GENERIC PSEUDOGROUPS ON $(\mathbb{C}, 0)$ AND THE TOPOLOGY OF LEAVES

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Abstract. We show that generically a pseudogroup generated by holomorphic diffeomorphisms defined about $0 \in \mathbb{C}$ is free in the sense of pseudogroups even if the class of conjugacy of the generators is fixed. This result has a number of consequences on the topology of leaves for a (singular) holomorphic foliation defined on a neighborhood of an invariant curve. In particular, in the classical and simplest case arising from local nilpotent foliations possessing a unique separatrix which is given by a cusp of the form $\{y^2 - x^{2n+1} = 0\}$, our results allow us to settle the problem of showing that a generic foliation possesses only countably many non-simply connected leaves.

Key-words: pseudogroups on $(\mathbb{C}, 0)$ - free groups in Diff $(\mathbb{C}, 0)$ - nilpotent singularities

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1. Introduction

This paper is motivated by several difficulties concerning to greater or lesser extent the "topology of leaves" that are encountered in the study of some well-known problems about (singular) holomorphic foliations. Yet, most of these problems are essentially concerned with pseudogroups generated by certain local holomorphic diffeomorphisms defined on a neighborhood of $0 \in \mathbb{C}$. For this reason, we shall begin our discussion by stating our results in this context. First consider the group Diff $(\mathbb{C}, 0)$ of germs of holomorphic diffeomorphisms fixing $0 \in \mathbb{C}$, where the group law is induced by composition. Let Diff $(\mathbb{C}, 0)$ be equipped with the *analytic topology* introduced by Takens [T]. The precise definition of this topology will be given in Section 2, for the time being, it suffices to know that it possesses the Baire property. The reader is then reminded that a G_δ-dense set, sometimes also called a residual set, is nothing but a countable intersection of open and dense sets in a Baire space. All the "generic results" stated in this part of the Introduction concern G_δ-dense sets for this topology. It is however worth pointing out that, once they are established, it is easy to derive the "generic" character in other contexts, including suitable topologies involving the coefficients of Taylor series at $0 \in \mathbb{C}$ and/or in the sense of measure, cf. Remark 2.8. Next, consider a k-tuple of local holomorphic diffeomorphisms $f_1, \ldots, f_k$ fixing $0 \in \mathbb{C}$. The first theorem proved here states that we can perturb the f_i's *without altering their classes of holomorphic conjugacy in* Diff $(\mathbb{C}, 0)$ so that the group they generate is the free product of the corresponding cyclic groups. Motivation for keeping the conjugacy class fixed will be clear when discussing applications to singular foliations and, especially, Theorem B below. For the time being, note only that this condition is equivalent to preserving the order of a diffeomorphism provided that this order is finite. Also, in the case of a diffeomorphism having a hyperbolic fixed point at $0 \in \mathbb{C}$, it follows from Poincaré linearization theorem that the condition of preserving the conjugacy class amounts to fixing the corresponding multiplier. Here, it may be convenient to recall that a fixed point $p \in \mathbb{C}$ of a holomorphic local diffeomorphism h is said to be *hyperbolic* if $|h'(p)| \neq 1$. More generally, the *multiplier* of the fixed point p of h is simply the value of the derivative $h'(p) \in \mathbb{C}$.

As usual, given an element $f \in \text{Diff}\,(\mathbb{C}, 0)$, we shall denote by f^j its j^{th}-iterate, i.e. for $j > 0$, f^j is the element of $\text{Diff}\,(\mathbb{C}, 0)$ induced by the composition $f \circ \cdots \circ f$, j-times. Also $f^0 = \text{id}$ and, for $j < 0$, $f^j = (f^{|j|})^{-1}$. At level of germs, this definition does not pose any further difficulty whereas it requires some comments at level of *pseudogroups* as it will be seen later.

Denote by $\text{Diff}_\alpha(\mathbb{C}, 0)$ the normal subgroup of $\text{Diff}\,(\mathbb{C}, 0)$ consisting of those germs of diffeomorphisms that are tangent to the identity to order $\alpha \in \mathbb{N}$ (if $\alpha = 0$ then $\text{Diff}_\alpha(\mathbb{C}, 0) = \text{Diff}\,(\mathbb{C}, 0)$). The subgroup $\text{Diff}_\alpha(\mathbb{C}, 0) \subseteq \text{Diff}\,(\mathbb{C}, 0)$ is closed for the analytic topology (cf. Section 2) in $\text{Diff}\,(\mathbb{C}, 0)$. Besides the analytic topology of $\text{Diff}\,(\mathbb{C}, 0)$ naturally restricts to $\text{Diff}_\alpha(\mathbb{C}, 0)$ and, indeed, can directly be defined on $\text{Diff}_\alpha(\mathbb{C}, 0)$. Also, note that for every $\alpha \in \mathbb{N}$, $\text{Diff}_\alpha(\mathbb{C}, 0)$ equipped with the analytic topology possesses the Baire property. Next, let $(\text{Diff}_\alpha(\mathbb{C}, 0))^k$ denote the product of k-copies of $\text{Diff}_\alpha(\mathbb{C}, 0)$, endowed with the product analytic topology and viewed as a group for the composition law. Suppose we are given k elements $f_1, \ldots, f_k$ in $\text{Diff}\,(\mathbb{C}, 0)$ and denote by G_i the cyclic group generated by f_i, $i = 1, \ldots, k$. Naturally, the group G_i may or may not be finite and its order is the order of the germ f_i which is denoted by r_i. In other words, r_i is the smallest strictly positive integer for which $f_i^{\mathsf{r}_i} = \text{id}$. If this integer does not exist, then we set $\mathsf{r}_i = \infty$ and, in this case, the group G_i turns out to be infinite and isomorphic to $\mathbb{Z}$.

Consider the free group F_k on k generators $a_1, \ldots, a_k$ and consider the natural evaluation morphism from F_k to $\text{Diff}\,(\mathbb{C}, 0)$ consisting of making the substitutions $a_i \mapsto f_i$ (and interpreting the "concatenation of letters" as composition of germs). Let N be the *normal subgroup* of F_k generated by $\{a_1^{\mathsf{r}_1}, \ldots, a_k^{\mathsf{r}_k}\}$, with the convention that $a_i^\infty = \text{id}$. The quotient group F_k/N is isomorphic to the *free product* $G_1 * \cdots * G_k$ of the groups $G_1, \ldots, G_k$. Furthermore, the above mentioned evaluation morphism factors through the quotient F_k/N so as to induce a homomorphism $\mathcal{E}$ from $G_1 * \cdots * G_k$ to $\text{Diff}\,(\mathbb{C}, 0)$.

An alternative way to construct the homomorphism $\mathcal{E}$ consists of using the fact that every element in the free product $G_1 * \cdots * G_k$ is represented by a *unique* reduced word in the letters $a_1, \ldots, a_k$, where the empty-word represents the identity (cf. Section 2 for further details). Therefore, the elements of $G_1 * \cdots * G_k$ are naturally identified to reduced words $W(a_1, \ldots, a_k)$. With this notation, $\mathcal{E}(W(a_1, \ldots, a_k))$ is simply the element of $\text{Diff}\,(\mathbb{C}, 0)$ obtained by substituting $a_i \mapsto f_i$ in the spelling of $W(a_1, \ldots, a_k)$ (where again the "concatenation of letters" becomes composition of germs). In the sequel, the element of $\text{Diff}\,(\mathbb{C}, 0)$ given by $\mathcal{E}(W(a_1, \ldots, a_k))$ is going to be denoted by $W(f_1, \ldots, f_k)$.

To state Theorem A, recall that a local diffeomorphism f fixing $0 \in \mathbb{C}$ is linearizable if and only it is conjugated to the linear map $z \mapsto f'(0).z$ by a local holomorphic change of coordinates, where $f'(0)$ stands for the derivative of f at $0 \in \mathbb{C}$. This local diffeomorphism is said to have a *Cremer point* (at $0 \in \mathbb{C}$) if it is not linearizable and $f'(0)$ has norm 1 but it is not a root of unit. Now, we have:

Theorem A. *Fixed $\alpha \in \mathbb{N}$, let $f_1, \ldots, f_k$ be given elements in* $\text{Diff}\,(\mathbb{C}, 0)$ *and consider the corresponding cyclic groups $G_1, \ldots, G_k$. Then, there exists a G_δ-dense set $\mathcal{V} \subset (\text{Diff}_\alpha(\mathbb{C}, 0))^k$ such that, whenever $(h_1, \ldots, h_k) \in \mathcal{V}$, the following holds:*

(1) *The group generated by* $h_1^{-1} \circ f_1 \circ h_1, \ldots, h_k^{-1} \circ f_k \circ h_k$ *induces a group in* $\mathrm{Diff}\,(\mathbb{C}, 0)$ *that is isomorphic to the free product*

$$G_1 * \cdots * G_k \, .$$

(2) *Let* $f_1, \ldots, f_k$ *and* $h_1, \ldots, h_k$ *be identified to local diffeomorphisms defined about* $0 \in \mathbb{C}$*. Suppose that none of the local diffeomorphisms* $f_1, \ldots, f_k$ *has a Cremer point at* $0 \in \mathbb{C}$*. Denote by* Γ^h *the pseudogroup defined on a neighborhood* V *of* $0 \in \mathbb{C}$ *by the mappings* $h_1^{-1} \circ f_1 \circ h_1, \ldots, h_k^{-1} \circ f_k \circ h_k$*, where* $(h_1, \ldots, h_k) \in \mathcal{V}$*. Then* V *can be chosen so that, for every non-empty reduced word* $W(a_1, \ldots, a_k)$*, the element of* Γ^h *associated to* $W(h_1^{-1} \circ f_1 \circ h_1, \ldots, h_k^{-1} \circ f_k \circ h_k)$ *does not coincide with the identity on any connected component of its domain of definition.*

Note that the assumption that none of the fixed diffeomorphisms $f_1, \ldots, f_k$ has a Cremer point at $0 \in \mathbb{C}$ is not necessary for the first conclusion of Theorem A. This assumption is, however, indispensable for the second item due to certain examples of dynamics near Cremer points that were constructed by Perez-Marco, cf. Section 4.

Theorem A touches on an issue previously developed in a number of works. Namely, the realization of certain groups as subgroups of $\mathrm{Diff}\,(\mathbb{C}, 0)$ or of $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$, where the latter stands for the group of formal diffeomorphisms of $(\mathbb{C}, 0)$. In this direction, the papers [Br-C-LN] and [C-L] may respectively be quoted as being the first work to realize rank 2 free groups in $\mathrm{Diff}\,(\mathbb{C}, 0)$ and as the first paper to realize the free product of two finite cyclic groups in $\mathrm{Diff}\,(\mathbb{C}, 0)$. In [E-V] several groups are realized in $\mathrm{Diff}\,(\mathbb{C}, 0)$ and/or $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0)$ whereas in [N-Y] an interesting study about the possibility of breaking relations at level of formal diffeomorphisms is conducted.

It is natural to expect the preceding statement to have consequences on the topology of the leaves of a foliation on a neighborhood of an invariant curve and/or on a neighborhood of a singular point. Recall that a local (singular) holomorphic foliation on a neighborhood of $(0, 0) \in \mathbb{C}^2$ is nothing but the (singular) foliation induced by the local orbits of a holomorphic vector field having isolated singularities and defined on the mentioned neighborhood. In particular, two representatives of a same foliation differ by an invertible multiplicative holomorphic function. A singular foliation is then said to have a *nilpotent singularity* if it can be represented by a vector field whose linear part at $(0, 0) \in \mathbb{C}^2$ is nilpotent and different from *zero*. To abridge notations, this situation will often be referred to by saying that the origin is a nilpotent singularity (with the corresponding foliation being left implicit) or by saying that $\mathcal{F}$ is a nilpotent foliation. In terms of applications of Theorem A to the topology of singular foliations, we shall content ourselves of providing an answer to a long-standing question on nilpotent singularities leaving invariant a cusp of the form $\{y^2 + x^{2n+1} = 0\}$. By a small abuse of language, by a cusp of the form $\{y^2 + x^{2n+1} = 0\}$ it will always be meant a (local) curve analytically equivalent to the cusp in question. This choice will help us to explain most of the relevant ideas without making the discussion too technical. To begin with, consider a singular foliation $\mathcal{F}$ defined about $(0, 0) \in \mathbb{C}^2$ by a nilpotent vector field $y\partial/\partial x + \cdots$ with isolated singularities and possessing a cusp $\{y^2 + x^{2n+1} = 0\}$ as its *unique separatrix*, i.e. the cusp in question is the only local analytic curve containing the origin and invariant by the foliation. This much studied class of nilpotent singularities corresponds to Arnold's singularities of type A^{2n+1}. Whereas several works were devoted to these nilpotent singularities, and in particular to the description of suitable normal forms (cf. [Lo], [S-Z] and references therein), the question about the topology of most leaves for the "generic foliation" remained unsettled. Our Theorem B below states that for a generic foliation in the class A^{2n+1} there is only countably many non-simply connected leaves.

Generic theorems about foliations, as it is the case of Theorem B, are more commonly expressed in terms of Krull topology since the corresponding formulations automatically bear a meaning in terms of "generic coefficients" for the corresponding differential equation. This explains why Theorem B will be stated with respect to Krull topology rather than being "parameterized" by G_δ-dense sets in $\mathrm{Diff}\,(\mathbb{C},0)$ (yet a formulation in terms of G_δ-dense sets is possible, cf. Section 5 for details).

Let $X \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ be a holomorphic vector field with an isolated singularity at the origin and defining a germ of nilpotent foliation $\mathcal{F}$ of type A^{2n+1}, in particular $\mathcal{F}$ possesses one unique separatrix. Then, we have:

Theorem B (Cusps). *For arbitrarily large $N \in \mathbb{N}$, there exists a vector field $X' \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ defining a germ of a foliation $\mathcal{F}'$ and satisfying the following conditions:*

- *$J_0^N X' = J_0^N X$ (i.e. the vector fields X, X' are tangent to order N at the origin).*
- *The foliations $\mathcal{F}$ and $\mathcal{F}'$ have S as a common separatrix.*
- *There exists a fundamental system of open neighborhoods $\{U_j\}_{j\in\mathbb{N}}$ of S, inside a closed ball $\bar{B}(0,R)$, such that for all $j \in \mathbb{N}$, the leaves of the restriction of $\mathcal{F}'$ to $U_j \setminus S$ are simply connected except for a countable set of them.*

To apply Theorem A to the topology of leaves of foliations in more general settings is a quite subtle problem for which the theory developed in [Ma-M] becomes a powerful tool. Concerning our Theorem B, a self-contained proof is given in Section 5. This proof, however, amounts to applying the techniques of [Ma-M] to an elementary case. Another comment about Theorem B is that, though it is naturally constructed, the systems of neighborhoods U_j cannot be arbitrary. In fact, for an arbitrary neighborhood U_j it may happen, for example, that intersections between leaves with the boundary of U_j may create "holes" in the corresponding leaves therefore making them non-simply connected. In this regard, it is to be pointed out that a result slightly more accurate than Theorem B will be stated and proved in Section 5. This section also contains further information and details about these foliations.

To close this Introduction let us make some comments concerning the standard condition that we have considered in Theorem A namely, the fact that the analytic conjugacy class of the initial local diffeomorphisms is always kept fixed. For this, it is interesting to look at a foliation $\widetilde{\mathcal{F}}$ defined on a neighborhood of a rational curve C (in turn embedded in some complex surface). The singularities of $\widetilde{\mathcal{F}}$ in C are denoted by $p_1, \ldots, p_k$ and they are supposed to be irreducible with two eigenvalues different from zero, i.e. if they are represented by a vector field X with isolated singularities, then the linear part of X has two eigenvalues λ_1, λ_2 different from zero and such that neither λ_1/λ_2 nor λ_2/λ_1 is a positive integer. It is then natural to consider perturbations of $\widetilde{\mathcal{F}}$ satisfying our standard condition: the analytic class of the local holonomy map σ_k defined by a small loop around p_k is fixed. Since the singularities are irreducible with two eigenvalues different from zero, this condition is equivalent to saying that the *analytic types* of the singularities of $\widetilde{\mathcal{F}}$ are fixed. In fact, when the quotient of the eigenvalues of the corresponding singularity belongs to $\mathbb{C} \setminus \mathbb{R}_-$, i.e. when the singularity belongs to the Poincaré domain, the last claim follows from Poincaré linearization theorem (note that resonances are ruled out by the assumption that the singularities are irreducible).

On the other hand, when the mentioned quotient belongs to $\mathbb{R}_-^*$, i.e. when the singularity belongs to the Siegel domain, the statement follows from a classical lemma in [M-Mo]. In turn, in the case of irreducible singularities belonging to the Poincaré domain, our context becomes equivalent to the context of *isospectral deformations* i.e. deformations preserving the eigenvalues of each singular point. However, for singularities in the Siegel domain, our condition is far stronger than the isospectral one and, in fact, it is expected to be the natural "good" condition for developing a (global) moduli theory for holomorphic foliations. Finally, when a singularity in the Siegel domain gives rise to a local holonomy of finite order, then the condition becomes equivalent to deforming the foliation while keeping fixed the order of the mentioned local holonomy maps. This last case is precisely the situation that emerges in the analysis of singularities of type A^{2n+1} and it will play a role in the proof of Theorem B.

In any event, in a suitable sense, a generic foliation as above will still have all but countably many leaves simply connected. This assertion may be justified by a construction similar to the construction carried out in Section 5. Alternatively, we can resort to more general results obtained in [Ma-M]. In particular, the preceding theorems can be viewed as a step towards Anosov's conjecture stating a global result about the existence of only countably many non-simply connected leaves for a generic foliation of the projective plane.

Concerning foliations on the projective plane leaving a projective line invariant, Il'yashenko and Pyartli [Il-P] proved that, in the class of foliations with degree d of the projective plane leaving the line at infinity invariant, those for which the holonomy group of (the regular part of) the line at infinity is free are generic. This very interesting result has a different nature if compared to statements provided in this work and deserves further comments. Whereas Il'yashenko and Pyartli do not worry about how the singular points change in their considerations about "generic foliations", one of the main differences between the two works stems from the fact that their theorem is stated for global foliations whose space of parameters is far more restrictive and of finite dimension. Therefore, their result does not apply in a singular context, for example in the study of foliations leaving a cusp invariant, not only because the "parameter space" is totally different but also because singularities are often "deformed" in their procedure. Similarly, our construction does not apply in their global context since it is unclear whether our "perturbations" can be realized within the natural parameter space associated to (global) foliations of degree d. Another issue that needs to be pointed out is that, unfortunately, Il'yashenko and Pyartli's theorem works only at the "infinitesimal" level of the group of germs of diffeomorphisms fixing $0 \in \mathbb{C}$. Due to the reasons explained above (cf. also Section 4), it therefore does not imply the existence of simply connected leaves (apart from a countable set) in a fixed neighborhood of the line at infinity. In this respect, Firsova [F] has obtained interesting results about simply connected leaves for generic foliations in $\mathbb{C}^2$ by exploiting convexity properties of Stein manifolds. In fact, she introduced a method to "split" a dead-loop in a chosen leaf. Though this clearly goes in the same direction as Theorem A, again new difficulties arise from fixing the analytic type of singularities. Other issues related to Firsova's method and the problems discussed in this paper involve the "localization" of the convexity techniques in the context of singularities (i.e. the choice of "preferred" neighborhoods U_j as above) and the countable character of dead-loops. In view of the interest of this type of question and given the several links among these different approaches, it is natural to wonder whether a suitable blend of ideas in these papers may lead us to fill in some of the gaps mentioned above and provide further insight into the general case of Anosov's conjecture.

Finally a word about the structure of the paper. In Section 2, the analytic topology is introduced in the context adapted to our needs and it is also proved that the analytic topology is associated to a structure of complete metric space. Some relevant additional properties of the analytic topology are also put forward since they play a role in the subsequent discussion. The second part of Section 2 contains the general lines of our approach to Theorem A as well as a detailed discussion of reduced words and of the use of Baire property. By building in this material, the first conclusion of Theorem A, namely the generic nature of free products at germ level, is established in Section 3. Since some potential readers of this paper might primarily be interested in this result, we felt it was useful to single it out in the presentation. The argument is then naturally continued in Section 4, with a suitable discussion of pseudogroups of maps and of the size of their corresponding domains of definitions. Note that, in both Sections 3 and 4, we deal exclusively with the case where the corresponding groups are generated by only two local diffeomorphisms (i.e. $k = 2$ in the statements of Theorem A). This serves only to abridge notations since the general case does not offer additional difficulties. Finally, in Section 5, some consequences of these theorems to the topology of leaves of foliations are discussed, and Theorem B is proved.

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2. General set up

2.1. **The Analytic topology and some of its properties.** In the sequel, let $\mathrm{Diff}\,(\mathbb{C},0)$ stand for the group of local holomorphic diffeomorphisms fixing $0 \in \mathbb{C}$ whereas $\mathrm{Diff}_{\alpha}(\mathbb{C},0)$ stands for its normal subgroup consisting of elements tangent to the identity to order α (for $\alpha = 0$, the group $\mathrm{Diff}\,(\mathbb{C},0)$ is recovered). The group of formal series $\sum_{i=1}^{\infty} c_i x^i$, $c_i \in \mathbb{C}$, is going to be denoted by $\widehat{\mathrm{Diff}}\,(\mathbb{C},0)$ while $\widehat{\mathrm{Diff}}_{\alpha}(\mathbb{C},0)$ is the corresponding normal subgroups of series tangent to the identity to order α. There is an obvious injection of $\mathrm{Diff}\,(\mathbb{C},0)$ in $\widehat{\mathrm{Diff}}\,(\mathbb{C},0)$ (resp. $\mathrm{Diff}_{\alpha}(\mathbb{C},0)$ in $\widehat{\mathrm{Diff}}_{\alpha}(\mathbb{C},0)$) which associates to an element of $\mathrm{Diff}\,(\mathbb{C},0)$ (resp. $\mathrm{Diff}_{\alpha}(\mathbb{C},0)$) its Taylor series about $0 \in \mathbb{C}$. Also, let $\mathrm{Hol}\,(\mathbb{C},0)$ denote the space of (germs of) holomorphic functions defined about $0 \in \mathbb{C}$. Clearly $\mathrm{Diff}\,(\mathbb{C},0) \subset \mathrm{Hol}\,(\mathbb{C},0)$ and an element $f \in \mathrm{Hol}\,(\mathbb{C},0)$ belongs to $\mathrm{Diff}\,(\mathbb{C},0)$ if and only if $f'(0) \neq 0$.

Let us begin by defining the so-called *analytic topology* (or C^{ω}-topology) in either $\mathrm{Hol}\,(\mathbb{C},0)$ or $\mathrm{Diff}\,(\mathbb{C},0)$. To the best of our knowledge, this type of topology was first considered by Takens [T] in the general situation of groups of real analytic diffeomorphisms where he also observed that it possesses the Baire property. The definition, however, can immediately be adapted to $\mathrm{Hol}\,(\mathbb{C},0)$ or to $\mathrm{Diff}\,(\mathbb{C},0)$. Besides $\mathrm{Diff}\,(\mathbb{C},0)$ naturally becomes an open and dense subset of $\mathrm{Hol}\,(\mathbb{C},0)$. Also, we are going to show that, in $\mathrm{Hol}\,(\mathbb{C},0)$, this topology is induced by a metric turning $\mathrm{Hol}\,(\mathbb{C},0)$ into a *complete metric space*, cf. Proposition 2.1 below. Proposition 2.1 also ensures that, for $\alpha \geq 1$, $\mathrm{Diff}_\alpha(\mathbb{C},0)$ is a closed subgroup of $\mathrm{Diff}\,(\mathbb{C},0)$ and, therefore, it is itself a complete metric space with the induced metric (topology).

Given $r > 0$, let $B(r) \subset \mathbb{C}$ denote the open disc of radius r about $0 \in \mathbb{C}$. Consider a holomorphic function h defined about $0 \in \mathbb{C}$ and taking values in $\mathbb{C}$. If h possesses a holomorphic extension (still denoted by h) to $B(r)$, then set $\|h\|_r = \sup_{z\in B(r)} |h(z)|$. Otherwise, we pose $\|h\|_r = +\infty$.

Next for $r, \varepsilon > 0$ and $f \in \mathrm{Hol}\,(\mathbb{C},0)$ (resp. $f \in \mathrm{Diff}\,(\mathbb{C},0)$) chosen, let $(f+\mathcal{U}_r^\varepsilon) \subseteq \mathrm{Hol}\,(\mathbb{C},0)$ (resp. $f \in \mathrm{Diff}\,(\mathbb{C},0)$) be the set defined by

$$(f+\mathcal{U}_r^\varepsilon) = \{g \in \mathrm{Hol}\,(\mathbb{C},0) \ ; \ \|g-f\|_r < \varepsilon\}$$

(resp. $(f+\mathcal{U}_r^\varepsilon) = \{g \in \mathrm{Diff}\,(\mathbb{C},0) \ ; \ \|g-f\|_r < \varepsilon\}$ where, in any event, $(g-f)$ is interpreted simply as a holomorphic function that need not be a local diffeomorphism at $0 \in \mathbb{C}$). The quickest way to define the *analytic topology* on $\mathrm{Hol}\,(\mathbb{C},0)$ (resp. $\mathrm{Diff}\,(\mathbb{C},0)$) consists of declaring that the analytic topology is the topology generated by the sets $(f+\mathcal{U}_r^\varepsilon)$. In other words, the sets $(f+\mathcal{U}_r^\varepsilon)$ form a *basis of open sets* for the analytic topology. An immediate consequence of this definition is that a sequence $\{f_i\}_{i\in\mathbb{C}} \subset \mathrm{Hol}\,(\mathbb{C},0)$ (resp. $\{f_i\}_{i\in\mathbb{C}} \subset \mathrm{Diff}\,(\mathbb{C},0)$) is convergent in the analytic topology if and only if, to every pair $r, \varepsilon > 0$, there corresponds $N \in \mathbb{N}$ such that $\|f_i - f_j\|_r < \varepsilon$ whenever both i, j are greater than N.

Note that, if needed, when defining the basis of neighborhoods for the identity, the value of ε can be set equal to $1/r$ so as to have only "one parameter" to deal with. Nonetheless we preferred to allow for "free" r and ε, as done by Takens, since it makes the definition somehow closer to the most commonly used definitions of basis of neighborhoods for standard topologies in functional spaces.

The preceding definition deserves some comments. First, it is immediate to check that the analytic topology in $\mathrm{Diff}\,(\mathbb{C},0)$ coincides with the topology induced from the analytic topology in $\mathrm{Hol}\,(\mathbb{C},0)$ through the embedding $\mathrm{Diff}\,(\mathbb{C},0) \subset \mathrm{Hol}\,(\mathbb{C},0)$. It is also clear that $\mathrm{Diff}\,(\mathbb{C},0)$ is then identified to an open dense subset of $\mathrm{Hol}\,(\mathbb{C},0)$.

Another point that should be made about $\mathrm{Diff}\,(\mathbb{C},0)$ endowed with this analytic topology is that the composition map is not continuous. In other words, $\mathrm{Diff}\,(\mathbb{C},0)$ is not a topological group with the analytic topology. More precisely the mapping from $\mathrm{Diff}\,(\mathbb{C},0)$ to $\mathrm{Diff}\,(\mathbb{C},0)$ that associates to a chosen $h \in \mathrm{Diff}\,(\mathbb{C},0)$ the element $f \circ h$, where $f \in \mathrm{Diff}\,(\mathbb{C},0)$ is fixed, is not continuous in general. In fact, a sequence $\{h_i\} \subset \mathrm{Diff}\,(\mathbb{C},0)$ converges in the analytic topology to h if and only if $h_i = h + r_i$ where $r_i \in \mathcal{U}_r^\varepsilon$ for every fixed $r, \varepsilon > 0$ and sufficiently large i. However, if f has a bounded domain of definition, this does not guarantee that $f \circ (h + r_i) - f \circ h$ admits a holomorphic extension to arbitrarily large discs. This remark was once communicated to the second author by L. Lempert to whom we wish to thank.

A direct proof that $\mathrm{Diff}\,(\mathbb{C},0)$ endowed with the analytic topology is a Baire space can be obtained by specifying to $\mathrm{Diff}\,(\mathbb{C},0)$ the argument given in [T]. This result can, however, be recovered through the classical theorem of Baire since we are going to show that the analytic topology in $\mathrm{Hol}\,(\mathbb{C},0)$ is induced by a structure of complete metric space. Hence $\mathrm{Hol}\,(\mathbb{C},0)$ possesses the Baire property and so does $\mathrm{Diff}\,(\mathbb{C},0)$ since $\mathrm{Diff}\,(\mathbb{C},0)$ is an open and dense subset of $\mathrm{Hol}\,(\mathbb{C},0)$ (so that every open and dense subset U of $\mathrm{Diff}\,(\mathbb{C},0)$ is automatically an open and dense subset of $\mathrm{Hol}\,(\mathbb{C},0)$). As to $\mathrm{Diff}_\alpha(\mathbb{C},0)$, $\alpha \geq 1$, it will be seen that these groups also become complete metric spaces (with the restriction of the metric) and therefore possess the Baire property as well.

Following a suggestion made by the anonymous referee, let us first show that the analytic topology in $\mathrm{Hol}\,(\mathbb{C},0)$ is *metrizable* i.e. it is induced by a certain metric "d_A". To define the metric d_A, suppose that f,g in $\mathrm{Hol}\,(\mathbb{C},0)$ are given and consider the holomorphic function $f-g$ which is defined on a neighborhood of $0\in\mathbb{C}$. Denote by $c_1x+c_2x^2+\cdots$ the Taylor series of $f-g$ at $0\in\mathbb{C}$. Finally, set

$$d_A(f,g)=\sup_{k\in\mathbb{N}}\|c_k\|^{1/k}\,.$$

Note that d_A is a well-defined metric on $\mathrm{Hol}\,(\mathbb{C},0)$. Formally speaking, we clearly have $d_A(f,g)\geq 0$ and $d_A(f,g)=d_A(g,f)$. Also $d_A(f,g)=0$ if and only if $f=g$, thanks to analytic continuation. Nonetheless, it remains to show that $d_A(f,g)$ is well-defined or, equivalently, that the supremum considered in its definition is actually finite. For this, observe first that the radius of convergence of the power series $c_1x+c_2x^2+\cdots$ is strictly positive. Since this radius is precisely given by $1/\limsup_{n\to\infty}\|c_n\|^{1/n}$, it follows that $d_A(f,g)<\infty$. Finally, to conclude that d_A is a metric on $\mathrm{Hol}\,(\mathbb{C},0)$, it only remains to check the triangle inequality. This is however an easy consequence of the well-known inequality $(a+b)^{1/n}\leq a^{1/n}+b^{1/n}$ for positive reals a,b and $n\in\mathbb{N}^*$ (to check the inequality just raise both sides to the n^{th}-power).

We can now show that $\mathrm{Hol}\,(\mathbb{C},0)$ endowed with the metric d_A is complete.

Proposition 2.1. *When endowed with the metric d_A,* $\mathrm{Hol}\,(\mathbb{C},0)$ *becomes a complete metric space. Moreover d_A induces the analytic topology in* $\mathrm{Hol}\,(\mathbb{C},0)$*.*

Proof. Let us first show that d_A induces the analytic topology in $\mathrm{Hol}\,(\mathbb{C},0)$. For this, it suffices to check that a sequence of elements f_i in $\mathrm{Hol}\,(\mathbb{C},0)$ converges to a certain $f\in\mathrm{Hol}\,(\mathbb{C},0)$ in the analytic topology if and only if $d_A(f,f_i)$ goes to *zero* as $i\to\infty$. To begin with, let us suppose that $f_i\to f$ in the analytic topology. Thus, for positive r and $\varepsilon<1$, and modulo taking i large enough, the difference f_i-f possesses a holomorphic extension (still denoted by f_i-f) to the disc $B(r)$ of radius r about the origin and, besides, this extension satisfies

$$\sup_{z\in B(r)}\|f_i(z)-f(z)\|\leq\varepsilon\,. \tag{1}$$

Denote by $\sum_{n=1}^{\infty}c_n^iz^n$ the Taylor series of f_i-f based at $0\in\mathbb{C}$. Thanks to (1), Cauchy estimates applied to the Taylor coefficients c_n^i show that $\|c_n^i\|\leq\varepsilon/r^n$. Since $\varepsilon<1$, we conclude that $\|c_n^i\|^{1/n}\leq 1/r$ for every $n\in\mathbb{N}^*$. Next, by choosing r arbitrarily large, it follows that $d_A(f_i,f)\to 0$ as desired. To show the converse, let again $(f_i-f)(z)=\sum_{n=1}^{\infty}c_n^iz^n$. Given $\varepsilon>0$, modulo choosing i very large, we have $\sup_{n\in\mathbb{N}}\|c_n^i\|^{1/n}<\varepsilon$. In particular, the radius of convergence of the series $\sum_{n=1}^{\infty}c_n^iz^n$ is at least $1/\varepsilon$. Therefore, as i increases, the functions f_i-f admit holomorphic extensions to arbitrarily large discs. It remains to show that, fixed a radius r_0, the holomorphic extension of f_i-f to $B(r_0)$ converges uniformly to *zero*. This is however easy: note that, for $\|z\|<r_0$, we have $\|f_i(z)-f(z)\|\leq\sum_{n=1}^{\infty}\|c_n^i\|r_0^n$. Setting $d_A(f_i,f)=\tau_i$, it follows that $\tau_i\to 0$ and that $\|c_n^i\|\leq\tau_i^n$. Therefore, for $\|z\|<r_0$,

$$\|f_i(z)-f(z)\|\leq\sum_{n=1}^{\infty}(\tau_ir_0)^n\leq\frac{\tau_ir_0}{1-\tau_ir_0}\,.$$

The claim follows since $\tau_i \to 0$.

To finish the proof, we still need to check that the metric space $(\mathrm{Hol}\,(\mathbb{C},0),\ d_A)$ is complete. Let then $\{f_i\}$ be a Cauchy sequence for d_A and set $f_i = \sum_{n=1}^{\infty} a_n^i z^n$. Given a small $\tau \in (0,1)$, there is N such that $d_A(f_i, f_j) < \tau$ provided that $i, j > N$. The definition of the metric d_A then implies that

$$\|a_n^i - a_n^j\| < \tau^n \leq \tau \tag{2}$$

whenever $i,\, j > N$. Therefore, for every $n \in \mathbb{N}$ fixed, the sequence $\{a_n^i\}_{i\in\mathbb{N}}$ is a Cauchy sequence and, hence, it converges towards a certain $b_n \in \mathbb{C}$. In particular, the limit of $\{f_i\}$ must be *unique*, provided that it exists. Consider then the power series $\sum_{n=1}^{\infty} b_n z^n$ and denote by ρ its convergence radius. Assume for the time being that $\rho > 0$ (strictly) so that $\sum_{n=1}^{\infty} b_n z^n$ defines an element $f \in \mathrm{Hol}\,(\mathbb{C},0)$. By considering Estimate (2) and letting $j \to \infty$, we conclude that

$$\|a_n^i - b_n\| \leq \tau^n$$

for every $n \in \mathbb{N}^*$ as long as $i > N$. Thus $d_A(f_i, f) \leq \tau$ for $i > N$ and, since $\tau > 0$ can be chosen arbitrarily small, it follows that $\{f_i\}$ converges to f proving that the metric d_A is complete.

It only remains to check that the convergence radius ρ of $\sum_{n=1}^{\infty} b_n z^n$ is strictly positive. For this, we proceed as follows. Since $\{f_i\}$ is Cauchy, the definition of d_A implies the existence of $i_0 \in \mathbb{N}$ such that for every $i, j \geq i_0$ and all $n \in \mathbb{N}$, the estimate

$$\|a_n^i - a_n^j\|^{1/n} \leq 1$$

holds. In particular, $\|a_n^{i_0} - b_n\|^{1/n} \leq 1$ and thus $\|b_n\| \leq \|a_n^{i_0}\| + 1$ for every n. The (possibly null) radius of convergence of $\sum_{n=1}^{\infty} b_n z^n$ being given by the $(\limsup_{n\to\infty} \|b_n\|^{1/n})^{-1}$, it follows that

$$\frac{1}{\limsup_{n\to\infty} \|b_n\|^{1/n}} \geq \frac{1}{\limsup_{n\to\infty} \|a_n^{i_0} + 1\|^{1/n}} \geq \frac{1}{1 + \limsup_{n\to\infty} \|a_n^{i_0}\|^{1/n}}\,.$$

Since $f^{i_0} \in \mathrm{Diff}\,(\mathbb{C},0)$, $\limsup_{n\to\infty} \|a_n^{i_0}\|^{1/n}$ is finite so that the radius of convergence of $\sum_{n=1}^{\infty} b_n z^n$ is strictly positive. The proposition is proved. $\square$

It is immediate from the definition of the metric d_A that $\mathrm{Diff}_\alpha(\mathbb{C},0)$ is a closed subset of $\mathrm{Hol}\,(\mathbb{C},0)$ provided that $\alpha \geq 1$. In particular, for $\alpha \geq 1$, $\mathrm{Diff}_\alpha(\mathbb{C},0)$ is a complete metric space and hence possesses the Baire property itself.

Remark 2.2. A curious phenomenon involving the analytic topology is the fact that, given $f \in \mathrm{Diff}\,(\mathbb{C},0)$ with finite convergence radius (about $0 \in \mathbb{C}$), every other element $g \in \mathrm{Diff}\,(\mathbb{C},0)$ sufficiently close to f must have the same convergence radius. This may be seen by directly comparing the convergence radii of the Taylor series of f, g by means of the definition of d_A or, alternatively, by resorting to the definition of the analytic topology in terms of basis of neighborhoods. Indeed, for a sequence $\{g_i\}$ to converge to f, the difference $f - g_i$ must admit a holomorphic extension to arbitrarily large discs provided that i is large as well. Thus, for i very large, $f - g_i$ becomes holomorphic on a disc of radius greater than the convergence radius of the Taylor series of f at $0 \in \mathbb{C}$. It then follows that the Taylor series at $0 \in \mathbb{C}$ of these local diffeomorphisms g_i must have the same convergence radius as f.

For an additional useful property of the analytic topology, let J^m denote the vector space consisting of the m-jets of holomorphic functions at $0 \in \mathbb{C}$, for $m \in \mathbb{N}$ given. The space J^m can naturally be identified to the quotient $\mathbb{C}\{z\}/(z^{m+1})$ of the ring of convergent power series $\mathbb{C}\{z\}$ by the principal ideal generated by z^{m+1}. Given a holomorphic function f defined about $0 \in \mathbb{C}$, viewed as an element of $\mathbb{C}\{z\}$, let $j^m f$ denote its projection on $\mathbb{C}\{z\}/(z^{m+1})$. Conversely, this projection admits a natural set consisting of the map $\sigma : J^m \to \mathbb{C}\{z\}$, defined on $j^m(\mathrm{Hol}\,(\mathbb{C}, 0))$, that assigns to a m-jet its unique representative consisting of a polynomial of degree (at most) m. In other words, $\sigma(j^m(\sum_{i=1}^{\infty} c_i z^i)) = \sum_{i=1}^{m} c_i z^i$. The space J^m is naturally identified to $\mathbb{C}^m$ and it is endowed with the standard topology. Nonetheless let us consider on J^m the norm defined by $\|j^m(\sum_{i=1}^{\infty} c_i z^i)\| = \max_{i=1,\ldots,m}\{|c_i|^{1/i}\}$ instead of the more common Euclidean norm. Finally let $\mathrm{Hol}\,(\mathbb{C}, 0)$ be equipped with the distance d_A associated to the analytic topology.

Proposition 2.3. *The map $j^m : \mathrm{Hol}\,(\mathbb{C}, 0) \to J^m$ is continuous and open. Besides the section σ yields an isometric embedding of J^m in $\mathrm{Hol}\,(\mathbb{C}, 0)$ in the sense that it satisfies $d_A(\sigma(j^m f), \sigma(j^m g)) = \|j^m f - j^m g\|$.*

Proof. The fact that σ is an isometry as indicated is clear from the definitions of d_A and of $\| \, \|$. Also, these same definitions provide the general estimate

$$\|j^m f - j^m g\| \le d_A(f, g) \tag{3}$$

for every pair $f, g \in \mathrm{Hol}\,(\mathbb{C}, 0)$. The continuity of j^m follows at once whereas the continuity of σ follows from its isometric nature. It remains only to check that j^m is an open map. For this let $f \in \mathrm{Hol}\,(\mathbb{C}, 0)$ and $R \in (0, 1)$ be given and denote by $B_{d_A}(f, R) \subset \mathrm{Hol}\,(\mathbb{C}, 0)$ the (open) ball of center f and radius R (with respect to the metric d_A). Estimate (3) then shows that $j^m(B_{d_A}(f, R))$ is contained in the ball $B_{\| \|}(j^m f, R) \subset J^m$ of center $j^m f$ and radius R (w.r.t. the metric associated to the norm $\| \, \|$). Conversely, the isometric nature of σ ensures that $\sigma(B_{\| \|}(j^m f, R)) \subset B_{d_A}(f, R)$ so that $j^m(B_{d_A}(f, R)) = B_{\| \|}(j^m f, R)$ showing that j^m is open as desired. $\square$

Remark 2.4. For $\alpha \ge 1$ given, the same proof above also shows that the restriction of j^m to $\mathrm{Diff}_{\alpha}(\mathbb{C}, 0) \subset \mathrm{Hol}\,(\mathbb{C}, 0)$ is clearly continuous. Moreover this map takes values on the subset of J^m consisting of those jets that are tangent to the identity to order α (assuming that $m > \alpha$ for otherwise the map is constant). The latter set can, in turn, be identified to $J^{m-\alpha}$ and the restriction of j^m to $\mathrm{Diff}_{\alpha}(\mathbb{C}, 0)$ will still admit an isometric section defined on $J^{m-\alpha}$. In particular, with these identifications, the map j^m from $\mathrm{Diff}_{\alpha}(\mathbb{C}, 0)$ to $J^{m-\alpha}$ is also open.

2.2. **Strategy of proof.** In this paragraph, we shall briefly describe our strategy for proving the statement of Theorem A *at germ level*. The argument will further be developed in the next section. The structure of the proof will also be followed, to a good extent, to settle the remainder of the statement of Theorem A. Let then $f_1, \ldots, f_k$ be elements in $\mathrm{Diff}\,(\mathbb{C}, 0)$ and denote by G_i the cyclic group generated by f_i. As previously mentioned, Section 3 will be devoted to showing that, for a generic choice of $h_1, \ldots, h_k$, the subgroup of $\mathrm{Diff}\,(\mathbb{C}, 0)$ (viewed as germs of holomorphic diffeomorphisms) generated by $h_1^{-1} \circ f_1 \circ h_1, \ldots, h_k^{-1} \circ f_k \circ h_k$ is isomorphic to the free product $G_1 * \cdots * G_k$. The part of Theorem A involving pseudogroups will be deferred to Section 4 since it requires a more detailed discussion which may be skipped by readers who are only interested in groups of converging power series. Also, in both Sections 3 and 4, we shall content ourselves of dealing with the case $k = 2$ to avoid needlessly cumbersome notations. The passage from $k = 2$ to the general case does not pose any new difficulty as the reader will not fail to notice.

This said, let $f, g \in \text{Diff}\,(\mathbb{C}, 0)$ be two holomorphic diffeomorphisms fixing the origin of $\mathbb{C}$ and assume that both f, g are distinct from the identity. Denote by r (resp. s) the order of f (resp. g), namely $\mathsf{r} \in \mathbb{N}^*$ (resp. $\mathsf{s} \in \mathbb{N}^*$) is the smallest strictly positive integer for which $f^{\mathsf{r}} = \text{id} \in \text{Diff}\,(\mathbb{C}, 0)$ (resp. $g^{\mathsf{s}} = \text{id} \in \text{Diff}\,(\mathbb{C}, 0)$). If r (resp. s) does not exist, then the order of f (resp. g) is said to be ∞. We shall write $\mathsf{r} = \infty$ (resp. $\mathsf{s} = \infty$) to refer to the latter case and $\mathsf{r} < \infty$ (resp. $\mathsf{s} < \infty$) to indicate the former one. If r (resp. s) equals ∞, then, by convention, $\mathbb{Z}/\mathsf{r}\mathbb{Z}$ (resp. $\mathbb{Z}/\mathsf{s}\mathbb{Z}$) is isomorphic to $\mathbb{Z}$.

With the previous notations, let us consider the free product $\mathbb{Z}/\mathsf{r}\mathbb{Z}*\mathbb{Z}/\mathsf{s}\mathbb{Z}$ between $\mathbb{Z}/\mathsf{r}\mathbb{Z}$ and $\mathbb{Z}/\mathsf{s}\mathbb{Z}$. In terms of presentation, this group is isomorphic to the group defined by $\langle a, b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \text{id}\rangle$, where the relation $a^{\mathsf{r}} = \text{id}$ (resp. $b^{\mathsf{s}} = \text{id}$) *is understood to be void if* $\mathsf{r} = \infty$ *(resp.* $\mathsf{s} = \infty$*)*. In other words, we keep the convention $a^{\infty} = b^{\infty} = \text{id}$. In terms of the mentioned presentation, a *reduced word* in the letters a, b (sometimes also said in the letters a, a^{-1}, b, b^{-1}) is a word $W(a, b)$ whose spelling has the form $\vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$ with the following rules being respected:

1. ϑ_i takes on the values $\{a, b\}$.
2. If ϑ_{i_0} takes on the value a (resp. b) then ϑ_{i_0-1} and ϑ_{i_0+1} take on the value b (resp. a) provided that $i_0 - 1$ and $i_0 + 1$ are defined.
3. If ϑ_i takes on the value a, then r_i takes values in the set $\{1, \ldots, \mathsf{r} - 1\}$ provided that $\mathsf{r} < \infty$. If $\mathsf{r} = \infty$, then r_i takes values in $\mathbb{Z}^*$ (it is understood that, for $r_i < 0$, a^{r_i} means $(a^{-1})^{|r_i|}$).
4. Similarly, if ϑ_i takes on the value b, then r_i takes values in the set $\{1, \ldots, \mathsf{s} - 1\}$ provided that $\mathsf{s} < \infty$. If $\mathsf{s} = \infty$, then r_i takes values in $\mathbb{Z}^*$ (where b^{r_i} means $(b^{-1})^{|r_i|}$ whenever $r_i < 0$).

Remark 2.5. With the above definitions, note that talking about reduced words only makes sense with a previously fixed (finitely presented) group in the background.

As mentioned in the Introduction, the interest of considering reduced words lies in the fact that every element in the free product $\langle a, b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \text{id}\rangle$ is represented by a *unique reduced word* $W(a, b)$ (by convention the neutral element corresponds to the empty word).

Now let us go back to the initially chosen local diffeomorphisms $f, g \in \text{Diff}\,(\mathbb{C}, 0)$ generating a subgroup of $\text{Diff}\,(\mathbb{C}, 0)$ denoted by G. Every (reduced) word $W(a, b)$ induces an element of G by means of the substitutions $f \mapsto a$ and $g \mapsto b$. If $W(a, b)$ is fixed, the element of G obtained through these substitutions is going to be denoted by $W(f, g)$. Furthermore, the assignment of $W(f, g)$ to $W(a, b)$ actually induces a homomorphism from $\langle a, b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \text{id}\rangle$ to $\text{Diff}\,(\mathbb{C}, 0)$ which was denoted by $\mathcal{E}$.

In the sequel, fixed the group $\mathbb{Z}/\mathsf{r}\mathbb{Z} * \mathbb{Z}/\mathsf{s}\mathbb{Z} = \langle a, b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \text{id}\rangle$, every word $W(a, b)$ considered *is supposed to be non-empty and reduced unless otherwise stated.* Recall from the Introduction that this group is also isomorphic to the quotient of the free group F_2 on two generators a, b by the normal subgroup generated by a^{r}, b^{s}, where it is understood that both $a^{\infty} = b^{\infty} = \text{id}$, in other words, there is not "a" (resp. "b") if $\mathsf{r} = \infty$ (resp. $\mathsf{s} = \infty$).

The problem that needs to be considered is as follows. Assume we are given two holomorphic diffeomorphisms f, g and a (non-empty reduced as it will always be the case) word $W(a, b)$ such that $W(f, g) = \text{id}$ (at level of germs). Recalling that $\text{Diff}_{\alpha}(\mathbb{C}, 0) \times \text{Diff}_{\alpha}(\mathbb{C}, 0) =$

$(\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ is equipped with the product analytic topology, we want to show the existence of an open dense set $\mathcal{U}_W \subset (\mathrm{Diff}\,(\mathbb{C},0))^2$ so that, whenever $(h_1,h_2) \in \mathcal{U}_W$, the pair $h_1^{-1} \circ f \circ h_1$, $h_2^{-1} \circ g \circ h_2$ satisfies $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2) \neq \mathrm{id}$, where $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ stands for the element of $\mathrm{Diff}\,(\mathbb{C},0)$ obtained through the substitutions $a^{\pm 1} = h_1^{-1} \circ f^{\pm 1} \circ h_1$ and $b^{\pm 1} = h_2^{-1} \circ g^{\pm 1} \circ h_2$. In the next section, the following will be proved:

Proposition 2.6. *Suppose that $W(a,b)$ is a (non-empty reduced) word in a,b. Suppose also that f,g are given elements of* $\mathrm{Diff}\,(\mathbb{C},0)$ *of orders respectively equal to r, $\mathsf{s} \in \mathbb{N}^* \cup \{\infty\}$. Then, for all $\alpha \in \mathbb{N}$, there exists an open dense set $\mathcal{U}_W \subset (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ such that $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2) \neq \mathrm{id} \in \mathrm{Diff}\,(\mathbb{C},0)$ for every $(h_1,h_2) \in \mathcal{U}_W$.*

The above proposition holds at germ level or, equivalently, in terms of power series based at $0 \in \mathbb{C}$. A version of it for pseudogroup, which is required for the full statement of Theorem A, will be worked out in Section 4. Proposition 2.6 also yields the following result.

Theorem 2.7. *Suppose we are given f, $g \in \mathrm{Diff}\,(\mathbb{C},0)$ of orders respectively equal to r, $\mathsf{s} \in \mathbb{N}^* \cup \{\infty\}$ (with the preceding terminology). Given $\alpha \in \mathbb{N}$ let $(\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ be endowed with the product analytic topology. Then there exists a G_δ-dense set $\mathcal{U} \subseteq (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ such that, whenever $(h_1,h_2) \in \mathcal{U}$, the pair of diffeomorphisms $h_1^{-1} \circ f \circ h_1$, $h_2^{-1} \circ g \circ h_2$ generates a subgroup of* $\mathrm{Diff}\,(\mathbb{C},0)$ *isomorphic to the free product $\langle a,b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \mathrm{id}\rangle$.*

Proof. Let $\alpha \in \mathbb{N}$ be fixed. In particular $\mathrm{Diff}\,(\mathbb{C},0)$ corresponds to $\alpha = 0$. Consider a word $W(a,b)$ in the letters a,b. According to Proposition 2.6, there is an open dense set $\mathcal{U}_W \subset (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ such that $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2) \neq \mathrm{id} \in \mathrm{Diff}\,(\mathbb{C},0)$ whenever $(h_1,h_2) \in \mathcal{U}_W$. Next, let us form the intersection

$$\mathcal{U} = \bigcap_{\substack{W(a,b)\,;\,W(a,b)\ \text{non-empty}\\ \text{and reduced}}} \mathcal{U}_W\,.$$

Since there are only countably many reduced words $W(a,b)$ in the letters a,b, the Baire property of the analytic topology guarantees that $\mathcal{U}$ is dense (in particular, it is not empty). Besides, if $(h_1,h_2) \in \mathcal{U}$, then for every reduced word $W(a,b)$ as above the germ of diffeomorphism induced by $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ at $0 \in \mathbb{C}$ is different from the identity. In other words, the kernel of the homomorphism $\mathcal{E}$ from $\langle a,b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \mathrm{id}\rangle$ to $\mathrm{Diff}\,(\mathbb{C},0)$ that associates to an (non-empty, reduced) word $W(a,b)$ the element $W(h_1^{-1} \circ f \circ h_1, h_2^{-1} \circ f \circ h_2)$ is trivial. The statement then follows at once. □

Remark 2.8. The analytic topology considered in this section is certainly very strong. Rather than a drawback, this becomes of an advantage for the statement of Theorem A since it allows us to derive the "generic" character of the corresponding local diffeomorphisms $h_1, \ldots, h_k$ in several other contexts. In particular, a general useful remark concerning Proposition 2.6 is as follows. Note that when a formal relation in $\widehat{\mathrm{Diff}}\,(\mathbb{C},0)$ is broken, this fact can be read off from a finite (and thus converging) part of the corresponding series. Fix then a large K and consider the corresponding equation in the first K coefficients $a_1 z + \cdots + a_K z^K$. Identifying $a_1 z + \cdots + a_K z^K$ to $\mathbb{C}^K$ in the obvious way, it follows that the set of coefficients $(a_1, \ldots, a_K)$ that do break the relation in question is a *non-empty Zariski-open set of* $\mathbb{C}^K$. In particular, it is automatically dense and is also large in the sense of measure (its complement has zero Lebesgue measure). It is by exploiting this idea that the generic character of the

k-tuples $(h_1, \ldots, h_k)$ in the sense of Baire (for the analytic topology) yields their generic character in other natural settings as well. These issues will further be detailed in the next section, cf. Lemma 3.1.

3. Destroying relations for groups of germs

The purpose of this section is to prove Proposition 2.6. As mentioned, this proposition concerns the case of only two generators in the statement of Theorem A. The adaptations to larger number of generators being straightforward, they will be left to the reader. In what follows, we therefore consider a pair of local diffeomorphisms f, g as in the statement of Proposition 2.6.

Recall that, unless otherwise stated, all words $W(a, b)$ are supposed to be non-empty and reduced with respect to the group $\langle a, b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \mathrm{id}\rangle$. Let a word $W(a, b)$ be fixed once and for all. Following the notations of Section 2, we let $W(a, b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$ according to the rules (1) through (4) in Section 2. In view of the definition of the orders r, s of f, g, respectively, there is nothing to be proved if $l = 1$. Therefore, we assume without loss of generality that $l \geq 2$. In other words, if ϑ is thought of as taking on the values $\{a, b\}$ (where negative exponents are allowed when r or s equals ∞), then both sets $\{a\}$ and $\{b\}$ effectively appear in the spelling of $W(a, b)$.

Consider the context of Proposition 2.6. We have a fixed word $W(a, b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$, $l \geq 2$. Also $\alpha \in \mathbb{N}$ is fixed and $(\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ is endowed with the (product) analytic topology. Let $\mathcal{U}_W \subset (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ be the set of pairs $(h_1, h_2) \in (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ for which $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2) \neq \mathrm{id} \in \mathrm{Diff}\,(\mathbb{C}, 0)$ (i.e. at germ level). The lemma below provides a rather convenient reduction in the proof of Proposition 2.6 and goes along the ideas proposed in Remark 2.8.

Lemma 3.1. *With the above notations, the set* $\mathcal{U}_W \subset (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ *is either empty or open and dense in* $(\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$.

Proof. Let us consider the case $\alpha = 0$. Recall that f, g are fixed. Given $N > 0$, the N-jet of $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ at $0 \in \mathbb{C}$ depends solely on the N-jets of h_1, h_2 at $0 \in \mathbb{C}$. Therefore the map from $(\mathrm{Diff}\,(\mathbb{C}, 0))^2$ to J^N that assigns to a pair $(h_1, h_2) \in (\mathrm{Diff}\,(\mathbb{C}, 0))^2$ the N-jet at $0 \in \mathbb{C}$ of $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ induces a well-defined map $T_W^N : J^N \times J^N \to J^N$ by means of the formula

$$T_W^N(j^N h_1, j^N h_2) = j^N(W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2))\,.$$

Moreover the map T_W^N has polynomial coordinates. Hence, the pre-image $(T_W^N)^{-1}[J^N \setminus \{j^N z\}]$ of the complement in J^N of the N-jet associated to the identity is a Zariski-open subset of $J^N \times J^N$. In particular, $(T_W^N)^{-1}[J^N \setminus \{j^N z\}]$ is either empty or open and dense in $J^N \times J^N$.

Suppose now that $\mathcal{U}_W \subset (\mathrm{Diff}\,(\mathbb{C}, 0))^2$ is not empty. Thus, for sufficiently large N, $(T_W^N)^{-1}[J^N \setminus \{j^N z\}] \neq \emptyset$. From now on, let N be a fixed integer for which $(T_W^N)^{-1}[J^N \setminus \{j^N z\}] \neq \emptyset$. Therefore $(T_W^N)^{-1}[J^N \setminus \{j^N z\}]$ is open and dense in $J^N \times J^N$. However, in view of Proposition 2.3 the map $\tilde{j}^N : (\mathrm{Diff}\,(\mathbb{C}, 0))^2 \to J^N \times J^N$ defined by

$$\tilde{j}^N(h_1, h_2) = (j^N h_1, j^N h_2)$$

is continuous and open. Being continuous the pre-image $(\tilde{j}^N)^{-1}((T_W^N)^{-1}[J^N \setminus \{j^N z\}]) \subset (\mathrm{Diff}\,(\mathbb{C}, 0))^2$ is open since $(T_W^N)^{-1}[J^N \setminus \{j^N z\}]$ is open in $J^N \times J^N$. Furthermore, since $\tilde{j}^N$

is an open map and $(T_W^N)^{-1}[J^N \setminus \{j^N z\}]$ is dense in $J^N \times J^N$, it follows that the pre-image $(\tilde{j}^N)^{-1}((T_W^N)^{-1}[J^N \setminus \{j^N z\}])$ is also dense in $(\mathrm{Diff}\,(\mathbb{C},0))^2$. The lemma is proved for $\alpha = 0$.

The general case is totally analogous with slightly heavier notations, cf. Remark 2.4. Details are left to the reader.

□

Remark 3.2. In the above lemma, the assumption that $\mathcal{U}_W$ is not empty was exploited to conclude that $\mathcal{U}_W$ is, actually, dense. In other words, assuming that a "relation" is not *always* satisfied, then it can be broken by elements in an open and dense set. Whereas our assumption was the existence of a pair of elements in $\mathrm{Diff}\,(\mathbb{C},0)$ "breaking" the relation in question, the reader will easily check that the existence of a pair of merely formal diffeomorphism would suffice. This seems to indicate that another possible use of the analytic topology, not developed in this work, concerns the possibility of turning (faithful) representations of group in $\widehat{\mathrm{Diff}}\,(\mathbb{C},0)$ into (faithful) representations in $\mathrm{Diff}\,(\mathbb{C},0)$. In other words, when it is possible to "break up" a relation at formal level, it should also be possible to do it with convergent power series. Moreover, the elements of $\mathrm{Diff}\,(\mathbb{C},0)$ breaking the relation must also form a dense set for the analytic topology. Though this remark will not be exploited in this paper, it may have non-trivial implications in view of the papers [E-V], [N-Y] and their interesting results about representations of groups in $\widehat{\mathrm{Diff}}\,(\mathbb{C},0)$.

Thanks to Lemma 3.1, the proof of Proposition 2.6 is reduced to show that $\mathcal{U}_W \subset (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ is not empty whenever $W(a,b)$ is a word as in the statement of the proposition in question. In fact, a word $W(a,b)$ for which $\mathcal{U}_W \subset (\mathrm{Diff}_\alpha(\mathbb{C},0))^2 = \emptyset$ may be named a *universal relation (w.r.t. α)*. Hence, the preceding can be rephrased by saying that the proof of Proposition 2.6 amounts to showing that there is no universal relation $W(a,b)$ regardless of the fixed value of α. Reminding the reader that all words are supposed to be non-empty and reduced, we state:

Lemma 3.3. *With the preceding notations, and for arbitrarily given $\alpha \in \mathbb{N}$, no word $W(a,b)$ represents a universal relation in* $\mathrm{Diff}_\alpha(\mathbb{C},0)$.

To begin the discussion, consider the spelling of $W(a,b)$ in the form $\vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$, $l \geq 1$, as indicated in the end of Section 2. The proof that $\mathcal{U}_W \neq \emptyset$ will be carried out by finding an element in $\mathcal{U}_W$ having either the form (h, id) or the form (id, h), depending on the spelling of $W(a,b)$ (where h belongs to $\mathrm{Diff}_\alpha(\mathbb{C},0)$). Recall that every element different from the identity in $\langle a,b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \mathrm{id}\rangle$ has a unique representative in the form of a word $W(a,b)$. The value of $l \geq 1$ in the spelling $W(a,b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$ is going to be called the *length* of $W(a,b)$.

Before starting the proof of Lemma 3.3, let us indicate some normalizations that will be assumed to hold throughout the rest of the section. These conditions will further be developed in Section 4 where we shall deal with pseudogroups defined on a fixed open set.

Let then $W(a,b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$ be a fixed word and consider, for every $j \in \{1,\ldots,l\}$ and every $e(j) \in \{1,\ldots,r_j\}$, the word $W_j^{e(j)}(a,b) = \vartheta_j^{e(j)} * \vartheta_{j-1}^{r_{j-1}} * \cdots * \vartheta_1^{r_1}$ along with the corresponding local diffeomorphism $W_j^{e(j)}(f,g)$. Modulo re-scaling coordinates and choosing a sufficiently small *connected* neighborhood U of $0 \in \mathbb{C}$, the following conditions necessarily hold:

C.1: f, g and their inverses are one-to-one maps defined on the unit disc $B(1) \subset \mathbb{C}$.

C.2: For every j and every $e(j)$ as above, the local diffeomorphism $W_j^{e(j)}(f,g)$ is defined on U.

C.3: For every j and every $e(j)$ as above, $W_j^{e(j)}(f,g)(U) \subset B(1)$.

The proof of Lemma 3.3 will be carried out by induction on the length l of the words $W(a,b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$. Clearly no universal relation of length 1 may exist since identities of the form $f^m = \mathrm{id}$ or $g^m = \mathrm{id}$ are codified in the very definition of the orders r and s, and hence are automatically considered in the definition of the group $\langle a, b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \mathrm{id}\rangle$. The next lemma shows the non-existence of universal relations with length 2. Note that this statement is not formally indispensable in the sense that the induction argument can be initialized with words of length 1. It appears, however, that working out the details in the case of words with length 2 helps to clarify the ideas which can then quickly be generalized to handle words of arbitrary length.

Lemma 3.4. *No word of length two, $W(a,b) = \vartheta_2^{r_2} * \vartheta_1^{r_1}$, is a universal relation in* $\mathrm{Diff}_\alpha(\mathbb{C},0)$.

Proof. Let us consider the element of $\mathrm{Diff}\,(\mathbb{C},0)$ defined by $W(f,g)$. Modulo permuting the roles of f,g and also permuting them with their inverses, there is not loss of generality in supposing that $W(a,b) = g^{r_2} \circ f^{r_1}$ with $r_1, r_2 > 0$. In fact, as just indicated, the remaining combination are totally analogous. By assumption $r_1 < \mathsf{r}$ and $r_2 < \mathsf{s}$. In particular, for $n \in \{1, \ldots, r_1\}$, the local diffeomorphism f^n does not coincide with the identity in $\mathrm{Diff}\,(\mathbb{C},0)$. Similarly, for $m \in \{1, \ldots, r_2\}$, g^m does not coincide with the identity in $\mathrm{Diff}\,(\mathbb{C},0)$.

Let a neighborhood U as in Conditions C.1, C.2 and C.3 be fixed. Let h be a local holomorphic diffeomorphism, tangent to the identity to order α and yielding an one-to-one map defined on the disc $B(2)$ of radius 2. Set $\tilde{g} = h^{-1} \circ g \circ h \in \mathrm{Diff}\,(\mathbb{C},0)$ and set $\tilde{g}^j(z) = (h^{-1} \circ g \circ h)^j(z)$ for points z for which the local diffeomorphism $h^{-1} \circ g \circ h$ can be iterated in the natural sense (more details on these notions reminiscent from general pseudogroups can be found in Section 4). Suppose also that $\sup_{z \in B(2)} \|h(z) - z\| < \epsilon$. Then, if ϵ is sufficiently small, the following holds:

- For every $j \in \{1,2\}$ and $e(j) \in \{1, \ldots, r_j\}$, $W_j^{e(j)}(f,\tilde{g})$ is defined on U.
- For every $j \in \{1,2\}$ and $e(j) \in \{1, \ldots, r_j\}$, $W_j^{e(j)}(f,\tilde{g})(U) \subset B(1)$.
- Both h, h^{-1} are defined and are one-to-one on $B(1)$.

Suppose that h as above is such that $\tilde{g}^{r_2} \circ f^{r_1}(p) \neq p$ for some point $p \in U$. Then $\tilde{g}^{r_2} \circ f^{r_1}$ does not coincide with the identity on U. Recalling that U is connected, it follows that $\tilde{g}^{r_2} \circ f^{r_1}$ does not represent the identity in $\mathrm{Diff}\,(\mathbb{C},0)$ (i.e. at level of germs). Thus $W(a,b)$ is not a universal relation in $\mathrm{Diff}_\alpha(\mathbb{C},0)$ since the pair (id, h) belongs to the corresponding set $\mathcal{U}_W$.

Summarizing what precedes, to prove the lemma, it suffices to find an element $h \in \mathrm{Diff}_\alpha(\mathbb{C},0)$, arbitrarily close to the identity on $B(2)$ and such that, whenever defined, $\tilde{g}^{r_2} \circ f^{r_1}$ does not coincide with the identity on U. To do this, we can suppose that $g^{r_2} \circ f^{r_1}$ does coincide with the identity on U, otherwise simply take $h = \mathrm{id}$.

To "break up" the relation $g^{r_2} \circ f^{r_1} = \mathrm{id} \in \mathrm{Diff}\,(\mathbb{C},0)$, we proceed as follows. Let $z_0 \in U$ be such that $f^{r_1}(z_0) = z_1 \neq z_0$. Note that z_0 exists for f^{r_1} cannot coincide with the identity on U since $r_1 \in \{1, \ldots, \mathsf{r}-1\}$. Thus $g^{r_2}(z_1) = z_0$. We are going to construct local diffeomorphism $h \in \mathrm{Diff}_\alpha(\mathbb{C},0)$, arbitrarily close to the identity on $B(2)$, and such that $\tilde{g}^{r_2} \circ f^{r_1}(z_0) \neq z_0$, where $\tilde{g} = h^{-1} \circ g \circ h$. The lemma will then immediately follows. To construct h consider a polynomial P such that $P(z_0) = 0$ and $P(z_1) \neq 0$. Next for $t \in (0,1)$, let $h_t(z) = z + tz^{\alpha+1}P(z)$. In particular $h_t(0) = 0$ for every $t \in [0,1]$ and, in fact, for every $t \in [0,1]$, h_t lies in $\mathrm{Diff}_\alpha(\mathbb{C},0)$. Besides, for t sufficiently small, both h_t is one-to-one on a neighborhood of $B(1)$ since it is close to the identity map. From this we conclude that the inverse h_t^{-1} of h_t is also defined and one-to-one on neighborhood of $B(1)$ (modulo reducing t). Furthermore it is also clear that, indeed, h_t converges uniformly to the identity on $B(2)$ as $t \to 0$. Setting $\tilde{g}_t = h_t^{-1} \circ g \circ h_t$, the lemma is reduced to proving the following claim:

Claim. For $t > 0$ sufficiently small $\tilde{g}_t^{r_2} \circ f^{r_1}(z_0) \neq z_0$.

Proof of the Claim. Clearly $\tilde{g}_t^{r_2} \circ f^{r_1}(z_0) = \tilde{g}_t^{r_2}(z_1)$. However, for t small enough, $\tilde{g}_t^{r_2}(z_1) = h_t^{-1} \circ g^{r_2} \circ h_t(z_1)$ and, on the other hand $h_t(z_1) \neq z_1$. Besides, modulo reducing t, $h_t(z_1)$ and $\tilde{z} = g^{r_2} \circ h_t(z_1)$ lies in $B(1)$. However, $\tilde{z} \neq z_0$. Indeed, since $h_t(z_1) \neq z_1$ are both contained in $B(1)$ where g^{r_2} is one-to-one, it follows that $\tilde{z} = g^{r_2} \circ h_t(z_1) \neq z_0 = g^{r_2}(z_1)$. Thus, to deduce the claim, it suffices to check that $h_t^{-1}(\tilde{z}) \neq z_0$. For this, recall again that both $z_0, \tilde{z}$ belong to $B(1)$ where h_t^{-1} is one-to-one. Therefore we have $h_t^{-1}(\tilde{z}) \neq h_t^{-1}(z_0)$ since $\tilde{z} \neq z_0$. Finally, by construction, $h_t^{-1}(z_0) = z_0$ so that we obtain $h_t^{-1}(\tilde{z}) \neq h_t^{-1}(z_0) = z_0$ implying the claim. □

The proof of the lemma is also over. □

The preceding lemma contains the basic ideas that will be used in the general proof of Lemma 3.3, and hence of Proposition 2.6. Given Lemma 3.3, let us suppose by induction that no word of length $1, 2, \ldots, l-1$ represents a universal relation in $\mathrm{Diff}_\alpha(\mathbb{C},0)$ (where α is fixed). By relying in this assumption, we must conclude that no word of length l may represent a universal relation in $\mathrm{Diff}_\alpha(\mathbb{C},0)$ either.

Let $W(a,b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$ be a (reduced as always) word of length l. The local diffeomorphism $W(f,g)$ will then be denoted by $W(f,g) = F_l^{r_l} \circ \cdots \circ F_1^{r_1}$, where F_i takes on the value f (resp. g) if ϑ_i takes on the value a (resp. b). Assume that $W(f,g) = \mathrm{id} \in \mathrm{Diff}\,(\mathbb{C},0)$. Let us also suppose for a contradiction that $W(a,b)$ represents a universal relation in $\mathrm{Diff}_\alpha(\mathbb{C},0)$. Given a point $z \in \mathbb{C}$ sufficiently close to $0 \in \mathbb{C}$, by the *itinerary* of z under $W(f,g)$, it is meant the sequence of points $z = z_0, z_1, \ldots, z_l$ obtained as follows: first $z_0 = z$. Besides, if z_i is defined for $i = 0, \ldots, l-1$, then $z_{i+1} = F_{i+1}^{r_{i+1}}(z_i)$. This sequence of points is clearly well-defined for z sufficiently close to $0 \in \mathbb{C}$. Moreover, the fact that $W(f,g)$ represents the identity in $\mathrm{Diff}\,(\mathbb{C},0)$ ensures that $z_l = z_0$. Now we have:

Lemma 3.5. *Without loss of generality, we can assume that the points $z_0, \ldots, z_{l-1}$ are pairwise distinct provided that $z_0 \neq 0$.*

Proof. Consider the local diffeomorphisms associated to the sub-words $W_{j_1,j_2}(f,g) = F_{j_2}^{r_{j_2}} \circ \cdots \circ F_{j_1}^{r_{j_1}}$ with $j_1 \leq j_2$ and j_1, j_2 in $\{1, \ldots, l-1\}$. Since all these words have length at most $l-1$, the induction assumption allows us to suppose that none of them represents the identity in $\mathrm{Diff}\,(\mathbb{C},0)$. In fact, to check the claim note that $W(f,g)$ must be a universal relation for every pair $(\tilde{f}, \tilde{g})$ having the form $\tilde{f} = h_1^{-1} \circ f \circ h_1$ and $\tilde{g} = h_2^{-1} \circ f \circ h_2$, with $h_1, h_2 \in \mathrm{Diff}_\alpha(\mathbb{C},0)$. On the other hand, none of the previously considered words are universal relations for (f,g) so that, for each of them, we can find an open and dense subset of $(\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ whose elements break the corresponding word. By intersecting these finitely many open dense sets, we find elements $(\tilde{h}_1, \tilde{h}_2) \in (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ so that none of the above words $W_{j_1,j_2}(\tilde{h}_1^{-1} \circ f \circ \tilde{h}_1,\, \tilde{h}_2^{-1} \circ f \circ \tilde{h}_2)$ represents the identity in $\mathrm{Diff}\,(\mathbb{C},0)$. Finally, all that need to be done is to substitute f, g by $\tilde{f} = \tilde{h}_1^{-1} \circ f \circ \tilde{h}_1$, $\tilde{g} = \tilde{h}_2^{-1} \circ f \circ \tilde{h}_2$. In other words, we can suppose without loss of generality that no sub-word $W_{j_1,j_2}(f,g)$ represents the identity in Diff$(\mathbb{C},0)$.

Next, modulo reducing U in the normalizations C.1, C.2 and C.3, all these maps are defined on U. Thus the solutions of $W_{j_1,j_2}(f,g)(z) = z$ in U are isolated points. Hence, if $V \subset U$ is very small, then $W_{j_1,j_2}(f,g)(z) \neq z$ for every $z \neq 0$ in V. It is now clear that every point $z_0 \neq 0$ in V has itinerary $z_0, \ldots, z_{l-1}, z_l$ where the points $z_0, \ldots, z_{l-1}$ are pairwise distinct as desired. $\square$

We are now ready to prove Lemma 3.3 what, in turn, completes the proof of Proposition 2.6.

Proof of Lemma 3.3. Let $W(f,g)$ be obtained as before by means of a word $W(a,b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$ of length l. To fix notations, we assume without loss of generality that ϑ_l takes on the value b.

Recall that conditions C.1, C.2 and C.3 are supposed to be verified. By using Lemma 3.5, every point $z_0 \in U \setminus \{0\}$ sufficiently close to $0 \in \mathbb{C}$ has itinerary $z_0, \ldots, z_{l-1}, z_l$ such that the points $z_0, \ldots, z_{l-1}$ are pairwise distinct (naturally $z_l = z_0$, otherwise there is nothing to be proved). Let then one of these points z_0 be fixed.

Again, given a local diffeomorphism $h \in \mathrm{Diff}_\alpha(\mathbb{C},0)$, defined on $B(2)$, set $\tilde{g} = h^{-1} \circ g \circ h$. There is $\epsilon > 0$ such that, whenever $\sup_{B(2)} \|h(z) - z\| < \epsilon$, the following conditions hold:

C'.1: For every $j, e(j)$ as before, $W_j^{e(j)}(f, \tilde{g})$ is defined on U.

C'.2: For every $j, e(j)$ as before, $W_j^{e(j)}(f, \tilde{g})(U) \subset B(1)$.

C'.3: Both h, h^{-1} are defined and are one-to-one on $B(1)$.

Consider now a polynomial P such that $P(z_0) = P(z_1) = \cdots = P(z_{l-2}) = 0$ and $P(z_{l-1}) \neq 0$. Again, for $t \in [0,1]$, let h_t be defined by $h_t(z) = z + tz^{\alpha+1}P(z)$ so that $h_t \in \mathrm{Diff}_\alpha(\mathbb{C},0)$ for every $t \in [0,1]$ (and in particular $h_t(0) = 0$). For sufficiently small $t > 0$, it is clear that $\sup_{B(2)} \|h_t(z) - z\| < \epsilon$ so that Conditions C'.1, C'.2 and C'.3 will be satisfied for $\tilde{g}_t = h_t^{-1} \circ g \circ h_t$. As before, to finish the proof of the lemma, it suffices to prove the following claim:

Claim. For sufficiently small $t > 0$, we have $W(f, \tilde{g}_t)(z_0) \neq z_0$.

Proof of the Claim. Consider the itinerary of z_0 under $W(f,g)$ written as $z_0, z_1 \ldots, z_{l-1}, z_l = z_0$. The construction of P makes it clear that the itinerary of z_0 under $W(f, \tilde{g}_t)$ can similarly be written under the form $z_0, \ldots, z_{l-1}, (\tilde{g}_t)^{r_l}(z_{l-1})$. Therefore, it suffices to prove that $(\tilde{g}_t)^{r_l}(z_{l-1}) \neq z_0$. For this recall that, for what concerns $W(f,g)$, we have $z_l = z_0 = g^{r_l}(z_{l-1})$. Now for t very small, the points $h_t(z_{l-1})$ and $g^{r_l} \circ h_t(z_{l-1})$ belong to $B(1)$ where h_t^{-1} is injective. Thus, since $h_t^{-1}(z_0) = z_0$, to conclude that $(\tilde{g}_t)^{r_l}(z_{l-1}) \neq z_0$ it suffices to check that $g^{r_l} \circ h_t(z_{l-1}) \neq z_0$. However, g^{r_l} is still one-to-one on $B(1)$. Since both z_{l-1} and $h_t(z_{l-1})$ belong to $B(1)$, the fact that $z_{l-1} \neq h_t(z_{l-1})$ ensures that $g^{r_l} \circ h_t(z_{l-1}) \neq g^{r_l}(z_{l-1}) = z_0$. The claim is proved. $\square$

The proofs of Lemma 3.3 and of Proposition 2.6 are now comppleted. $\square$

4. Proof of Theorem A

The proof of Theorem A is going to be completed in this section. The fundamental object involved in the subsequent discussion is the notion of pseudogroup generated by local diffeomorphisms f and g about $0 \in \mathbb{C}$. In fact, already an intrinsic difficulty arising from dealing with pseudogroups, as opposed to groups of germs, has to do with the following fact: while in the proof of Proposition 2.6 we were allowed to reduce neighborhoods and choose z_0 very close to $0 \in \mathbb{C}$, in what follows all neighborhoods will be fixed and we shall need to work with points that are "far from $0 \in \mathbb{C}$" in a sense to be made accurate. It is then natural to use the setting provided by pseudogroups.

Let us begin by recalling the notion of *pseudogroup* as it will be needed for our discussion. Consider local diffeomorphisms f, f^{-1}, g, g^{-1} that are defined and one-to-one on an open disc D of $0 \in \mathbb{C}$. We want to consider the *pseudogroup* $\Gamma = \Gamma(f, g, D)$ *generated by* f, f^{-1}, g, g^{-1} *on* D (in the sequel this pseudogroup will be referred to as being generated by f, g and their inverses or simply by f, g, when no confusion is possible). Let us make the definition of Γ precise. Recall that $\mathsf{r} \in \mathbb{N}^* \cup \{\infty\}$ (resp. $\mathsf{s} \in \mathbb{N}^* \cup \{\infty\}$) stands for the order of f (resp. g) and that words are always reduced with respect to the group $\langle a, b\, ;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \mathrm{id}\rangle$, with the conventions of Section 3 if r or s equals ∞. Consider then a fixed word $W(a, b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$. In the cases where r or s equals ∞, the exponents r_j can be negative so that we also consider the spelling of $W(a, b)$ resulting from splitting the components $\vartheta_i^{r_i}$. More precisely, if both r, s equals ∞, then we also consider $W(a, b)$ under the form $\theta_s * \cdots * \theta_1$, where:

- θ_j takes on one of the values a, b, a^{-1}, b^{-1}
- if θ_j takes on the value a (resp. a^{-1}) then, whenever defined, neither θ_{j-1} nor θ_{j+1} takes on the value a^{-1} (resp. a). A similar rule applies to b, b^{-1}.

In the cases where both $\mathsf{r}, \mathsf{s} < \infty$, θ_j takes only on the values a, b and every sequence $\theta_i, \theta_{i+1}, \ldots$ of "θ_i" with the same value is contained in the split of some $\vartheta_j^{r_j}$ in the natural sense. Adaptations to the mixed cases $\mathsf{r} < \infty$, $\mathsf{s} = \infty$ or $\mathsf{r} = \infty$, $\mathsf{s} < \infty$ are straightforward and left to the reader. In any event, we obtain $s = \sum_{i=1}^{l} |r_i|$. Now, consider the corresponding local diffeomorphism $W(f, g)$ written under the form $H_s \circ \cdots \circ H_1$ where each H_i, $i \in \{1, \ldots, s\}$, belongs to the set $\{f^{\pm 1}, g^{\pm 1}\}$. In other words, H_i replaces θ_i by means of the substitutions $f^{\pm 1} \mapsto a^{\pm 1}$, $g^{\pm 1} \mapsto b^{\pm 1}$. The *domain of definition* of $W(f, g) = H_s \circ \cdots \circ H_1$ as an element of Γ can be introduced by recursively defining the domains of definitions of each element $H_i \circ \cdots \circ H_1$ of Γ, $i = 1, \ldots, s$, as follows:

- The domain of definition of H_1 is all of D and, for every $z \in D$, $H_1(z)$ is defined in the obvious way.
- Suppose that the domain of definition $\mathrm{Dom}_{H_i \circ \cdots \circ H_1}$ of $H_i \circ \cdots \circ H_1$ is already known along with the points $H_i \circ \cdots \circ H_1(z)$, for $z \in \mathrm{Dom}_{H_i \circ \cdots \circ H_1}$. Then the domain of definition $\mathrm{Dom}_{H_{i+1} \circ \cdots \circ H_1}$ of $H_{i+1} \circ \cdots \circ H_1$ is obtained by setting
$$\mathrm{Dom}_{H_{i+1} \circ \cdots \circ H_1} = \{z \in \mathrm{Dom}_{H_i \circ \cdots \circ H_1} \;\; ; \;\; H_i \circ \cdots \circ H_1(z) \in D\}\,.$$

In particular, $\mathrm{Dom}_{H_{i+1} \circ \cdots \circ H_1} \subseteq \mathrm{Dom}_{H_i \circ \cdots \circ H_1}$ and hence the domain of definition of every element in Γ is naturally contained in D. Besides, for $z \in \mathrm{Dom}_{H_{i+1} \circ \cdots \circ H_1}$, the value of $H_{i+1} \circ \cdots \circ H_1(z)$ is defined by setting $H_{i+1} \circ \cdots \circ H_1(z) = H_{i+1} \circ [H_i \circ \cdots \circ H_1](z)$.

Consider now the local diffeomorphisms f, g and a small open disc D about $0 \in \mathbb{C}$ such that all the elements f, g, f^{-1}, g^{-1} yield one-to-one maps defined on an open neighborhood of $\overline{D}$. Here $\overline{D}$ denotes the closure of D whereas ∂D will stand for the boundary of D. With the notations of Section 3, consider a word $W(a,b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1} = \theta_s * \cdots * \theta_1$ which, as always, is supposed to be non-empty and reduced (w.r.t. the group $\langle a, b\,;\, a^{\mathsf{r}} = b^{\mathsf{s}} = \mathrm{id} \rangle$). Denote by Γ the pseudogroup generated by f, g on D. The domain of definition of $W(f,g)$ as element of Γ is going to be denoted by $\mathrm{Dom}_W(D)$. To be able to take advantage of Baire property, we would like to have a statement of type "for an open dense set of local diffeomorphisms $(h_1, h_2) \in (\mathrm{Diff}\,(\mathbb{C}, 0))^2$, the element $W(h_1^{-1} \circ f \circ h_1, h_2^{-1} \circ g \circ h_2)$ of the pseudogroup generated by $h_1^{-1} \circ f \circ h_1$, $h_2^{-1} \circ g \circ h_2$ does not coincide with the identity on any connected component of $\mathrm{Dom}_W(D)$". This statement, however, makes no sense since the domain of definition of the given local diffeomorphisms h_1, h_2 may be smaller than D so that the pseudogroup generated by $h_1^{-1} \circ f \circ h_1$, $h_2^{-1} \circ g \circ h_2$ does not naturally act on the whole D. This is the main reason explaining why a more careful formulation of our statements is needed.

Consider local diffeomorphisms $\tilde{f}$, $\tilde{g}$ having the form $\tilde{f} = h_1^{-1} \circ f \circ h_1$ and $\tilde{g} = h_2^{-1} \circ g \circ h_2$. The corresponding elements of the pseudogroup they generate (on some suitable open set) will then be denoted by $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$. The point to be made here concerns the domain of definition of $\tilde{f}$, $\tilde{g}$ and, therefore, the domains of definitions of all elements $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ in the pseudogroup generated by $\tilde{f}$, $\tilde{g}$ on an open neighborhood of $0 \in \mathbb{C}$ to be fixed later. With the preceding notations, we have:

Definition 4.1. (1) *The domain of definition of h_1 is defined as follows: let ρ be the radius of the maximal open disc about $0 \in \mathbb{C}$ in which h_1 is defined and injective. Then the open domain of definition of h_1 is defined to be the open disc of radius $9\rho/10$. The closed domain of definition of h_1 will also be considered and this will be nothing but the closed disc of radius $9\rho/10$. Analogous definitions apply to each of the local diffeomorphisms: h_1^{-1}, h_2, h_2^{-1}.*

(2) *The domain of definition of $\tilde{f} = h_1^{-1} \circ f \circ h_1$ consists of those points p verifying all the following conditions: p belongs to the open domain of definition of h_1, $h_1(p)$ belongs to the domain of definition of f, i.e. to D. Besides $f \circ h_1(p)$ must belong to the open domain of definition of h_1^{-1}. Analogous considerations apply to the domain of definition of $\tilde{g} = h_2^{-1} \circ g \circ h_2$ and to $\tilde{f}^{-1}$, $\tilde{g}^{-1}$.*

(3) *Finally, considering the pseudogroup generated by $\tilde{f}$, $\tilde{g}$ on some suitable open set, the domain of definition of its element $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2) = W(\tilde{f}, \tilde{g})$ is obtained according to the above given general definitions concerning pseudogroups, with $\tilde{f}$, $\tilde{g}$ in the place of f, g.*

Concerning the second item above, we shall have occasion of considering closed domains of definitions not only for h_1, h_1^{-1}, h_2, h_2^{-1} but also for more general elements in the pseudogroup generated by $\tilde{f}$, $\tilde{g}$ as above. When doing so, the domains of definitions for both f, g will be understood to be the closed disc $\overline{D}$. Then the domains of definition of $\tilde{f}, \tilde{g}$ will be obtained as in item (2) above, except that each corresponding domain of definition will be closed. Finally, the closed domain of definition of a general element $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2) = W(\tilde{f}, \tilde{g})$ will be obtained by means of the closed domains of $\tilde{f}$, $\tilde{g}$ by following the general pseudogroup rules.

Since $0 \in \mathbb{C}$ is fixed by f, g, we conclude that every word has a non-empty domain of definition as element of Γ. Similarly, for every h_1, h_2 as before, every word in the pseudogroup generated by $\tilde{f}$, $\tilde{g}$ has a non-empty domain of definition. Furthermore, since non-constant holomorphic maps are open maps, the domains of definition of general elements in this pseudogroup is an open set provided that we start with open domains of definitions for h_1, h_2 and f, g (as mentioned, later closed domains of definitions will also be considered). They may, however, be *disconnected*. Therefore, Proposition 2.6 only has a bear in the connected component containing $0 \in \mathbb{C}$ of these domains. More precisely, suppose that f, g as above are given. Then, modulo conjugating f, g by a generic elements h_1, $h_2 \in \text{Diff}\,(\mathbb{C}, 0)$, it can be assumed that the element $W(\tilde{f}, \tilde{g})$ is different from the identity *on the connected component containing* $0 \in \mathbb{C}$ *of its domain of definition*, for every word $W(a, b)$. Since the domain of definition of $W(\tilde{f}, \tilde{g})$ may have more than one connected component, the preceding results do not rule out the possibility of having an element $W(\tilde{f}, \tilde{g})$ coinciding with the identity on some non-empty open set. In particular, if we are dealing with the pseudogroup associated to a foliation, we cannot yet derive the conclusions about the topology of the leaves stated in Theorem B.

Considering the pseudogroup Γ, there is already a point to be made about the above defined powers f^j, g^j, of f, g, which should themselves be understood as elements of Γ defined on some fixed neighborhood of $0 \in \mathbb{C}$. More generally, given $F \in \text{Diff}\,(\mathbb{C}, 0)$ and fixed a neighborhood W of $0 \in \mathbb{C}$ where F is defined, the notation F^j, where $j \in \mathbb{Z}^*$, refers to the element F^j viewed as an element of the pseudogroup generated by F on W.

Recall also that $F \in \text{Diff}\,(\mathbb{C}, 0)$ is said to have a *Cremer point* (at $0 \in \mathbb{C}$) if F is not linearizable at $0 \in \mathbb{C}$ and verifies $F'(0) = e^{2\pi\sqrt{-1}\beta}$ with $\beta \in \mathbb{R} \setminus \mathbb{Q}$. The lemma below concerns the behavior of the powers of an element $F \in \text{Diff}\,(\mathbb{C}, 0)$ on sufficiently small neighborhoods of $0 \in \mathbb{C}$.

Lemma 4.2. *Suppose that* $F \in \text{Diff}\,(\mathbb{C}, 0)$ *is a local diffeomorphism that does not have a Cremer point at* $0 \in \mathbb{C}$. *Then there is a neighborhood* $\mathcal{W}$ *of* $0 \in \mathbb{C}$ *where, for every* $j \in \mathbb{Z}^*$, F^j *has no fixed point unless* F^j *coincides with the identity on all of its domain of definition. In particular, if* F^j, $j \in \mathbb{Z}$, *coincides with the identity on some connected component of its domain of definition, then it coincides with the identity on all of its domain of definition.*

Proof. The statement clearly holds if F is linearizable about $0 \in \mathbb{C}$. In particular, it holds provided that $|F'(0)| \neq 1$ thanks to Poincaré Linearization Theorem. Thus, we can assume that $|F'(0)| = 1$ and that F is not linearizable. Since, by assumption, F does not have a Cremer point at the origin, it follows that $F'(0) = e^{2\pi\sqrt{-1}\beta}$ with $\beta \in \mathbb{Q}$. Therefore the local dynamics of F at $0 \in \mathbb{C}$ is closely related to the special case of the "Leau flower" corresponding to $\alpha = 0$. These dynamics are well-understood and their topological description, cf. for example [C-G], ensures that the statement of the lemma holds. □

Remark 4.3. According to Yoccoz and Perez-Marco, cf. [Y], the assumption that F does not have a Cremer point is, indeed, necessary for the statement of Lemma 4.2 to hold. In fact, there are local diffeomorphisms $F \in \text{Diff}\,(\mathbb{C}, 0)$ exhibiting a Cremer point at $0 \in \mathbb{C}$ for which there exists a sequence of points $\{q_i\}$ accumulating to $0 \in \mathbb{C}$ along with a sequence of *periods* $\{n_i\}$, $n_i \neq 0$, going to infinity such that $F^{n_i}(q_i) = q_i$ for every $i \in \mathbb{N}$. Moreover, the dynamics of F^{n_i} about its fixed point q_i may arbitrarily be fixed: in particular, it can be chosen so that F^{n_i} coincides with the identity on some (very small) neighborhood of q_i.

Note however that this type of phenomenon cannot play any role at infinitesimal level: fixed $n \in \mathbb{Z}^*$, and assuming that $F'(0) = e^{2\pi\sqrt{-1}\beta}$ with $\beta \in \mathbb{R} \setminus \mathbb{Q}$, there always exists a sufficiently small neighborhood of $0 \in \mathbb{C}$ on which F^n has no fixed point other than the origin itself. This explains why the first conclusion of Theorem A does not require any additional condition concerning Cremer points.

Whereas the previous lemma will be used only later, we assume from now on that neither f nor g has a Cremer point at $0 \in \mathbb{C}$. Therefore, modulo reducing the radius of the disc D, the statement of Lemma 4.2 can be supposed to hold for both f, g on a neighborhood of $\overline{D}$. In the sequel D is fixed and the reader is reminded that f, g and their inverses yield one-to-one maps defined on a neighborhood of $\overline{D}$. Given h_1, h_2 with (open and closed) domains of definitions as in Definition 4.1, set $\tilde{f} = h_1^{-1} \circ f \circ h_1$, $\tilde{g} = h_2^{-1} \circ g \circ h_2$, where the domains of definition of $\tilde{f}$, $\tilde{g}$ are again as in Definition 4.1. Now, we have:

Definition 4.4. *With the preceding notations, the pseudogroup generated by $\tilde{f}$, $\tilde{g}$ on the closed disc $\overline{D}$ is the pseudogroup of maps between subsets of $\overline{D}$ where the domains of definition for $\tilde{f}^{\pm 1}$, $\tilde{g}^{\pm 1}$ are obtained by considering closed domains of definitions for h_1, h_1^{-1}, h_2, h_2^{-1} and by setting the domains of definition of both $f^{\pm 1}$, $g^{\pm 1}$ equal to $\overline{D}$.*

Similarly the pseudogroup generated by $\tilde{f}$, $\tilde{g}$ on the open disc D is the pseudogroup of maps between subsets of D where the domains of definition for $\tilde{f}^{\pm 1}$, $\tilde{g}^{\pm 1}$ are obtained by considering open domains of definitions for h_1, h_1^{-1}, h_2, h_2^{-1} and by setting the domain of definition of both $f^{\pm 1}$, $g^{\pm 1}$ equal to D.

Recalling that non-constant holomorphic maps are open maps, it follows from the above definition that the domain of definition of an element $W(\tilde{f}, \tilde{g})$ as element of the pseudogroup generated by $\tilde{f}$, $\tilde{g}$ on the open disc D (resp. on the closed disc $\overline{D}$) is an open set (resp. closed set).

Next choose and fix once and for all a sequence $\{p_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$ dense in $\mathbb{C}$ and such that no point p_j lies in ∂D. Let also $\alpha \in \mathbb{N}$ be fixed. For a chosen point p_j and a given word $W(a, b)$, let $\mathcal{U}_{W,\alpha}^{(j)}$ denote the set formed by those pairs of local diffeomorphisms $(h_1, h_2) \in (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ for which one of the two possibilities below is verified:

(1) p_j does not belong to the domain of definition of $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ viewed as an element of the pseudogroup generated by $\tilde{f}$, $\tilde{g}$ *on the closed disc* $\overline{D}$.

(2) p_j belongs to the domain of definition of $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ viewed as an element of the pseudogroup generated by $\tilde{f}$, $\tilde{g}$ *on the open disc* D. Furthermore $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ is required *not to* coincide with the identity on a neighborhood of p_j.

Lemma 4.5. *For every word $W(a, b)$ as above, the set $\mathcal{U}_{W,\alpha}^{(j)} \subset (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ is open for the (product) analytic topology in* $(\mathrm{Diff}\,(\mathbb{C}, 0))^2$.

Proof. The set $\mathcal{U}_{W,\alpha}^{(j)}$ is open as consequence of the fact that domains of definition move "continuously" with respect to the analytic topology. To be more precise, suppose that (h_1, h_2) lies in $\mathcal{U}_{W,\alpha}^{(j)}$. Then one possibility is that p_j does not belong to the domain of definition of $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2) = W(\tilde{f}, \tilde{g})$ viewed as an element of the pseudogroup generated on $\overline{D}$ by $\tilde{f}$, $\tilde{g}$. In this case, the domain of definition of $W(\tilde{f}, \tilde{g})$ is a closed set and therefore p_j lies at a strictly positive distance from it. Because convergence in the analytic topology implies uniform convergence on fixed domains, it then follows that, for $(\overline{h}_1, \overline{h}_2) \in (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ sufficiently close to (h_1, h_2), p_j will still belong to the complement of the domain of definition of $W(\overline{h}_1^{-1} \circ f \circ \overline{h}_1,\, \overline{h}_2^{-1} \circ g \circ \overline{h}_2)$. Hence $(\overline{h}_1, \overline{h}_2) \in \mathcal{U}_{W,\alpha}^{(j)}$ and (h_1, h_2) is an interior point of $\mathcal{U}_{W,\alpha}^{(j)}$.

The other possibility for having $(h_1, h_2) \in \mathcal{U}_{W,\alpha}^{(j)}$ is to have p_j in the domain of definition of $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2) = W(\tilde{f},\, \tilde{g})$ viewed as an element of the pseudogroup generated on D by $\tilde{f}$, $\tilde{g}$. In this case, the domain of definition of $W(\tilde{f},\, \tilde{g})$ is an open set but $W(\tilde{f},\, \tilde{g})$ cannot coincide with the identity on a neighborhood of p_j. Since the domain of definition of $W(\tilde{f},\, \tilde{g})$ is open, it follows from the preceding discussion that p_j belongs to the domain of definition of $W(\overline{h}_1^{-1} \circ f \circ \overline{h}_1,\, \overline{h}_2^{-1} \circ g \circ \overline{h}_2)$ provided that $(\overline{h}_1, \overline{h}_2) \in (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ is close enough to (h_1, h_2). Modulo taking $(\overline{h}_1, \overline{h}_2)$ closer to (h_1, h_2) it can similarly be ensured that $W(\overline{h}_1^{-1} \circ f \circ \overline{h}_1,\, \overline{h}_2^{-1} \circ g \circ \overline{h}_2)$ does not coincide with the identity on a neighborhood of p_j since convergence in the analytic topology implies convergence of Taylor coefficients (at every a priori fixed order). Therefore every pair $(\overline{h}_1, \overline{h}_2) \in (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ sufficiently close to (h_1, h_2) belongs to $\mathcal{U}_{W,\alpha}^{(j)}$ and this shows that $\mathcal{U}_{W,\alpha}^{(j)}$ is an open set. □

The following lemma is a useful tool that will enable us to prove that $\mathcal{U}_{W,\alpha}^{(j)} \subset (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ is, in addition, “almost” dense.

Lemma 4.6. *With the preceding notations, consider the element $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2) = W(\tilde{f},\, \tilde{g})$ viewed as an element of the pseudogroup generated on D by $\tilde{f}$, $\tilde{g}$, where $(h_1, h_2) \in (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$. Suppose that q, lying in the domain of definition of $W(\tilde{f},\, \tilde{g})$, satisfies $W(\tilde{f},\, \tilde{g})(q) = q$. Then, there is $(\overline{h}_1, \overline{h}_2) \in (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ arbitrarily close to (h_1, h_2) and such that the following holds:*

- *q lies in the domain of definition of $W(\overline{h}_1^{-1} \circ f \circ \overline{h}_1,\, \overline{h}_2^{-1} \circ g \circ \overline{h}_2)$ viewed as element of the pseudogroup generated by $\overline{h}_1^{-1} \circ f \circ \overline{h}_1$, $\overline{h}_2^{-1} \circ g \circ \overline{h}_2$ on D.*
- *$W(\overline{h}_1^{-1} \circ f \circ \overline{h}_1,\, \overline{h}_2^{-1} \circ g \circ \overline{h}_2)(q) \neq q$.*

Proof. What precedes shows that the first condition above is always satisfied provided that $(\overline{h}_1, \overline{h}_2)$ is very close to (h_1, h_2). Thus we only need to prove that $(\overline{h}_1, \overline{h}_2) \in (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ can be obtained so as to satisfy, in addition, $W(\overline{h}_1^{-1} \circ f \circ \overline{h}_1,\, \overline{h}_2^{-1} \circ g \circ \overline{h}_2)(q) \neq q$.

Let then the word $W(a,b)$ be given by $W(a,b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$. The proof of the existence of $(\overline{h}_1, \overline{h}_2)$ satisfying the second condition above and arbitrarily close to (h_1, h_2) is going to be carried out by induction on l. Suppose first that l equals to 1. In this case, the statement follows at once from Lemma 4.2, where it is shown that a local diffeomorphism as in the corresponding statement does not have periodic points.

By inducting on the length of the words, the proposition can be assumed to hold for words of length $1, \ldots, l-1$. We need to show it also holds for words of length l. First consider the itinerary $q = q_0, \ldots, q_{l-1}, q_l$ of q under $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ and suppose that $q = q_0 = q_l$ for otherwise there is nothing to be proved. The induction assumption allows us to suppose that the points $q_0, \ldots, q_{l-1}$ are pairwise distinct. Indeed, given $0 \leq i_1 < i_2 < l$, we have that $q_{i_2} = W'(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)(q_{i_1})$ where $W'(a,b)$ is a word whose length is at most $l-1$. Thus, by the induction assumption, (h_1, h_2) can be perturbed into $(h_{1,*}, h_{2,*}) \in (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ so as to satisfy $q_{i_2} = W'(h_{1,*}^{-1} \circ f \circ h_{1,*},\, h_{2,*}^{-1} \circ g \circ h_{2,*})(q_{i_1}) \neq q_1$. Since, once obtained, the condition $q_{i_2} = W'(h_{1,*}^{-1} \circ f \circ h_{1,*},\, h_{2,*}^{-1} \circ g \circ h_{2,*})(q_{i_1}) \neq q_1$ is open, the fact that there are only finitely many words $W'(a, b)$ that need to considered allows us to construct a first perturbation $(\overline{h}_{1,*}, \overline{h}_{2,*}) \in (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ of (h_1, h_2) so that the itinerary of $q = q_0$ by $W(\overline{h}_{1,*}^{-1} \circ f \circ \overline{h}_{1,*},\, \overline{h}_{2,*}^{-1} \circ g \circ \overline{h}_{2,*})$ satisfies the required condition. In other words, we can assume without loss of generality that the itinerary $q = q_0, \ldots, q_{l-1}, q_l$ of q under $W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)$ is such that the points $q_0, \ldots, q_{l-1}$ are pairwise distinct.

Let us now construct pairs of local diffeomorphisms $(\overline{h}_1, \overline{h}_2) \in (\mathrm{Diff}\,(\mathbb{C}, 0))^2$ arbitrarily close to (h_1, h_2) and such that $W(\overline{h}_1^{-1} \circ f \circ \overline{h}_1,\, \overline{h}_2^{-1} \circ g \circ \overline{h}_2)(q) \neq q$. First, since $W(a, b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$, with $l \geq 2$, we shall assume that ϑ_1 takes on the value a (with $r_1 > 0$) and that ϑ_l takes on the value b (with $r_l > 0$). The purpose of this assumption is only to abridge notations since the other cases are analogue and can be handled by a very straightforward adaptation of the discussion below. In particular, if the word $W(a, b)$ is such that ϑ_1, ϑ_l takes on the same value (a or b), then $W(f, g)$ is conjugate to a word of smaller length and the desired conclusion can quickly be derived.

As in Section 3, let P be a polynomial such that $P(q_0) = \cdots = P(q_{l-2}) = 0$ and $P(q_{l-1}) \neq 0$. Since ϑ_l takes on the value b, we set

$$h_{1,t} = h_1 \text{ and } h_{2,t} = h_2 + t z^{\alpha+1} P$$

where $t \in [0, 1]$. Clearly $h_{2,t}$ converges to h_2 in the analytic topology when $t \to 0$ and $h_{2,t} \in \mathrm{Diff}_\alpha(\mathbb{C}, 0)$ for every $t \in [0, 1]$. Therefore, to conclude the proof, it suffices to show that $(h_{1,t}, h_{2,t})$ lies in $\mathcal{U}_{W,\alpha}^{(j)}$ for arbitrarily small $t > 0$ (strictly). As already observed, for t sufficiently small q belongs to the domain of definition of $W(h_{1,t}^{-1} \circ f \circ h_{1,t},\, h_{2,t}^{-1} \circ g \circ h_{2,t})$ viewed as of the pseudogroup generated on the open disc D by $h_{1,t}^{-1} \circ f \circ h_{1,t}$, $h_{2,t}^{-1} \circ g \circ h_{2,t}$. The corresponding itinerary is going to be denoted by $q = q_{0,t}, \ldots, q_{l-2,t}, q_{l-1,t}$ and $q_{l,t}^l = h_{2,t}^{-1} \circ g^{r_l} \circ h_{2,t}(q_{l-1,t})$ By construction, it follows that $q_i = q_{i,t}$ for $i = 0, 1, \ldots, l-1$. However, $h_{2,t}(q_{l-1}) = h_{2,t}(q_{l-1,t}) \neq h_2(q_{l-1,t})$. Now the assumption concerning the injective character of both $h_2^{\pm 1}, g^{\pm 1}$ on the domains in question allows us to conclude that $q = q_0 = q_l = W(h_1^{-1} \circ f \circ h_1,\, h_2^{-1} \circ g \circ h_2)(q_0) \neq W(h_{1,t}^{-1} \circ f \circ h_{1,t},\, h_{2,t}^{-1} \circ g \circ h_{2,t})(q_0)$ for every $t > 0$ sufficiently small. The lemma is proved. $\square$

Consider again a word $W(a, b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$ and choose a point p_j in the dense sequence fixed at the beginning. If ϑ_1 takes on the value a, let $C_{W,\alpha}^{(j)}$ be defined as the subset of $(\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ consisting of those pairs (h_1, h_2) for which p_j lies in the boundary of the domain of definition of h_1 in the sense of Definition 4.1. Similarly, if ϑ_1 takes on the value b, then $C_{W,\alpha}^{(j)}$ is constituted by those pairs $(h_1, h_2) \in (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ such that p_j lies in the boundary of the domain of definition of h_2 in the sense of Definition 4.1. With this definition we can state:

Proposition 4.7. *For every word $W(a, b)$ as above, the set $\mathcal{U}_{W,\alpha}^{(j)} \subset (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2$ is dense for the (product) analytic topology in $(\mathrm{Diff}_\alpha(\mathbb{C}, 0))^2 \setminus C_{W,\alpha}^{(j)}$.*

To prove Proposition 4.7 it only remains to check that $\mathcal{U}^{(j)}_{W,\alpha}$ is dense in $(\mathrm{Diff}_\alpha(\mathbb{C},0))^2 \backslash C^{(j)}_{W,\alpha}$. The structure of this proof is similar to the structure of the proof of Lemma 3.3 but with a more direct argument.

Proof of Proposition 4.7. Let $W(a,b) = \vartheta_l^{r_l} * \cdots * \vartheta_1^{r_1}$ as above be fixed along with a point p_j. We already know that the set $\mathcal{U}^{(j)}_{W,\alpha} \subset (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ is open so that it only remains to check it is also dense. Consider then a pair of local diffeomorphisms $(h_1,h_2) \in (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ lying in the complement of $\mathcal{U}^{(j)}_{W,\alpha}$. Since $(h_1,h_2) \notin \mathcal{U}^{(j)}_{W,\alpha}$, it follows from the construction of $\mathcal{U}^{(j)}_{W,\alpha}$ that p_j belongs to the domain of definition of $W(h_1^{-1}\circ f\circ h_1,\, h_2^{-1}\circ g\circ h_2)$ viewed as an element of the pseudogroup generated on *the closed disc* $\overline{D}$ by $\tilde{f} = h_1^{-1}\circ f\circ h_1$ and $\tilde{g} = h_2^{-1}\circ g\circ h_2$. Furthermore one of the following possibilities must hold:

- There is a point P in the itinerary of p_j by $W(h_1^{-1}\circ f\circ h_1,\, h_2^{-1}\circ g\circ h_2)$ that satisfies one of the following conditions:
  - $h_1(P)$ or $h_2(P)$ lies in the boundary ∂D of D (the choice between $h_1(P)$ or $h_2(P)$ depends on whether a power of $h_1^{-1}\circ f\circ h_1$ or of $h_2^{-1}\circ g\circ h_2$ is applied next).
  - P lies in the boundary of the domain of definition of h_1 or of h_2.
  - $f\circ h_1(P)$ lies in the boundary of the domain of definition of h_1^{-1} or $g\circ h_2(P)$ lies in the boundary of the domain of definition of h_2^{-1}. However, in this case, the possibility of having $P = p_j$ lying in the boundary of the domain of h_1 (resp. h_2) provided that ϑ_1 takes on the value a (resp. b) is ruled out by the fact that we are working in the complement of $C^{(j)}_{W,\alpha}$
- p_j belongs to the domain of definition of $W(h_1^{-1}\circ f\circ h_1,\, h_2^{-1}\circ g\circ h_2)$ viewed as an element of the pseudogroup generated on the open disc D and, in addition, $W(h_1^{-1}\circ f\circ h_1,\, h_2^{-1}\circ g\circ h_2)$ coincides with the identity on a neighborhood of p_j.

Assume the first alternative holds. To prove the statement it then suffices to find a sequence of elements $\{(\overline{h}_{1,k},\overline{h}_{2,k}\})_{k\in\mathbb{N}} \subset (\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ converging to (h_1,h_2) and such that p_j does not belong to the domain of definition of $W(\overline{h}_{1,k}^{-1}\circ f\circ\overline{h}_{1,k},\, \overline{h}_{2,k}^{-1}\circ g\circ\overline{h}_{2,k})$, viewed as an element of the pseudogroup generated by $\overline{h}_{1,k}^{-1}\circ f\circ\overline{h}_{1,k}$, $\overline{h}_{2,k}^{-1}\circ g\circ\overline{h}_{2,k}$ on *the closed disc* $\overline{D}$. Indeed, by construction, all the pairs $(\overline{h}_{1,k},\overline{h}_{2,k})$ in this sequence belong to $\mathcal{U}^{(j)}_{W,\alpha}$ so that the proposition follows in this first case. On the other hand, note that, to construct the desired sequence is enough to slightly perturb the local diffeomorphisms h_1, h_2 by using some easy version of the transversality principle, what can be done without changing their domains of definitions. For this only reason, it is important to rule out the case where p_j itself belongs to the boundary of the domain of definition of h_1 or h_2 (depending on the value of ϑ_1) since, to eliminate this condition, we would need to change the domain of definition of the corresponding h_i what is much harder due to the phenomenon pointed out in Remark 2.2.

On the other hand, if the second case above occurs, then the conclusion follows immediately from Lemma 4.6. This ends the proof of Proposition 4.7. □

Theorem A is now essentially reduced to Theorem 4.8 below.

Theorem 4.8. *Assume that neither f nor g has a Cremer point at $0 \in \mathbb{C}$. Then there is a G_δ-dense set $\mathcal{U}_\alpha$ of $(\mathrm{Diff}_\alpha(\mathbb{C},0))^2$ such that, for every pair $(h_1,h_2) \in \mathcal{U}_\alpha$ and for every reduced word $W(a,b)$, the element $W(h_1^{-1}\circ f\circ h_1,\, h_2^{-1}\circ g\circ h_2)$ of the pseudogroup generated by*

$h_1^{-1} \circ f \circ h_1$, $h_2^{-1} \circ g \circ h_2$ *on* D *does not coincide with the identity on any connected component of its domain of definition.*

Proof. Fix a word $W(a,b)$ and a point p_j. It follows from Proposition 4.7 and from Lemma 4.5 that the only obstruction for the open set $\mathcal{U}_{W,\alpha}^{(j)}$ to be dense is the set $C_{W,\alpha}^{(j)}$ to have non-empty interior $\mathrm{Int}\,[C_{W,\alpha}^{(j)}]$. Therefore, letting $\mathcal{V}_{W,\alpha}^{(j)} = \mathcal{U}_{W,\alpha}^{(j)} \cup \mathrm{Int}\,[C_{W,\alpha}^{(j)}]$, it becomes clear that $\mathcal{V}_{W,\alpha}^{(j)}$ is an open and dense subset of $(\mathrm{Diff}_\alpha(\mathbb{C},0))^2$. Therefore, recalling that a countable union of countable sets is itself countable, the intersection

$$\mathcal{U}_\alpha = \bigcap_{\substack{W(a,b)\,;\,W(a,b)\ \text{non-empty}\\ \text{and reduced}}} \left[\bigcap_{j=1}^{\infty} \mathcal{V}_{W,\alpha}^{(j)} \right].$$

is a G_δ-dense subset of $(\mathrm{Diff}_\alpha(\mathbb{C},0))^2$.

Now, suppose we are given $(h_1,h_2) \in \mathcal{U}_\alpha$ and consider the pseudogroup generated on open subsets of D by $\tilde{f} = h_1^{-1} \circ f \circ h_1$ and by $\tilde{g} = h_2^{-1} \circ g \circ h_2$ (according to Definition 4.1 and Definition 4.4). This pseudogroup may be denoted by Γ_{h_1,h_2}.

Given a word $W(a,b)$, consider the element $W(\tilde{f},\,\tilde{g})$ of Γ_{h_1,h_2} whose domain of definition will be denoted by $\mathrm{Dom}_{W_{h_1,h_2}}(D)$. Then, note that $\mathrm{Dom}_{W_{h_1,h_2}}(D)$ is clearly an open set and so are its connected components. Let U_1 be one of these connected components. Since $\{p_j\}$ is dense in $\mathbb{C}$, there exists j_1 such that $p_{j_1} \in U_1$. Furthermore, p_{j_1} can be chosen away from the boundaries of the domains of definitions of h_1, h_2 so that $(h_1,h_2) \notin C_{W,\alpha}^{(j_1)}$. Because (h_1,h_2) belongs to $\mathcal{U}_\alpha$, it also belongs to $\mathcal{V}_{W,\alpha}^{(j_1)}$ and, hence, to $\mathcal{U}_{W,\alpha}^{(j_1)}$. It follows that $W(\tilde{f},\,\tilde{g})$ does not coincide with the identity on a neighborhood of p_{j_1}. Therefore $W(\tilde{f},\,\tilde{g})$ does not coincide with the identity on U_1. Since U_1 is an arbitrary connected component of $\mathrm{Dom}_{W_{h_1,h_2}}(D)$, the statement results at once. □

Proof of Theorem A. The statement follows at once from assembling Theorem 2.7 and Theorem 4.8. □

5. An application to nilpotent foliations

As indicated in the Introduction, the problem of perturbing the generators of a subgroup of $\mathrm{Diff}\,(\mathbb{C},0)$ inside their conjugacy classes arises naturally in the study of germs of singular foliations at the origin of $\mathbb{C}^2$. Probably the most typical example where this situation can be found corresponds to the class of nilpotent foliations of type A^{2n+1}. More precisely, these are local foliations $\mathcal{F}_X$ defined by a (germ of) vector field X having nilpotent linear part, i.e. $X = y\partial/\partial x + \cdots$, and a unique separatrix S that happens to be a curve analytically equivalent to $\{y^2 - x^{2n+1} = 0\}$. In other words, there are local coordinates where S is given by the equation $\{y^2 - x^{2n+1} = 0\}$. For this type of foliation, the desingularization of the separatrix coincides with the reduction of the foliation itself. More precisely, the map associated to the desingularization of the separatrix $E_S : M \to \mathbb{C}^2$ reduces also the foliation $\mathcal{F}_X$. The corresponding exceptional divisor $\mathcal{D} = E_S^{-1}(0)$ consists of a chain of $n+2$ rational curves whose dual graph is as in Figure 1.

The vertices of this graph correspond to the irreducible components of $\mathcal{D}$. The weight of each irreducible component equals its self-intersection. In turn, the edges correspond to the

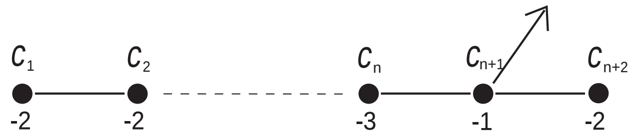

FIGURE 1. The desingularization diagram of the foliation

intersection of two irreducible components whereas the arrow corresponds to the intersection point of the (unique) component C_{n+1} of self-intersection -1 with the transform $\tilde{S}$ of S. The component C_{n+1} contains three singular points s_0, s_1 and s_2 where s_0 is the point determined by the intersection of C_{n+1} with $\tilde{S}$. Finally s_1 (resp. s_2) is the intersection point of C_{n+1} with C_{n+2} (resp. C_n).

Denote by $\widetilde{\mathcal{F}}$ the transform of $\mathcal{F}_X$ and note that the singular points of $\widetilde{\mathcal{F}}$ are the intersection points of two consecutive components in the chain $C_1, \ldots, C_{n+2}$ along with the point s_0. All these singular points are *simple* in the sense that they possess two eigenvalues different from *zero*. The corresponding eigenvalues can precisely be determined by using the weights of the various components of the exceptional divisor. In particular, it follows that the holonomy associated to the component C_{n+2}, i.e. the holonomy map associated to the regular leaf $C_{n+2} \setminus \{s_1\}$ of $\widetilde{\mathcal{F}}$, coincides with the identity since this leaf is simply connected. Therefore the germ of $\widetilde{\mathcal{F}}$ at s_1 admits a holomorphic first integral. Since the corresponding eigenvalues are 1, 2, we conclude that the local holonomy map g associated to a small loop around s_1 and contained in C_{n+1}, has order equal to 2. A similar discussion applies to the component C_1 and leads to the conclusion that the local holonomy map f associated to a small loop around s_2 and contained in C_{n+1} has order equal to $2n+1$. Since $C_{n+1} \setminus \{s_0, s_1, s_2\}$ is a regular leaf of $\widetilde{\mathcal{F}}$, we conclude that the (image of the) holonomy representation of the fundamental group of $C_{n+1} \setminus \{s_0, s_1, s_2\}$ in $\mathrm{Diff}\,(\mathbb{C}, 0)$ is nothing but the group generated by f, g. Note that this conclusion depends only on the configuration of the reduction tree which, in turn, is determined by some finite order jet of X. Hence, if the coefficients of Taylor series of the vector field X are perturbed starting from a sufficiently high order, the new resulting vector field X' will still give rise to a foliation whose singularity is reduced by the same blow-up map associated to the divisor of Figure 1. In particular, the holonomy representation of the fundamental group of $C_{n+1} \setminus \{s_0, s_1, s_2\}$ in $\mathrm{Diff}\,(\mathbb{C}, 0)$, obtained from this new foliation, is still generated by two elements of $\mathrm{Diff}\,(\mathbb{C}, 0)$ having finite orders respectively equal to 2 and to $2n+1$. Since every local diffeomorphism of finite order is conjugate to the corresponding rotation, it follows that the mentioned perturbations are made inside the conjugacy classes of f and g. This also justifies the fact that in Theorem A only perturbations of local diffeomorphisms that do not alter the corresponding conjugation classes were allowed.

Conversely, given two local diffeomorphism f, g of orders respectively 2, $2n+1$, they can be realized (up to simultaneous conjugation) as the holonomy of the corresponding component C_{n+1} for some local foliation $\mathcal{F}_X$ (or $\widetilde{\mathcal{F}}$). This is done through a well-known gluing procedure for which precise references will be provided later. Therefore, the set of all foliations $\mathcal{F}_X$, up to conjugation, can also be "parameterized" by the pair of elements $h_1^{-1} \circ f \circ h_1$ and $h_2^{-1} \circ g \circ h_2$ for some $(h_1, h_2) \in (\mathrm{Diff}\,(\mathbb{C}, 0))^2$. In what follows this procedure will be refined in order to preserve finite orders jets.

We can now state a sharper version of Theorem B.

Theorem 5.1. *Let $X \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ be a vector field with an isolated singularity at the origin and defining a germ of nilpotent foliation $\mathcal{F}$ of type A^{2n+1}. Then, for each $N \in \mathbb{N}$, there exists a vector field $X' \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ defining a germ of foliation $\mathcal{F}'$ and satisfying the following conditions:*

(a) $J_0^N X' = J_0^N X$.
(b) *$\mathcal{F}$ and $\mathcal{F}'$ have S as a common separatrix.*
(c) *there exists a fundamental system of open neighborhoods $\{U_j\}_{j\in\mathbb{N}}$ of S, inside a closed ball $\bar{B}(0,R)$, such that the following holds for every $j \in \mathbb{N}$:*
  (c1) *The leaves of the restriction of $\mathcal{F}'$ to $U_j \setminus S$, $\mathcal{F}'|_{(U_j\setminus S)}$ are simply connected except for a countable number of them.*
  (c2) *all leaves of $\mathcal{F}'|_{(U_j\setminus S)}$ are incompressible, i.e. their fundamental groups inject in the fundamental group of $U_j \setminus S$.*
  (c3) *the morphism $\pi_1(U_j \setminus S, .) \to \pi_1(\bar{B}(0,R) \setminus S, .)$ induced by the inclusion map is an isomorphism.*

The proof of the above theorem will follow from Theorem A combined to two specific lemmas. However, before stating these lemmas, let us make accurate the construction of the relevant holonomy maps.

Suppose we are given a nilpotent foliation $\mathcal{F}$ with separatrix $S = \{y^2 - x^{2n+1}\}$ which is defined by a vector field $X = y\partial/\partial x + \cdots$. Fix then a germ of (smooth) transverse section $(\Sigma, t_0) \simeq (\mathbb{C}, 0)$ through a point $t_0 \in C_{n+1}$, with $t_0 \notin \{s_0, s_1, s_2\}$. Choose conformal open discs $D_k \subset C_{n+1}$ containing s_k and such that their closures $\overline{D}_k$ are pairwise disjoint, $k = 0, 1, 2$. The next step consists of constructing two paths γ_1, γ_2 issued from t_0, contained in

$$C^*_{n+1} = C_{n+1} \setminus (D_0 \cup D_1 \cup D_2)$$

and such that their homotopy classes generate the fundamental group of C^*_{n+1}. To do this, let us choose, for $k = 1, 2$, the following objects:

- A simple path θ_k going from t_0 to a point $\theta_k(1) \in \partial\overline{D}_0$, where $\partial\overline{D}_0$ stands for the boundary of $\overline{D}_0$.
- A simple path σ_k going from $\theta_k(1)$ to some point in ∂D_k.
- A simple loop δ_k based at $\sigma_k(1)$ and contained in ∂D_k.

All the above paths are chosen differentiable and such that their pairwise intersections are contained in the corresponding endpoints (these choices are summarized by Figure 2). Finally we set

$$\gamma_k = \theta_k^{-1}\sigma_k^{-1}\delta_k\sigma_k\theta_k \ , \quad k = 1, 2 \, . \tag{4}$$

Let $\xi : \mathcal{T} \mapsto C_{n+1}$ denote a locally trivial C^∞-fibration whose fibers are discs, where $\mathcal{T}$ stands for an open neighborhood of C_{n+1}. The main difficulty involved in deriving Theorem B from Theorem A lies in the following issue: if μ is a loop contained in $\xi^{-1}(C_{n+1})$ and in a leaf of $\widetilde{\mathcal{F}}$, a homotopy $(\zeta_t)_{t\in[0,1]}$, contained in C_{n+1} and beginning at $\zeta_0 = \xi \circ \mu$, may fail to lift (for every value of the parameter) in a homotopy $(\mu_t)_{t\in[0,1]}$, beginning at $\mu_0 = \mu$ and contained in the corresponding leaf of $\widetilde{\mathcal{F}}$.

Consider now a fundamental system of neighborhoods $\{U_j\}_j$ for S satisfying the properties (c2) and (c3) of Theorem 5.1. Note that the existence of $\{U_j\}_j$ is guaranteed by the main theorem in [Ma-M]. More precisely, U_j is nothing but the open set constructed in the

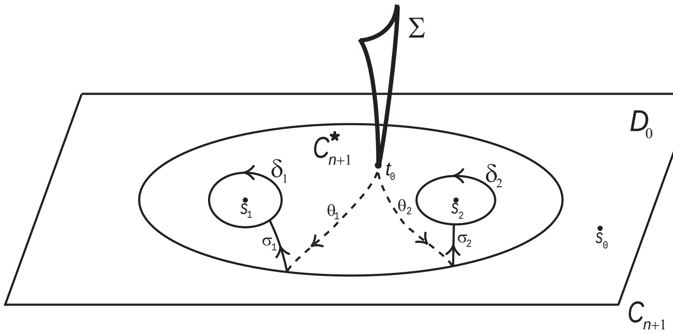

FIGURE 2

proof of the mentioned theorem. Apart from properties (c2) and (c3), the construction of the open sets U_j ensures that they also satisfy certain additional conditions. To state these conditions, we set $\widetilde{U}_j = E_S^{-1}(U_j) \subset M$ and denote by $|\gamma_1|$ (resp. $|\gamma_2|$) the image of the path γ_1 (resp. γ_2). With these notations, we have:

($\imath$) $\widetilde{U}_j \cap \Sigma$ is a closed conformal disc. Moreover f, g are defined and holomorphic on a neighborhood of $\widetilde{U}_j \cap \Sigma$ and, in addition, for every $p \in \widetilde{U}_j \cap \Sigma$, the points $p, f(p)$ (resp. $p, g(p)$) are the endpoints of a unique path λ_1 (resp. λ_2) contained in $L \cap \xi^{-1}(|\gamma_1| \cup |\gamma_2|)$ and such that $\xi \circ \lambda_1 = \gamma_1$ (resp. $\xi \circ \lambda_2 = \gamma_2$), where L denotes the leaf through p.

($\imath\imath$) Every leaf of the restriction of $\widetilde{\mathcal{F}}$ to $\widetilde{U}_j$ not contained in the total transform $E_S^{-1}(S)$ intersects Σ.

($\imath\imath\imath$) Every loop based at a point in $\widetilde{U}_j \cap \Sigma$ and contained in a leaf L of the restriction of $\widetilde{\mathcal{F}}$ to $\widetilde{U}_j$ is homotopic inside L to a loop contained in $\xi^{-1}(|\gamma_1| \cup |\gamma_2|)$.

($\imath v$) Every loop μ contained in the intersection of $\xi^{-1}(|\gamma_1| \cup |\gamma_2|)$ with a leaf L of the restriction of $\widetilde{\mathcal{F}}$ to $\widetilde{U}_j$ is homotopic inside L to a point provided that $\xi \circ \mu$ is equivalent to γ_1^2 or to γ_2^{2n+1}.

Lemma 5.2. *Let μ be a loop contained in the intersection of $\xi^{-1}(|\gamma_1| \cup |\gamma_2|)$ with a leaf L of the restriction of $\widetilde{\mathcal{F}}$ to $\widetilde{U}_j$. Then the following holds:*

(1) *If μ is not homotopic to a point inside L, then it is homotopic inside L to a loop $\widetilde{\mu}$ contained in $\xi^{-1}(|\gamma_1| \cup |\gamma_2|)$ and such that $\xi \circ \widetilde{\mu} = W(\gamma_1, \gamma_2)$ where $W(a, b)$ is a reduced word in two letters in the sense of Section 2.*

(2) *If $\xi \circ \widetilde{\mu} = W(\gamma_1, \gamma_2)$, where $W(a, b)$ is a word as above, then the initial point of μ belongs to the domain of definition* Dom (W, Σ) *of $W(f, g)$ in the sense of pseudogroups introduced in Section 4. Moreover this initial point is actually a fixed point of the element $W(f, g)$ in question.*

Proof. The loop $\mu \subset \xi^{-1}(|\gamma_1| \cup |\gamma_2|)$ is a concatenation of paths μ_l, $l = 1, \ldots, m$ such that each $\xi(|\mu_l|)$ is contained in the image $|\zeta|$ of a path ζ coinciding with one of the paths θ_i, σ_i, δ_i or with their inverses (where as usual $|\mu_l|$ stands for the image of μ_l). Furthermore $\xi(\mu_l(0))$ and $\xi(\mu_l(1))$ are endpoints of ζ. It is then clear that each μ_l is homotopic, inside its own image $|\mu_l|$, to a certain (one-to-one) path μ_l' verifying $\xi \circ \mu_l' = \zeta$ so long $\xi(\mu_l(0)) \neq \xi(\mu_l(1))$. Note that, in this case, ζ coincides with one of the paths θ_i, σ_i, θ_i^{-1}, σ_i^{-1}. In the case where $\xi(\mu_l(0)) = \xi(\mu_l(1))$, it follows that μ_l is always homotopic to a constant except when $\xi(|\mu_l|)$

is contained in δ_i and $\xi \circ \mu_l$ has winding number around s_i equal to ± 1. Therefore μ is homotopic to $\mu' = \mu'_m * \cdots * \mu'_1$. Modulo performing homotopies supported in $|\mu|$, all successive concatenations of the form $\mu_{l+1} * \mu_l$ with $\mu_{l+1} = \mu_l^{-1}$ can be eliminated from the previous expression. Similarly, all constant paths μ'_l can be eliminated as well. Having eliminated all these terms and performed the appropriate re-groupings, we obtain a decomposition $\mu' = \nu_{\overline{m}} * \cdots * \nu_1$ where each ν_i is a path contained in $|\mu|$ and such that $\xi \circ \nu_i$ coincides with either γ_1^a, γ_2^a or with their inverses, where γ_1^a, γ_2^a stand for the a^{th}-power of γ_1, γ_2, for certain $a \in \mathbb{Z}^*$. Finally, by using property $(\imath v)$, we perform all needed homotopies in the leaf L to eliminate terms of the form ν_i^2, with $\xi \circ \nu_i = \gamma_1$, as well as terms of the form ν_i^{2n+1} with $\xi \circ \nu_i = \gamma_2$. Continuing this procedure, we shall eventually obtain a loop $\tilde{\mu}$ in $L \cap \xi^{-1}(|\gamma_1| \cup |\gamma_2|)$ homotopic inside L to μ and possessing a decomposition $\xi \circ \widetilde{\mu} = W(\gamma_1, \gamma_2)$ such that $W(\gamma_1, \gamma_2)$ verifies one of the following conditions: $W(\gamma_1, \gamma_2)$ is empty, in which case $\widetilde{\mu}$ is a constant loop, or $W(\gamma_1, \gamma_2)$ is a reduced word (as in Section 2) spelled out in two letters a, b. The latter possibility, however, cannot occur since μ is not homotopic to a point inside L. This establishes the first conclusion in the statement. In turn, the proof of the second conclusion follows at once from property $(\imath)$ and $(\imath v)$. The lemma is proved. $\square$

Next we have

Lemma 5.3 (Tangential realization). *There is a constant $K \in \mathbb{N}$ such that, for every integer $r > 1$ and every pair $(h_1, h_2) \in (\mathrm{Diff}_{r+K}(\Sigma, t_0))^2$ of diffeomorphism tangent to the identity to order $r + K$, the following holds: there exists a foliation $\mathcal{F}'$ whose separatrix is exactly the curve S, which is defined by a nilpotent vector field X' satisfying the conditions below:*

- $J_0^r X = J_0^r X'$.
- *The local diffeomorphisms arising as holonomy maps associated to the transform $\widetilde{\mathcal{F}}' = E_S^* \mathcal{F}'$ over the loops γ_1 and γ_2 are given by $h_1^{-1} \circ f \circ h_1$ and $h_2^{-1} \circ g \circ h_2$.*

Proof. The proof relies on the techniques developed in [M-S] and, more precisely, in Theorem (2.3.7) and Theorem (6.2.2) which also appear, in conditions similar to those used here, in [LF]. We shall only provide the corresponding main steps.

Consider a divisor $\mathcal{D}$ in a manifold M and suppose that $\mathcal{D}$ contains at most nodal singular points. Two holomorphic foliations $\mathcal{G}$ and $\mathcal{G}'$ defined on an open set Ω of M are said to be *r-tangent over the divisor $\mathcal{D}$* if, for every point $p \in \Omega \cap \mathcal{D}$, they can locally be represented by respective vector fields Y and Y' having isolated singularities and such that the following holds: in suitable local coordinates (z_1, z_2) the divisor $\mathcal{D}$ is given by $u = 0$ where u is either z_1 or $z_1 z_2$. Furthermore, the vector field $Y' - Y$ must take on the form $u^{r+1} Y''$, where Y'' is holomorphic. The following is well-known:

($*$) *There exists $K_1 \in \mathbb{N}$ such that whenever the transforms $E_S^* \mathcal{G}'$ and $E_S^* \mathcal{G}''$ of two germs $\mathcal{G}'$ and $\mathcal{G}''$ of nilpotent foliations at $(0, 0) \in \mathbb{C}^2$ are $(r + K_1)$-tangent over $\mathcal{D}$, then the initial foliations $\mathcal{G}'$ and $\mathcal{G}''$ are defined by 1-forms sharing the same r-jet at the origin.*

On the other hand, fix a simple loop δ_0 going around s_0 exactly once, i.e. with winding number around s_0 equal to ± 1, and based at $t'_0 \in C_{n+1}$, "near to s_0". The loop δ_0 can be thought to be the boundary of D_0. Then choose a path β starting at t_0 and ending at t'_0. Assuming δ_0 to be ∂D_0, then β may be supposed to be contained in C_{n+1}^*. Modulo reversing the orientation of δ_0, there is no loss of generality in assuming that the loop $\gamma_0 = \beta^{-1} * \delta_0 * \beta$

is homotopic in C^*_{n+1} to $\gamma_2 * \gamma_1$. Also let a local transverse section (Σ', t'_0) through t'_0 be fixed. We shall then proceed as follows. Consider a covering Λ of $\mathcal{D}$ constituted by the following open sets:

(a) The "*large open sets*" U_j, $j = 1, \ldots, n+2$, whose closures $\overline{\mathsf{U}}_j$ are equal to C_j, that are the connected components of the complement of the singular set of $\widetilde{\mathcal{F}}$ in $\mathcal{D}$.

(b) The "*small open sets*" U_s, where s is in natural correspondence with the singularities of $\widetilde{\mathcal{F}}$ on $\mathcal{D}$, obtained by intersecting $\mathcal{D}$ with open balls $\Omega_s \subset M$ about s. These balls are chosen to be small enough to ensure that their closures are pairwise disjoint.

Let us fix $K \in \mathbb{N}$.

Step 1 (Realization over the punctured divisor). Over an open neighborhood Ξ_{n+1} of U_{n+1}, we construct a holomorphic foliation $\mathcal{H}_{n+1}$, $(r+K-1)$-tangent to $\widetilde{\mathcal{F}}$ over U_{n+1} and whose holonomy diffeomorpisms $\tilde{f}$, $\tilde{g} : (\Sigma, t_0) \to (\Sigma, t_0)$ induced by the loops γ_1, γ_2 coincide respectively with $h_1^{-1} \circ f \circ h_1$, $h_2^{-1} \circ g \circ h_2$.

Step 2 (Realization at s_0). On the open ball Ω_{s_0}, the results of [Ma-Ra] allows us to construct a (reduced) holomorphic foliation $\mathcal{H}_{s_0}$ having an isolated singularity at s_0 and satisfying the conditions below:

(c) $\mathcal{H}_{s_0}$ is $(r+K-1)$-tangent to $\widetilde{\mathcal{F}}$, and hence to $\mathcal{H}_{n+1}$, at every point of $\mathsf{U}_{n+1} \cap \mathsf{U}_{s_0}$.

(d) The local holonomy diffeomorphisms H_0, $F_0 : (\Sigma', t'_0) \to (\Sigma', t'_0)$ induced by the loop δ_0 for respectively $\mathcal{H}_{s_0}$ and $\widetilde{\mathcal{F}}$ are conjugate, $H_0 = \overline{h}_0^{-1} \circ F_0 \circ \overline{h}_0$ by a diffeomorphism $\overline{h}_0$ tangent to the identity to order $r+K$ at the origin.

Step 3 (Gluing procedure). Consider the intersections $\mathsf{U}_{n+1} \cap \mathsf{U}_{s_k}$, $k = 1, 2$. Over these intersections the foliations $\widetilde{\mathcal{F}}$ and $\mathcal{H}_{n+1}$ are conjugate by a germ $\Phi_{n+1,\, s_k}$ of automorphism of $(M,\, \mathsf{U}_{n+1} \cap \mathsf{U}_{s_k})$, i.e. $\Phi^*_{n+1,\, s_k}\mathcal{F} = \mathcal{H}_{n+1}$. On $\mathsf{U}_{n+1} \cap \mathsf{U}_{s_0}$ the foliations $\mathcal{H}_{s_0}$ and $\mathcal{H}_{n+1}$ are also conjugate by a germ $\Phi_{n+1,\, s_0}$ of automorphism i.e. $\Phi^*_{n+1,\, s_0}\mathcal{H}_{s_0} = \mathcal{H}_{n+1}$. In addition, the conjugating automorphisms $\Phi_{n+1,\, s_k}$, $k = 0, 1, 2$ are all tangent to the identity to order $r + K - 1$ at every point of $\mathsf{U}_{n+1} \cap \mathsf{U}_{s_k}$, $k = 0, 1, 2$. They define a Čech cocycle on a suitable sheaf. Denote by $\mathbf{G}_v$, $v \geq 1$, the sheaf of groups with base $\mathcal{D}$ such that $\mathbf{G}_v(U)$, $U \subset \mathcal{D}$, is the group of germs of holomorphic diffeomorphisms defined on open neighborhoods $\Omega \subset M$ of U which, furthermore, are tangent to the identity to order v at every point of $\mathcal{D}$. Finally if, for $m \neq n+1$ and s being a singular point of $\widetilde{\mathcal{F}}$ on C_m, we let $\Phi_{m,s}$ to be the identity, then we obtain a 1-cocycle $\mathbf{\Phi} = (\Phi_{n,s}) \in Z^1(\Lambda, \mathbf{G}_{r+K-1})$. However, there is an integer $K_2 > 0$ such that, for every $v \in \mathbb{N}$, the natural map between non-commutative Čech cohomologies

$$H^1(\Lambda; \mathbf{G}_{v+K_2}) \longrightarrow H^1(\Lambda; \mathbf{G}_v)$$

is constant equal to the cocycle constituted by the identity maps. This fact is a version (without parameters) of the "Théorème de détermination finie (1.4.8)" in [M-S]. For $K > K_1 + K_2$, this result applied to the cohomology class of $\mathbf{\Phi}$ yields germs of holomorphic automorphisms $\Phi_\varrho : (M, \mathsf{U}_\varrho) \to (M, \mathsf{U}_\varrho)$, $(r+K_1)$-tangent to the identity over $\mathcal{D}$, such that $\Phi_{\varrho_1,\varrho_2} = \Phi_{\varrho_1} \circ \Phi_{\varrho_2}^{-1}$, and where $\varrho \in \{1, \ldots, n+2\} \cup \mathrm{Sing}\,(\widetilde{\mathcal{F}})$. Therefore over the intersections $\mathsf{U}_{\varrho_1} \cap \mathsf{U}_{\varrho_1} \neq \emptyset$, we have

$$\Phi^*_{n+1}\mathcal{H}_{n+1} = \Phi^*_{s_1}\widetilde{\mathcal{F}}\,, \quad \Phi^*_{n+1}\mathcal{H}_{n+1} = \Phi^*_{s_2}\widetilde{\mathcal{F}}\,, \quad \Phi^*_{n+1}\mathcal{H}_{n+1} = \Phi^*_{s_0}\mathcal{H}_{s_0}\,,$$

$$\Phi^*_j\widetilde{\mathcal{F}} = \Phi^*_s\widetilde{\mathcal{F}} \quad \text{for} \quad j \neq n+1 \text{ and } s \in C_j \cap \mathrm{Sing}(\widetilde{\mathcal{F}})\,.$$

The foliations $\Phi_j^* \widetilde{\mathcal{F}}$ with $j \neq n+1$, $\Phi_{n+1}^* \mathcal{H}_{n+1}$, $\Phi_{s_i}^* \widetilde{\mathcal{F}}$ with $i \neq 0$ and $\Phi_{s_0}^* \mathcal{H}_{s_0}$ can then be glued together. This gluing yields a global foliation $\widetilde{\mathcal{F}}'$ defined on a neighborhood of $\mathcal{D}$ which, by construction, is $(r+K_1)$-tangent to the foliation $\widetilde{\mathcal{F}}$. Naturally $\widetilde{\mathcal{F}}'$ is the pull-back by E_S of a foliation $\mathcal{F}'$ defined about $(0,0) \in \mathbb{C}^2$. In turn, Condition $(*)$ ensures that the latter foliation is tangent to $\mathcal{F}$ to order r. The lemma is proved. □

Proof of Theorem 5.1. We shall keep the preceding notations. In particular $S = \{y^2 - x^{2n+1} = 0\}$ is the (unique) separatrix of the foliation $\mathcal{F}$. Similarly f, g denote the holonomy maps associated to the foliation $\widetilde{\mathcal{F}} = E_s^* \mathcal{F}$ over the manifold M arising from the desingularization of S. Finally r and K will stand for certain integers to be chosen later on. Thanks to Theorem A, there is a pair $(h_1, h_2) \in (\mathrm{Diff}_{r+K}(\mathbb{C},0))^2$ such that, on every sufficiently small neighborhood $V \subset \Sigma$ of t_0, the map $W(h_1^{-1} \circ f \circ h_1, h_2^{-1} \circ g \circ h_2) : \mathrm{Dom}\,(V,W) \to \Sigma$ has only isolated fixed points, provided that $W(a,b)$ is a reduced word as in Section 2. Modulo choosing K sufficiently large, Lemma 5.3 yields a nilpotent foliation $\mathcal{F}'$ having S as separatrix and such that the holonomy maps associated to its transform $\widetilde{\mathcal{F}}' = E_S^* \mathcal{F}'$ over the loops γ_1, γ_2 are given by the local diffeomorphisms $\tilde{f} = h_1^{-1} \circ f \circ h_1$, $\tilde{g} = h_2^{-1} \circ g \circ h_2$, respectively. Besides the vector field X' defining $\mathcal{F}'$ has the same r-jet as X at the origin.

Let now the fundamental system $\{U_j\}_j$ be chosen by applying to $\mathcal{F}'$ the main result in [Ma-M]. Consider also the statement of Lemma 5.2 applied to $\mathcal{F}'$. Thus all the points in $\Sigma \cap U_j$ that happen to be the base point of a loop contained in a leaf L of the restriction $\mathcal{F}'_{|U_j}$ of $\mathcal{F}'$ to U_j that is not homotopic to a point inside L are necessarily fixed by one application of the form $W(h_1^{-1} \circ f \circ h_1, h_2^{-1} \circ g \circ h_2) : \mathrm{Dom}\,(\Sigma \cap U_n, w) \to \Sigma$, where $W(a,b)$ is a suitable reduced word. The set formed by all fixed points of this application is countable since Theorem A asserts that each of these fixed points are isolated. Since the set of possible (reduced) words $W(a,b)$ is countable as well, it follows that only countably many non-simply connected leaves of $\mathcal{F}'_{|U_j}$ may intersect Σ. Nonetheless every leaf of $\mathcal{F}'_{|U_j}$ intersects Σ thanks to property $(\imath\imath)$ of U_j above. Therefore property $(\imath\imath\imath)$ allows us to conclude the statement. □

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