E-LOCAL PSEUDOVARIETIES

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ABSTRACT. Generalizing a property of the pseudovariety of all aperiodic semigroups observed by Tilson, we call E-*local* a pseudovariety V which satisfies the following property: for a finite semigroup, the subsemigroup generated by its idempotents belongs to V if and only if so do the subsemigroups generated by the idempotents in each of its regular $\mathcal{D}$-classes. In this paper, we characterize the E-local pseudovarieties. We also determine several examples of the smallest E-local pseudovariety containing a given pseudovariety and present some necessary or sufficient conditions for a pseudoidentity to define an E-local pseudovariety.

1. INTRODUCTION

The motivation for this work came from an exercise suggested by Pin [5] about a result from Tilson [8]. With the aim of finding a method for computing the complexity of a finite semigroup in terms of the structure of its subsemigroups, Tilson started by establishing a useful method for computing the group-complexity of a finite semigroup with at most two non-zero $\mathcal{D}$-classes. This led him to prove the following result: given a finite semigroup S, the subsemigroup $\langle E(S)\rangle$ is aperiodic if and only if, for every regular $\mathcal{D}$-class D of S, the subsemigroup $\langle E(D)\rangle$ is aperiodic.

As a consequence of the work of Fitz-Gerald [4], we have that a regular semigroup is orthodox if and only if the product of idempotents of every regular $\mathcal{D}$-class of S is idempotent. Thus, it suffices to analyze the property of the product of idempotents to be an idempotent on every regular $\mathcal{D}$-class to conclude the property for an arbitrary product of idempotents.

Much work has been done on the structure of idempotent-generated semigroups. So, it becomes interesting to determine the pseudovarieties V satisfying the following property: given $S \in \mathsf{S}$, $\langle E(S)\rangle \in \mathsf{V}$ if and only if $\langle E(D)\rangle \in \mathsf{V}$, for each regular $\mathcal{D}$-class D of S. We call *E-local* a pseudovariety with this property.

In this paper, we characterize the E-local pseudovarieties. We start by recalling, in Section 2, some basics of the theory of pseudovarieties of semigroups, in particular, some results concerning the block operator $\mathsf{B}\_$ and the idempotent-generated subsemigroups of a semigroup. Section 3, which is the main section, concerns the study of E-local pseudovarieties: we observe some properties and examples, we make a complete characterization of E-local pseudovarieties, and we introduce a new operator, $\_^{\mathsf{E}}$, where V^{E} is the smallest E-local pseudovariety containing a pseudovariety V. Finally, in Section 4, we present some necessary or sufficient conditions for a pseudoidentity to define an E-local pseudovariety.

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2. Preliminaries

We briefly recall some basics of the theory of pseudovarieties of semigroups. We recommend [1, 5, 7] for a better understanding of this area.

Let S be a semigroup. We denote by $E(S)$ the set of idempotents of S and by $\langle E(S)\rangle$ the subsemigroup of S generated by $E(S)$. More generally, $\langle X\rangle$ denotes the subsemigroup of S generated by $X \subseteq S$. In case S is finite, s^ω denotes the unique idempotent in the subsemigroup generated by a given $s \in S$.

Let S be a finite semigroup and let D be a regular $\mathcal{D}$-class of S. Consider the equivalence relation $\sim$ on the set of group elements of a regular $\mathcal{D}$-class D of S defined in the following way: given two group elements a and b of D, $a \sim b$ if and only if there exists an *idempotent-chain* $e_0, e_1, \ldots, e_{n-1}, e_n$ such that $a \mathrel{\mathcal{H}} e_0$, $b \mathrel{\mathcal{H}} e_n$, and either $e_i \mathrel{\mathcal{R}} e_{i-1}$ or $e_i \mathrel{\mathcal{L}} e_{i-1}$, for all $i \in \{1, \ldots, n\}$. A *block of* D is the Rees quotient of the subsemigroup of S generated by a $\sim$-class modulo the ideal consisting of the elements that are not in D. The *blocks of* S are the blocks of its regular $\mathcal{D}$-classes.

A class of finite semigroups that is closed under taking subsemigroups, homomorphic images and finite direct products is called a *pseudovariety* and generally denoted by V. For example, S denotes the pseudovariety of all finite semigroups.

We may construct new pseudovarieties from known ones by applying operators to pseudovarieties. In this paper, we use the following operators on pseudovarieties:

- EV consists of all $S \in \mathsf{S}$ such that $\langle E(S)\rangle \in \mathsf{V}$;
- DV consists of all $S \in \mathsf{S}$ such that, for every regular $\mathcal{D}$-class D of S, $D \in \mathsf{V}$;
- for a pseudovariety H of groups, $\bar{\mathsf{H}}$ consists of all $S \in \mathsf{S}$ such that every subgroup G of S belongs to H;
- BV consists of all $S \in \mathsf{S}$ such that, for every block B of S, $B \in \mathsf{V}$;
- $\mathsf{V_E}$ is the pseudovariety generated by the idempotent-generated semigroups of V.

The last operator was introduced in Almeida and Moura [2] and we refer the reader to that paper as needed, but we opt to present here an easy lemma that will be used frequently in this paper:

Lemma 2.1 (Almeida and Moura [2])**.** *The operator* $\_\mathsf{E}$ *has the following properties, where* V *and* W *are arbitrary pseudovarieties:*

1. $\mathsf{V} \subseteq \mathsf{W}$ *implies* $\mathsf{V_E} \subseteq \mathsf{W_E}$*;*
2. $(\mathsf{V} \cap \mathsf{W})_\mathsf{E} \subseteq \mathsf{V_E} \cap \mathsf{W_E}$*;*
3. $(\mathsf{V_E})_\mathsf{E} = \mathsf{V_E}$*;*
4. $(\mathsf{EV})_\mathsf{E} = \mathsf{V_E}$*;*
5. $\mathsf{E}(\mathsf{V_E}) = \mathsf{EV}$.

The main aim of our study is the characterization of the E-local pseudovarieties. For this purpose, we need some results concerning idempotent-generated subsemigroups and blocks of such subsemigroups.

Lemma 2.2. *For every pseudovariety* V*,* $\mathsf{BBV} = \mathsf{BV}$.

Lemma 2.3. *Let* $S \in \mathsf{S}$ *and* $X \subseteq E(S)$*. Then* $\langle E\langle X\rangle\rangle = \langle X\rangle$*.*

To prove that the idempotent-generated subsemigroup of a regular semigroup is also regular, Fitz-Gerald [4] uses a technique that consists in writing a product of idempotents of $\langle E(S)\rangle$ as a product of idempotents of $\langle E(D)\rangle$, for a regular $\mathcal{D}$-class

D of S. As a consequence, we have the following lemma whose statement and proof may be found in [7, Lemma 4.13.1], for example. It enables us to easily conclude the statement presented in the introduction that a regular semigroup S is orthodox if and only if the product of idempotents of every regular $\mathcal{D}$-class of S is idempotent.

Lemma 2.4. *Let S be a semigroup and let $s \in \langle E(S)\rangle$ be an element of a regular $\mathcal{J}$-class J of S. Then, there exists an idempotent-chain $e_1, e_2, \ldots, e_m \in E(J)$ such that $s = e_1 e_2 \cdots e_m$. Hence $\langle E(S)\rangle \cap J = \langle E(J)\rangle \cap J$.*

Corollary 2.5. *Every finite semigroup S has the following properties:*

1. *Let a be a regular element of $\langle E(S)\rangle$. Then a is in a block of D_a, where D_a is the regular $\mathcal{D}$-class of S containing a.*
2. *Let B be a block of $\langle E(S)\rangle$. Then B is also a block of $\langle E(D)\rangle$, for some regular $\mathcal{D}$-class D of S.*
3. *Given $X \subseteq E(S)$, the regular $\mathcal{D}$-classes of $\langle X\rangle$ have only one block.*

Proof. (1) and (2) follow immediately from Lemma 2.4 and from the definition of block of S. Now, by Lemma 2.3 and by (1), we have that every regular element of $\langle E\langle X\rangle\rangle = \langle X\rangle$ is in a block of $\langle E\langle X\rangle\rangle = \langle X\rangle$ and we have (3). □

A *pseudoidentity* is a formal equality $u = v$, where $u, v \in \overline{\Omega}_A \mathsf{S}$, the set of A-ary implicit operations. We say that $S \in \mathsf{V}$ satisfies $u = v$, and we write $S \models u = v$, if $u_S = v_S$. Recall that an A-ary operation $u_S : S^A \to S$ has the following property: for every homomorphism $\varphi : S \to T$, with $S, T \in \mathsf{V}$, the following diagram commutes:

$$\begin{array}{ccc} S^A & \xrightarrow{u_S} & S \\ {\scriptstyle \varphi^A}\downarrow & & \downarrow{\scriptstyle \varphi} \\ T^A & \xrightarrow[u_T]{} & T \end{array}$$

Reiterman's Theorem [6] says that every pseudovariety is defined by some set of finitary pseudoidentities, in the sense that it is the class of finite semigroups satisfying this set of pseudoidentities. The converse of the theorem is easily verified.

In this paper, we use, in particular, the pseudovarieties that we list below together with some corresponding bases of pseudoidentities defining them:

$\mathsf{I} = [\![x = y]\!]$	trivial semigroups;
$\mathsf{J} = [\![(xy)^\omega x = (xy)^\omega = y(xy)^\omega]\!]$	$\mathcal{J}$-trivial semigroups;
$\mathsf{R} = [\![(xy)^\omega x = (xy)^\omega]\!]$	$\mathcal{R}$-trivial semigroups;
$\mathsf{L} = [\![y(xy)^\omega = (xy)^\omega]\!]$	$\mathcal{L}$-trivial semigroups;
$\mathsf{A} = [\![x^{\omega+1} = x^\omega]\!]$	aperiodic (or $\mathcal{H}$-trivial) semigroups;
$\mathsf{G} = [\![x^\omega = 1]\!]$	groups;
$\mathsf{LG} = [\![(x^\omega y)^\omega x^\omega = x^\omega]\!]$	local groups;
$\mathsf{CR} = [\![x^{\omega+1} = x]\!]$	completely regular semigroups;
$\mathsf{CS} = [\![x^{\omega+1} = x, (xyx)^\omega = x^\omega]\!]$	completely simple semigroups;
$\mathsf{RB} = [\![x^2 = x, xyx = x]\!]$	rectangular bands;

$\mathsf{LZ} = [\![xy = x]\!]$	left-zero semigroups;
$\mathsf{DA} = [\![((xy)^\omega x)^2 = (xy)^\omega x]\!]$	regular $\mathcal{D}$-classes are aperiodic semigroups;
$\mathsf{DG} = [\![(xy)^\omega = (yx)^\omega]\!]$	regular $\mathcal{D}$-classes are groups;
$\mathsf{DO} = [\![(xy)^\omega (yx)^\omega (xy)^\omega = (xy)^\omega]\!]$	regular $\mathcal{D}$-classes are orthodox semigroups;
$\mathsf{DS} = [\![((xy)^\omega x)^{\omega+1} = (xy)^\omega x]\!]$	regular $\mathcal{D}$-classes are semigroups.

3. E-local pseudovarieties

We start this section by observing some properties of E-local pseudovarieties and several examples of pseudovarieties having this property. In particular, we prove, in Example 3.8, the result of Tilson [8] that the pseudovariety A is E-local. After that, we characterize the E-local pseudovarieties. We finish with the introduction of the operator $\_^{\mathsf{E}}$, where V^{E} denotes the smallest E-local pseudovariety containing V.

3.1. **Properties and examples.** We start by noting that the property of being E-local is preserved under intersection. Next, we relate the E-locality of V, EV and $\mathsf{V_E}$.

Lemma 3.1. *Let* V *be a pseudovariety and let* $S \in \mathsf{S}$. *The following conditions are equivalent:*

1. *for every regular* $\mathcal{D}$*-class* D *in* S, $\langle E(D)\rangle \in \mathsf{V}$;
2. *for every regular* $\mathcal{D}$*-class* D *in* S, $\langle E(D)\rangle \in \mathsf{EV}$;
3. *for every regular* $\mathcal{D}$*-class* D *in* S, $\langle E(D)\rangle \in \mathsf{V_E}$.

Proof. (3) $\Rightarrow$ (1) $\Rightarrow$ (2): This follows immediately from $\mathsf{V_E} \subseteq \mathsf{V} \subseteq \mathsf{EV}$.

(2) $\Rightarrow$ (3): Note that $\langle E(D)\rangle \in \mathsf{EV}$ if and only if $\langle E\langle E(D)\rangle\rangle = \langle E(D)\rangle \in \mathsf{V}$, by Lemma 2.3. By the same lemma and by the definition of $\mathsf{V_E}$, we deduce that $\langle E\langle E(D)\rangle\rangle = \langle E(D)\rangle \in \mathsf{V_E}$. □

Similarly, we may prove the following lemma:

Lemma 3.2. *The following conditions are equivalent for every pseudovariety* V *and every finite semigroup* S:

1. $\langle E(S)\rangle \in \mathsf{V}$;
2. $\langle E(S)\rangle \in \mathsf{EV}$;
3. $\langle E(S)\rangle \in \mathsf{V_E}$. □

The equivalence of E-locality for the pseudovarieties V, EV, and $\mathsf{V_E}$ follows directly from the previous lemmas.

Corollary 3.3. *Let* V *be a pseudovariety. The following conditions are equivalent:*

1. V *is* E*-local;*
2. EV *is* E*-local;*
3. $\mathsf{V_E}$ *is* E*-local.*

The properties of the operator $\_\mathsf{_E}$ (see Lemma 2.1) together with the previous corollary enable us to identify intervals consisting of E-local pseudovarieties.

Proposition 3.4. *Let* V *be an* E*-local pseudovariety. Then any pseudovariety* $\mathsf{U} \in [\mathsf{V_E}, \mathsf{EV}]$ *is* E*-local.*

Proof. Applying Lemma 2.1, we obtain $\mathsf{EV} = \mathsf{E}(\mathsf{V_E}) \subseteq \mathsf{EU} \subseteq \mathsf{EV}$. The result now follows from Corollary 3.3. □

In an attempt to identify all E-local pseudovarieties, we start by determining several families of pseudovarieties satisfying this property.

Proposition 3.5. *Let* V *and* H*, with* $\mathsf{H} \subseteq \mathsf{G}$*, be pseudovarieties. Then:*

(1) BV *is* E*-local;*
(2) DV *is* E*-local;*
(3) $\bar{\mathsf{H}}$ *is* E*-local.*

Proof. (1) follows directly from item (2) of Corollary 2.5.

(2) By items (2) and (3) from Corollary 2.5, we have that, for every regular $\mathcal{D}$-class D of $\langle E(S)\rangle$, there exists a regular $\mathcal{D}$-class D' of S such that D is a $\mathcal{D}$-class of $\langle E(D')\rangle$. Let S be a semigroup such that, for every regular $\mathcal{D}$-class D, $\langle E(D)\rangle \in \mathsf{DV}$. Then, every regular $\mathcal{D}$-class of $\langle E(D)\rangle$ is a semigroup in V. It follows that every regular $\mathcal{D}$-class of $\langle E(S)\rangle$ is a semigroup in V.

(3) Let $S \in \mathsf{S}$ be such that, for every regular $\mathcal{D}$-class D, $\langle E(D)\rangle \in \bar{\mathsf{H}}$. Let T be a subgroup of $\langle E(S)\rangle$. By Lemma 2.4, $T \subseteq \langle E(D_T)\rangle$, where D_T is the $\mathcal{D}$-class of S containing T. Hence $T \in \mathsf{H}$ and $\langle E(S)\rangle \in \bar{\mathsf{H}}$. □

Example 3.6. Since $\mathsf{J} = \mathsf{DI}$ (see Pin [5, Proposition III.4.1]), it follows from Proposition 3.5 that J is E-local.

Example 3.7. Since $\mathsf{R} = \mathsf{DLZ}$ (see Pin [5, Proposition III.4.1]), it follows from Proposition 3.5 that R is E-local.

Example 3.8. To conclude the result from Tilson [8], it suffices to note that $\mathsf{A} = \bar{\mathsf{I}}$. So, the conclusion that the pseudovariety is E-local follows immediately from Proposition 3.5.

3.2. **Characterizations.** The properties of the operators $\mathsf{E}_{\_}$ and $\mathsf{B}_{\_}$ are useful to characterize the E-local pseudovarieties, as follows:

Theorem 3.9. *The following conditions are equivalent:*

(1) V *is* E*-local;*
(2) $\mathsf{EV} = \mathsf{EBEV}$*;*
(3) *there exists* W *such that* $\mathsf{EV} = \mathsf{EBW}$*;*
(4) *there exists* W *such that* $(\mathsf{EBW})_{\mathsf{E}} \subseteq \mathsf{V} \subseteq \mathsf{EBW}$*;*
(5) $\mathsf{EV} = \mathsf{BEV}$*;*
(6) *there exists* W *such that* $\mathsf{EV} = \mathsf{BW}$*.*

Proof. (1) ⇒ (2): The direct inclusion follows from $\mathsf{EV} \subseteq \mathsf{BEV} \subseteq \mathsf{EBEV}$. For the converse, let $S \in \mathsf{EBEV}$, i.e., for every block B of $\langle E(S)\rangle$, $\langle E(B)\rangle \in \mathsf{V}$. We want to show that $S \in \mathsf{EV}$, i.e., $\langle E(S)\rangle \in \mathsf{V}$. Using the E-locality of V, it suffices to show that, for every regular $\mathcal{D}$-class D of S, $\langle E(D)\rangle \in \mathsf{V}$. Let D be a regular $\mathcal{D}$-class of S. Using again the E-locality of V and Lemma 2.3, we prove that, for every regular $\mathcal{D}$-class D' of $\langle E(D)\rangle$, $\langle E(D')\rangle \in \mathsf{V}$. Recall that, by item (3) from Corollary 2.5, D' has a unique block. Hence $D' \subseteq B$, for some block B of $\langle E(D)\rangle$ and, therefore, $\langle E(D')\rangle \leq \langle E(B)\rangle \in \mathsf{V}$. Hence $\langle E(S)\rangle \in \mathsf{V}$ and $S \in \mathsf{EV}$.

(2) ⇒ (3) ⇒ (1): The first implication is trivial and the second follows from item (1) of Proposition 3.5 and Corollary 3.3.

(3) ⇔ (4): Given W such that $\mathsf{EV} = \mathsf{EBW}$, it follows, by Lemma 2.1, that $(\mathsf{EBW})_{\mathsf{E}} = (\mathsf{EV})_{\mathsf{E}} = \mathsf{V}_{\mathsf{E}} \subseteq \mathsf{V} \subseteq \mathsf{EV} = \mathsf{EBW}$. Conversely, if $(\mathsf{EBW})_{\mathsf{E}} \subseteq \mathsf{V} \subseteq \mathsf{EBW}$ for some W, applying the increasing operator $\mathsf{E}_{\_}$, we obtain, by the same lemma, $\mathsf{E}((\mathsf{EBW})_{\mathsf{E}}) = \mathsf{EBW} \subseteq \mathsf{EV} \subseteq \mathsf{EEBW} = \mathsf{EBW}$, i.e., $\mathsf{EV} = \mathsf{EBW}$.

(2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1): Since $\mathsf{EV} \subseteq \mathsf{BEV} \subseteq \mathsf{EBEV}$, if $\mathsf{EV} = \mathsf{EBEV}$, then $\mathsf{EV} = \mathsf{BEV}$ and we have the first implication. The second implication is trivial. Finally, if there exists W such that $\mathsf{EV} = \mathsf{BW}$, it follows, by Proposition 3.5 and Corollary 3.3, that V is E-local. □

Corollary 3.10. *Let* $\mathsf{V} \subseteq \mathsf{EDS}$ *be such that* $\mathsf{V} = \mathsf{EV}$. *Then* V *is* E*-local if and only if there exists* $\mathsf{W} \subseteq \mathsf{CS}$ *such that* $\mathsf{V} = \mathsf{BW}$.

Proof. Suppose that V is E-local. Since $\mathsf{V} = \mathsf{EV}$ and by item (5) from Theorem 3.9, we have $\mathsf{V} = \mathsf{EV} = \mathsf{BEV} = \mathsf{BV}$ with $\mathsf{V} \subseteq \mathsf{EDS}$. Given $S \in \mathsf{V} = \mathsf{BV}$, we have that every block B of S is such that $B \in \mathsf{V} \subseteq \mathsf{EDS}$ and, therefore, $\langle E(B)\rangle \in \mathsf{DS}$, i.e., $\langle E(B)\rangle \in \mathsf{CS}$. Note that $\langle E(B)\rangle$ has the same structure in $\mathcal{R}$-classes and $\mathcal{L}$-classes as B, but it may have less elements in the $\mathcal{H}$-classes. Hence $B \in \mathsf{CS}$. It follows that $B \in \mathsf{V} \cap \mathsf{CS}$ and $S \in \mathsf{B}(\mathsf{V} \cap \mathsf{CS})$. The converse follows from $\mathsf{B}\_$ being an increasing operator and from $\mathsf{V} = \mathsf{BV}$. Thus, we have $\mathsf{V} = \mathsf{BV} = \mathsf{B}(\mathsf{V} \cap \mathsf{CS})$. The converse implication follows directly from Proposition 3.5. □

Theorem 3.9 gives several characterizations of E-local pseudovarieties. It is an easy exercise to verify the E-locality of the pseudovarieties J, R, DS and A, for example, using such characterizations.

3.3. **The operator $\_^{\mathsf{E}}$.** Because there are pseudovarieties V which are not E-local, it is natural to consider the smallest E-local pseudovariety containing V, which we denote V^{E}.

In this subsection, we determine some pseudovarieties of the form V^{E}. For that purpose, we also use the operator $\__{\mathsf{E}}$ which is studied in detail in [2].

Proposition 3.11. *Let* $\mathsf{V} \subseteq \mathsf{CS}$ *be such that* $\mathsf{V}_{\mathsf{E}} = \mathsf{V}$. *Then* $\mathsf{V}^{\mathsf{E}} = (\mathsf{DV})_{\mathsf{E}}$.

Proof. Let $S = \langle E(S)\rangle \in \mathsf{DV}$ and let W be any E-local pseudovariety containing V. Then $\langle E(D)\rangle \in \mathsf{V}$, for every regular $\mathcal{D}$-class D of S, since D is in V and $\langle E(D)\rangle \leq D$. Since W is E-local and $\mathsf{V} \subseteq \mathsf{W}$, it follows that $S = \langle E(S)\rangle \in \mathsf{W}$. Hence $(\mathsf{DV})_{\mathsf{E}} \subseteq \mathsf{W}$ and, therefore, $(\mathsf{DV})_{\mathsf{E}} \subseteq \mathsf{V}^{\mathsf{E}}$.

For the direct inclusion, since $\mathsf{V} \subseteq \mathsf{CS}$, we have $\mathsf{V} \subseteq \mathsf{DV}$. As $\__{\mathsf{E}}$ is an increasing operator, it follows that $\mathsf{V} = \mathsf{V}_{\mathsf{E}} \subseteq (\mathsf{DV})_{\mathsf{E}}$. By Proposition 3.5, DV is E-local and so is $(\mathsf{DV})_{\mathsf{E}}$, by Corollary 3.3. This yields the inclusion $\mathsf{V}^{\mathsf{E}} \subseteq (\mathsf{DV})_{\mathsf{E}}$. □

Corollary 3.12. *The class* J *is the smallest* E*-local pseudovariety.*

Proof. Let V be an E-local pseudovariety. Since $\mathsf{I} \subseteq \mathsf{V}$, we have $\mathsf{I}^{\mathsf{E}} \subseteq \mathsf{V}^{\mathsf{E}} = \mathsf{V}$. By Proposition 3.11, we have $\mathsf{I}^{\mathsf{E}} = (\mathsf{DI})_{\mathsf{E}} = \mathsf{J}_{\mathsf{E}} = \mathsf{J}$, where the last equality follows from [2, Corollary 5.6]. Hence $\mathsf{J} \subseteq \mathsf{V}$. By Example 3.6, J is E-local, which establishes the corollary. □

Example 3.13. It follows from Corollary 3.12 that the pseudovarieties LG and CR are not E-local.

Example 3.14. By Proposition 3.11, we conclude that $(\mathsf{RB})^{\mathsf{E}} = (\mathsf{DRB})_{\mathsf{E}} = (\mathsf{DA})_{\mathsf{E}} = \mathsf{DA}$, where the last equality follows by [2, Corollary 5.6], and $(\mathsf{CS})^{\mathsf{E}} = (\mathsf{DCS})_{\mathsf{E}} = (\mathsf{DS})_{\mathsf{E}}$.

Example 3.15. By Example 3.14, we have $(\mathsf{DS})_{\mathsf{E}} = (\mathsf{CS})^{\mathsf{E}} \subseteq (\mathsf{CR})^{\mathsf{E}}$. Conversely, since $\mathsf{CR} \subseteq \mathsf{DS}$, by [2, Proposition 3.16] and Lemma 2.1, we deduce that $\mathsf{CR} = (\mathsf{CR})_{\mathsf{E}} \subseteq (\mathsf{DS})_{\mathsf{E}}$. Note that $(\mathsf{DS})_{\mathsf{E}}$ is E-local, by Proposition 3.5 and Corollary 3.3. Hence $(\mathsf{CR})^{\mathsf{E}} \subseteq (\mathsf{DS})_{\mathsf{E}}$, which establishes the equality $(\mathsf{CR})^{\mathsf{E}} = (\mathsf{DS})_{\mathsf{E}}$.

Example 3.16. Let H be a pseudovariety of groups. By [2, Example 3.7], we have $(\mathsf{DH})_{\mathsf{E}} = \mathsf{J} \subseteq \mathsf{J} \vee \mathsf{H} \subseteq \mathsf{DH}$. Since, by Proposition 3.5, DH is E-local, it follows, by Proposition 3.4, that $\mathsf{J} \vee \mathsf{H}$ is E-local. That $\mathsf{J} \vee \mathsf{H}$ is the smallest E-local pseudovariety containing H, is an immediate consequence from the fact that J is the smallest E-local pseudovariety. Thus $\mathsf{H}^{\mathsf{E}} = \mathsf{J} \vee \mathsf{H}$.

Example 3.17. Let H be a pseudovariety of groups. By Example 3.14, we have $\mathsf{DA} = (\mathsf{RB})^{\mathsf{E}} \subseteq (\mathsf{RB} \vee \mathsf{H})^{\mathsf{E}}$ and, therefore, $\mathsf{DA} \vee \mathsf{H} \subseteq (\mathsf{RB} \vee \mathsf{H})^{\mathsf{E}}$. Now, from [2, Lemma 3.1, Example 3.8, Corollary 4.2], we obtain $(\mathsf{DO} \cap \bar{\mathsf{H}})_{\mathsf{E}} \subseteq (\mathsf{DO})_{\mathsf{E}} \cap \bar{\mathsf{H}}_{\mathsf{E}} = \mathsf{DA} \cap \bar{\mathsf{H}} \subseteq \mathsf{DA}$. Therefore, $(\mathsf{DO} \cap \bar{\mathsf{H}})_{\mathsf{E}} \subseteq \mathsf{DA} \subseteq \mathsf{DA} \vee \mathsf{H} \subseteq \mathsf{DO} \cap \bar{\mathsf{H}}$. As an intersection of E-local pseudovarieties, the pseudovariety $\mathsf{DO} \cap \bar{\mathsf{H}}$ is E-local. By Proposition 3.4, $\mathsf{DA} \vee \mathsf{H}$ is E-local. Thus $(\mathsf{RB} \vee \mathsf{H})^{\mathsf{E}} = \mathsf{DA} \vee \mathsf{H}$.

Example 3.18. Since $\mathsf{LG} \subseteq \mathsf{DS}$ and DS is E-local by Proposition 3.5, we have $(\mathsf{LG})^{\mathsf{E}} \subseteq \mathsf{DS}$. On the other hand, by Example 3.14 and by [2, Example 3.17], we obtain $(\mathsf{DS})_{\mathsf{E}} = (\mathsf{CS})^{\mathsf{E}} = ((\mathsf{LG})_{\mathsf{E}})^{\mathsf{E}} \subseteq (\mathsf{LG})^{\mathsf{E}}$. Thus the equality $(\mathsf{DS})_{\mathsf{E}} \subseteq (\mathsf{LG})^{\mathsf{E}} \subseteq \mathsf{DS}$ holds.

If we prove that $(\mathsf{DS})_{\mathsf{E}} = \mathsf{DS}$, we will have the equality in the previous example. This provides additional motivation for the calculation of $(\mathsf{DS})_{\mathsf{E}}$ which remains an open problem (see [2]).

We end this subsection by noting that $(\mathsf{V} \cap \mathsf{W})^{\mathsf{E}} \subseteq \mathsf{V}^{\mathsf{E}} \cap \mathsf{W}^{\mathsf{E}}$, for all pseudovarieties V and W. However, we do not know whether equality holds.

4. E-local pseudoidentities

We call E-*local* a pseudoidentity which defines an E-local pseudovariety. Note that a pseudovariety defined by a set of E-local pseudoidentities is E-local, since it is the intersection of the E-local pseudovarieties defined by each pseudoidentity of the set. We do not know whether the converse is valid.

In 3.2 we obtained several characterizations of E-local pseudovarieties that enable us to also characterize E-local pseudoidentities. However, some of the results that we obtained, like some techniques developed allow us to give a different characterization of several pseudoidentities with this property.

For $u \in \overline{\Omega}_A \mathsf{S}$, let $\mathrm{first}(u)$ and $\mathrm{last}(u)$ be, respectively, the first and last letters of u. We relate the E-locality of the pseudoidentities of the form $u = v$, where $\mathrm{first}(u) \neq \mathrm{first}(v)$ or $\mathrm{last}(u) \neq \mathrm{last}(v)$, with the condition $\mathsf{V} \subseteq [\![u = v]\!]$, where V is one of the pseudovarieties R, L and J. We also obtain some results concerning the pseudovariety DA.

Proposition 4.1. *The following properties are verified by every pseudoidentity* $u = v$.

(1) *If* $\mathrm{last}(u) \neq \mathrm{last}(v)$ *and* $\mathsf{R} \models u = v$, *then* $u = v$ *is* E-*local.*
(2) *If* $\mathrm{first}(u) \neq \mathrm{first}(v)$ *and* $\mathsf{L} \models u = v$, *then* $u = v$ *is* E-*local.*
(3) *If* $\mathrm{first}(u) \neq \mathrm{first}(v)$, $\mathrm{last}(u) \neq \mathrm{last}(v)$ *and* $\mathsf{J} \models u = v$, *then* $u = v$ *is* E-*local.*

Proof. Let $u = v$ be a pseudoidentity such that $\mathrm{last}(u) \neq \mathrm{last}(v)$ and suppose that $\mathsf{R} \models u = v$. We claim that $\mathsf{R} \subseteq [\![u = v]\!] \subseteq \mathsf{ER}$. So that, by Example 3.7 and by Proposition 3.4, $[\![u = v]\!]$ is E-local. The first inclusion is assumed by hypothesis. To prove the second inclusion, let S be a semigroup satisfying $u = v$ and suppose that $S \notin \mathsf{ER}$, i.e., $\langle E(S) \rangle \notin \mathsf{R}$. Then, by [1, cf. Exercise 5.2.8], there exist two distinct idempotents such that $ef = f$ and $fe = e$. Evaluating the last letter of u by e, the last letter of v by f and the other letters by e or f, we obtain $S \models e = u = v = f$, which is a contradiction.

Similarly, we obtain (2) and (3). □

Since, by [2], the pseudovarieties R, L, J and DA satisfy the equality $\mathsf{V}_{\mathsf{E}} = \mathsf{V}$, it is easy to obtain the following results:

Theorem 4.2. *Let $u = v$ be an arbitrary pseudoidentity.*

(1) *If* $\mathrm{first}(u) = \mathrm{first}(v)$ *and* $\mathrm{last}(u) \neq \mathrm{last}(v)$*, then $u = v$ is E-local if and only if $\mathsf{R} \models u = v$.*
(2) *If* $\mathrm{first}(u) \neq \mathrm{first}(v)$ *and* $\mathrm{last}(u) = \mathrm{last}(v)$*, then $u = v$ is E-local if and only if $\mathsf{L} \models u = v$.*
(3) *If* $\mathrm{first}(u) \neq \mathrm{first}(v)$ *and* $\mathrm{last}(u) \neq \mathrm{last}(v)$*, then $u = v$ is E-local if and only if $\mathsf{J} \models u = v$.*
(4) *If* $\mathrm{first}(u) = \mathrm{first}(v)$, $\mathrm{last}(u) = \mathrm{last}(v)$ *and $u = v$ is E-local, then $\mathsf{DA} \models u = v$.*

Proof. (1) Let $u = v$ be a pseudovariety such that $\mathrm{first}(u) = \mathrm{first}(v)$ and $\mathrm{last}(u) \neq \mathrm{last}(v)$. Suppose that it is E-local. Since $\mathsf{LZ} \subseteq \mathsf{CS}$ and $(\mathsf{LZ})_{\mathsf{E}} = \mathsf{LZ}$, it follows from Proposition 3.11 that $(\mathsf{LZ})^{\mathsf{E}} = (\mathsf{DLZ})_{\mathsf{E}} = \mathsf{R}_{\mathsf{E}} = \mathsf{R}$. Thus, as the pseudovariety $[\![u = v]\!]$ is E-local and it contains LZ, it also contains $(\mathsf{LZ})^{\mathsf{E}} = \mathsf{R}$. The converse follows from Proposition 4.1. Dually, we obtain (2).

(3) It follows directly from J being the smallest E-local pseudovariety (see Corollary 3.12) and from Proposition 4.1.

(4) In that case, we have $\mathsf{RB} \models u = v$, $\mathsf{RB} \subseteq \mathsf{CS}$ and $\mathsf{RB}_{\mathsf{E}} = \mathsf{RB}$. By Proposition 3.11, we have $(\mathsf{RB})^{\mathsf{E}} = (\mathsf{DRB})_{\mathsf{E}} = (\mathsf{DA})_{\mathsf{E}} = \mathsf{DA}$. As in (1), we deduce that $\mathsf{DA} = (\mathsf{RB})^{\mathsf{E}} \subseteq [\![u = v]\!]$. □

However, we do not have a characterization of all E-local pseudoidentities of the form $u = v$, with $\mathrm{first}(u) = \mathrm{first}(v)$ and $\mathrm{last}(u) = \mathrm{last}(v)$.

We finish this paper with another sufficient condition for a pseudoidentity to be E-local that follows from Lemma 2.4.

Theorem 4.3. *Let $u = v$ be a pseudoidentity such that $u, v \in \overline{\langle X \rangle}$, where all elements of $X \subseteq \overline{\Omega}_A\mathsf{S}$ lie in a same regular $\mathcal{D}$-class of $\overline{\Omega}_A\mathsf{S}$. Then $u = v$ is E-local.*

Proof. Let S be a finite semigroup and suppose that $\langle E(D) \rangle \models u = v$, for each regular $\mathcal{D}$-class D of S. We want to prove that $\langle E(S) \rangle \models u = v$. Let $\varphi : \overline{\Omega}_A\mathsf{S} \to S$ be a continuous surjective homomorphism such that, for every $x \in X$, $\varphi(x) \in \langle E(S) \rangle$. Since all elements of X lie in a same regular $\mathcal{D}$-class of $\overline{\Omega}_A\mathsf{S}$, then there exists a regular $\mathcal{D}$-class D of S such that $\varphi(x) \in D$, for all $x \in X$. By Lemma 2.4, it follows that $\varphi(x) \in \langle E(D) \rangle$, for all $x \in X$. Since $u, v \in \overline{\langle X \rangle}$, it follows that $\varphi(u), \varphi(v) \in \langle E(D) \rangle$ and, by hypothesis, they are equal. Thus $\langle E(S) \rangle \models u = v$ and $u = v$ is E-local. □

Note that several pseudoidentities considered in this paper are of this form. Specifically, the pseudoidentities that we used in Section 2 to define the pseudovarieties J, R, L, A, DA, DG, DO and DS are all of this form. Another example is the pseudoidentity $(x^\omega y^\omega)^\omega = (y^\omega x^\omega)^\omega$ which defines the pseudovariety BG. As a last example, Almeida and Volkov [3] showed that, if $u_i = v_i$, with $i \in I$, is a basis of pseudoidentities for a pseudovariety of groups H, then $u_i' = v_i'$ is a basis of pseudoidentities for $\bar{\mathsf{H}}$, where u_i' and v_i' result from the substitution of each letter

$x_j \in A$ of u_i and v_i by ex_je where e is a fixed idempotent in the minimum ideal of $\overline{\Omega}_A\mathsf{S}$. These pseudoidentities are also of the form of Theorem 4.3.

In the last result, we identify all E-local pseudoidentities with only one variable.

Corollary 4.4. *The* E*-local pseudoidentities in one variable are those of the form* $x^\alpha = x^\beta$*, with both* α *and* β *infinite.*

Proof. If α or β are finite and are not equal, then DA does not satisfy the pseudoidentity $u = v$ since DA contains all finite monogenic aperiodic semigroups. Thus, $u = v$ is not E-local, by item (4) of Theorem 4.2.

On the other hand, if α and β are infinite, then x^α and x^β are in a same group with neutral element x^ω. Thus, by Theorem 4.3, the pseudoidentity is E-local. □

We do not know if every E-local pseudovariety is defined by a set of pseudoidentities satisfying the condition of Theorem 4.3.

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