_0) then the optimal profit A_l is a C^k-function around p_0.*

By Theorem 1, $A_l(p) = c(p)$ where $c(p)$ is the unique solution of equation

$$\frac{P(p,c)}{T(p,c)} - c = 0. \tag{2}$$

At a level providing the maximum averaged profit $\frac{\partial}{\partial c}(P/T)$ must be zero and consequently $\frac{\partial}{\partial c}(\frac{P(p,c)}{T(p,c)} - c) = -1$. So the implicit function theorem can be used to study the singularities of the best averaged profit A_l from the singularities of T and P.

Computing the period and the profit corresponding to a level cycle (see [1] or [9] for details), it is easy to understand that one of the obstructions for the functions T and P to be smooth (leading to singularities of A_l) is the existence of points of the *Maxwell set* of the extremal velocities, that is, points where at least one of the extremal velocities is not smooth. In fact, the situations leading to singularities (loss of differentiability) of T and P are divided in the following three types:

1. The optimal level is a regular value of the profit density and there exist points of the Maxwell set of the extremal velocities inside the domains where they are used (*passing through a point of the Maxwell set*);

2. The optimal level is a regular value of the profit density and there exist switching points (points where the extremal velocities are interchanged) on the Maxwell set of the extremal velocities (*switching at a point of the Maxwell set*);

3. The optimal averaged profit coincides with a critical value of the profit density (*transition through a critical value of the profit*).

So, first of all, we are led to study the Maxwell set of the extremal velocities. Recall that the control space U is of the form $U = \{1, 2, .., n\}$ with $n \geq 2$. So the admissible velocities are in finite number $\#U = n$ and consequently each of the extremal velocities ($v_{\max}$ and $v_{\min}$) just can lose differentiability at points of coincidence of some of the admissible velocities. Using transversality theorems, it is easy to see that generically, for the 2-dimensional parameter case, there are no points of coincidence of more than four admissible velocities. A point of the Maxwell set is called a *double point* if one of the extremal velocities is not smooth due to the coincidence of exactly two admissible velocities. Analogously we define *triple* and *quadruple points* for the coincidence of exactly three and four admissible velocities, respectively.

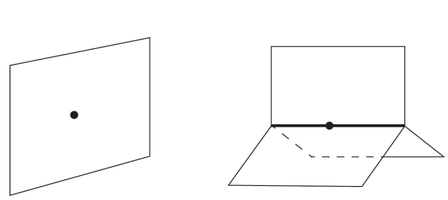


Figure 1: The Maxwell set in the three dimensional space (x, p) around a double, a triple and a quadruple point

Because we will use $\mathcal{F}$-equivalence when treating objects defined on $S^1 \times P$, we must distinguish double, triple and quadruple points also in what concerns the local projection on the parameter space of the Maxwell set. So, a point of the Maxwell set is called *tangent* for the maximum/minimum velocity if at that point the natural fibration of the product space $S^1 \times P$ over the parameter space P is *tangent* to the set of points where the maximum/minimum velocity is not smooth; otherwise, it is called a *regular point* for the maximum/minimum velocity. A tangent point for the maximum/minimum velocity has *tangency of order k* if k is the highest order of contact among all pairs of admissible velocities coinciding at that point and providing that velocity.

Again using transversality theorems it is easy to conclude that generically, for the 2-dimensional parameter case, the tangency of a double point can be of first or second order, the tangency of a triple point just can be of first order and a quadruple point is regular.

There can also exist points where both the extremal velocities are not smooth. Those points are called self intersection points of the Maxwell set.

To analyze all generic singularities of the optimal averaged profit for level cycles, one must now assemble the three situations above with the generic possibilities for the Maxwell set. In the following theorem we list all situations that have to be considered.

Theorem 2 ([6]) *Consider a 2-parameter family of polydynamical systems on the circle. Generically, the optimal averaged profit A_l for level cycles can be nonsmooth only in the following situations:*
1. passing through a tangent double point - 1^{st} order tangency
2. passing through a regular triple point
3. switching at a regular double point that is not a self-intersection point of the Maxwell set
$4_\pm$. transition through a local minimum/maximum of the profit density provided by a point out of the Maxwell set
5. passing through a tangent triple point - 1^{st} order tangency
6. passing through a regular quadruple point
7. switching at a regular double point that is a self-intersection point of the Maxwell set
8. switching at a tangent double point - 1^{st} order tangency and $\#U > 2$
9. switching at a tangent double point - 1^{st} order tangency and $\#U = 2$
10. switching at a regular triple point and $\#U > 3$;
$11_\pm$. transition through a local minimum/maximum of the profit density provided by a regular double point and $\#U > 2$;

$12_\pm$. transition through a local minimum/maximum of the profit density provided by a regular double point and $\#U = 2$;
13. switching at a regular triple point and $\#U = 3$;
14. passing through a tangent double point - 2^{nd} order tangency
15. transition through a critical value of the profit density which is not a minimum nor a maximum.

Remark 3 Situations 1-4 correspond to codimension 1 singularities in the parameter space ([3]) and situations 5-12 correspond to codimension 2 singularities in the parameter space ([6]).

4 Singularities of the optimal averaged profit at transition values

Recall that the aim of this paper is to classify all generic, up to codimension two, singularities of the optimal averaged profit when it is necessary to switch between optimal strategies to get the optimal averaged profit. A parameter value is called a *transition value* if in any neighborhood of it, the maximum averaged profit can not be provided by one and only one type of strategy, namely, either by a level cycle or by a stationary strategy.

In this section we deduce firstly all generic cases that have to be considered to obtain all generic singularities of the optimal averaged profit at transition values and then we present the classification's theorem (Theorem 4). We leave the technically proofs to the last section. The following two lemmas play an important role in the identification of those generic cases.

Lemma 2 ([4]) *At a transition value the optimal averaged profit is provided by a stationary strategy. Moreover, for a transition value p_0, if the optimal averaged profit A_s for stationary strategies is continuous at p_0 and is defined in a neighborhood of it then $A_s(p_0) = A_l(p_0)$, where $A_l(p_0)$ is the upper limit of the averaged profits provided by level cycles as $p \to p_0$.*

Lemma 3 *Generically, if p_0 is a transition value then the profit $A_s(p_0)$*

1. *just can be attained at points of the stationary domain of the following types: I^2, ${A_0}^1$, ${A_1}^1$, ${A_{0,0}}^1$, ${A_0}^2$, ${A_1}^2$;*

2. *can not be attained at more than two points. Moreover, if it is attained at exactly two points then they just can be of type I^2 or of type ${A_0}^1$.*

The proof of the previous lemma is easy to understand. In fact, due to Lemma 1, generically $A_s(p_0)$ just can be attained at points of type I^2, ${A_0}^1$, ${A_1}^1$, ${A_{0,0}}^1$, ${A_0}^2$, I^4, ${A_2}^1$, ${A_{0,1}}^1$, ${A_{0,0,0}}^1$, ${A_1}^2$, ${A_{0,0}}^2$, ${A_0}^3$. However, points of type I^4, ${A_2}^1$, ${A_{0,1}}^1$, ${A_{0,0,0}}^1$, ${A_{0,0}}^2$ and ${A_0}^3$ provide codimension 2 singularities which are continuous at the origin defined in a neighborhood of it [10] and so, due to Lemma 2, an independent condition has to be satisfied. For this reason they have to be excluded and the first part of the lemma is proved. The second part follows from Multijet Transversality Theorem.

Theorem 3 *Let p_0 be a parameter value. If the profit $A_s(p_0)$ is attained at a point of type ${A_1}^1$, ${A_{0,0}}^1$ or ${A_1}^2$ then generically p_0 is not a transition value.*

This theorem is proved in Section 5. From the results presented in this section we conclude that at a transition value p_0 the profit $A_s(p_0)$ just can be attained at points of type I^2, A_0^1 or A_0^2. Besides, as in these three cases A_s is continuous at p_0 and is defined in a neighborhood of it ([10]), the additional condition $A_s(p_0) = A_l(p_0)$ has to be satisfied. So to obtain all generic singularities of the optimal averaged profit at transition values we must analyze exactly the following situations:

1 $A_s(p_0)$ attained at exactly one point, type I^2, and no points (x, p_0) leading to a nonsmoothness situation of Theorem 2.

2 $A_s(p_0)$ attained at exactly one point, type ${A_0}^1$, and no points (x, p_0) leading to a nonsmoothness situation of Theorem 2.

3 $A_s(p_0)$ attained at exactly one point, type I^2, and existence of another point (x, p_0) in situation 1 of Theorem 2.

4 $A_s(p_0)$ attained at exactly one point, type I^2, and existence of another point (x, p_0) in situation 2 of Theorem 2.

5 $A_s(p_0)$ attained at exactly one point, type I^2, and existence of another point (x, p_0) in situation 3 of Theorem 2.

$6_\pm$ $A_s(p_0)$ attained at exactly one point, type I^2, and existence of another point (x, p_0) in situation $4_\pm$ of Theorem 2.

7 $A_s(p_0)$ attained at exactly one point, type ${A_0}^1$, and existence of another point (x, p_0) in situation 1 of Theorem 2.

8 $A_s(p_0)$ attained at exactly one point, type ${A_0}^1$, and existence of another point (x, p_0) in situation 2 of Theorem 2.

9 $A_s(p_0)$ attained at exactly one point, type ${A_0}^1$, and existence of another point (x, p_0) in situation 3 of Theorem 2.

10$_\pm$ $A_s(p_0)$ attained at exactly one point, type ${A_0}^1$, and existence of another point (x, p_0) in situation $4_\pm$ of Theorem 2.

11 $A_s(p_0)$ attained at exactly two points, both type I^2 and no points (x, p_0) leading to a nonsmoothness situation of Theorem 2.

12 $A_s(p_0)$ attained at exactly two points, types I^2 and ${A_0}^1$ and no points (x, p_0) leading to a nonsmoothness situation of Theorem 2.

13 $A_s(p_0)$ attained at exactly two points, both type ${A_0}^1$ and no points (x, p_0) leading to a nonsmoothness situation of Theorem 2.

14 $A_s(p_0)$ attained at exactly one point, type ${A_0}^2$ and no points (x, p_0) leading to a nonsmoothness situation of Theorem 2.

The classification of all generic codimension two singularities of the optimal averaged profit at transition values can now be obtained analyzing (case by case) only the situations above. The results are presented in the following theorem, whose proof we leave for the next section.

Theorem 4 *Consider a k-parameter family of pairs of polydynamical systems and profit densities on the circle. Generically, the germ of the optimal averaged profit at a transition value is equivalent (up to the equivalence described) to the germ at the origin of one of the functions in:*

- *Table 2, if $k = 1$,*
- *Tables 2 and 3, if k=2.*

Table 2:

N.	Sing.	Equiv.
1	$\max\{0; p_1\}$	R^+
2	$\max\{0; -\frac{p_1}{\ln p_1}(1 + H_1)\}$	R^+

Remark 4 Singularities of Table 2 are already known [4]. Besides,

- In singularities 2, 7-10$_\pm$ and 12, $H_1 = h_1\left(p, \frac{1}{\ln p_1}, \frac{\ln|\ln p_1|}{\ln p_1}\right)$, where h_1 is a smooth function with $h_1(p, 0, 0) \equiv 0$

Table 3:

N.	Sing.	Eq.
3	$\max\{0; p_1; {p_2}^{3/2}\}$	Γ
4	$\max\{0; p_1; {p_2}^2, p_2 \geq 0\}$	R^+
5	$\max\{0; p_1; {p_2}^3, p_2 \geq 0\}$	R^+
$6_\pm$	$\max\{0; p_1; {p_2}^{3/2} \pm {p_2}^2\}$	Γ
7	$\max\left\{0;\ c(p) = \begin{cases} -\frac{p_1}{\ln p_1}(1+H_1), & p_2 \leq 0 \\ -\frac{p_1+{p_2}^{3/2}}{\ln(p_1+{p_2}^{3/2})}(1+H_2), & p_2 \geq 0 \end{cases}\right\}$	R^+
8	$\max\left\{0;\ c(p) = \begin{cases} -\frac{p_1}{\ln p_1}(1+H_1), & p_2 \leq 0 \\ -\frac{p_1+{p_2}^2}{\ln(p_1+{p_2}^2)}(1+H_2), & p_2 \geq 0 \end{cases}\right\}$	R^+
9	$\max\left\{0;\ c(p) = \begin{cases} -\frac{p_1}{\ln p_1}(1+H_1), & p_2 + c \leq 0 \\ -\frac{p_1+{p_2}^3}{\ln(p_1+{p_2}^3)}(1+H_2), & p_2 + c \geq 0 \end{cases}\right\}$	R^+
$10_\pm$	$\max\left\{0;\ c(p) = \begin{cases} -\frac{p_1}{\ln p_1}(1+H_1), & p_2 \pm c \leq 0 \\ -\frac{p_1+{p_2}^{3/2}}{\ln(p_1+{p_2}^{3/2})}(1+H_2), & p_2 \pm c \geq 0 \end{cases}\right\}$	R^+
11	$\max\{0; p_1; p_2\}$	R^+
12	$\max\{0; p_2; -\frac{p_1}{\ln p_1}(1+H_1)\}$	R^+
13	$\max\left\{\lvert p_1\rvert; \lvert p_1\rvert - \frac{G(p,q)}{\ln G(p,q)}\left(1/[\gamma_1+\gamma_2-q]\ + H_q\right)\right\}$ with $G(p,q) = \begin{cases} p_2 - \lvert p_1\rvert + q\lvert 2p_1\rvert \ln\lvert 2p_1\rvert, & p_1 \neq 0 \\ p_2, & p_1 = 0 \end{cases}$, $q = \begin{cases} \gamma_2(p), & p_1 < 0 \\ 0, & p_1 = 0 \\ \gamma_1(p), & p_1 > 0 \end{cases}$	R^+
14	$\begin{cases} 0, & p_1 \leq 0 \\ p_1, & p_1 \geq 0,\ p_2 \geq 0 \\ p_1 - {p_2}^2, & p_1 \geq {p_2}^2 B,\ p_2 \leq 0 \\ c(p), & p_1 < {p_2}^2 B,\ p_1 \geq 0,\ p_2 \leq 0 \end{cases}$ where c is the unique solution vanishing at the origin of equation $c\varphi(p,c) - 2p_2(p_1-c)^{1/2} + (p_1-c)^{3/2}\psi(p,c) = (p_1 - {p_2}^2 - c)\ln\left(\frac{(p_1-c)^{1/2}-p_2}{-(p_1-c)^{1/2}-p_2}\right)$	R^+

- In singularity 7, $H_2 = h_2\left(p, {p_2}^{3/2}, \frac{1}{\ln(p_1+{p_2}^{3/2})}, \frac{\ln\lvert\ln(p_1+{p_2}^{3/2})\rvert}{\ln(p_1+{p_2}^{3/2})}\right)$, where h_2 is a smooth function with $h_2(p, {p_2}^{3/2}, 0, 0) \equiv 0$

- In singularity 8, $H_2 = h_2\left(p, \frac{1}{\ln(p_1+{p_2}^2)}, \frac{\ln\lvert\ln(p_1+{p_2}^2)\rvert}{\ln(p_1+{p_2}^2)}\right)$, where h_2 is a smooth function with $h_2(p,0,0) \equiv 0$

- In singularity 9, $H_2 = h_2\left(p, \frac{1}{\ln(p_1+{p_2}^3)}, \frac{\ln\lvert\ln(p_1+{p_2}^3)\rvert}{\ln(p_1+{p_2}^3)}\right)$, where h_2 is a smooth function with $h_2(p,0,0) \equiv 0$

- In singularity 10, $H_2 = h_2\left(p, \frac{1}{\ln(p_1+{p_2}^{3/2})}, \frac{\ln|\ln(p_1+{p_2}^{3/2})|}{\ln(p_1+{p_2}^{3/2})}\right)$, where h_2 is a continuous function with $h_2(p,0,0) \equiv 0$

- In singularity 13, $H_q = h_q\left(p, \frac{1}{\ln G(p,q)}, \frac{\ln|\ln G(p,q)|}{\ln G(p,q)}\right)$, where h_q is a smooth function with $h_q(p,0,0) \equiv 0$, and γ_i are smooth functions of the parameter

- In singularity 14, B is a smooth function of the parameter with $B(0) > 1$.

5 Proofs

5.1 Proof of Theorem 3

Let p_0 be a parameter value. We prove that when $A_s(p_0)$ is attained at a point of type ${A_1}^1$, ${A_{0,0}}^1$ or ${A_1}^2$, then in a generic case the inequality $A_l(p_0) > A_s(p_0)$ holds, contradicting the fact that p_0 is a transition value (Lemma 2). Lemma 3 implies that in all these cases the profit $A_s(p_0)$ is attained at a unique point (x_0, p_0) of the stationary domain.

Situation 1: (x_0, p_0) is a point of type ${A_1}^2$

In this situation there are three conditions, namely, $f_x = 0$ and $v = v_x = 0$ at (x_0, p_0), for some admissible velocity v, and so, in a generic case, no other independent condition can be satisfied. In particular, due to Lemma 2, A_s can not be continuous and defined in a neighborhood of (x_0, p_0). So A_s must have one of the following singularities ([10]):

$$\max\{-x^2 + p_1 x : \, x^2 + p_2 \leq 0\} \quad \text{or} \quad \sqrt{p_1}|p_2|.$$

Moreover, due to Multijet Transversality Theorem, generically there can not appear a point distinct of (x_0, p_0) satisfying one of the situations listed on Theorem 2.

We can consider ([10]) a fibered local coordinate system with origin at (x_0, p_0) where

$$v_{\min}(x,p) = (x^2 - p_1) \cdot V(x,p),$$

for some smooth function V positive at the origin, and the family of profit densities f is, up to $\mathcal{F}^+$-equivalence, one of the functions $\pm x^2 + p_2 x$. In this coordinate system we consider a sufficiently small neighborhood $[-a, a] \times [-\varepsilon, \varepsilon]^2$ of the origin where these normal forms take place.

(1) The normal form for f is $x^2 + p_2 x$
For $p = 0$ we have $f(x) = x^2$ and $v_{\min}(x) = x^2 h(x)$, for some smooth function h positive at the origin. All positive values of the profit density are cyclic values and the corresponding averaged profit is given by is

$$A(c) = \frac{P(c) + \int_{-a}^{-\sqrt{c}} \frac{1}{h(x)} dx + \int_{-\sqrt{c}}^{\sqrt{c}} \frac{x^2}{v_{\max}(x)} dx + \int_{\sqrt{c}}^{a} \frac{1}{h(x)} dx}{T(c) + \int_{-a}^{-\sqrt{c}} \frac{1}{x^2 h(x)} dx + \int_{-\sqrt{c}}^{\sqrt{c}} \frac{1}{v_{\max}(x)} dx + \int_{\sqrt{c}}^{a} \frac{1}{x^2 h(x)} dx}$$

where $P(c)$ and $T(c)$ are the profit and the period, respectively, of the c-level cycle outside the neighborhood $[-a, a]$ of x_0. Is it easy to see that $\lim_{c\to 0^+} A(c) = 0$ and so $A_l(0) \geq 0$. The vanishing of $A_l - A_s$ at the origin gives an excessive independent condition on the transition and so, it does not take place in a generic case. Therefore, because $A_s(0) = 0$, we have $A_l(0) > 0$ and 0 is not a transition value.

(2) The normal form for f is $-x^2 + p_2 x$
For $p_2 = 0$ we have $f(x, p_1) = -x^2$ and $v_{\min}(x, p_1) = (x^2 - p_1)h(x, p_1)$, for some smooth function h positive at the origin. When $p_1 < 0$ the optimal averaged profit has to be provided by cyclic strategies and so in a neighborhood of $c = 0$ all values are cyclic. We fix a negative value of c and we study the averaged profit $A(p_1, c)$ when $p_1 \to 0^-$. The averaged profit $A(p_1, c)$ is given by

$$\frac{P(p_1, c) + \int_{-a}^{-\sqrt{-c}} \frac{-x^2}{v_{\max}(x,p_1)} dx + \int_{-\sqrt{-c}}^{\sqrt{-c}} \frac{-x^2}{(x^2-p_1)h(x,p_1)} dx + \int_{\sqrt{-c}}^{a} \frac{-x^2}{v_{\max}(x,p_1)} dx}{T(p_1, c) + \int_{-a}^{-\sqrt{-c}} \frac{1}{v_{\max}(x,p_1)} dx + \int_{-\sqrt{-c}}^{\sqrt{-c}} \frac{1}{(x^2-p_1)h(x,p_1)} dx + \int_{\sqrt{-c}}^{a} \frac{1}{v_{\max}(x,p_1)} dx}$$

where $P(p_1, c)$ and $T(p_1, c)$ are defined as previously. It is easy to see that the limit of $A(p_1, c)$ when $p_1 \to 0^-$ and $c \to 0^-$ is equal to 0 implying that $A_l(0) \geq 0$ and so, as in the previous case, we conclude that 0 is not a transition value.

Situation 2: (x_0, p_0) is a point of type $A_{0,0}{}^1$
In this situation $v_1 = v_2 = 0$ at (x_0, p_0), for some admissible velocities v_1 and v_2, and because A_s is well defined and continuous around p_0 ([10]),

we must have the extra condition $A_s(p_0) = A_l(p_0)$. So, due to Multijet Transversality Theorem, generically there can not appear a point distinct of (x_0, p_0) satisfying one of the situations listed on Theorem 2.

We can consider ([10]) a fibered local coordinate system with origin at (x_0, p_0) where

$$v_{\min}(x,p) = x(x - p_1) \cdot V(x,p),$$

for some smooth function V positive at the origin, and the family of profit densities f is, up to $\mathcal{F}^+$-equivalence, the function x. In this coordinate system we consider a sufficiently small neighborhood $[-a,a] \times [-\varepsilon, \varepsilon]^2$ of the origin where these normal forms take place.

For $p = 0$ we have $f(x) = x$ and $v_{\min}(x) = x^2 h(x)$, for some smooth function h positive at the origin. All positive values of the profit density are cyclic values. Then,

$$\lim_{c \to 0^+} A(c) = \lim_{c \to 0^+} \frac{P(c) + \int\limits_c^a \frac{1}{xh(x)} dx}{T(c) + \int\limits_c^a \frac{1}{x^2 h(x)} dx}$$

where $P(c)$ and $T(c)$ are the profit and the period, respectively, of the c-level cycle outside $[c,a]$. It is easy (L'Hôpital's Rule) to conclude that such limit is equal to 0^+. So, $A_l(0) > 0$ and 0 is not a transition value.

Situation 3: (x_0, p_0) is a point of type ${A_1}^1$

In this situation there are two conditions, namely, $v = v_x = 0$ at (x_0, p_0), for some admissible velocity v. Using Multijet Transversality Theorem in a generic case there can appear exactly one point (x_1, p_0) distinct from (x_0, p_0) leading to one of the nonsmoothness situations 1-4 listed on Theorem 2. Hence, this situation is divided in five cases:

Case 1: There are no points (x, p_0) leading to a nonsmoothness situation listed on Theorem 2.

Case 2: There is a point (x_1, p_0) distinct from (x_0, p_0) leading to situation 1 of Theorem 2. In this case there are two additional conditions: $v_1 - v_2 = (v_1 - v_2)_x = 0$ at (x_1, p_0), for some admissible velocities v_1 and v_2.

Case 3: There is a point (x_1, p_0) distinct from (x_0, p_0) leading to situation 2 of Theorem 2. In this case there are two additional conditions: $v_1 - v_2 = v_1 - v_3 = 0$ at (x_1, p_0), for some admissible velocities v_1, v_2 and v_3.

Case 4: There is a point (x_1, p_0) distinct from (x_0, p_0) leading to situation 3 of Theorem 2. In this case there are two additional conditions: $f = 0$ and $v_1 - v_2 = 0$ at (x_1, p_0), for some admissible velocities v_1 and v_2.

Case 5: There is a point (x_1, p_0) distinct from (x_0, p_0) leading to situation $4_\pm$ of Theorem 2. In this case there are two additional conditions: $f = f_x = 0$ at (x_1, p_0).

In cases 2-5, generically (x_1, p_0) can not satisfy any other independent condition, due to Multijet Transversality Theorem. All cases are proved by the same process.

We can consider ([10]) a fibered local coordinate system with origin at (x_0, p_0) where

$$v_{\min}(x, p) = (x^2 - p_1) \cdot V(x, p),$$

for some smooth function V positive at the origin, and the family of profit densities f is, up to $\mathcal{F}^+$-equivalence, the function x. In this coordinate system we consider a sufficiently small neighborhood $[-a, a] \times [-\varepsilon, \varepsilon]^2$ of the origin where these normal forms take place. For $p = 0$ we have $f(x) = x$ and $v_{\min}(x) = x^2 h(x)$, for some smooth function h positive at the origin. All positive values of the profit density are cyclic values.

In case 1,

$$\lim_{c \to 0^+} A(c) = \lim_{c \to 0^+} \frac{P(c) + \int\limits_{-a}^{c} \frac{x}{v_{\max}(x)} dx + \int\limits_{c}^{a} \frac{1}{x h(x)} dx}{T(c) + \int\limits_{-a}^{c} \frac{1}{v_{\max}(x)} dx + \int\limits_{c}^{a} \frac{1}{x^2 h(x)} dx}$$

where $P(c)$ and $T(c)$ are the profit and the period, respectively, of the c-level cycle outside the neighborhood $[-a, a]$ of x_0. As in the previous situation, this limit is equal to 0^+. So, $A_l(0) > 0$ and 0 is not a transition value.

In all the other cases we can apply transversality theorems and obtain in a generic case the normal forms presented in Figure 2 ([6]) in a fibered local coordinate system (y, p) with origin at $(x_1, 0)$.

For these cases we consider, in these new coordinate systems, that these normal forms take place in the neighborhood $[-a, a] \times [-\varepsilon, \varepsilon]^2$ of the origin, obtaining that

$$A(c) = \frac{P(c) + \int\limits_{-a}^{c} \frac{x}{v_{\max}(x)} dx + \int\limits_{c}^{a} \frac{1}{x h(x)} dx + P_1(c)}{T(c) + \int\limits_{-a}^{c} \frac{1}{v_{\max}(x)} dx + \int\limits_{c}^{a} \frac{1}{x^2 h(x)} dx + T_1(p)}$$

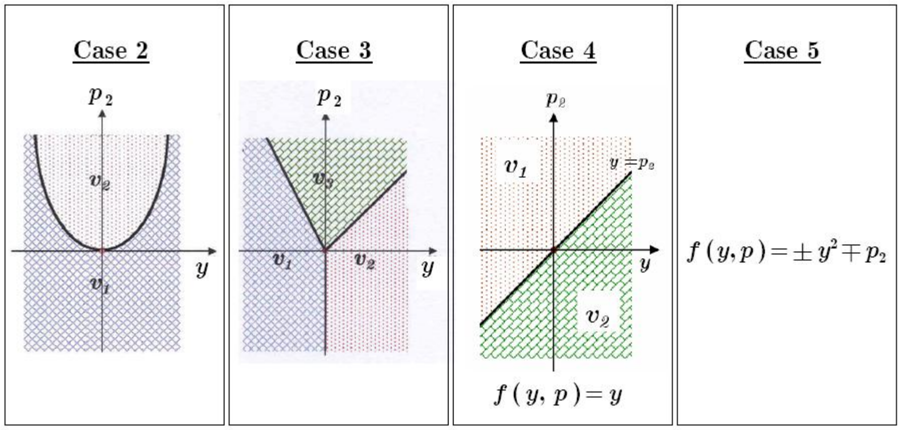


Figure 2: Normal forms around (x_1, p_0)

where $P(c)$ and $T(c)$ are the profit and the period, respectively, of the c-level cycle outside the neighborhoods $[-a, a]$ of x_0 and x_1, and P_1 and T_1 depend on the case:

- In case 2, for $p = 0$ we just use v_1 in a neighborhood of x_1 and so

$$P_1(c) = \int_{-a}^{a} \frac{f}{v_1}(y)dy, \qquad T_1(c) = \int_{-a}^{a} \frac{1}{v_1}(y)dy.$$

- In case 3, for $p = 0$ we just have to switch from velocity v_1 to velocity v_2 at $y = 0$ and so

$$P_1(c) = \int_{-a}^{0} \frac{f}{v_1}(y)dy + \int_{0}^{a} \frac{f}{v_2}(y)dy,$$

$$T_1(c) = \int_{-a}^{0} \frac{1}{v_1}(y)dy + \int_{0}^{a} \frac{1}{v_2}(y)dy.$$

- In case 4, we assume that it is the maximum velocity that is not smooth (the other case is similar). For $p = 0$, we just have to switch from velocity v_1 to velocity v_2 at $y = 0$ and from velocity v_2 to $v_{\min}$ at $y = c$ and so

$$P_1(c) = \int_{-a}^{0} \frac{y}{v_1(y)}dy + \int_{0}^{c} \frac{y}{v_2(y)}dy + \int_{c}^{a} \frac{y}{v_{\min}(y)}dy,$$

$$T_1(c) = \int_{-a}^{0} \frac{1}{v_1(y)} dy + \int_{0}^{c} \frac{1}{v_2(y)} dy + \int_{c}^{a} \frac{1}{v_{\min}(y)} dy.$$

- In case 5,

$$P_1(c) = \int_{-a}^{a} \frac{y^2}{v_{\min}(y)} dy + \int_{-\sqrt{c}}^{\sqrt{c}} \left(\frac{y^2}{v_{\max}(y)} - \frac{y^2}{v_{\min}(y)} \right) dy$$

$$T_1(c) = \int_{-a}^{a} \frac{1}{v_{\min}(y)} dy + \int_{-\sqrt{c}}^{\sqrt{c}} \left(\frac{1}{v_{\max}(y)} - \frac{1}{v_{\min}(y)} \right) dy$$

and

$$P_1(c) = \int_{-a}^{a} \frac{-y^2}{v_{\max}(y)} dy, \qquad T_1(c) = \int_{-a}^{a} \frac{1}{v_{\max}(y)} dy$$

for the minimum (5_+) and maximum (5_-) cases, respectively.

In cases 2, 3, 4 and 5_- we have that P_1 and T_1 are smooth and so the conclusion is exactly the same as for the first case. In case 5_+ we have that both integrals depending on c vanish when $c \to 0^+$. In fact, these integrals are of the form $\int_0^{\sqrt{c}} H(y^2)dy$, with H smooth, and so, because $M(z) = \int_0^z H(y^2)dy$ is an odd function we conclude that $\int_0^{\sqrt{c}} H(y^2)dy = \sqrt{c}N(c)$, for some smooth function N. So in this case, the conclusion is also the same as for the first case. ■

5.2 Proof of Theorem 4

Consider a transition value p_0.

Situation 1: Suppose that the profit $A_s(p_0)$ is attained at a unique point (x_0, p_0) of the stationary domain of type I^2. We can consider ([10]) a fibered local coordinate system with origin at (x_0, p_0) where the family of profit densities is, up to $\mathcal{F}^+$-equivalence, the function $-x^2$. By R^+-equivalence $A_s \equiv 0$ around the origin. Note that 0 is not the global maximum of the profit density $f(\,\cdot\,, p_0)$ and so, all positive values of the density are cyclic. Then, the averaged profit A along level cycles is defined for all (p, c) with p sufficiently close to p_0 and $c > 0$.

In this situation there must be satisfied two conditions, namely, $f_x(x_0, p_0) = 0$ and $A_s(p_0) = A_l(p_0)$. Therefore, using Multijet Transversality Theorem, in a generic case there can appear exactly one point (x_1, p_0) distinct from (x_0, p_0) leading to one of the nonsmoothness situations 1-4 listed on Theorem 2. So, this situation is divided in exactly the same five cases listed in situation 3 of the previous section on page 18. Note that in cases 2-5, generically (x_1, p_0) can not satisfy any other independent condition, due to Multijet Transversality Theorem. All cases are proved by the same process.

In case 1,

$$A(p,c) = \frac{P}{T}(p,c)$$

where P and T, both smooth, are the profit and the period, respectively, of the c-level cycle.

In all the other cases we can obtain the formulas of the averaged profit A along level cycles considering a fibered local coordinate system (y, p) with origin at $(x_1, 0)$ ([6]). For example, in case 2

$$A(p,c) = \begin{cases} \dfrac{P(p,c)}{T(p,c)}, & p_2 \leq 0 \\ \\ \dfrac{P(p,c)+P_1(p)}{T(p,c)+T_1(p)}, & p_2 \geq 0 \end{cases}$$

where P and T, both smooth, are the profit and period, respectively, when around x_1 we just use velocity v_1 and

$$P_1(p) = \int\limits_{-\sqrt{p_2}}^{\sqrt{p_2}} \left[f \cdot \left(\frac{1}{v_2} - \frac{1}{v_1} \right) \right] (y,p)dy, \; T_1(p) = \int\limits_{-\sqrt{p_2}}^{\sqrt{p_2}} \left(\frac{1}{v_2} - \frac{1}{v_1} \right) (y,p)dy.$$

Now, for all cases 1-5 we consider an extension of the A function to all (p, c) around the origin considering the previous expressions also for $c \leq 0$. Equation $c = (P/T)(p, c)$ has now a unique solution $c = c_1(p)$, where c_1 is a smooth function defined around the origin with $c_1(0) = 0$, due to the Implicit Function Theorem. Note that this solution is the optimal averaged profit for level cycles only when it is positive.

Subtracting this function from the family of profit densities we obtain $A_s = -c_1$ and so the optimal averaged profit is

$$\max\{-c_1(p), A_l(p)\},$$

where A_l is the optimal averaged profit for level cycles. Using results of [6], it is possible to obtain, for the fixed coordinate systems, the following normal forms for A_l:

- In case 2, $A_l(p) = \max\{0; {p_2}^{3/2} A_1(p) + {p_2}^3 A_2(p), p_2 \geq 0\}$

- In case 3, $A_l(p) = \max\{0; {p_2}^2 A_3(p), p_2 \geq 0\}$

- In case 4, $A_l(p) = \max\{0; {p_2}^3 A_4(p), p_2 \geq 0\}$

- In case 5, $A_l(p) = \max\{0; {p_2}^{3/2} A_5(p) \pm {p_2}^2 A_6(p), p_2 \geq 0\}$

where all A_i are smooth functions positive at the origin.

Transversality theorems guarantee that generically $\frac{\partial c_1}{\partial p_1}(0) \neq 0$ (in case 1 we must eventually permute firstly p_1 and p_2) and so we choose a new coordinate $\tilde{p}_1 = -c_1(p)$. Now it is easy to choose new coordinates on the parameter space for each case to get the following normal forms:

- In case 1, $\max\{0; p_1\}$

- In case 2, $\max\{0; p_1; {p_2}^{3/2}\}$

- In case 3, $\max\{0; p_1; {p_2}^2, p_2 \geq 0\}$

- In case 4, $\max\{0; p_1; {p_2}^3, p_2 \geq 0\}$

- In case 5, $\max\{0; p_1; {p_2}^{3/2} \pm {p_2}^2\}$.

Situation 2: Suppose that the profit $A_s(p_0)$ is attained at a unique point (x_0, p_0) of the stationary domain of type ${A_0}^1$. We can consider ([10]) a fibered local coordinate system with origin at (x_0, p_0) where

$$v_{\min}(x, p) = x \cdot V(x, p),$$

for some smooth function V positive at the origin, and the family of profit densities f is, up to $\mathcal{F}^+$-equivalence, the function x. By R^+-equivalence $A_s \equiv 0$ around the origin. In this coordinate system we consider a sufficiently small neighborhood $[-a, a] \times [-\varepsilon, \varepsilon]^2$ of the origin where these normal forms take place. In this situation two conditions must be satisfied, namely, $v_{\min}(x_0, p_0) = 0$ and $A_s(p_0) = A_l(p_0)$. Therefore, using Multijet Transversality Theorem, in a generic case there can appear exactly one point (x_1, p_0) distinct from (x_0, p_0) leading to one of the nonsmoothness situations 1-4 listed on Theorem 2. Hence, this situation is divided in exactly the same five cases listed in situation 3 of the previous section on page 18. In cases 2-5, generically (x_1, p_0) can not satisfy any other independent condition, due to Multijet Transversality Theorem. All cases are proved by the same process.

In case 1, for $c \in (0,a)$,

$$P(p,c) = \tilde{P}(p,c) + \int_c^a h(x,p)dx$$

$$T(p,c) = T^*(p,c) + \int_c^a \frac{1}{x} h(x,p)dx = \tilde{T}(p,c) - \gamma(p)\ln c,$$

where $h = 1/V$ and γ is a smooth function positive at the origin, and P and $\tilde{T}$ are smooth near $(p,c) = (0,0)$. Therefore, equation $c = A(p,c)$ can be written as

$$c = \frac{P(p,c)}{\tilde{T}(p,c) - \gamma(p)\ln c}.$$

Function γ is easily removed considering new smooth functions P and $\tilde{T}$. Transversality theorems justify that generically $P_{p_1}(0) \neq 0$ or $P_{p_2}(0) \neq 0$. Without loss of generality, we assume the first case. Writing $P(p,c) = P(p,0) + c\tilde{P}(p,c)$, for some smooth function $\tilde{P}$ and choosing $\tilde{p}_1 = P(p,0)$ as a new coordinate we get the equation:

$$c[H(p,c) - \ln c] = p_1, \tag{3}$$

where $H = \tilde{T} - \tilde{P}$ is a smooth function. Now, because

$$\lim_{c \to 0^+} c[H(p,c) - \ln c] = 0^+$$

we conclude that near the origin equation (3) has no solution for $p_1 \leq 0$. For $p_1 > 0$ it has a solution of the form $c(p) = -\frac{p_1}{\ln p_1} A(p)$, for some function A positive at the origin [4]. Replacing it in (3) we obtain

$$A(p)\left[\frac{1}{\ln p_1} H\left(p, -\frac{p_1}{\ln p_1} A(p)\right) - \left(1 + \frac{\ln A(p)}{\ln p_1} - \frac{\ln|\ln p_1|}{\ln p_1}\right)\right] + 1 = 0. \tag{4}$$

Choosing new coordinates $z = A(p)$, $r = 1/\ln p_1$ and $s = \ln|\ln p_1|/\ln p_1$, Equation (4) can be written as $F(p,r,s,z) = 0$, where

$$F(p,r,s,z) = z[rH(p, -p_1 r z) - (1 + r\ln z - s)] + 1$$

is a smooth function around $(0,0,0,A(0))$. Due to the Implicit Function Theorem we conclude that, around the considered point, Equation (4) is equivalent to $z = Z(p,r,s)$, for some smooth function Z with $Z(p,0,0) \equiv 1$. Then,

$$c(p) = -\frac{p_1}{\ln p_1} Z\left(p, \frac{1}{\ln p_1}, \frac{\ln|\ln p_1|}{\ln p_1}\right) = -\frac{p_1}{\ln p_1}\left[1 + H\left(p, \frac{1}{\ln p_1}, \frac{\ln|\ln p_1|}{\ln p_1}\right)\right],$$

where H is a smooth function with $H(p,0,0) \equiv 0$ and, because the optimal averaged profit is $\max\{A_s(p); c(p), c > 0\}$, we obtain singularity 2.

In all the other cases (2-5) we can obtain the formulas of the averaged profit A along level cycles considering a fibered local coordinate system (y,p) with origin at $(x_1, 0)$ ([6]) and conclude that equation $c = A(p,c)$ takes the form

$$\begin{array}{lll}
\text{Case 2:} & c[T(p,c) - \gamma(p)\ln c] = P(p,c), & p_2 \leq 0 \\
 & c[T(p,c) + {p_2}^{3/2}\tilde{A}(p) - \gamma(p)\ln c] = P(p,c) + {p_2}^{3/2}A(p), & p_2 \geq 0 \\
\text{Case 3:} & c[T(p,c) - \gamma(p)\ln c] = P(p,c), & p_2 \leq 0 \\
 & c[T(p,c) + {p_2}^{2}\tilde{A}(p) - \gamma(p)\ln c] = P(p,c) + {p_2}^{2}A(p), & p_2 \geq 0 \\
\text{Case 4:} & c[T(p,c) - \gamma(p)\ln c] = P(p,c), & c - p_2 \leq 0 \\
 & c[T(p,c) - \gamma(p)\ln c] = P(p,c) + (c-p_2)^3 B(p,c), & c - p_2 \geq 0 \\
\text{Case 5:} & c[T(p,c) - \gamma(p)\ln c] = P(p,c), & p_2 \pm c \leq 0 \\
 & c[T(p,c) - \gamma(p)\ln c] = P(p,c) + (p_2 \pm c)^{3/2} B(p,c), & p_2 \pm c \geq 0
\end{array}$$

where all functions are smooth and, at the origin, A, B and γ are positive and P vanishes. After this we remove γ considering new smooth functions P, T, A, $\tilde{A}$ and B and write these equations as

$$\begin{array}{lll}
\text{Case 2:} & c[H_1(p,c) - \ln c] = P(p,0), & p_2 \leq 0 \\
 & c[H_2(p, {p_2}^{3/2}, c) - \ln c] = P(p,0) + {p_2}^{3/2}A(p), & p_2 \geq 0 \\
\text{Case 3:} & c[H_1(p,c) - \ln c] = P(p,0), & p_2 \leq 0 \\
 & c[H_2(p,c) - \ln c] = P(p,0) + {p_2}^{2}A(p), & p_2 \geq 0 \\
\text{Case 4:} & c[H_1(p,c) - \ln c] = P(p,0), & c - p_2 \leq 0 \\
 & c[H_2(p,c) - \ln c] = P(p,0) - {p_2}^{3}A(p), & c - p_2 \geq 0 \\
\text{Case 5:} & c[H_1(p,c) - \ln c] = P(p,0), & p_2 \pm c \leq 0 \\
 & c[B(p,c) - \ln c] = P(p,0) + {p_2}^{3/2}A(p), & p_2 \pm c \geq 0
\end{array}$$

where B is a continuous function and all the other functions are smooth with A positive at the origin. Finally, transversality theorems guarantee that generically $P_{p_1}(0) \neq 0$ and so, for each case it is easy to choose new coordinates on the parameter space in such a way that equation $c = A(p,c)$ takes the form

$$\begin{array}{lll}
\text{Case 2:} & c[H_1(p,c) - \ln c] = p_1, & p_2 \leq 0 \\
 & c[H_2(p, {p_2}^{3/2}, c) - \ln c] = p_1 + {p_2}^{3/2}, & p_2 \geq 0 \\
\text{Case 3:} & c[H_1(p,c) - \ln c] = p_1, & p_2 \leq 0 \\
 & c[H_2(p,c) - \ln c] = p_1 + {p_2}^{2}, & p_2 \geq 0 \\
\text{Case 4:} & c[H_1(p,c) - \ln c] = p_1, & p_2 + c \leq 0 \\
 & c[H_2(p,c) - \ln c] = p_1 + {p_2}^{3}, & p_2 + c \geq 0 \\
\text{Case 5:} & c[H_1(p,c) - \ln c] = p_1, & p_2 \pm c \leq 0 \\
 & c[B(p,c) - \ln c] = p_1 + {p_2}^{3/2}, & p_2 \pm c \geq 0
\end{array}$$

where B is a continuous function and all the other functions are smooth. Therefore, proceeding as in case 1 we obtain singularities 7-10.

Situation 3: Suppose that the profit $A_s(p_0)$ is attained at exactly two points (x_0, p_0) and (x_1, p_0) of the stationary domain, both of type I^2. In this situation four conditions must be satisfied, namely, $f_x(x_0, p_0) = 0$, $f_x(x_1, p_0) = 0$, $f(x_0, p_0) - f(x_1, p_0) = 0$ and $A_s(p_0) = A_l(p_0)$. Due to transversality theorems, in a generic case no other independent condition can be satisfied.

Due to the Implicit Function Theorem, there are smooth functions γ_0 and γ_1 defined around p_0 such that

$$f_x(\gamma_i(p), p) = 0 \text{ and } \gamma_i(p_0) = x_i \qquad i = 1, 2.$$

Therefore, for every p around p_0 the profit A_s is given by

$$\max\{f(\gamma_0(p), p), f(\gamma_1(p), p)\}.$$

In a generic case, because no other independent condition can be satisfied, none of the situations listed on Theorem 2 occurs. So, the averaged profit A along level cycles is a smooth function defined for all (p, c) with p sufficiently close to p_0 and $c > A_s(p_0)$ as

$$A(p, c) = \frac{P(p, c)}{T(p, c)},$$

where P and T are the profit and the period, respectively, of the c-level cycle. Note that around x_0 and x_1 it is the maximum velocity that is used. As in situation 1, we consider an extension of the A function to all (p, c) around the origin considering the previous expression also for $c \leq 0$. Now, equation $c = A(p, c)$ has a unique solution $c = C(p)$, where C is a smooth function defined around p_0 with $C(p_0) = A_s(p_0)$, due to the Implicit Function Theorem. Note that this solution is the optimal averaged profit for level cycles only when it is greater than $A_s(p_0)$.

Therefore, for every p around p_0, the optimal averaged profit is given by

$$\max\{f(\gamma_0(p), p), f(\gamma_1(p), p), C(p)\}$$

which is R^+-equivalent to

$$\max\{0, f(\gamma_1(p), p) - f(\gamma_0(p), p), C(p) - f(\gamma_0(p), p)\}.$$

Using Mutijet Transversality Theorem we can choose $\tilde{p}_1 = f(\gamma_1(p), p) - f(\gamma_0(p), p)$ and $\tilde{p}_2 = C(p) - f(\gamma_0(p), p)$ as new coordinates, obtaining so singularity 11.

Situation 4: Suppose that the profit $A_s(p_0)$ is attained at exactly two points (x_0, p_0) and (x_1, p_0) of the stationary domain, one of type I^2 and another of type ${A_0}^1$, respectively. In this situation four conditions must be satisfied, namely, $f(x_0, p_0) - f(x_1, p_0) = 0$, $f_x(x_0, p_0) = 0$, $v(x_1, p_0) = 0$ and $A_s(p_0) = A_l(p_0)$, for some admissible velocity v. Due to transversality theorems, in a generic case no other independent condition can be satisfied.

We can consider ([10]) fibered local coordinate systems (x, p) and (y, p) with origin at (x_0, p_0) and (x_1, p_0), respectively, where

$$v_{\min}(y, p) = y \cdot V(y, p),$$

for some smooth function V positive at the origin, and the family of profit densities is written as $-x^2 + \alpha(p)$ around (x_0, p_0) and as $y + \beta(p)$ around (x_1, p_0), for some smooth functions α and β with $\alpha(0) = \beta(0)$. By R^+-equivalence the function β can be removed and using Multijet Transversality Theorem we can choose a new coordinate p_1 in such a way that $f(x, p) = -x^2 + p_1$ around (x_0, p_0). In the given coordinate systems, consider a neighborhood $[-a, a] \times [-\varepsilon, \varepsilon]^2$ of the respective origins where the previous normal forms take place. Observe that, for every p around the origin, $A_s(p) = \max\{0, p_1\}$.

Due to transversality theorems, in a generic case no other independent condition can be satisfied and so, none of the situations listed on Theorem 2 occurs. So, the averaged profit A along level cycles is defined for all (p, c) with p sufficiently close to 0 and $c > \max\{0, p_1\}$ as

$$A(p,c) = \frac{P(p,c) + \int\limits_{-a}^{c} \frac{y}{v_{\max}(y,p)} dy + \int\limits_{c}^{a} h(y,p) dy + \int\limits_{-a}^{a} \frac{-x^2+p_1}{v_{\max}(x,p)} dx}{T(p,c) + \int\limits_{-a}^{c} \frac{1}{v_{\max}(y,p)} dy + \int\limits_{c}^{a} \frac{1}{y} h(y,p) dy + \int\limits_{-a}^{a} \frac{1}{v_{\max}(x,p)} dx}$$

where $h = 1/V$ and $P(p, c)$ and $T(p, c)$ are the profit and the period, respectively, of the c-level cycle outside the neighborhoods $[-a, a]$ of x_0 and x_1. As in situation 1, we consider an extension of the A function to all (p, c) around the origin considering the previous expression also for $c \leq \max\{0, p_1\}$. Now, equation $c = A(p, c)$ has a unique solution $c = C(p)$, where C is a smooth function defined around the origin with $C(0) = 0$, due to the Implicit Function Theorem. Note that this solution is the optimal averaged profit for level cycles only when it is positive. As it was seen in situation 2, this equation takes the form

$$c[H(p,c) - \ln c] = P(p,0)$$

for some smooth function H. The vanishing of the derivative P_{p_2} at the origin gives an excessive independent condition on the transition and so, it does not take place in a generic case. Therefore, generically, we can consider $\tilde{p}_2 = P(p,0)$. As it was seen in the cited situation, this equation only has solution if $p_2 > 0$ which takes the form

$$C(p) = -\frac{p_2}{\ln p_2}\left[1 + H\left(p, \frac{1}{\ln p_2}, \frac{\ln|\ln p_2|}{\ln p_2}\right)\right],$$

for some smooth function H. So the optimal averaged profit is $\max\{0, p_1, C(p)\}$ and we obtain singularity 12.

Situation 5: Suppose that the profit $A_s(p_0)$ is attained at exactly two points $(x_0,p_0),(x_1,p_0)$ of the stationary domain, both of type ${A_0}^1$. In this situation four conditions must be satisfied, namely, $v_1(x_0,p_0) = 0$, $v_2(x_1,p_0) = 0$, $f(x_0,p_0) - f(x_1,p_0) = 0$ and $A_s(p_0) = A_l(p_0)$, for some admissible velocities v_1 and v_2. Due to transversality theorems, in a generic case no other independent condition can be satisfied.

We can consider ([10]) local coordinate systems (x,p) and (y,p) with origin at (x_0,p_0) and (x_1,p_0), respectively, where

$$v_{\min}(x,p) = x \cdot V_1(x,p) \quad \text{and} \quad v_{\min}(y,p) = y \cdot V_2(y,p)$$

for some smooth functions V_1 and V_2 positive at the corresponding origin, and the family of profit densities f is written as $x + \alpha(p)$ around (x_0,p_0) and as $y + \beta(p)$ around (x_1,p_0), for some smooth functions α and β with $\alpha(0) = \beta(0)$. By R^+-equivalence, we add $-1/2(\alpha+\beta)$ to the family of profit densities and using Multijet Transversality Theorem we can consider $\tilde{p}_1 = 1/2(\beta-\alpha)(p)$ and write $f(x,p) = x - p_1$ and $f(y,p) = y + p_1$. In the given coordinate systems, consider a neighborhood $[-a,a] \times [-\varepsilon,\varepsilon]^2$ of the origin where the previous normal forms take place. Note that for every p around the origin, $A_s(p) = |p_1|$.

Due to transversality theorems, in a generic case no other independent condition can be satisfied and so, none of the situations listed on Theorem 2 occurs. Then, the averaged profit A along level cycles is defined for all (p,c) with p sufficiently close to 0 and $c > |p_1|$ and equation $c = A(p,c)$ is written as

$$c = \frac{P_1(p,c) + \int\limits_{c+p_1}^{a} \frac{x-p_1}{x} H_1(x,p)dx + \int\limits_{c-p_1}^{a} \frac{y+p_1}{y} H_2(y,p)dy}{T_1(p,c) + \int\limits_{c+p_1}^{a} \frac{1}{x} H_1(x,p)dx + \int\limits_{c-p_1}^{a} \frac{1}{y} H_2(y,p)dy}$$

where all functions are smooth and, at the origin, T_1, H_1 and H_2 are positive and P_1 vanishes. This equation can be simplified to the form

$$c{\cdot}T_1(p,c) - P(p,c) = \gamma_1(p)(c+p_1)\ln(c+p_1) + \gamma_2(p)(c-p_1)\ln(c-p_1), \quad c > |p_1|,$$

where all functions are smooth and, at the origin, T_1, γ_1 and γ_2 are positive and P vanishes. Now, $P(p,c) = P(p,0) + c\tilde{P}(p,c)$ and generically the derivative P_{p_2} does not vanish at the origin. So, after a suitable coordinate change, we reduce the last equation to:

$$c{\cdot}T(p,c) - p_2 = \gamma_1(p)(c+p_1)\ln(c+p_1) + \gamma_2(p)(c-p_1)\ln(c-p_1), \quad c > |p_1|, \quad (5)$$

where T is a smooth function. The region where this equation has solution and the respective normal form can be obtained proceeding as in the previous situation 2. We obtain the following solution

$$c(p) = |p_1| - \frac{G}{\ln G}\left(1/(\gamma_1 + \gamma_2 - q) + H_q\left(p, \frac{1}{\ln G}, \frac{\ln|\ln G|}{\ln G}\right)\right)$$

where G and q are defined as

$$G(p,q) = \begin{cases} p_2 - |p_1| + q|2p_1|\ln|2p_1|, & p_1 \neq 0 \\ p_2, & p_1 = 0 \end{cases}, \quad q = \begin{cases} \gamma_2(p), & p_1 < 0 \\ 0, & p_1 = 0 \\ \gamma_1(p), & p_1 > 0 \end{cases}.$$

This solution is valid for $G(p,q) > 0$. Therefore, the optimal profit is

$$A(p) = \max\{|p_1|;\ c(p),\ p_2 > 0 \wedge G(p,\gamma_1(p)) > 0 \wedge G(p,\gamma_2)(p) > 0\}.$$

Situation 6: Suppose that the profit $A_s(p_0)$ is attained at a unique point (x_0, p_0) of the stationary domain of type ${A_0}^2$. In this situation, four conditions must be satisfied, namely, $f = f_x = 0$ and $v = 0$ at (x_0, p_0), for some admissible velocity v, and $A_s(p_0) = A_l(p_0)$. Due to transversality theorems, in a generic case no other independent condition can be satisfied.

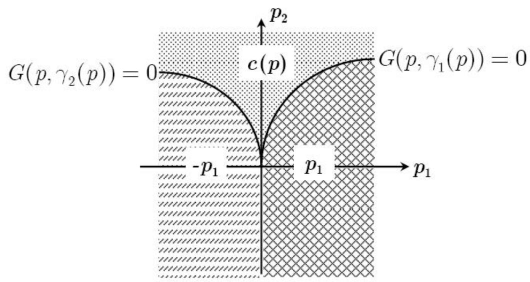


Figure 3: Singularity 13

We can consider ([10]) a fibered local coordinate system with origin at (x_0, p_0) where

$$v_{\min}(x,p) = (x - p_2) \cdot V(x,p),$$

for some smooth function V positive at the origin, and the family of profit densities f is, up to $\mathcal{F}^+$-equivalence, the function $-x^2 + p_1$. Note that for every p around the origin,

$$A_s(p) = \begin{cases} p_1 - {p_2}^2, & p_2 \leq 0 \\ p_1, & p_2 \geq 0 \end{cases} .$$

In order to understand the optimal averaged profit provided by level cycles, we consider equation

$$c = \frac{P_1}{T_1}(p,c),$$

where P_1 and T_1 are the profit and the period, respectively, when around the origin we just use the maximum velocity. Due to the Implicit Function Theorem, this equation has a unique solution $c = c_1(p)$, for some smooth function c_1 vanishing at the origin. Subtracting this function from the family of profit densities we reduce this solution to zero and the profit P_1 takes the form $P_1(p,c) = c^2 h(p,c)$, for some smooth function h. The effect of this subtraction on A_s disappears after considering a suitable change of coordinates on the parameter space, because due to Multijet Transversality Theorem $\frac{\partial(p_1 - c_1(p))}{\partial p_1} \neq 0$. Observe that the switching to the minimum velocity just can give a better cyclic profit when $p_1 \geq 0$ and $p_2 \leq 0$. Therefore, the optimal averaged profit is

$$A(p) = \begin{cases} 0, & p_1 \leq 0 \\ p_1, & p_1 \geq 0, p_2 \geq 0 \\ \max\{c(p), p_1 - {p_2}^2\}, & p_1 \geq 0, p_2 \leq 0 \end{cases}$$

where c is the unique solution vanishing at the origin of equation

$$c = \frac{c^2 h(p,c) + \int\limits_{-\sqrt{p_1-c}}^{\sqrt{p_1-c}} (-x^2+p_1)\left(\frac{1}{v_{\min}} - \frac{1}{v_{\max}}\right)(x,p)dx}{T(p,c) + \int\limits_{-\sqrt{p_1-c}}^{\sqrt{p_1-c}} \left(\frac{1}{v_{\min}} - \frac{1}{v_{\max}}\right)(x,p)dx}. \tag{6}$$

Observe that

$$\left(\frac{1}{v_{\min}} - \frac{1}{v_{\max}}\right)(x,p) = \frac{H(x,p)}{x-p_2},$$

where H is a smooth function positive at the origin. After some calculations it is possible to reduce (6) to the form

$$c\varphi(p,c) - 2p_2(p_1-c)^{1/2} + (p_1-c)^{3/2}\psi(p,c) = (p_1 - {p_2}^2 - c)\ln\left(\frac{(p_1-c)^{1/2} - p_2}{-(p_1-c)^{1/2} - p_2}\right), \tag{7}$$

where φ and ψ are smooth functions with $\varphi(0) > 0$.

It is easy to see that this equation has no solution for $p_1 \geq {p_2}^2 B(p_2)$, for some smooth function B with $B(0) > 1$. For $p_1 < {p_2}^2 B(p)$ its unique solution $c(p)$ vanishing at the origin is only defined for $c > p_1 - {p_2}^2$. Therefore, $c(p) > p_1 - {p_2}^2$ and we conclude that the optimal averaged profit is

$$A(p) = \begin{cases} 0, & p_1 \leq 0 \\ p_1, & p_1 \geq 0,\ p_2 \geq 0 \\ p_1 - {p_2}^2, & p_1 \geq {p_2}^2 B(p),\ p_2 \leq 0 \\ c(p), & p_1 < {p_2}^2 B(p),\ p_1 \geq 0,\ p_2 \leq 0. \end{cases}$$

where c is the unique solution vanishing at the origin of equation (7) and B is a smooth function with $B(0) > 1$. ■

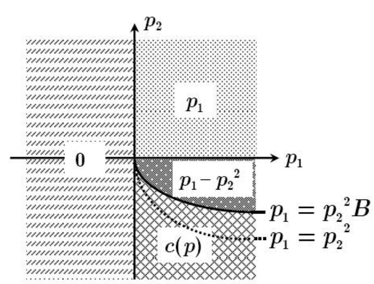


Figure 4: Singularity 14

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