

# NOTES ON THE LINK BETWEEN DEPENDENCE AND INDEPENDENCE IN EXTREME VALUE THEORY FOR DYNAMICAL SYSTEMS

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ABSTRACT. It is well known that under some conditions on the dependence structure we can relate the asymptotic distribution of the partial maximum of a stationary stochastic process with the maximum of an associated independent sequence of random variables with the same distribution function of the dependent one. These conditions are known as  $D(u_n)$  and  $D'(u_n)$ . Although  $D(u_n)$  is of mixing type, when studying stochastic processes arising from a dynamical system with good mixing properties, verifying  $D(u_n)$  is not straightforward. We propose a reformulation of  $D(u_n)$  so that its validity may follow easily if we have a certain decay of correlations for the dynamical system in consideration.

## 1. INTRODUCTION

Let  $(\Omega, \mathcal{A}, P)$  be a probability space and  $X : \Omega \rightarrow \mathbb{R}$  a random variable (r.v.) with distribution function (d.f.) given by  $F(x) = P(X \leq x)$ . Also, let  $E(\cdot)$  denote expectation with respect to  $P$  so that  $E(X) = \int X dP = \int x dF(x)$ . Consider a stationary stochastic process  $X_0, X_1 \dots$  of r.v. defined on  $\Omega$  with common d.f.  $F$ . For notational simplicity we assume that  $X = X_0$ . For every  $i, j, n \in \mathbb{N}$ , set

$$M_{i,j} = \max \{X_i, X_{i+1}, \dots, X_{i+j-1}\} \quad \text{and} \quad M_n = M_{0,n}. \quad (1.1)$$

The study of the limiting behavior for maxima of a stationary process can be reduced, under adequate conditions on the dependence structure, to the Classical Extreme Value Theory for sequences of i.i.d. r.v. Hence, to the stationary process  $X_0, X_1, \dots$  we associate an independent sequence of r.v. denoted by  $Z_1, Z_2, \dots$  with common d.f.  $F$ . We also set for each  $n \in \mathbb{N}$

$$\hat{M}_n = \max \{Z_0, \dots, Z_{n-1}\}. \quad (1.2)$$

Let us focus on the conditions that allow us to relate the asymptotic distribution of  $M_n$  with that of  $\hat{M}_n$ . Following [LLR83] we refer to these conditions as  $D(u_n)$  and  $D'(u_n)$ , where  $u_n$  is a suitable sequence of thresholds converging to  $\sup_{\omega \in \Omega} X(\omega) = 1$ , as  $n$  goes to  $\infty$ , that will be defined below.  $D(u_n)$  imposes a certain type of distributional mixing property. Essentially, it says that the dependence between some special type of events fades away as they become more and more apart in the time line.  $D'(u_n)$  restricts the

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*Date:* September 25, 2007.

*2000 Mathematics Subject Classification.* 37A50, 37C40, 60G10, 60G70.

*Key words and phrases.* Dependence, Extreme Value Theory, Stationary Stochastic Processes, Ergodic Theory, Decay of Correlations.

appearance of clusters, that is, it makes the occurrence of consecutive ‘exceedances’ of the level  $u_n$  an unlikely event.

The purpose of these notes is to verify that condition  $D(u_n)$  can be stated in a weaker form and the result still prevails. The advantage of having this weaker requirement is that, in the context of Dynamical Systems,  $D(u_n)$  should follow from decay of correlations.

We say that an *exceedance* of the level  $u_n$  occurs at time  $i$  if  $X_i > u_n$ . The probability of such an exceedance is  $1 - F(u_n)$  and so the mean value of the number of exceedances occurring up to  $n$  is  $n(1 - F(u_n))$ . The sequences of levels  $u_n$  we consider are such that  $n(1 - F(u_n)) \rightarrow \tau$  as  $n \rightarrow \infty$ , for some  $\tau \geq 0$ , which means that, in a time period of length  $n$ , the expected number of exceedances is approximately  $\tau$ .

Condition  $D(u_n)$  is a type of mixing requirement specially adapted to Extreme Value Theory. In this context, the events of interest are those of the form  $\{X_i \leq u\}$  and their intersections. Observe that  $\{M_n \leq u\}$  is just  $\{X_0 \leq u, \dots, X_{n-1} \leq u\}$ . Motivated by Collet’s work [Co01], we propose:

**Condition** ( $D(u_n)$ ). We say that  $D(u_n)$  holds for the sequence  $X_0, X_1, X_2, \dots$  if for any integers  $\ell, t$  and  $n$

$$P(\{X_0 > u_n\} \cap \{M_{t,\ell} \leq u_n\}) - P(\{X_0 > u_n\})P(\{M_\ell \leq u_n\}) \leq \gamma(n, t),$$

where  $\gamma(n, t)$  is nonincreasing in  $t$  for each  $n$  and  $n\gamma(n, t_n) \rightarrow 0$  as  $n \rightarrow \infty$  for some sequence  $t_n = o(n)$ , which means that  $t_n/n \rightarrow 0$  as  $n \rightarrow \infty$ .

We remark that the actual definition of  $D(u_n)$  appearing in [LLR83, Section 3.2] is, in some sense, a stronger requirement.

The sequence  $u_n$  is such that the average number of exceedances in the time interval  $\{0, \dots, [n/k]\}$  is approximately  $\tau/k$ , which goes to zero as  $k \rightarrow \infty$ . However, the exceedances may have a tendency to be concentrated in the time period following the first exceedance at time 0. To avoid this we introduce

**Condition** ( $D'(u_n)$ ). We say that  $D'(u_n)$  holds for the sequence  $X_0, X_1, X_2, \dots$  if

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{j=1}^{[n/k]} P(\{X_0 > u_n\} \cap \{X_j > u_n\}) = 0. \quad (1.3)$$

Notice that Condition 1.3 forbids the concentration of exceedances by bounding the probability of more than one exceedance in the time interval  $\{0, \dots, [n/k]\}$ . This guarantees that the exceedances should appear scattered through the time period  $\{0, \dots, n-1\}$ .

Our goal is to show that  $M_n$  and  $\hat{M}_n$  have the same asymptotic distribution under  $D(u_n)$  and  $D'(u_n)$ .

**Proposition 1.1.** *Let  $(u_n)_{n \in \mathbb{N}}$  be such that  $nP(X > u_n) = n(1 - F(u_n)) \rightarrow \tau$ , as  $n \rightarrow \infty$ , for some  $\tau \geq 0$ . Assume that conditions  $D(u_n)$  and  $D'(u_n)$  hold. Then*

$$\lim_{n \rightarrow \infty} P(M_n \leq u_n) = \lim_{n \rightarrow \infty} P(\hat{M}_n \leq u_n).$$

## 2. EXTREMES ON DYNAMICAL SYSTEMS

In this section, we observe that the study of the partial maximum of deterministic processes arising from a certain dynamical system that possesses an invariant measure, with sufficient decay of correlations, can be reduced to prove  $D'(u_n)$  and investigate the domain of attraction of maximum of the associated independent process.

Let  $f : M \rightarrow M$  be a smooth map on the manifold  $M$  admitting an invariant Borel probability measure  $\mu$ . Assume that there exists a nonincreasing  $g : \mathbb{N} \rightarrow \mathbb{R}$  such that for all  $\varphi, \psi : M \rightarrow \mathbb{R}$  with bounded variation, there is  $C > 0$  independent of  $\varphi, \psi$  and  $n$  such that

$$\left| \int \varphi \cdot (\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq C \text{Var}(\varphi) \|\psi\|_\infty g(n), \quad \forall n \geq 0, \quad (2.1)$$

where  $\text{Var}(\varphi)$  denotes the total variation of  $\varphi$  and  $ng(t_n) \rightarrow 0$ , as  $n \rightarrow \infty$  for some  $t_n = o(n)$ .

Consider now the r.v.  $X : M \rightarrow \mathbb{R}$  of bounded variation on the probability space  $(M, \mathcal{B}, \mu)$ , where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra on  $M$ , with d.f.  $F(x) = \mu(\{X^{-1}(-\infty, x]\})$ . Define the stationary stochastic process  $X_0, X_1, X_2, \dots$  by  $X_i = X \circ f^i$ . Let

$$M_n = \max\{X_0, \dots, X_{n-1}\},$$

for each  $n \in \mathbb{N}$ . Denote by  $Z_0, Z_1, Z_2, \dots$  an independent sequence of r.v. with common d.f.  $F$  and set, for each  $n \in \mathbb{N}$ ,

$$\hat{M}_n = \max\{Z_0, \dots, Z_{n-1}\}.$$

Taking  $\varphi = \mathbf{1}_{\{X > u_n\}}$  and  $\psi = \mathbf{1}_{\{M_\ell \leq u_n\}}$ , then (2.1) implies that Condition  $D(u_n)$  holds with  $\gamma(n, t) = \gamma(t) = C \text{Var}(\mathbf{1}_{\{X > u_n\}}) \|\mathbf{1}_{\{M_\ell \leq u_n\}}\|_\infty g(t) \leq C(\text{Var}(X) + 1)g(t)$  and for the sequence  $t_n$  such that  $t_n/n \rightarrow 0$  and  $ng(t_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

This means that if we are also able to prove  $D'(u_n)$  for the sequence  $X_0, X_1, \dots$  then the study of the limiting behavior of  $M_n$  is reduced to study the domain of attraction for maximums of  $F$  using the usual tools of Classical Extreme Value Theory, available, for example, in [LLR83, Section 1.6].

*Remark 2.1.* Observe that often decay of correlations is known for Hölder continuous functions instead. In these situations one can use an Hölder continuous approximation of  $\mathbf{1}_{\{X > u_n\}}$  and try to prove that the error introduced does not affect much, just as Collet did in [Co01, Lemma 3.3 and Section 3].

## 3. PROOF OF THE RESULT

We follow closely the arguments in [Co01, Section 3] and adapt them to our needs. We begin with stating two abstract Lemmas.

**Lemma 3.1.** *For any  $\ell \in \mathbb{N}$  and  $u \in \mathbb{R}$  we have*

$$\sum_{j=0}^{\ell-1} P(X_j > u) \geq P(M_\ell > u) \geq \sum_{j=0}^{\ell-1} P(X_j > u) - \sum_{j=0}^{\ell-1} \sum_{i \neq j, i=0}^{\ell-1} P(\{X_j > u\} \cap \{X_i > u\})$$

*Proof.* This is a straightforward consequence of the formula for the probability of a multiple union on events. See for example first Theorem of Chapter 4 in [Fe52].  $\square$

**Lemma 3.2.** *For any integers  $t, r, m, \ell, s \geq 0$  we have*

$$0 \leq P(M_r \leq u) - P(M_{r+\ell} \leq u) \leq \ell \cdot P(X > u) \quad (3.1)$$

and

$$\begin{aligned} & \left| P(M_{s+t+m} \leq u) - P(M_m \leq u) + \sum_{j=0}^{s-1} P(\{X > u\} \cap \{M_{s+t-j,m} \leq u\}) \right| \\ & \leq 2s \sum_{j=1}^{s-1} P(\{X > u\} \cap \{X_j > u\}) + tP(X > u). \end{aligned} \quad (3.2)$$

*Proof.* We have of course

$$0 \leq P(M_r \leq u) - P(M_{r+\ell} \leq u) \leq \sum_{j=0}^{\ell-1} (P(M_{r+j} \leq u) - P(M_{r+j+1} \leq u)).$$

On the other hand, for any  $i \geq 0$

$$P(M_i \leq u) = P(M_{i+1} \leq u) + P(\{M_i \leq u\} \cap \{X_i > u\}) \leq P(M_{i+1} \leq u) + P(X_i > u)$$

and the first statement of the Lemma follows by stationarity.

We now observe that

$$\{M_{s+t+m} \leq u\} = \{M_s \leq u\} \cap \{M_{s,t} \leq u\} \cap \{M_{s+t,m} \leq u\}.$$

It follows easily from this identity that

$$(\{M_s \leq u\} \cap \{M_{s+t,m} \leq u\}) \setminus \{M_{s+t+m} \leq u\} \subset \{M_{s,t} > u\}.$$

Therefore, using Lemma 3.1 we get

$$\left| P(\{M_s \leq u\} \cap \{M_{s+t,m} \leq u\}) - P(M_{s+t+m} \leq u) \right| \leq tP(X > u).$$

Using stationarity and the first inequality in Lemma 3.1 we have

$$\begin{aligned} P(\{M_s \leq u\} \cap \{M_{s+t,m} \leq u\}) &= P(M_{s+t,m} \leq u) - P(\{M_s > u\} \cap \{M_{s+t,m} \leq u\}) \\ &= P(M_m \leq u) - P(\{M_s > u\} \cap \{M_{s+t,m} \leq u\}) \\ &\geq P(M_m \leq u) - \sum_{j=0}^{s-1} P(\{X_j > u\} \cap \{M_{s+t,m} \leq u\}). \end{aligned}$$

Now, by the second inequality in Lemma 3.1 and stationarity we have

$$\begin{aligned} P(\{M_s \leq u\} \cap \{M_{s+t,m} \leq u\}) &\leq P(M_m \leq u) - \sum_{j=0}^{s-1} P(\{X_j > u\} \cap \{M_{s+t,m} \leq u\}) \\ &\quad + \sum_{j=0}^{s-1} \sum_{j \neq \ell, \ell=0}^{s-1} P(\{X_j > u\} \cap \{X_\ell > u\} \cap \{M_{s+t,m} \leq u\}). \end{aligned}$$

Hence, stationarity and the last three inequalities give

$$\begin{aligned} & \left| P(\{M_s \leq u\} \cap \{M_{s+t,m} \leq u\}) - P(M_m \leq u) + \sum_{j=0}^{s-1} P(\{X > u\} \cap \{M_{s+t-j,m} \leq u\}) \right| \\ & \leq 2s \sum_{j=1}^{s-1} P(\{X > u\} \cap \{X_j > u\}), \end{aligned}$$

and the result follows.  $\square$

*Proof of Proposition 1.1.* Let  $\ell = \ell_n = \lfloor \frac{n}{k} \rfloor$  where  $\lfloor \frac{n}{k} \rfloor$  is the integer part of  $\frac{n}{k}$ . We begin by replacing  $P(M_n \leq u_n)$  by  $P(M_{k(\ell+t)} \leq u_n)$  for some  $t > 1$ . According to (3.1) of Lemma 3.2, we have

$$|P(M_n \leq u_n) - P(M_{k(\ell+t)} \leq u_n)| \leq ktP(X > u_n). \quad (3.3)$$

We now estimate recursively  $P(M_{i(\ell+t)} \leq u_n)$  for  $i = 0, \dots, k$ . Using (3.2) of Lemma 3.2 and stationarity, we have for any  $1 \leq i \leq k$

$$|P(M_{i(\ell+t)} \leq u_n) - (1 - \ell P(X > u_n))P(M_{(i-1)(\ell+t)} \leq u_n)| \leq \Gamma_{n,i},$$

where

$$\begin{aligned} \Gamma_{n,i} &= \left| \ell P(X > u_n)P(M_{(i-1)(\ell+t)} \leq u_n) - \sum_{j=0}^{\ell-1} P(\{X_j > u_n\} \cap \{M_{\ell+t, (i-1)(\ell+t)} \leq u_n\}) \right| \\ &+ tP(X > u_n) + 2\ell \sum_{j=1}^{\ell-1} P(\{X > u_n\} \cap \{X_j > u_n\}). \end{aligned}$$

Using stationarity,  $D(u_n)$  and, in particular, that  $\gamma(n, t)$  is nonincreasing in  $t$  for each  $n$  we conclude

$$\begin{aligned} \Gamma_{n,i} &\leq \sum_{j=0}^{\ell-1} |P(X_0 > u_n)P(M_{(i-1)(\ell+t)} \leq u_n) - P(\{X_0 > u_n\} \cap \{M_{\ell+t-j, (i-1)(\ell+t)} \leq u_n\})| \\ &+ tP(X > u_n) + 2\ell \sum_{j=1}^{\ell-1} P(\{X > u_n\} \cap \{X_j > u_n\}) \\ &\leq \ell\gamma(n, t) + tP(X > u_n) + 2\ell \sum_{j=1}^{\ell-1} P(\{X > u_n\} \cap \{X_j > u_n\}). \end{aligned}$$

Define  $\Upsilon_n = \ell\gamma(n, t) + tP(X > u_n) + 2\ell \sum_{j=1}^{\ell-1} P(\{X > u_n\} \cap \{X_j > u_n\})$ . Then for every  $1 < i \leq k$  we have

$$|P(M_{i(\ell+t)} \leq u_n) - (1 - \ell P(X > u_n))P(M_{(i-1)(\ell+t)} \leq u_n)| \leq \Upsilon_n$$

and for  $i = 1$

$$\left| P(M_{(\ell+t)} \leq u_n) - (1 - \ell P(X > u_n)) \right| \leq \Upsilon_n.$$

Assume that  $k$  and  $n$  are large enough in order to have  $\ell P(X > u_n) < 2$ , which implies that  $|1 - \ell P(X > u_n)| < 1$ . A simple inductive argument allows to conclude

$$\left| P(M_{k(\ell+t)} \leq u_n) - (1 - \ell P(X > u_n))^k \right| \leq k\Upsilon_n.$$

Recalling (3.3), we have

$$\left| P(M_n \leq u_n) - (1 - \ell P(X > u_n))^k \right| \leq ktP(X > u_n) + k\Upsilon_n. \quad (3.4)$$

Since  $nP(X > u_n) = n(1 - F(u_n)) \rightarrow \tau$ , as  $n \rightarrow \infty$ , for some  $\tau \geq 0$ , we have

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (1 - [\frac{n}{k}]P(X > u_n))^k = \lim_{k \rightarrow \infty} (1 - \frac{\tau}{k})^k = e^{-\tau}.$$

Now, observe that  $nP(X > u_n) = n(1 - F(u_n)) \rightarrow \tau$  is equivalent to  $P(\hat{M}_n \leq u_n) = (F(u_n))^n \rightarrow e^{-\tau}$ , where the limits are taken when  $n \rightarrow \infty$  and  $\tau \geq 0$  (see [LLR83, Theorem 1.5.1] for a proof of this fact). Hence,

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (1 - [\frac{n}{k}]P(X > u_n))^k = \lim_{n \rightarrow \infty} P(\hat{M}_n \leq u_n). \quad (3.5)$$

It is now clear that, according to (3.4) and (3.5),  $M_n$  and  $\hat{M}_n$  share the same limiting distribution if

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} (ktP(X > u_n) + k\Upsilon_n) = 0,$$

that is

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} 2ktP(X > u_n) + n\gamma(n, t) + 2n \sum_{j=1}^{\ell} P(\{X > u_n\} \cap \{X_j > u_n\}) = 0. \quad (3.6)$$

Assume that  $t = t_n$  where  $t_n = o(n)$  is given by Condition  $D(u_n)$ . Then, for every  $k \in \mathbb{N}$ , we have  $\lim_{n \rightarrow \infty} kt_n P(X > u_n) = 0$ , since  $nP(X > u_n) \rightarrow \tau \geq 0$ . Finally, we use  $D(u_n)$  and  $D'(u_n)$  to obtain that the two remaining terms in (3.6) also go to 0.  $\square$

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