Guarding Thin Orthogonal Polygons is Hard

Ana Paula Tomás⋆

DCC & CMUP, Faculdade de Ciências
Universidade do Porto, Portugal
apt@dcc.fc.up.pt

Abstract. An orthogonal polygon of P is called "thin" if the dual graph of the partition obtained by extending all edges of P towards its interior until they hit the boundary is a tree. We show that the problem of computing a minimum guard set for either a thin orthogonal polygon or only its vertices is NP-hard, indeed APX-hard, either for guards lying on the boundary or on vertices of the polygon. For guards lying anywhere in the polygon, we show that computing an optimal guard set for the vertices of such a polygon is NP-hard.

1 Introduction

Advances in communication technologies have brought renewed attention to guarding and sensor coverage problems, so called art gallery problems [4, 5, 7, 9]. The classical art gallery problem for a polygon P asks for a minimum set of points $\mathcal{G}$ in P such that every point in P is seen by at least one point in $\mathcal{G}$ (the guard set). Many variations of art gallery problems have been studied over the years to deal with various types of constraints on guards and different notions of visibility. In the general visibility model, two points p and q in a polygon P *see* each other if the line segment $\overline{pq}$ contains no points of the exterior of P. The set $V(p)$ of all points of P visible to $p \in P$ is the *visibility region* of p. A guard set $\mathcal{G}$ for a set S is a set of points of P such that $S \subseteq \cup_{g\in\mathcal{G}} V(g)$. If $V(p)\cap S \subset V(q)\cap S$ then q strictly dominates p, and q can replace p in an optimal guard set for S. If $V(p) \cap S = V(q) \cap S$, the two points are equivalent for guarding S. Guards that may lie anywhere in P are called *point guards* whereas *vertex* or *boundary guards* are restricted to lie on vertices or on the boundary. Combinatorial upper and lower bounds on the number of necessary guards are known for specific settings (for surveys, we refer to e.g. [18, 21]). The fact that some art gallery problems are NP-hard [13, 14, 20] motivates the design of heuristic and metaheuristic methods for finding approximate solutions and the study of more specific classes of polygons where some guarding problems may be tractable [1, 4, 5, 7, 15]. In this paper, we address the set of *thin orthogonal polygons* (TOPs, for short). These are the orthogonal polygons for which the dual graph of the *grid partition* $\Pi_{HV}(P)$ is a tree. $\Pi_{HV}(P)$ is obtained by adding all horizontal and vertical cuts incident to the reflex vertices of P (see Fig. 1).

⋆ Partially funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT – Fundação para a Ciência e Tecnologia under the project PEst-C/MAT/UI0144/2011.

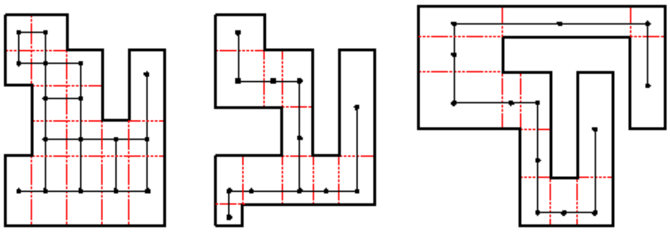

Fig. 1. Orthogonal polygons, grid partitions and dual graphs: $\Pi_{HV}(P)$ and its dual graph in general, a thin orthogonal polygon and a path orthogonal polygon.

We show that finding an optimal guard set for *the vertices* of a TOP is NP-hard, which was known for generic orthogonal polygons [12]. Although our proof is inspired in [12], the need to obtain a TOP led to novel aspects in the construction. In addition, we adapt it to show that guarding a TOP is NP-hard for vertex or boundary guards. We remark that the proofs developed previously for polyominoes [4] and generic orthogonal polygons [20] do not apply to this class as the dual graph of the corresponding partition is not a tree. We note also that the class of TOPs strictly contains the class of thin polyomino trees introduced in [4], for which the authors conjecture that the guarding problem under the general visibility model has a polynomial-time exact algorithm. To the best of our knowledge, the complexity of this problem remains open.

In [23], we give a linear-time algorithm for computing an optimal vertex guard set for any given *path* orthogonal polygon (i.e., a TOP such that the dual graph of $\Pi_{HV}(P)$ is a path graph), and prove tight lower and upper bounds of $\lceil n/6 \rceil$ and $\lfloor n/4 \rfloor$ for the optimal solution for the subclass where all horizontal and vertical cuts intersect the boundary at Steiner points. We show also that a minimum guard set for the vertices of a path orthogonal polygon can be found in linear-time. This work extends [15], as the *thin grid orthogonal polygons* are path orthogonal polygons. Our motivation for studying these classes comes also from previous work on generation and guarding [22] and the empirical observation that for random grid orthogonal polygons, the minimum number of vertex guards is often less than the theoretical bound of $\lfloor \frac{n}{4} \rfloor$, and often around $\frac{n}{6}$, e.g., for the sample instances of [7]. Since the grid orthogonal polygons have been used in recent works for the evaluation of heuristics and exact methods, e.g. [5, 7], we found it worthwhile trying to understand the structure of these related classes.

In rest of the paper, in sections 2 to 4, we show that computing a minimum guard set for *the vertices* of a thin orthogonal polygon (GVTP) is NP-hard, either for boundary, vertex or point guards. In section 5, we show that computing a minimum guard set for *the polygon* (GTP) is NP-hard either for boundary or vertex guards. For vertex and boundary guards, our reductions are based on the vertex cover problem in graphs, which is known to be APX-complete, even for graphs with bounded degree [2]. The constructions are still valid if the graph has bounded degree. Hence, in Section 6, we show that the corresponding guarding problems are APX-hard, as well as for generic orthogonal polygons.

2 GVTP for Boundary Guards

Theorem 1. GVTP *is NP-hard for boundary guards.*

For the proof, we define a reduction directly from the VERTEX-COVER problem in graphs to GVTP with *boundary-guards*, instead of from the minimum 2-interval piercing problem used in [12]. In this way, we can control the aperture of visibility cones and the structure of the thin orthogonal polygon we obtain. A vertex-cover of $G = (V, E)$ is a subset $S \subseteq V$ such that for each edge $(u, v) \in E$, either $u \in S$, or $v \in S$, or both. To decide whether G has a vertex-cover S of size $|S| \leq k$, for k integer, is a NP-complete problem. Without loss of generality, we assume that $E \neq \emptyset$ and G contains no isolated vertices. Now, the TOP we construct for a given graph G is essentially a large square with $|E|$ tiny gadgets attached to its bottom. In Fig. 2 we sketch this construction. We fix the side-

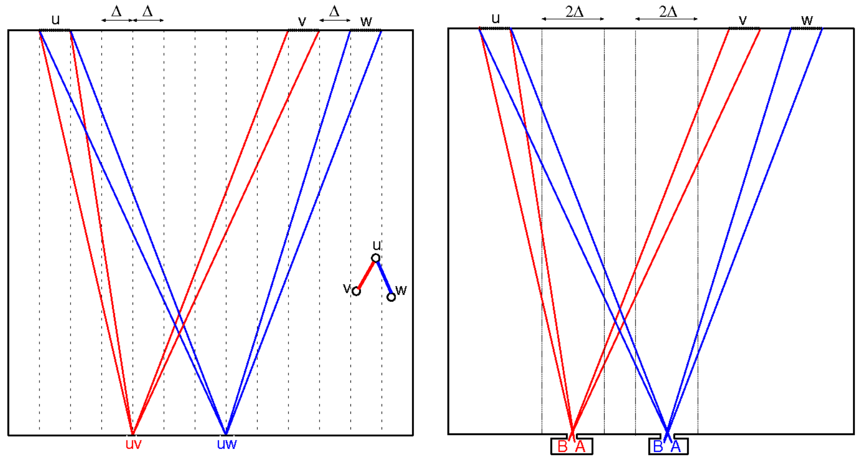

Fig. 2. The reduction from VERTEX-COVER to GVTP with boundary guards for $G = (\{u, v, w\}, \{(u, v), (u, w)\})$. The edges of G are mapped to points uv and uw that will be replaced by tiny d-gadgets. The vertices are mapped to the segments u, v and w.

length of this square to be $L\Delta$, with $L = 1 + 2|V| + 3|E|$ and $\Delta = 10L$. We consider $V = \{v_1, v_2, \dots, v_n\}$ sorted and denote by E_i^+ the subset of all edges $(v_i, v_j) \in E$ such that $i < j$, sorted by increasing value of j. In the construction we follow these orderings: for each i, we represent v_i by a segment of length Δ on the top edge of the square and the edges in E_i^+ as middle points of $|E_i^+|$ consecutive segments of length 2Δ on the bottom edge, placed between the projections of v_i and v_{i+1}, and with separation gaps of length Δ between each other. The square is implicitly divided into L slabs of length Δ, and we leave the first slab empty and an empty slab between consecutive items. Fig. 3 presents the *double gadget* (d-gadget) we defined for the proof. The vertices on the left side are N_1

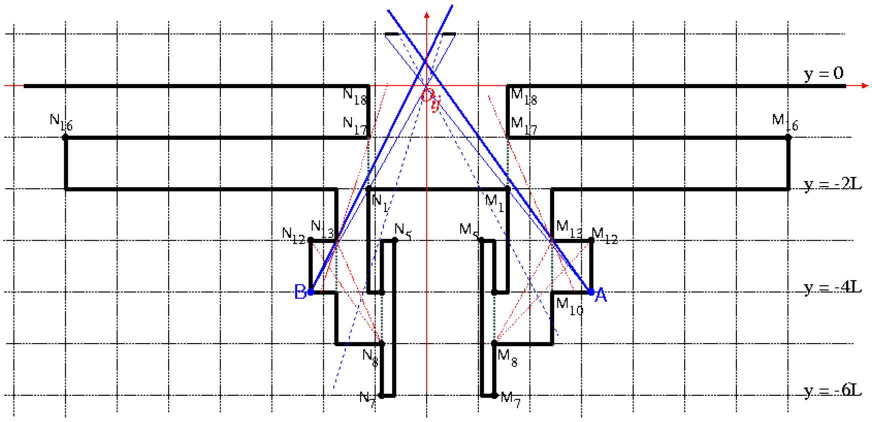

Fig. 3. A sketch of the d-gadget $\varXi_{ij}$ defined for the edge (v_i, v_j).

to N_{18} in CW order and on the right side are M_1 to M_{18} in CCW order. The idea for the d-gadget associated to an edge $(v_i, v_j) \in E_i^+$, denoted by $\varXi_{ij}$, is as follows. Let O_{ij} be the point that stands for the edge (v_i, v_j) and $\overline{A_iB_i}$ and $\overline{A_jB_j}$ the segments associated to v_i and v_j. Together with O_{ij}, these segments define two visibility cones with apex O_{ij}. By a slight perturbation, we decouple the two cones and move the new apexes to the distinguished vertices (B and A) of a tiny d-gadget $\varXi_{ij}$. The structure of $\varXi_{ij}$ fixes segment $\overline{A_iB_i}$ (resp. $\overline{A_jB_j}$) as the portion of the boundary of the polygon that A (resp. B) sees above line $y = 0$ (i.e, above the gadget). Some of the vertices of a d-gadget can only be guarded by a *local guard* (i.e., a guard below line $y = 0$), e.g., M_{16}, M_{12}, M_8, M_7 and M_5 on its right part and N_{16}, N_{12}, N_8, N_7 and N_5 on the left part. For every d-gadget, at least three local boundary-guards are needed to guard these vertices and no three such guards can see both A and B if they see all these vertices. Moreover, one can always locate three local boundary-guards that see all the gadget vertices other than A (namely, at N_8, N_1 and M_8) or other than B (namely, at N_8, M_1 and M_8). Another guard is required to guard the unguarded vertex but it does not need to be local. As we will see, this guard can be located on the portion of the top edge of the polygon seen from the unguarded vertex.

We define the coordinates of the vertices of $\varXi_{ij}$ w.r.t. a cartesian coordinate system $\mathcal{R}_{O_{ij}}$ with origin at O_{ij}. First we remark that, by construction, the x-coordinates of the points A_i, B_i and O_{ij} w.r.t. a cartesian system fixed at the bottom left corner of the large square are given by $x'_{A_i} = (2i-1+3\sum_{k<i}|E_k^+|)\varDelta$, $x'_{B_i} = x'_{A_i} + \varDelta$, and $x'_{O_{ij}} = x'_{B_i} + 2\varDelta + 3\varDelta\,|E_i^+ \cap \{(v_i, v_{j'}) : j' < j\}|$. As a result, if we define x_i and x_j as $x_i = (x'_{O_{ij}} - x'_{B_i})/\varDelta$ and $x_j = (x'_{A_j} - x'_{O_{ij}})/\varDelta$, then, w.r.t. the cartesian system $\mathcal{R}_{O_{ij}}$, we have

$$
\begin{aligned}
A_i &= (-(x_i+1)\varDelta,\ L\varDelta) \qquad & A_j &= (x_j\varDelta,\ L\varDelta)\\
B_i &= (-x_i\varDelta,\ L\varDelta) \qquad & B_j &= ((x_j+1)\varDelta,\ L\varDelta)
\end{aligned}
$$

for integers $x_i \geq 2$ and $x_j \geq 2$. Then, we define A and B as the intersection points of $\overrightarrow{O_{ij}A_i}$ and $\overrightarrow{O_{ij}B_j}$ with the straight line $y = -4L$, that is, as $A = (4x_i + 4, -4L)$ and $B = (-4x_j - 4, -4L)$. So, the rays $\overrightarrow{AA_i}$ and $\overrightarrow{BB_j}$ share the supporting lines of the initial rays $\overrightarrow{O_{ij}A_i}$ and $\overrightarrow{O_{ij}B_j}$. The aperture of the visibility cone $\mathcal{C}_A = cone(A, \overline{A_iB_i})$ is fixed by M_1 and M_{13}, being M_1 the intersection of $\overrightarrow{AA_i}$ with the line $y = -2L$ and M_{13} the intersection of $\overrightarrow{AB_i}$ with the line $y = -3L$. Therefore, $M_1 = (2x_i + 2, -2L)$ and $M_{13} = (\tau_i, -3L)$, with $\tau_i = 3x_i + 3 + \frac{\Delta}{\Delta+4}$, because the straight lines AA_i and AB_i are given by the following equations.

$$AA_i: \quad y = \frac{-L}{x_i + 1}x; \qquad AB_i: \quad y = \frac{-L(\Delta + 4)}{x_i(\Delta + 4) + 4}x + \frac{4L\Delta}{x_i(\Delta + 4) + 4}\ .$$

Similarly, N_1 and N_{13} determine the aperture of $\mathcal{C}_B = cone(B, \overline{A_jB_j})$, being N_1 the intersection of $\overrightarrow{BB_j}$ with $y = -2L$ and N_{13} the intersection of $\overrightarrow{BA_j}$ with $y = -3L$. The supporting lines of these rays are given by similar equations, and $N_1 = (-2x_j - 2, -2L)$ and $N_{13} = (\tilde{\tau}_j, -3L)$, with $\tilde{\tau}_j = -3x_j - 3 - \frac{\Delta}{\Delta+4}$. The coordinates of the vertices of Ξ_{ij} are

$$\begin{array}{lll}
M_1 = (2x_i + 2, -2L) & M_2 = (2x_i + 2, -4L) & M_3 = (2x_i + 1, -4L) \\
M_4 = (2x_i + 1, -3L) & M_5 = (2x_i, -3L) & M_6 = (2x_i, -6L) \\
M_7 = (2x_i + 1, -6L) & M_8 = (2x_i + 1, -5L) & M_9 = (\tau_i, -5L) \\
M_{10} = (\tau_i, -4L) & A = (4x_i + 4, -4L) & M_{12} = (4x_i + 4, -3L) \\
M_{13} = (\tau_i, -3L) & M_{14} = (\tau_i, -2L) & M_{15} = (7L, -2L) \\
M_{16} = (7L, -L) & M_{17} = (2x_i + 2, -L) & M_{18} = (2x_i + 2, 0)
\end{array}$$

with $N_k = (-\alpha x_j - \beta, \gamma)$ iff $M_k = (\alpha x_i + \beta, \gamma)$ for $1 \leq k \leq 18$. Thus, the coordinates are defined by rational numbers given as pairs of integers bounded by a quadratic polynomial function on the size of the graph.

Correctness. We note the dual graph of the grid partition of the resulting polygon is a tree, as required. Moreover, it is not difficult to conclude that M_{16}, M_{12}, M_8, M_7, M_5, and N_{16}, N_{12}, N_8, N_7 and N_5 require local guards. Still, we have to prove that the boundary of Ξ_{ij} imposes no restriction on the propagation of the corresponding visibility cones $\mathcal{C}_A$ and $\mathcal{C}_B$. We present the proof for $\mathcal{C}_A$, since it can be straightforwardly adapted to $\mathcal{C}_B$. First we observe that, by construction, $\mathcal{C}_A$ is to the right of $\overrightarrow{AA_i}$, and therefore, the cone stands to the right of O_{ij} until it leaves the gadget. This means that the left part of Ξ_{ij} cannot obstruct $\mathcal{C}_A$. We show now that $\mathcal{C}_A$ is not obstructed by the right part either. Actually, the point where $\overrightarrow{AB_i}$ intersects the line $y = -L$ is always to the left of $M_{17} = (2x_i + 2, -L)$, as depicted in Fig. 3. Indeed, the x-coordinate of such point is given by $x = x_i + 1 + \frac{3\Delta}{\Delta+4}$ and, being $x_i \geq 2$, we have $0 < x < 2x_i + 2$. Moreover, we can check that M_1 is the unique point on the boundary of Ξ_{ij} that sees both A and N_{16} (on the left), as well as the other local vertices on the top part of the d-gadget. In a similar way, we can check that the visibility cone $\mathcal{C}_B$ is not obstructed and conclude that N_1 is the unique point on the boundary of Ξ_{ij} that sees both B and M_{16} (as well as the other local vertices on the top part). Finally, to conclude that one can always locate three local boundary-guards that see all the gadget vertices other than A (namely, at N_8, N_1 and M_8) or other than B (namely, at N_8, M_1 and M_8), we have to check that M_8 sees M_{12} and, similarly, that N_8 sees N_{12}. Being $M_8 = (2x_i + 1, -5L)$ and $M_{12} = (4x_i + 4, -3L)$, the straight line M_8M_{12} is defined by the equation

$$M_8M_{12}: \qquad y = \frac{2L}{2x_i+3}x - \frac{(14x_i+17)L}{2x_i+3}$$

and, so, it intersects the line $y = -4L$ at point $(3x_i + 5/2, -4L)$. This point stands between $M_2 = (2x_i + 2, -4L)$ and $M_{10} = (\tau_i, -4L)$ because

$$2x_i + 2 < 3x_i + \frac{5}{2} < 3x_i + 3 + \frac{\Delta}{\Delta+4} = \tau_i$$

and, therefore, M_8 sees M_{12}. In a similar way, we conclude that N_8 sees N_{12}. We can check that M_8 sees M_{13} (and that N_8 sees N_{13}), but it is not too relevant. If A is seen from M_1 in an optimal solution, then M_1 guards M_{13} also, and if the guard is at the segment $\overline{A_iB_i}$ instead, then M_{13} is in its visibility region, as the visibility region of M_{13} above the d-gadget contains the visibility region of A. Lemma 1 states the final result we need to conclude the proof.

Lemma 1. *The resulting* TOP *can be guarded by* $3|E| + k$ *boundary guards iff there is a vertex-cover of size* k *for* $G = (V, E)$.

Proof. Given a vertex-cover S of G of size k, we place a guard anywhere in the segment associated to each $v \in S$. These k guards see all the vertices of P that are not below the line $y = 0$. Moreover, for each d-gadget Ξ_{ij}, at least one of two distinguished vertices (A or B) is seen by one of these k guards. The other distinguished vertex and all the remaining local vertices of Ξ_{ij} can be guarded by three local guards: N_8 and M_8, and either N_1 (which sees also B) or M_1 (which sees also A). Therefore, if there is a vertex-cover S of G of size k, there is a guard set for the vertices of P of size $3|E| + k$. Reciprocally, let us assume that there is a guard set of size $3|E| + k$. Then, the idea is to replace the local guard set of each Ξ_{ij} that has more than three guards by three local guards at N_8, M_8 and N_1 and to locate a guard at $\overline{A_iB_i}$ to guard vertex A (if this segment has no guard yet). We end up with at most k guards located on the top edge of P, which define a vertex-cover for G, since each d-gadget Ξ_{ij} will have just three local guards and, consequently, there must be a guard either on $\overline{A_iB_i}$ or $\overline{A_jB_j}$ to guard the unguarded vertex of Ξ_{ij}. □

3 GVTP for Vertex Guards

Theorem 2. GVTP *is NP-hard for vertex guards.*

For the proof, we adapt the previous construction as sketched in Fig. 4, following [12]. We consider the polygon obtained previously and attach a tiny

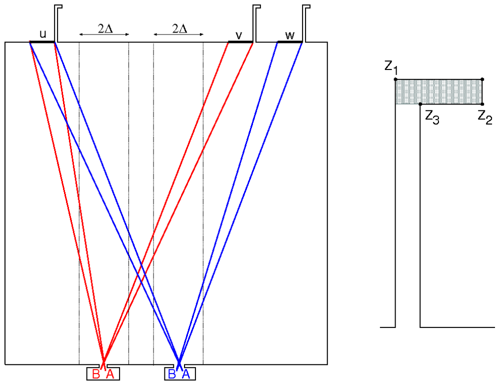

Fig. 4. The reduction from VERTEX-COVER to GVTP with vertex guards for the same graph: ear gadgets are attached to the right endpoints of the segments. Each ear gadget requires a local guard on a vertex of the shaded region (to guard Z_2).

ear gadget to the right endpoint of each line segment $\overline{A_iB_i}$, for each $v_i \in V$. The ear gadgets will be defined in such a way that the vertices denoted by A in each d-gadget Ξ_{ij} cannot see any vertex of an ear-gadget except for B_i. Otherwise, the vertex A would see points on the boundary of P arbitrarily close to B_i but to the right of B_i, which is impossible by the definition of the visibility cone $\mathcal{C}_A$. We shall see now how to adjust the height of each ear gadget to prevent also B from seeing any local vertex of the ear-gadget attached to A_jB_j. For each $j \geq 2$, it is sufficient to guarantee that, for all Ξ_{ij}, the intersection point of the ray $\overrightarrow{BB_j}$ with the vertical edge incident to the vertex labelled Z_3 is below Z_3. Actually, this result holds for all i, if we ensure that it holds for B in the rightmost d-gadget $\Xi_{i'j}$, since the rays $\overrightarrow{BB_j}$ are sorted by slope around B_j.

For each $j \geq 1$, we define the coordinates of the local vertices of the ear gadget attached to B_j w.r.t. the cartesian system fixed at the bottom left corner of the large square, as follows.

$$\begin{aligned} Z_1 &= ((x'_j+1)\Delta,\, L(\Delta+1)+1) \\ Z_3 &= ((x'_j+1)\Delta+1,\, L(\Delta+1)) \\ Z_2 &= ((x'_j+1)\Delta+L,\, L(\Delta+1)) \;. \end{aligned}$$

Clearly, the other vertex is $((x'_j+1)\Delta+L,\, L(\Delta+1)+1)$. To check that the construction is correct, we consider the coordinates of B_j and Z_3 w.r.t. the cartesian coordinate system $\mathcal{R}_{O_{i'j}}$ fixed at $O_{i'j}$, namely $B_j = ((x_j+1)\Delta, L\Delta)$ and $Z_3 = ((x_j+1)\Delta+1,\, L(\Delta+1))$. The y-coordinate of the intersection point of the line $x = (x_j+1)\Delta+1$ with the $y = L/(x_j+1)x$, which supports $\overrightarrow{BB_j}$, is given by $L\Delta + L/(x_j+1)$, and since $x_j \geq 2$, we conclude that the intersection point is below Z_3, as we stated. Each ear gadget is a TOP. Moreover, the dual graph of $\Pi_{HV}(P)$ for the new polygon P is still a tree, as required.

Each ear-gadget needs a local guard that must be located in one of the vertices of the shaded region and none of these vertices sees a local vertex of a d-gadget. This means that these guards cannot replace any guard located in a segment. On the other hand, since any guard located on a segment can move to the segment right endpoint to become a vertex-guard, without loss of visibility, we can show Lemma 2.

Lemma 2. *The resulting* TOP *can be guarded by* $|V|+3|E|+k$ *boundary guards iff there is a vertex-cover of size* k *for* $G=(V,E)$.

4 GVTP for Point Guards

Theorem 3. GVTP *is NP-hard for point guards.*

Now, we show that GVTP remains NP-hard when guards may lie anywhere in the polygon. As in [12], we will construct a reduction from the minimum line cover problem (MLCP). Given a set $\mathcal{L}=\{l_1,\ldots,l_n\}$ of lines in the plane, MLCP is the problem of finding a set of points of minimum cardinality such that each line $l\in\mathcal{L}$ contains at least one point in that set. This problem is known to be NP-hard [16] and APX-hard [6].

Without loss of generality, we assume that $\mathcal{L}$ contains neither vertical nor horizontal lines. The polygon is obtained by attaching *single-gadgets* (called *s-gadgets*) to a bounding box $\mathcal{B}(\mathcal{L})$ that contains all intersection points of pairs of lines in $\mathcal{L}$ in its interior. The idea is sketched in Fig. 5. To guarantee that a

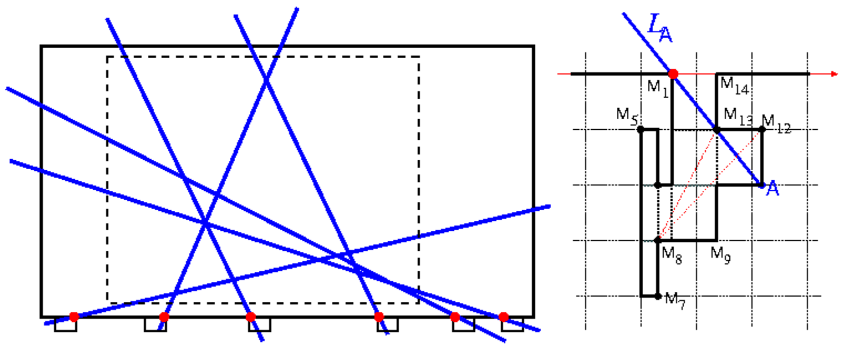

Fig. 5. The reduction from MLCP to GVTP with guards anywhere. Each tiny box on the bottom represents an s-gadget (note that not all lines intersected the bottom edge of the dashed bounding box). On the right, an s-gadget in detail.

TOP is obtained, we define an s-gadget, where M_1 and M_{13} reduce the visibility cone $\mathcal{C}_A$ to the line L_A. Moreover, we had to restrict the locations of s-gadgets to the bottom edge of $\mathcal{B}(\mathcal{L})$, in contrast to [12]. This can be done because, for a sufficiently large bounding box, all lines will intersect the bottom edge of $\mathcal{B}(\mathcal{L})$, as there are no horizontal lines in $\mathcal{L}$. At least a local guard is needed for each s-gadget. As for the d-gadgets, taking into account the relative positions of intersections of the lines with the bottom line (i.e., of vertices M_1), and their slopes, we can define the vertices of the tiny s-gadget in such a way that M_8 sees M_{12} and M_7, and all local vertices except for A. The vertices of P can be guarded by $n + k$ guards if and only if there is a cover for $\mathcal{L}$ of size k.

5 Guarding Thin Orthogonal Polygons

We adapt the d-gadget defined above to show that finding an optimal guard set for a thin orthogonal polygon (GTP) is NP-hard for boundary and vertex guards. For that, we change the two *legs* of the d-gadget, as shown in Fig. 6.

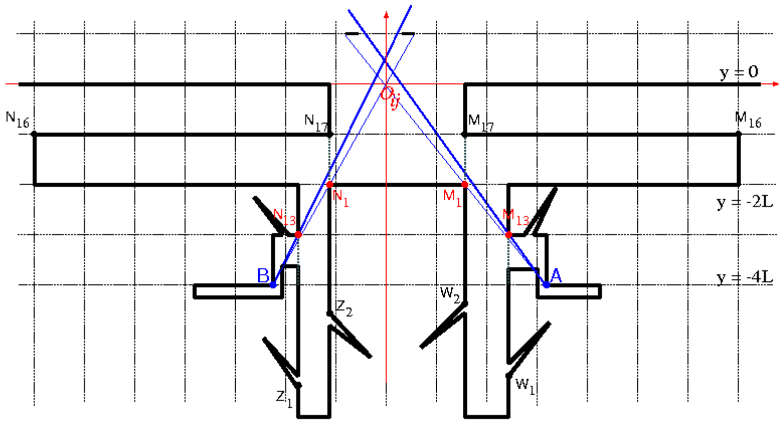

Fig. 6. The d-gadget for the reduction of Vertex-Cover to GTP, with boundary or vertex guards. Each spike represents a tiny staircase polygon.

We focus on the left leg (the right leg is similar). It can be checked that it is safe to define the edge e_B on the line $y = (-4 + \frac{1}{4})L$ and the spikes as follows. The intersection point B' of the ray $\overrightarrow{BN_{13}}$ with this line is the only vertex that B now sees on e_B. The segment $B'B''$ of e_B defines a critical region. B'' is the intersection of $\overrightarrow{BN_1}$ with e_B, and $C = (x_c, y_c)$ is restricted by B'' and R, being $x_C = x_{B''}$ and R the point where $\overrightarrow{BN_1}$ intersects the vertical chord incident at N_{13}. We define the slope of line CR to be $-L/(x_j + 1)$, which allows us to compute y_C. The vertex Z_2 is the intersection of $\overrightarrow{CR}$ with the vertical edge that contains N_1, and the ray incident to Z_1 is parallel to $\overrightarrow{CR}$. The staircases in Fig. 7, on the right, are regular, and their entry windows have width $\delta_y = L/(x_j + 1)\delta_x$, for $\delta_x = \Delta/(\Delta + 4)$. Based on the dominance relation, we can check that the four vertices labelled g must have a guard in any optimal solution, C and Z_1, in the two legs, and also A and B, and either N_1 (or M_1). It can be checked that the expansion of the aperture of the cone (the points that see $B'B''$ on the top edge of the large square) is on the safe slabs.

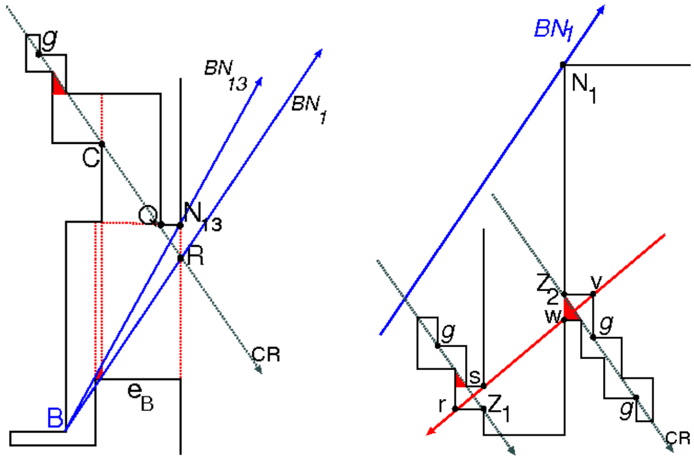

Fig. 7. The spikes in the left leg in more detail (not to scale nor accurate). Two guards in C and Z_1 see the critical regions in the three spikes (the tiny filled in regions).

Lemma 3. *The constructed* TOP *can be guarded by* $15|E|+k$ *boundary guards iff there is a vertex-cover of size* k *for* $G=(V,E)$.

As in section 3, by introducing suitable ear gadgets, we get a similar reduction for vertex-guards, and the following result.

Lemma 4. *The constructed* TOP *(with suitable ear gadgets) can be guarded by* $15|E|+|V|+k$ *vertex guards iff there is a vertex-cover of size* k *for* $G=(V,E)$.

6 Inapproximability Results

An NP-optimization problem is APX-hard if there is a constant $\epsilon>0$ such that an approximation ratio of $1+\epsilon$ cannot be guaranteed by any polynomial time algorithm, unless P = NP. We will use the L-reduction technique [19] to show the APX-hardness of GVTP and GTP, for vertex and boundary guards. Given two NP optimization problems $\mathcal{U}$ and $\mathcal{W}$ and a polynomial-time transformation f from instances of $\mathcal{U}$ to instances of $\mathcal{W}$, we say that f is an L-reduction if there are constants $\alpha,\beta>0$ such that for every instance I of $\mathcal{U}$: $\mathrm{opt}_{\mathcal{W}}(f(I))\leq\alpha\cdot\mathrm{opt}_{\mathcal{U}}(I)$; and for any solution of $f(I)$ with cost c_2, we can find in polynomial time a solution of I with cost c_1 such that $|\mathrm{opt}_{\mathcal{U}}(I)-c_1|\leq\beta\cdot|\mathrm{opt}_{\mathcal{W}}(f(I))-c_2|$. If $\mathcal{U}$ L-reduces to $\mathcal{W}$, and there is a polynomial-time approximation algorithm for $\mathcal{W}$ with worst case error ϵ, then, there is a polynomial-time approximation algorithm for $\mathcal{U}$ with worst-case error $\alpha\beta\epsilon$ [19].

Our reductions from the vertex cover problem are still valid if the graph has degree bounded by d and it is known that, for $d\geq3$, the minimum vertex cover is APX-complete [2, 3]. This allows us to show the following result.

Theorem 4. *Computing an optimal guard set for a* TOP *or for the vertices of a* TOP *is APX-hard, either for vertex or boundary guards.*

Proof. We consider the reduction from the minimum vertex cover for graphs $G = (V, E)$ with degree bounded by 3 (and without isolated vertices). For the TOPs constructed, the minimum number of guards is of the form $c|E| + k^\star$, for boundary guards, or $c|E| + |V| + k^\star$ for vertex guards, for the constants c given above ($c = 3$ for GVTP and $c = 15$ for GTP). Since each vertex of the graph can cover at most three edges, we have $|E|/3 \leq k^\star$, i.e., $|E| \leq 3k^\star$, being $k^\star$ the cardinality of a minimum vertex cover for G. Therefore, $c|E| + k^\star \leq 3ck^\star + k^\star = (3c + 1)k^\star$, and, since G has no isolated vertices, $|V|/2 \leq E$, and $c|E| + |V| + k^\star \leq (3(c + 2) + 1)k^\star = (3c + 7)k^\star$. The constants α and β for the L-reductions are $\beta = 1$ and $\alpha = 3c + 1$ for boundary guards, and $\beta = 1$ and $\alpha = (3c + 7)$ for vertex-guards. To conclude that we can take $\beta = 1$, we note that, as in the proof of Lemma 1, we can replace the local guard set of each Ξ_{ij} that has more than c guards by c local guards and locate a guard at $\overline{A_iB_i}$ (at B_i) to guard A (or the critical region), if this segment has no guard yet. This transformation gives us the required solution for the vertex cover problem. □

From [3], we known that, for every $\epsilon > 0$, it is NP-hard to approximate the minimum vertex cover problem for graphs of bounded degree 3 within factor $100/99 - \epsilon$. Thus, we deduce inapproximability factors of $(99\alpha + 1)/99\alpha - \epsilon$, for the guarding problems.

Corollary 1. *For every* $\epsilon > 0$*, it is hard to approximate* GVTP *within factor* $991/900-\epsilon$*, for boundary guards, and* $1585/1584-\epsilon$ *for vertex guards. For* GTP*, the corresponding factors are* $4555/4554 - \epsilon$ *and* $5149/5148 - \epsilon$*.*

Corollary 2. *The minimum guard covering problem for orthogonal polygons is APX-hard either for vertex or boundary guards (even if just the vertices were to be covered).*

By combining the construction given in [6] and Fig. 5, we can conclude that GVTP is APX-hard for point guards. We conjecture that the reduction of Fig. 5 can be adapted to show that GTP is APX-hard for point guards also.

7 Conclusion

We show that computing a minimum guard set for the vertices of a TOP is NP-hard, indeed APX-hard, either for boundary, vertex or point guards. We show that computing a minimum guard set for a TOP is NP-hard and APX-hard either for boundary or vertex guards. For thin polyomino trees [4], to the best of our knowledge, the complexity remains open. Our work implies that other properties need to explored, in addition to the tree structure.

Acknowledgments. The author would like to thank anonymous reviewers for insightful comments.

References

1. Abellanas, M., Canales, S., Hernández-Peñalver, G.: An Art Gallery Theorem for Pyramids. Inf. Process. Lett. 109, 719–721 (2009)
2. Alimonti, P., Kann, V.: Some APX-completeness Results for Cubic Graphs. Theoretical Computer Science 237, 123–134 (2000)
3. Chlebík, M., Chlebíková, J.: Complexity of approximating bounded variants of optimization problems. Theoretical Computer Science 354, 320–338 (2006)
4. Biedl, T., Irfan, M.T, Iwerks, J., Kim, J., Mitchell, J.S.B.: Guarding Polyominoes. In: 27th ACM Symp. on Computational Geometry, pp. 387–396, ACM (2011)
5. Bottino, A., Laurentini, A.: A Nearly Optimal Algorithm for Covering the Interior of an Art Gallery. Pattern Recognition 44, 1048–1056 (2011)
6. Brodén, B., Hammar, M., Nilsson, B.J.: Guarding Lines and 2-Link Polygons is APX-hard. In: 13th Canadian Conf. on Computational Geometry, CCCG 2001, pp. 45–48 (2001)
7. Couto, M.C., Rezende, P.J., Souza, C.C.: An Exact Algorithm for Minimizing Vertex Guards on Art Galleries. Int. T. Oper. Res. 18, 425–448 (2011)
8. Edelsbrunner, H., O'Rourke, J., Welzl, E.: Stationing Guards in Rectilinear Art Galleries. Computer Vision, Graphics, and Image Processing 27, 167–176 (1984)
9. Eidenbenz, S., Stamm, C., Widmayer, P.: Inapproximability Results for Guarding Polygons and Terrains. Algorithmica 31, 79–113 (2001)
10. Ghosh, S.K.: Approximation Algorithms for Art Gallery Problems in Polygons. Discrete Applied Mathematics 158, 718–722 (2010)
11. Kahn, J., Klawe, M., Kleitman, D.: Traditional Galleries Require Fewer Watchmen. SIAM J. Algebraic and Discrete Methods 4, 194–206 (1983)
12. Katz, M.J., Roisman, G.S.: On Guarding the Vertices of Rectilinear Domains. Comput. Geom. Theory Appl. 39, 219–228 (2008)
13. Krohn, E., Nilsson, B.J.: The Complexity of Guarding Monotone Polygons. 24th Canadian Conf. on Computational Geometry, CCCG 2012, pp. 167–172 (2012)
14. Lee, D.T., Lin, A.K.: Computational Complexity of Art Gallery Problems. IEEE Transactions on Information Theory 32, 276–282 (1986)
15. Martins, A.M., Bajuelos, A.L.: Vertex Guards in a Subclass of Orthogonal Polygons. Int. J. Computer Science and Network Security 6, 102–108 (2006)
16. Megiddo, N., Tamir, A.: On the Complexity of Locating Facilities in the Plane. Oper. Res. Lett. 1, 194–197 (1982)
17. Nilsson, B.J.: Approximate Guarding of Monotone and Rectilinear Polygons. In: Caires, L. et al. (eds.) ICALP 2005, LNCS, vol. 3580, pp. 1362–1373, Springer (2005)
18. O'Rourke, J.: Art Gallery Theorems and Algorithms. Oxford University Press, Inc. New York, USA (1987)
19. C. Papadimitriou, P., Yannakakis, M.: Optimization, Approximation and Complexity Classes. Journal of Computer and System Sciences 43, 425–440 (1991)
20. Schuchardt, D., Hecker, H.: Two NP-hard Problems for Ortho-Polygons. Math. Logiv Quart. 41, 261–267 (1995)
21. Urrutia, J.: Art Gallery and Illumination Problems. In: Sack, J.-R., Urrutia, J. (eds.), Handbook on Computational Geometry, Elsevier, North-Holland (2000)
22. Tomás, A.P., Bajuelos, A.L.: Quadratic-Time Linear-Space Algorithms for Generating Orthogonal Polygons with a Given Number of Vertices. In: Laganà, A. et al. (eds.) ICCSA 2004/CGA'04, LNCS, vol. 3045, pp. 117-126, Springer (2004)
23. Tomás, A.P.: Guarding Path Orthogonal Polygons. Internal report, DCC (2013)