

IDEMPOTENT-GENERATED SEMIGROUPS AND PSEUDOVARITIES

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ABSTRACT. The operator which constructs the pseudovariety generated by the idempotent-generated semigroups of a given pseudovariety is investigated. Several relevant examples of pseudovarieties generated by their idempotent-generated elements are given as well as some properties of this operator. Particular attention is paid to the pseudovarieties in $\{J, R, L, DA\}$ concerning this operator and their generator ranks and idempotent-generator ranks.

1. INTRODUCTION

Due to its applications in Computer Science, the theory of finite semigroups saw significant advancements in the 1960's driven by developments in the theory of finite automata. This connection with finite semigroups was firstly explored to obtain computational results. In parallel, combinatorial and algebraic properties of finite semigroups were studied. Eilenberg [4] established a correspondence between certain families of rational languages and certain classes of finite semigroups, called *pseudovarieties*, which provided a suitable framework for the bridges between the two theories.

There are many important pseudovarieties, often constructed from other ones by applying suitable operators. Some natural operators have been extensively studied. In this paper, we introduce a new one which constructs the subpseudovariety generated by the idempotent-generated semigroups of a given pseudovariety.

Several works have been dedicated to idempotent-generated semigroups. It is well-known that any finite semigroup embeds into a finite regular idempotent-generated semigroup, which was proved by Howie [6] using full transformations semigroups. Howie [7] also proved that the full transformations subsemigroup consisting of all order-preserving and contractive full transformations is idempotent-generated. Laradji and Umar [8] improved this result and showed that, in fact, every order-preserving and contractive full transformation is expressible as a product of idempotents of the same type and with the same range. The analogous result for the subsemigroup of contractive full transformations also holds [8].

On the other hand, Petrich and Reilly [14] proved that every completely simple semigroup embeds into an idempotent-generated one. Furthermore, Petrich [13] presents a concrete model of the embedding due to Pastijn and Yan [12] of a semigroup into an idempotent-generated Rees matrix semigroup that fixes some properties.

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In this paper, while we do not obtain a complete characterization of the pseudovarieties which are generated by their idempotent-generated semigroups, we prove that many familiar pseudovarieties have this property. The techniques used for this purpose include the representations of free profinite semigroups over \mathbf{R} , \mathbf{J} and \mathbf{DA} due to Almeida and Weil [3], Almeida [1], and Moura [10], respectively. In the cases of \mathbf{R} and \mathbf{J} , we also observe an alternative approach based on some results concerning transformations of a finite chain due to Pin [15] and Straubing [18] and the results concerning idempotent-generated subsemigroups of full transformations from Howie, and Laradji and Umar. On the other hand, the work of Petrich [13] allows us to show that the pseudovarieties $\bar{\mathbf{H}}$, where \mathbf{H} is a pseudovariety of groups, \mathbf{CS} and \mathbf{CR} also have this property.

The new approach in the case of the pseudovarieties \mathbf{R} and \mathbf{J} is justified as it gives a significant improvement in terms of the *generator rank* and *idempotent generator rank*. More generally, we show that both ranks are infinite for every pseudovariety in the interval $[\mathbf{J}, \mathbf{DS}]$. We also prove that every semigroup in the subpseudovariety generated by all n -generated members of any of the pseudovarieties \mathbf{J} , \mathbf{R} , \mathbf{L} , \mathbf{DA} divides a semigroup in the same pseudovariety generated by at most $n + 1$ idempotents. We compare these results with the works of Umar [19], and Laradji and Umar [8] concerning the ranks and idempotent-ranks of the subsemigroups of all contractive full transformations, and contractive and order-preserving full transformations, respectively. We observe that, in fact, we decrease the number of idempotent generators of the idempotent-generated semigroups when we use the embeddings of the semigroups of \mathbf{R} and \mathbf{J} presented in this paper.

The paper is organized as follows. In Section 2, we recall some basics of the theory of pseudovarieties of semigroups and profinite semigroups and we introduce some notation concerning operators on pseudovarieties. We also present a list of the pseudovarieties and bases of pseudoidentities defining them that will be used in our study. In Section 3, we observe some properties of the operator $\perp_{\mathbf{E}}$, we determine some pseudovarieties of the form $\mathbf{V}_{\mathbf{E}}$ and we make a short introduction of the main question addressed in the paper: what are the pseudovarieties that are generated by their idempotent-generated elements? We present in the following sections some pseudovarieties having this property: in Section 4 using the embedding in a Rees matrix semigroup constructed by Petrich, and in Section 5 using representations of the free profinite semigroup. Finally, in Section 6 we determine the generator rank and idempotent generator rank of every pseudovariety in the interval $[\mathbf{J}, \mathbf{DS}]$ and we also determine a lower bound for the idempotent generator rank of the subpseudovarieties generated by all n -generated members of any pseudovariety in the interval $[\mathbf{J}, \mathbf{DA}]$. Combining with the results of Section 5, we improve the last result showing that the lower bound is the exact value in the case of the pseudovarieties \mathbf{J} , \mathbf{R} , \mathbf{L} , \mathbf{DA} . To introduce some relevant results in our study, we develop some existing techniques that need to be recalled. Rather than including them in Section 2, we briefly introduce them along as need.

2. PRELIMINARIES

We assume acquaintance with notions concerning pseudovarieties of semigroups and profinite semigroups. We briefly recall some basics and we refer the reader to [1, 2, 15] for detailed information.

For a semigroup S , let S^1 be the monoid obtained by adjoining a neutral element 1 to S in case S does not already possess one, and $S^1 = S$ otherwise. We denote by $E(S)$ the set of idempotents of S and by $\langle E(S) \rangle$ the subsemigroup of S generated by $E(S)$. For $s \in S$, s^ω denotes the unique idempotent in the subsemigroup generated by s . We say that a semigroup S *divides* a semigroup T , and we write $S \prec T$, if there exists a surjective homomorphism of a subsemigroup of T onto S .

A *pseudovariety* of semigroups is a class of finite semigroups that is closed under taking subsemigroups, homomorphic images and finite direct products. Equivalently, a pseudovariety of semigroups is a class of finite semigroups closed under taking division and finite direct products. For example, \mathbf{S} is the pseudovariety of all finite semigroups.

There are many ways to construct new pseudovarieties from known ones, that is by applying operators to pseudovarieties. For example, given a pseudovariety \mathbf{V} , the following classes of finite semigroups are pseudovarieties:

- \mathbf{EV} consists of all $S \in \mathbf{S}$ such that $\langle E(S) \rangle \in \mathbf{V}$;
- \mathbf{DV} consists of all $S \in \mathbf{S}$ such that, for every regular \mathcal{D} -class D of S , $D \in \mathbf{V}$;
- for a pseudovariety \mathbf{H} of groups, \mathbf{H} consists of all $S \in \mathbf{S}$ such that every subgroup G of S belongs to \mathbf{H} .

We also have other types of operators that construct new pseudovarieties by describing the generators. The new pseudovariety is then the smallest pseudovariety containing such semigroups. In this way, we introduce the operator \mathbf{V}_E , which is the topic of this paper. Given a pseudovariety \mathbf{V} , we define \mathbf{V}_E as the pseudovariety generated by the subsemigroups generated by the idempotents of each member of \mathbf{V} , i.e.,

$$\mathbf{V}_E = \langle \langle E(S) \rangle \mid S \in \mathbf{V} \rangle.$$

Note that $\mathbf{V}_E \subseteq \mathbf{V}$ as the indicated generators of \mathbf{V}_E belong to \mathbf{V} .

Because it will be useful in our study, we present an obvious observation about the subsemigroup generated by a subset of idempotents of a given semigroup:

Lemma 2.1. *Let $S \in \mathbf{S}$ and $X \subseteq E(S)$. Then $\langle E\langle X \rangle \rangle = \langle X \rangle$. In particular, we have $\langle E\langle E(S) \rangle \rangle = \langle E(S) \rangle$ and $\langle E\langle E(D) \rangle \rangle = \langle E(D) \rangle$, for every regular \mathcal{D} -class D of S .*

A semigroup equipped with a topology for which the multiplication is a continuous function is called a *topological semigroup*. Finite semigroups are endowed with the discrete topology. A topological semigroup S is a *profinite semigroup* (respectively, a *pro- \mathbf{V} semigroup*) if it is a compact semigroup which is *residually finite* (respectively, *residually in \mathbf{V}*), which means that, for any two distinct elements of S , there exists a continuous homomorphism into a finite semigroup (respectively, into a member of \mathbf{V}) that separates them. Equivalently, profinite semigroups are compact *0-dimensional*, which means that the topology has an open basis consisting of clopen sets. The elements of a pseudovariety \mathbf{V} are pro- \mathbf{V} semigroups.

We denote by $\overline{\Omega}_A \mathbf{V}$ the *free pro- \mathbf{V} semigroup on A* , which is the unique (up to isomorphism of topological semigroups) pro- \mathbf{V} semigroup such that every mapping $\mu : A \rightarrow S$ into a pro- \mathbf{V} semigroup S can be extended to a unique continuous homomorphism $\hat{\mu} : \overline{\Omega}_A \mathbf{V} \rightarrow S$ such that $\hat{\mu} \circ \iota = \mu$, where $\iota : A \rightarrow \overline{\Omega}_A \mathbf{V}$ is the natural generating function (i.e., its image generates a dense subsemigroup of $\overline{\Omega}_A \mathbf{V}$). The elements of $\overline{\Omega}_A \mathbf{V}$ are called *implicit operations over \mathbf{V}* . For $u \in \overline{\Omega}_A \mathbf{V}$ the sequence

$(u^{n!})_n$ converges and we denote the limit by u^ω , which is the unique idempotent in the closed subsemigroup generated by u .

An equality of the form $u = v$, with $u, v \in \overline{\Omega}_A S$, is called a *pseudoidentity* and $|A|$ is its *arity*. The pseudoidentity is *valid* in a profinite semigroup S , and we write $S \models u = v$, if, for every continuous homomorphism $\varphi : \overline{\Omega}_A S \rightarrow S$, we have $\varphi(u) = \varphi(v)$. It is easy to see that the validity of a pseudoidentity in a finite semigroup is preserved under taking homomorphic images, subsemigroups and finite direct products. Conversely, Reiterman's Theorem [16] says that every pseudovariety is defined by some set of finitary pseudoidentities. We end this section with a list of pseudovarieties that will be used in this paper and some corresponding bases of pseudoidentities defining them:

$J = \llbracket (xy)^\omega x = (xy)^\omega = y(xy)^\omega \rrbracket$	\mathcal{J} -trivial semigroups;
$R = \llbracket (xy)^\omega x = (xy)^\omega \rrbracket$	\mathcal{R} -trivial semigroups;
$L = \llbracket y(xy)^\omega = (xy)^\omega \rrbracket$	\mathcal{L} -trivial semigroups;
$A = \llbracket x^{\omega+1} = x^\omega \rrbracket$	aperiodic (or \mathcal{H} -trivial) semigroups;
$G = \llbracket x^\omega = 1 \rrbracket$	groups;
$LG = \llbracket (x^\omega y)^\omega x^\omega = x^\omega \rrbracket$	local groups;
$CR = \llbracket x^{\omega+1} = x \rrbracket$	completely regular semigroups;
$CS = \llbracket x^{\omega+1} = x, (xyx)^\omega = x^\omega \rrbracket$	completely simple semigroups;
$RB = \llbracket x^2 = x, xyx = x \rrbracket$	rectangular bands;
$DA = \llbracket ((xy)^\omega x)^2 = (xy)^\omega x \rrbracket$	regular \mathcal{D} -classes are aperiodic semigroups;
$DG = \llbracket (xy)^\omega = (yx)^\omega \rrbracket$	regular \mathcal{D} -classes are groups;
$DO = \llbracket (xy)^\omega (yx)^\omega (xy)^\omega = (xy)^\omega \rrbracket$	regular \mathcal{D} -classes are orthodox semigroups;
$DS = \llbracket ((xy)^\omega x)^{\omega+1} = (xy)^\omega x \rrbracket$	regular \mathcal{D} -classes are semigroups.

3. PROPERTIES OF THE OPERATOR $_E$

We establish some basic properties of the operator $_E$. We start by observing that the definition given for V_E , where V is a pseudovariety, is equivalent to V_E being generated by the idempotent-generated semigroups of V .

Lemma 3.1. *The operator $_E$ has the following properties, where V and W are arbitrary pseudovarieties:*

- (1) $V_E = \langle T \in V \mid T = \langle E(T) \rangle \rangle$;
- (2) $V \subseteq W$ implies $V_E \subseteq W_E$;
- (3) $(V \cap W)_E \subseteq V_E \cap W_E$;
- (4) $(V_E)_E = V_E$;
- (5) $(EV)_E = V_E$;
- (6) $E(V_E) = EV$.

Proof. (1) Let $T = \langle E(S) \rangle$ with $S \in V$. Since $\langle E(S) \rangle$ is a subsemigroup of S , it follows that $T \in V$. By Lemma 2.1, we have that $\langle E(T) \rangle = \langle E\langle E(S) \rangle \rangle = \langle E(S) \rangle = T$. Hence the generators of the two pseudovarieties are the same.

(2) is immediate from the definition of V_E and (3) follows from (2).

(4) The direct inclusion follows from $V_E \subseteq V$ and (2). Conversely, since the generators of V_E are the semigroups $\langle E(S) \rangle$, with $S \in V$, it suffices to show that

$\langle E(S) \rangle \in (\mathbf{V}_E)_E$, for all $S \in \mathbf{V}$. Indeed, since $\langle E(S) \rangle \in \mathbf{V}_E$, by Lemma 2.1 and definition of $_E$ we have $\langle E(S) \rangle = \langle E\langle E(S) \rangle \rangle \in (\mathbf{V}_E)_E$.

(5) We have $(\mathbf{EV})_E = \langle \langle E(S) \rangle \mid S \in \mathbf{EV} \rangle = \langle \langle E(S) \rangle \mid \langle E(S) \rangle \in \mathbf{V} \rangle$. Let us see that the generators of $(\mathbf{EV})_E$ are in \mathbf{V}_E . In fact, as $\langle E(S) \rangle \in \mathbf{V}$, Lemma 2.1 yields $\langle E(S) \rangle = \langle E\langle E(S) \rangle \rangle \in \mathbf{V}_E$. The reverse inclusion follows from $\mathbf{V} \subseteq \mathbf{EV}$ and (2).

(6) Since $\mathbf{V}_E \subseteq \mathbf{V}$, applying the increasing operator \mathbf{E}_- , we obtain $\mathbf{E}(\mathbf{V}_E) \subseteq \mathbf{EV}$. If $S \in \mathbf{EV}$, i.e., $\langle E(S) \rangle \in \mathbf{V}$, then Lemma 2.1 gives that $\langle E(S) \rangle = \langle E\langle E(S) \rangle \rangle$ is one of the generators of \mathbf{V}_E so that, in particular, $S \in \mathbf{E}(\mathbf{V}_E)$. \square

A natural question, for which we have no answer, is whether we always have equality in part (3) of Lemma 3.1.

Corollary 3.2. *Let \mathbf{V} and \mathbf{W} be pseudovarieties such that $\mathbf{V}_E = \mathbf{V}$ and $\mathbf{EV} = \mathbf{EW}$. Then $\mathbf{V} \subseteq \mathbf{W}$.*

Proof. Applying the operator $_E$ to $\mathbf{EV} = \mathbf{EW}$ and using property (5) of Lemma 3.1, it follows that $\mathbf{V} = \mathbf{V}_E = (\mathbf{EV})_E = (\mathbf{EW})_E = \mathbf{W}_E \subseteq \mathbf{W}$. \square

Corollary 3.3. *Given two pseudovarieties \mathbf{V} and \mathbf{W} , the following conditions are equivalent:*

- (a) $\mathbf{V}_E = \mathbf{W}_E$;
- (b) $\mathbf{EV} = \mathbf{EW}$;
- (c) $\mathbf{V}_E \subseteq \mathbf{W} \subseteq \mathbf{EV}$.

Proof. (a) \Rightarrow (b) From (a) and property (6) of Lemma 3.1, it follows that $\mathbf{EV} = \mathbf{E}(\mathbf{V}_E) = \mathbf{E}(\mathbf{W}_E) = \mathbf{EW}$.

(b) \Rightarrow (c) The second inclusion in (c) follows from $\mathbf{W} \subseteq \mathbf{EW} = \mathbf{EV}$. To show the first inclusion, we recall that, by properties (4)–(6) of Lemma 3.1, $((\mathbf{EV})_E)_E = (\mathbf{EV})_E$ and $\mathbf{E}((\mathbf{EV})_E) = \mathbf{EV} = \mathbf{EW}$. Hence, by Corollary 3.2, $(\mathbf{EV})_E \subseteq \mathbf{W}$. Moreover, also by property (5) of Lemma 3.1, $\mathbf{V}_E = (\mathbf{EV})_E \subseteq \mathbf{W}$.

(c) \Rightarrow (a) Applying the operator $_E$ to (c), by properties (2), (4) and (5) of Lemma 3.1, we obtain $\mathbf{V}_E = (\mathbf{V}_E)_E \subseteq \mathbf{W}_E \subseteq (\mathbf{EV})_E = \mathbf{V}_E$ and, therefore, $\mathbf{W}_E = \mathbf{V}_E$. \square

In other words, given a pseudovariety \mathbf{V} , the equations $\mathbf{X}_E = \mathbf{V}_E$ and $\mathbf{EX} = \mathbf{EV}$ in the variable \mathbf{X} are equivalent and the class of its solutions is the interval $[\mathbf{V}_E, \mathbf{EV}]$.

It is natural to ask for which pseudovarieties \mathbf{V} , \mathbf{V}_E is equal to \mathbf{V} . As an obvious example, for every pseudovariety \mathbf{V} of bands, since its semigroups consist only of idempotents, we have $\mathbf{V}_E = \mathbf{V}$. But, there are pseudovarieties that do not satisfy the equality $\mathbf{V} = \mathbf{V}_E$. Let us see some examples:

Example 3.4. For every pseudovariety \mathbf{H} of groups, we have $\mathbf{H}_E = \mathbf{I}$, where $\mathbf{I} = \llbracket x = y \rrbracket$ is the trivial pseudovariety.

Example 3.5. It is well known that \mathbf{LG} is the class of all finite semigroups such that all idempotents are \mathcal{J} -equivalent and, therefore, they are in the minimal ideal of the semigroup. So we have $(\mathbf{LG})_E = \langle \langle E(S) \rangle \mid S \in \mathbf{LG} \rangle \subseteq \mathbf{CS} \subsetneq \mathbf{LG}$.

Example 3.6. It is well known that $\mathbf{RBV} \vee \mathbf{G} = \mathbf{CS} \cap \mathbf{O}$, where $\mathbf{O} = \llbracket (x^\omega y^\omega)^\omega = x^\omega y^\omega \rrbracket$ is the class of all finite orthodox semigroups. So we have $(\mathbf{RB} \vee \mathbf{G})_E = (\mathbf{CS} \cap \mathbf{O})_E = \mathbf{RB}$.

The notion of \mathbf{E} -local pseudovariety, introduced in [9], enables us to determine $(\mathbf{DO})_{\mathbf{E}}$ and $(\mathbf{DH})_{\mathbf{E}}$, as we see in the following examples. Recall that a pseudovariety \mathbf{V} is \mathbf{E} -local if it satisfies the following property: given $S \in \mathbf{S}$, $\langle E(S) \rangle \in \mathbf{V}$ if and only if $\langle E(D) \rangle \in \mathbf{V}$, for every regular \mathcal{D} -class D of S .

Example 3.7. Let $S \in \mathbf{DH}$. Since every regular \mathcal{D} -class D of S is a group, it follows that $\langle E(D) \rangle$ is trivial and, therefore, $\langle E(D) \rangle \in \mathbf{J}$. Since \mathbf{J} is \mathbf{E} -local (see [9, Example 3.6]), we have $\langle E(S) \rangle \in \mathbf{J}$. Hence $\mathbf{J} \subseteq \mathbf{DH} \subseteq \mathbf{EJ}$, where the first inclusion is trivial. Thus it follows from Corollary 3.3 that $(\mathbf{DH})_{\mathbf{E}} = \mathbf{J}_{\mathbf{E}} = \mathbf{J}$, where the last equality follows from Corollary 3.14, which is proved below.

Example 3.8. We observe that $\mathbf{DO} \subseteq \mathbf{EDA}$. Indeed, for $S \in \mathbf{DO}$ and a regular \mathcal{D} -class D of S , $\langle E(D) \rangle \in \mathbf{DA}$. Since \mathbf{DA} is \mathbf{E} -local (see [9, Proposition 3.5]), we have $\langle E(S) \rangle \in \mathbf{DA}$. Hence $S \in \mathbf{EDA}$. Since $\mathbf{DA} \subseteq \mathbf{DO} \subseteq \mathbf{EDA}$, it follows from Corollary 3.3 that $(\mathbf{DO})_{\mathbf{E}} = (\mathbf{DA})_{\mathbf{E}} = \mathbf{DA}$, where the last equality follows from Corollary 5.6 which is established in Section 5.

In an attempt to identify the pseudovarieties which are generated by their idempotent-generated elements, we present some results in the following sections. We start by suggesting, as an easy exercise, the result from Howie [6] which states that any finite semigroup embeds into a finite regular idempotent-generated semigroup, so that, in particular, we have the following result.

Proposition 3.9 (cf. Howie [6]). $\mathbf{S}_{\mathbf{E}} = \mathbf{S}$.

On the other hand, Pin [15] and Straubing [18] obtained the following representation theorems for \mathcal{R} -trivial monoids and \mathcal{J} -trivial monoids, respectively.

Theorem 3.10 (cf. Pin [15, Theorem IV.3.6]). *A finite monoid is \mathcal{R} -trivial if and only if it is a submonoid of \mathcal{E}_X , the submonoid consisting of all contractive total transformations of some finite chain X .*

Theorem 3.11 (cf. Straubing [18]). *A finite monoid is \mathcal{J} -trivial if and only if it divides \mathcal{C}_X , the submonoid of all order-preserving and contractive transformations of some finite chain X .*

Combining these theorems with the following results about idempotent-generated subsemigroups of total transformations due, respectively, to Laradji and Umar [8] and to Howie [7], we obtain Corollary 3.14.

Theorem 3.12 (cf. Laradji and Umar [8, Theorem 1.3]). *The monoid \mathcal{E}_X is idempotent-generated.*

Theorem 3.13 (cf. Howie [7, Theorem 3.2]). *The monoid \mathcal{C}_X is idempotent-generated.*

Corollary 3.14. *The equality $\mathbf{V}_{\mathbf{E}} = \mathbf{V}$ holds if \mathbf{V} is any of the pseudovarieties $\mathbf{R}, \mathbf{L}, \mathbf{J}$.*

Similarly, using embeddings into idempotent-generated semigroups of the same type from Petrich and Reilly [14] and from Pastijn and Yan [12] concerning, respectively, completely simple semigroups and completely regular semigroups, we obtain the following results.

Proposition 3.15 (cf. Petrich and Reilly [14, cf. Lemma III.2.11]). $(\mathbf{CS})_{\mathbf{E}} = \mathbf{CS}$.

Proposition 3.16 (cf. Pastijn and Yan [12]). $(\text{CR})_{\mathbf{E}} = \text{CR}$.

Using Proposition 3.15 we may establish an equality in Example 3.5, as we see below.

Example 3.17. By Proposition 3.15, property (2) of Lemma 3.1 and since $\text{CS} \subseteq \text{LG}$, it follows that $\text{CS} = (\text{CS})_{\mathbf{E}} \subseteq (\text{LG})_{\mathbf{E}}$. Thus, and by Example 3.5, we have $(\text{LG})_{\mathbf{E}} = \text{CS}$.

In Section 4 we return to these last two results and we show how to prove them using the general embedding of Petrich [13]. In Section 5, we prove that the pseudovarieties \mathbf{R} , \mathbf{L} and \mathbf{J} are fixed points of the $_{\mathbf{E}}$ operator by a different approach, namely by using implicit operations. While using transformation semigroups the number of idempotent generators of the idempotent-generated semigroup depends on the cardinality of the embedded semigroup, in this method the number of idempotent generators of the idempotent-generated semigroup is controlled by the number of generators of the embedded semigroup. However, for the case $\mathbf{V} = \mathbf{R}$, the first method enables us to show that, in fact, there exists an embedding of an \mathcal{R} -trivial semigroup into an idempotent-generated \mathcal{R} -trivial semigroup. In the other cases, we just prove a division property. The second method is also used to prove the equality for the pseudovariety \mathbf{DA} , while we do not know how to apply the first method.

4. THE PETRICH EMBEDDING INTO A REES MATRIX SEMIGROUP

Based on the embedding of Pastijn and Yan [12] (see also Pastijn [11]), Petrich [13] constructed an embedding of a semigroup S into an idempotent-generated semigroup in terms of a Rees matrix semigroup over S^1 . This embedding preserves some properties of the initial semigroup and it is this peculiarity that enables us to give an alternative proof of the results from Petrich and Reilly [14, Lemma III.2.11] and Pastijn and Yan [12], respectively, that every semigroup of CS and CR embeds into a finite idempotent-generated semigroup of the same type.

We briefly recall the construction of this embedding. Let S be a semigroup (not necessarily finite). We consider the Rees matrix semigroup $\Phi S = \mathcal{M}(S^1, S^1, \Sigma; Q)$, with $\Sigma = \{\sigma, \tau\}$, where σ and τ are two distinct symbols that are not in S , and $Q = (q_{\alpha s})$ is the sandwich matrix with entries

$$q_{\sigma s} = 1, \quad q_{\tau s} = s \quad (s \in S^1).$$

The mapping

$$\varphi_S : s \rightarrow (1, s, \sigma) \quad (s \in S)$$

embeds S into ΦS , although it is not the unique embedding from S into ΦS . Petrich determined the set of idempotents of ΦS , which is

$$(4.1) \quad E(\Phi S) = \{(s, t, \sigma) \in \Phi S \mid t \in E(S^1)\} \cup \{(s, t, \tau) \in \Phi S \mid t = tst\}$$

and described Green's relations on ΦS as follows.

Lemma 4.1 (Petrich [13, Lemma 4.3]). *Let $(s, t, \alpha), (u, v, \beta) \in \Phi S$. Then:*

- (1) $(s, t, \alpha) \mathcal{L} (u, v, \beta)$ if and only if $t \mathcal{L} v$ and $\alpha = \beta$;
- (2) $(s, t, \alpha) \mathcal{R} (u, v, \beta)$ if and only if $s = u$ and $t \mathcal{R} v$;
- (3) $(s, t, \alpha) \mathcal{H} (u, v, \beta)$ if and only if $s = u$, $t \mathcal{H} v$, and $\alpha = \beta$;
- (4) $(s, t, \alpha) \mathcal{D} (u, v, \beta)$ if and only if $t \mathcal{D} v$.

Thus, ΦS has the same number of \mathcal{D} -classes as S^1 and each \mathcal{D} -class D' of ΦS , which corresponds to a \mathcal{D} -class D of S , has the following number respectively of \mathcal{L} -classes and \mathcal{R} -classes: $2 \cdot |\mathcal{L}\text{-classes of } D|$ and $|S^1| \cdot |\mathcal{R}\text{-classes of } D|$. Each \mathcal{H} -class of ΦS has the same number of elements of the corresponding \mathcal{H} -class in S^1 .

It is obvious that if S is a finite semigroup, so is ΦS . Note that ΦS is generated by the set of idempotents $\{(s, 1, \sigma) \mid s \in S^1\} \cup \{(1, 1, \tau)\}$, which gives an immediate proof of the result from Howie (see Proposition 3.9). Petrich also showed that this embedding preserves other properties of S (see [13, Theorem 5.4]). In particular, he proves the following corollary.

Corollary 4.2 (cf. Petrich [13, Theorem 5.4]). *Every semigroup of $\bar{\mathbf{H}}$, where \mathbf{H} is a pseudovariety of groups, embeds into an idempotent-generated semigroup of $\bar{\mathbf{H}}$.*

Choosing some specific subsemigroups of ΦS , we can prove the following results:

Proposition 4.3. *Every semigroup of \mathbf{CS} embeds into an idempotent-generated semigroup of \mathbf{CS} .*

Proof. Note that, if $S \in \mathbf{CS}$, then ΦS has at most two \mathcal{D} -classes, the one corresponding to D , D' , and the other corresponding to the neutral element added to S . If we show that D' is generated by its idempotents, then it suffices to consider the embedding $\varphi'_S : s \mapsto (1, s, \sigma)$ from S into the semigroup D' .

Let $(1, s, \sigma)$, with $s \in S$, be any element of $\varphi'_S(S)$ and let $e \in E(S)$ be such that $e \mathcal{H} s$. Then one can compute $(1, e, \tau) \cdot (s, e, \sigma) = (1, s, \sigma)$ with $(1, e, \tau), (s, e, \sigma) \in E(D')$. Hence, the group \mathcal{H} -class $\{(1, s, \sigma) \mid s \in S\}$ is contained in $\langle E(D') \rangle$. Since all \mathcal{H} -classes of $\langle E(D') \rangle$ have the same number of elements, we conclude that $\langle E(D') \rangle = D'$. Hence S embeds into $\langle E(D') \rangle$, which is an idempotent-generated completely simple semigroup. \square

Proposition 4.4. *Every semigroup of \mathbf{CR} embeds into an idempotent-generated semigroup of \mathbf{CR} .*

Proof. Let $S \in \mathbf{CR}$. We want to determine an idempotent-generated completely regular subsemigroup of ΦS where S embeds. Let H be an \mathcal{H} -class of S . Since H is a group, then the \mathcal{H} -classes of ΦS of the form $\{s\} \times H \times \{\sigma\}$, with $s \in S^1$, and $\{1\} \times H \times \{\tau\}$ are groups. Let t be any element of S and H_t be the \mathcal{H} -class of S containing this element. We observe that the \mathcal{H} -classes of the form $\{s\} \times H_t \times \{\tau\}$, with $s \geq_{\mathcal{J}} t$, are groups. Let $e \in E(S)$ be such that $e \mathcal{H} t$. Then there exist $x, y \in S$ such that $e = xsy = exsye$. Hence $e \mathcal{L} sye$ and, since $L_e \cap R_{sye}$ is a group, we have that $esy e \mathcal{H} e$ and $es \mathcal{R} e$. By Green's Lemma, it follows that

$$\begin{aligned} \mu_s : H_t &\rightarrow H_{ts} \\ u &\mapsto us \end{aligned}$$

is a bijection. Let $v \in H_t$ be such that vs is the idempotent of H_{ts} . Since $vsv \mathcal{H} v$ and $\mu_s(vsv) = vsvs = vs = \mu_s(v)$, then $vsv = v$ and (s, v, τ) is the idempotent of $\{s\} \times H_t \times \{\tau\}$.

Now, we observe that if an \mathcal{H} -class of ΦS of the form $\{s\} \times H_t \times \{\tau\}$ is a group, then $s \geq_{\mathcal{J}} t$. Thus, if there exists $u \in H_t$ such that (s, u, τ) is an idempotent, then $usu = u$, and so $s \geq_{\mathcal{J}} u \mathcal{H} t$. So, we have identified all maximal subgroups of ΦS .

We consider the subsemigroup T of ΦS generated by the following idempotents:

$$\begin{aligned} (s, t, \sigma) &\text{ with } t \in E(S^1) \text{ and } s \geq_{\mathcal{J}} t; \\ (s, t, \tau) &\text{ with } t = tst. \end{aligned}$$

Basically, we chose all idempotents of the \mathcal{R} -classes whose \mathcal{H} -classes are groups. Thus, the product of any two idempotents on a same \mathcal{D} -class is also on this \mathcal{D} -class and, specifically, on a \mathcal{H} -class containing an idempotent of the set of generators. Let us see what is the product of two idempotents of the set of generators that are not in the same \mathcal{D} -class. Let (s, t, α) and (u, v, β) be two such idempotents. We have $(s, t, \alpha) \cdot (u, v, \beta) = (s, tq_{\alpha u}v, \beta)$. As $s \geq_{\mathcal{J}} t$, then $s \geq_{\mathcal{J}} tq_{\alpha u}v$. Hence this product is in an \mathcal{H} -class that contains an idempotent of the set of generators of T .

Note that the \mathcal{H} -classes of the form $\{1\} \times H \times \{\sigma\}$, where H is an \mathcal{H} -class of S^1 are in T . In fact, given $a \in S^1$, we have $(1, a, \sigma) = (1, e, \tau) \cdot (a, e, \sigma)$, where $e \in E(S^1)$ is such that $e \mathcal{H} a$, and $(1, e, \tau)$ and (a, e, σ) are idempotents of T . It follows that T is the subsemigroup of ΦS consisting of the \mathcal{R} -classes of ΦS whose \mathcal{H} -classes are groups.

Hence T is a completely regular semigroup and $\varphi''_S : s \mapsto (1, s, \sigma)$ is an embedding of S into T . \square

In the above proof, we may reduce the choice of the idempotents and we may consider the subsemigroup of T generated by the following idempotents:

$$(1, e, \tau), (a, e, \sigma) \text{ with } a \in S \text{ and } e \in E(S) \text{ such that } e \mathcal{H} a.$$

This subsemigroup is also a union of \mathcal{R} -classes of ΦS whose \mathcal{H} -classes are groups and the \mathcal{H} -classes of the form $\{1\} \times H \times \{\sigma\}$, where H is an \mathcal{H} -class of S^1 , are also in this subsemigroup. However, to simplify the proof, we considered the subsemigroup T .

Example 4.5. Consider the completely regular semigroup

$$S = \langle a, b, c, d \mid a^3 = a, b^2 = b, c^3 = c, d^2 = d, ab = ba, cb = bc, ada = a, \\ ac = ca = bd = db = cd = dc = 0 \rangle.$$

We present in Figure 1 a \mathcal{D} -class of ΦS to illustrate the distribution of the idempotents. We also observe that T consists of the \mathcal{R} -classes of ΦS whose \mathcal{H} -classes are groups, as we have mentioned previously.

Corollary 4.6. *The pseudovarieties \bar{H} , CS and CR satisfy the equality $\mathbb{V}_E = \mathbb{V}$.*

When we work with the pseudovarieties DS and DA, and since the regular \mathcal{D} -classes of the semigroups of these pseudovarieties are completely simple semigroups, one may ask whether the construction used in Proposition 4.4 may lead to a proof of existence of an embedding from every semigroup of any of these pseudovarieties into an idempotent-generated semigroup of the same pseudovariety. However, in the following example, we observe that this is not the case.

Example 4.7. Consider the semigroup $S = \langle a \mid a^3 = a^2 \rangle$. We look at the subsemigroup T of ΦS generated by the idempotents of the same type as those of Proposition 4.4 (see Figure 2). Note that neither the element $(1, a, \sigma)$ nor any element of the \mathcal{D} -class D_a is in T . We have to choose the idempotent $(a, 1, \sigma)$ to be a generator of T , but, in that case, T is no longer an element of DS (and, consequently, of DA).

We end this subsection with no answer for the question: *Does $\mathbb{V}_E = \mathbb{V}$ for any of the pseudovarieties DS or DA?* In the following section we see that, in fact, the pseudovariety DA satisfies such equality.

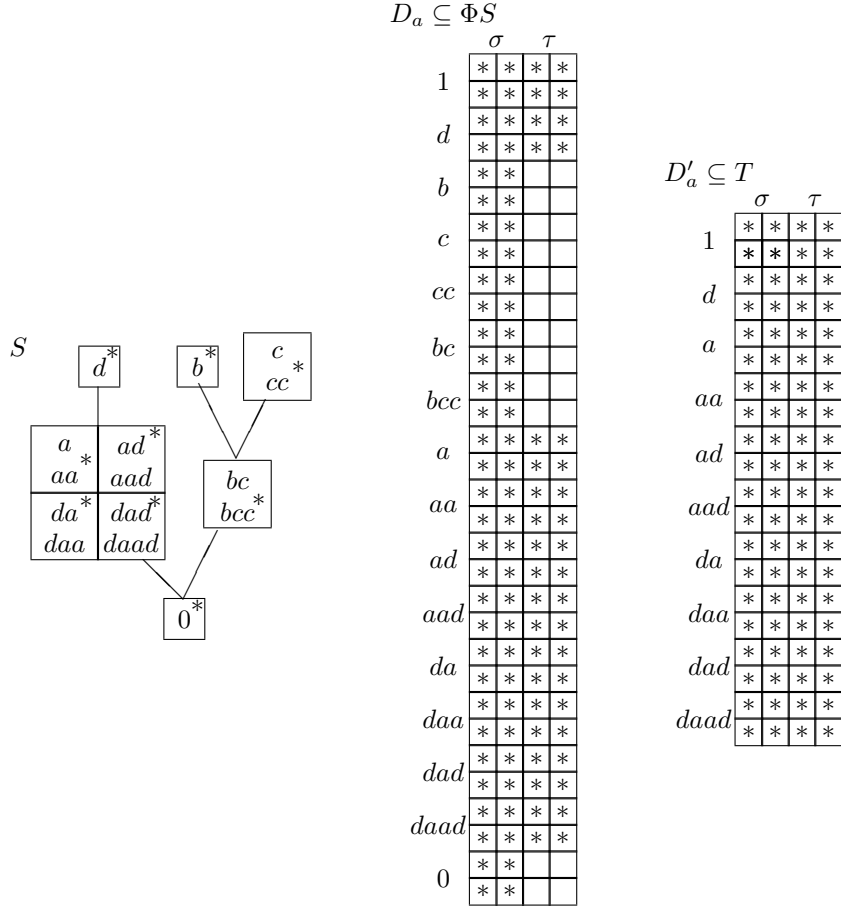


FIGURE 1. The Petrich embedding for a completely regular semigroup

5. REPRESENTATIONS BY IMPLICIT OPERATIONS

We refer the reader to [2] for detailed information about profinite semigroups and to standard references for the basics of topology. By an *embedding* of topological semigroups we mean a semigroup homomorphism that is simultaneously a homeomorphism with the image subspace. A *clopen* subset of a topological space is one that is simultaneously closed and open.

Theorem 5.1. *Let \mathbf{V} be a pseudovariety such that, for every n , there exists m such that $\overline{\Omega_n \mathbf{V}}$ embeds in $\overline{\langle X \rangle}$ for some $X \subseteq E(\Omega_m \mathbf{V})$. Then $\mathbf{V}_E = \mathbf{V}$.*

Proof. Let \mathbf{V} be a pseudovariety satisfying the above conditions. Let $S \in \mathbf{V}$ and $\varphi : \overline{\Omega_n \mathbf{V}} \rightarrow S$ be a continuous surjective homomorphism. Let $\mu_{\mathbf{V}} : \overline{\Omega_n \mathbf{V}} \rightarrow \overline{\langle X \rangle} \subseteq$

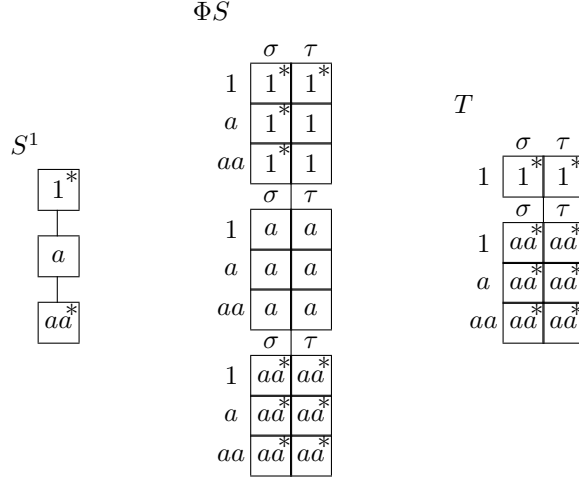


FIGURE 2. The Petrich embedding for a monogenic monoid

$\bar{\Omega}_m \mathbf{V}$ be an embedding, with $X \subseteq E(\bar{\Omega}_m \mathbf{V})$. We consider the following diagram:

$$\begin{array}{ccccc}
 \bar{\Omega}_n \mathbf{V} & \xrightarrow{\mu_V} & T & \hookrightarrow & \bar{\Omega}_m \mathbf{V} \\
 \varphi \downarrow & & \downarrow \phi|_T & & \downarrow \phi \\
 S \in \mathbf{V} & & \phi(T) \subseteq \langle E(F) \rangle & & F \in \mathbf{V}
 \end{array}$$

where T is the image of μ_V .

We claim that there is a family of clopen subsets $(U_s)_{s \in S}$ of $\bar{\Omega}_m \mathbf{V}$, pairwise disjoint, such that $U_s \cap T = \mu_V(\varphi^{-1}(s))$. We proceed to prove the claim. For each $s \in S$, let $A_s = \mu_V(\varphi^{-1}(s))$ and $A'_s = T \setminus A_s$. Since $\{s\}$ is a clopen subset of S , then $\varphi^{-1}(s)$ is a clopen subset of $\bar{\Omega}_n \mathbf{V}$ and $\mu_V(\varphi^{-1}(s))$ is a clopen subset of T . Since A_s and A'_s are closed sets of the closed subspace T of $\bar{\Omega}_m \mathbf{V}$, A_s and A'_s are closed subsets of $\bar{\Omega}_m \mathbf{V}$. Moreover, since $\bar{\Omega}_m \mathbf{V}$ is compact and 0-dimensional, then A_s and A'_s are separated by two disjoint clopen sets V_s and V'_s , respectively. We choose an arbitrary ordering for the elements of S : $s_1, \dots, s_{|S|}$. Let $U_{s_1} = V_{s_1}$ and recursively, for $i = 2, \dots, |S|$, let $U_{s_i} = V_{s_i} \setminus (\bigcup_{j < i} U_{s_j})$. Note that, for every i , U_{s_i} is also a clopen subset of $\bar{\Omega}_m \mathbf{V}$ and $(U_s)_{s \in S}$ is a family satisfying the claim.

Since $\bar{\Omega}_m \mathbf{V}$ is a pro- \mathbf{V} semigroup and, for all $s \in S$, U_s is a clopen subset, there exists a continuous homomorphism $\phi_s : \bar{\Omega}_m \mathbf{V} \rightarrow F_s$ with $F_s \in \mathbf{V}$ such that $U_s = \phi_s^{-1}(\phi_s(U_s))$ (cf. [1, 2]). Let $\phi : \bar{\Omega}_m \mathbf{V} \rightarrow F = \prod_s F_s$ be the continuous homomorphism such that $\phi = (\phi_s)_{s \in S}$. Then $U_s = \phi^{-1}(\phi(U_s))$ for all $s \in S$. We consider the diagram

$$\begin{array}{ccc}
 \bar{\Omega}_n \mathbf{V} & & \\
 \varphi \downarrow & \searrow \phi \circ \mu_V & \\
 S & \xleftarrow{\rho} \phi(T) \hookrightarrow & \langle E(F) \rangle.
 \end{array}$$

We show that S is a homomorphic image of $\phi(T)$, more precisely that, there exists $\rho : \phi(T) \rightarrow S$ such that the diagram commutes. It suffices to show that, for $w, z \in \overline{\Omega}_n \mathbf{V}$, if $(\phi \circ \mu_{\mathbf{V}})(w) = (\phi \circ \mu_{\mathbf{V}})(z)$, then $\varphi(w) = \varphi(z)$. Let $s_1 = \varphi(w)$, $s_2 = \varphi(z)$ and suppose that $s_1 \neq s_2$. Since $U_{s_1} \cap U_{s_2} = \emptyset$, we have that $\phi(U_{s_1}) \cap \phi(U_{s_2}) = \emptyset$. Now, $\mu_{\mathbf{V}}(w) \in \mu_{\mathbf{V}}(\varphi^{-1}(s_1)) = U_{s_1} \cap T$ and, therefore, $\phi(\mu_{\mathbf{V}}(w)) \in \phi(U_{s_1} \cap T)$. Similarly, we obtain $\phi(\mu_{\mathbf{V}}(z)) \in \phi(U_{s_2} \cap T)$. It follows that $\phi(\mu_{\mathbf{V}}(w)) \neq \phi(\mu_{\mathbf{V}}(z))$.

We conclude that S divides $\langle E(F) \rangle$. Since $F \in \mathbf{V}$, it follows that $\langle E(F) \rangle \in \mathbf{V}_{\mathbf{E}}$ and, therefore, $S \in \mathbf{V}_{\mathbf{E}}$. This shows that $\mathbf{V} \subseteq \mathbf{V}_{\mathbf{E}}$, while the reverse inclusion is always verified. \square

From Theorem 5.1, to conclude that $\mathbf{V}_{\mathbf{E}} = \mathbf{V}$, it suffices to exhibit an embedding $\mu_{\mathbf{V}} : \overline{\Omega}_n \mathbf{V} \rightarrow \overline{\langle X \rangle}$ with $X \subseteq E(\overline{\Omega}_m \mathbf{V})$, for every integer n . We do not know if, conversely, such an embedding always exists in case $\mathbf{V}_{\mathbf{E}} = \mathbf{V}$.

For $\mathbf{V} \in \{\mathbf{R}, \mathbf{L}, \mathbf{J}, \mathbf{DA}\}$, we consider the unique continuous homomorphism $\mu_{\mathbf{V}}$ such that

$$\begin{aligned} \mu_{\mathbf{V}} : \overline{\Omega}_n \mathbf{V} &\rightarrow \overline{\Omega}_{n+1} \mathbf{V} \\ x_i &\mapsto x_i^\omega y^\omega, \end{aligned}$$

where y is a new variable and we prove that $\mu_{\mathbf{V}}$ is an embedding. In each case, we depend heavily on a suitable representation of the profinite semigroup $\overline{\Omega}_n \mathbf{V}$.

Let us start with the pseudovariety \mathbf{R} . We use the representation of implicit operations over \mathbf{R} by means of labeled ordinals due to Almeida and Weil [3]. We briefly recall it. Let A be an alphabet with $|A| = n$ and let $\mathbf{rLO}(A)$ be the set of reduced A -labeled ordinals. Recall that an A -labeled ordinal is a pair (α, l) , where α is an ordinal and $l : \alpha \rightarrow A$ is a labeling function. The *content* of (α, l) , $c(\alpha, l)$, is the range of l . The *cumulative content* of a limit ordinal $\beta \leq \alpha$, $\overleftarrow{c}(\beta)$, is the set of all letters $a \in A$ such there exists a sequence $(\gamma_k)_k$ of ordinals with $\cup_k \gamma_k = \beta$, $\gamma_k < \beta$ and $l(\gamma_k) = a$ for all k . An A -labeled ordinal (α, l) is said to be *reduced* if $l(\beta) \notin \overleftarrow{c}(\beta)$ for each limit ordinal $\beta < \alpha$.

Let $(\alpha, l) \in \mathbf{rLO}(A)$. For each $a \in A$, let γ_a be the smallest ordinal such that $\gamma_a < \alpha$ and $l(\gamma_a) = a$ (i.e., γ_a is the position of the first occurrence of a). We set $\gamma_a = 0$ if $l(\gamma) \neq a$, for all $\gamma < \alpha$. Let $\alpha_1 = \max\{\gamma_a \mid a \in A\}$ (i.e., the first occurrence of the last appearing letter) and let β_1 be such that $\alpha = \alpha_1 + 1 + \beta_1$, with $(\alpha_1, l_1), (\beta_1, m_1) \in \mathbf{rLO}(A)$, $l_1 = l|_{\alpha_1}$ and $m_1(\gamma) = l(\alpha_1 + 1 + \gamma)$ where $\gamma < \beta_1$. We call the equality $\alpha = \alpha_1 + 1 + \beta_1$ the *left basic partition* of (α, l) . We iterate this process while $\beta_i \neq 0$. Let $\beta_0 = \alpha$, $m_0 = l$ and $\beta_i = \alpha_{i+1} + 1 + \beta_{i+1}$ with α_{i+1} and β_{i+1} constructed in the same way. While $\beta_i \neq 0$, we obtain ordinals (α_i, l_i) and (β_i, m_i) , with $i \geq 1$, where $l_{i+1} = m_i|_{\alpha_{i+1}}$ and $m_{i+1}(\gamma) = m_i(\alpha_{i+1} + 1 + \gamma)$ where $\gamma < \beta_{i+1}$. Almeida and Weil showed that $\alpha = \sum_{i \geq 1} (\alpha_i + 1)$ and they define the product in $\mathbf{rLO}(A)$ that follows. Let $(\alpha, l), (\beta, m) \in \mathbf{rLO}(A)$. If α is not a limit ordinal, then

$$(5.1) \quad (\alpha, l)(\beta, m) = (\alpha + \beta, l')$$

where $l'(\gamma) = l(\gamma)$ if $\gamma < \alpha$ and $l'(\alpha + \gamma) = m(\gamma)$ if $\gamma < \beta$. If α is a limit ordinal, then we write $\beta = \beta_1 + \beta_2$ where β_1 is the smallest ordinal such that $m(\beta_1) \notin \overleftarrow{c}(\alpha)$. The product is given by

$$(5.2) \quad (\alpha, l)(\beta, m) = (\alpha + \beta_2, l')$$

where $l'(\gamma) = l(\gamma)$ if $\gamma < \alpha$ and $l'(\alpha + \gamma) = m(\beta_1 + \gamma)$ if $\gamma < \beta_2$. Almeida and Weil proved that $\mathbf{rLO}(A)$ equipped with this operation is isomorphic to $\overline{\Omega}_n \mathbf{R}$.

Proposition 5.2. *The function $\mu_R : \overline{\Omega}_n \mathbf{R} \rightarrow \overline{\Omega}_{n+1} \mathbf{R}$ is an embedding.*

Proof. We denote by $\psi_A : \overline{\Omega}_n \mathbf{R} \rightarrow \mathbf{rLO}(A)$ the isomorphism defined by Almeida and Weil [3], where $|A| = n$. Let $B = \{a, b : a \in A\}$ with $b \notin A$. We consider the following diagram

$$\begin{array}{ccc} \overline{\Omega}_n \mathbf{R} & \xrightarrow{\mu_R} & \overline{\Omega}_{n+1} \mathbf{R} \\ \psi_A \downarrow & & \downarrow \psi_B \\ \mathbf{rLO}(A) - \bar{\nu} & \triangleright & \mathbf{LO}(B) \end{array}$$

with ν defined as follows:

$$\begin{aligned} \nu : \mathbf{rLO}(A) &\rightarrow \mathbf{LO}(B) \\ (\alpha, l) &\mapsto ((\omega + \omega)\alpha, l') \end{aligned}$$

where $\mathbf{LO}(B)$ is the set of B -labeled ordinals and

$$\begin{aligned} l' : (\omega + \omega)\alpha &\rightarrow B \\ \beta &\mapsto \begin{cases} l(\gamma) & \text{if } \beta = (\omega + \omega)\gamma + k \text{ with } \gamma < \alpha, k \in \omega \\ b & \text{if } \beta = (\omega + \omega)\gamma + \omega + k \text{ with } \gamma < \alpha, k \in \omega. \end{cases} \end{aligned}$$

We prove that the diagram commutes, i.e., that $\nu = \psi_B \circ \mu_R \circ \psi_A^{-1}$. Let $(\alpha = \sum_{i \geq 1} (\alpha_i + 1), l) \in \mathbf{rLO}(A)$. Since ψ_A^{-1} , ψ_B and μ_R are homomorphisms, we proceed by induction on $|c(\alpha_i, l_i)|$, which is finite and less than $|c(\alpha, l)|$, and we obtain:

$$\begin{array}{ccc} \prod_{i \geq 1} (\psi_A^{-1}(\alpha_i, l_i) a_i) & \xrightarrow{\mu_R} & \prod_{i \geq 1} (\mu_R(\psi_A^{-1}(\alpha_i, l_i)) a_i^\omega b^\omega) \\ \psi_A^{-1} \uparrow & & \downarrow \psi_B \\ (\alpha = \sum_{i \geq 1} (\alpha_i + 1), l) & & (\delta, m) = \sum_{i \geq 1} ((\omega + \omega)\alpha_i, l'_i) + (\omega + \omega, g_i) \end{array}$$

where $a_i = m_{i-1}(\alpha_i)$ and

$$\begin{aligned} g_i : \omega + \omega &\rightarrow B \\ \beta &\mapsto \begin{cases} a_i & \text{if } \beta < \omega \\ b & \text{if } \beta = \omega + \gamma \text{ with } \gamma < \omega. \end{cases} \end{aligned}$$

We want to prove that $(\delta, m) = \nu(\alpha, l) = ((\omega + \omega)\alpha, l')$. Indeed, we have

$$\begin{aligned} \delta &= \sum_{i \geq 1} ((\omega + \omega)\alpha_i + (\omega + \omega)) \\ &= (\omega + \omega)\alpha_1 + (\omega + \omega) + (\omega + \omega)\alpha_2 + (\omega + \omega) + \cdots \\ &= (\omega + \omega)(\alpha_1 + 1 + \alpha_2 + 1 + \cdots) \\ &= (\omega + \omega) \sum_{i \geq 1} (\alpha_i + 1) \\ &= (\omega + \omega)\alpha, \end{aligned}$$

where the third equality follows from [17, Exercise 1.41], and

$$m : (\omega + \omega)\alpha \rightarrow B$$

$$\beta \mapsto \begin{cases} l'_i(\gamma) & \text{if } \beta = (\omega + \omega)(\sum_{j=1}^{i-1}(\alpha_j + 1)) + \gamma \\ & \text{with } \gamma < (\omega + \omega)\alpha_i \\ g_i(\gamma) & \text{if } \beta = (\omega + \omega)(\sum_{j=1}^{i-1}(\alpha_j + 1)) + (\omega + \omega)\alpha_i + \gamma \\ & \text{with } \gamma < (\omega + \omega), \end{cases}$$

where we set $\sum_{j=1}^0(\alpha_j + 1) = 0$. In the first case, it follows that

$$\begin{aligned} m(\beta) &= l'_i(\gamma) \\ &= \begin{cases} l'_i(\delta) & \text{if } \gamma = (\omega + \omega)\delta + k, \text{ with } \delta < \alpha_i, k \in w \\ b & \text{if } \gamma = (\omega + \omega)\delta + \omega + k, \text{ with } \delta < \alpha_i, k \in w \end{cases} \\ &= \begin{cases} l(\alpha_1 + 1 + \dots + \alpha_{i-1} + 1 + \delta) & \text{if } \gamma = (\omega + \omega)\delta + k \\ b & \text{if } \gamma = (\omega + \omega)\delta + \omega + k \end{cases} \\ &= \begin{cases} l(\eta) & \text{if } \beta = (\omega + \omega)\eta + k \\ b & \text{if } \beta = (\omega + \omega)\eta + \omega + k \end{cases} \\ &= l'(\beta) \end{aligned}$$

where $\eta = \sum_{j=1}^{i-1}(\alpha_j + 1) + \delta$. In the second case, for $\eta = \sum_{j=1}^{i-1}(\alpha_j + 1) + \alpha_i$, we have

$$\begin{aligned} m(\beta) &= g_i(\gamma) \\ &= \begin{cases} a_i & \text{if } \gamma < w \\ b & \text{if } \gamma = w + \delta \text{ with } \delta < w \end{cases} \\ &= \begin{cases} l(\alpha_1 + 1 + \dots + \alpha_{i-1} + 1 + \alpha_i) & \text{if } \beta = (\omega + \omega)\eta + \delta \\ b & \text{if } \beta = (\omega + \omega)\eta + \omega + \delta \end{cases} \\ &= \begin{cases} l(\eta) & \text{if } \beta = (\omega + \omega)\eta + \delta \\ b & \text{if } \beta = (\omega + \omega)\eta + \omega + \delta \end{cases} \\ &= l'(\beta) \end{aligned}$$

and, therefore, $m = l'$. It follows that the diagram commutes and ν is a homomorphism from $\mathbf{rLO}(A)$ into $\mathbf{rLO}(B)$, where the product involved is the one defined by formulas (5.1) and (5.2). Thus, $\mu_{\mathbf{R}}$ is injective if and only if ν is injective. Let (α, l) and (β, m) be reduced labeled ordinals such that $\nu(\alpha, l) = \nu(\beta, m)$. By [17, Exercise 3.41], we have

$$(\omega + \omega)\alpha = (\omega + \omega)\beta \implies \alpha = \beta$$

and

$$l' = m' \implies l(\gamma) = m(\gamma) \text{ for all } \gamma < \alpha \implies l = m.$$

Hence $(\alpha, l) = (\beta, m)$ and ν is injective. \square

The dual result for the pseudovariety \mathbf{L} follows by duality.

Proposition 5.3. *The function $\mu_{\mathbf{L}} : \overline{\Omega}_n \mathbf{L} \rightarrow \overline{\Omega}_{n+1} \mathbf{L}$ is an embedding.* \square

Now, we consider the pseudovariety \mathbf{J} of \mathcal{J} -trivial semigroups. We use the representation by canonical form of implicit operations over \mathbf{J} obtained by the first author [1, Section 8.2]. Consider the variety \mathcal{V} of type $(2, 1)$ defined by the set of identities

$$\Sigma = \{(xy)z = x(yz), (xy)^\omega = (yx)^\omega = (x^\omega y^\omega)^\omega, x^\omega x = x^\omega = xx^\omega, (x^\omega)^\omega = x^\omega\}.$$

We may reduce any term in the variables x_1, x_2, \dots using the following Noetherian and confluent system of *reduction rules*:

- (**rr1**): to eliminate parentheses concerning the application of the operation of multiplication;
- (**rr2**): to substitute any subterm of the form t^ω by u^ω , where u is the product, in increasing order of the indices, of the variables occurring in t ;
- (**rr3**): to absorb in factors of the form u^ω any adjacent factors in which only occur variables of u .

A term of \mathcal{V} is called a *word* if it does not involve the unary operation $_{-}^\omega$, and it is called *idempotent* if it is of the form t^ω , for some term t . The *content* $c(t)$ of a term t is the set of variables occurring in t . The factorization in *canonical form* of a term t is $t = t_1 \cdots t_n$, where:

- (**cf1**): each t_i is a word or an idempotent;
- (**cf2**): each idempotent t_i is of the form u^ω , where u is a product of variables with the indices in strictly increasing order;
- (**cf3**): given two consecutive idempotents t_i and t_{i+1} , the sets $c(t_i)$ and $c(t_{i+1})$ are incomparable;
- (**cf4**): two consecutive terms t_i and t_{i+1} are not both words;
- (**cf5**): if t_i is a word and t_{i+1} is an idempotent, then the last letter of t_i is not in $c(t_{i+1})$;
- (**cf6**): if t_{i+1} is a word and t_i is an idempotent, then the first letter of t_{i+1} is not in $c(t_i)$.

Let $F_n \mathcal{V}$ be the \mathcal{V} -free algebra on $\{x_1, \dots, x_n\}$. The semigroup $\overline{\Omega}_n \mathbf{J}$ may be seen as an algebra of type $(2, 1)$, where all elements are constructed using the operations of multiplication and omega power and the variables $\{x_1, \dots, x_n\}$. Then we have a natural surjective homomorphism

$$\begin{aligned} \psi : F_n \mathcal{V} &\rightarrow \overline{\Omega}_n \mathbf{J} \\ x_i &\mapsto x_i \quad (i = 1, \dots, n), \end{aligned}$$

and [1, Theorem 8.2.7] establishes that ψ is, in fact, an isomorphism. We are now able to prove the desired proposition.

Proposition 5.4. *The function $\mu_{\mathbf{J}} : \overline{\Omega}_n \mathbf{J} \rightarrow \overline{\Omega}_{n+1} \mathbf{J}$ is an embedding.*

Proof. By the above, to show that $\mu_{\mathbf{J}}$ is injective is equivalent to establishing that

$$\begin{aligned} \nu : F_n \mathcal{V} &\rightarrow F_{n+1} \mathcal{V} \\ x_i &\mapsto x_i^\omega y^\omega \end{aligned}$$

is injective. Let $w, z \in F_n \mathcal{V}$ be such that $\nu(w) = \nu(z)$ and let $w = w_1 \cdots w_m$ and $z = z_1 \cdots z_n$ be the respective factorizations in canonical form. We want to determine the factorizations in canonical form of $\nu(w)$ and $\nu(z)$. Let $i \in \{1, \dots, m\}$. Suppose that w_i is a word, i.e., $w_i = x_{i_1} \cdots x_{i_k}$. Then $\nu(w_i) = x_{i_1}^\omega y^\omega \cdots x_{i_k}^\omega y^\omega$. Note that this factorization is in canonical form, because it is a product such that two consecutive

idempotents have incomparable contents. Suppose now that w_i is an idempotent, i.e., $w_i = (x_{i_1} \cdots x_{i_l})^\omega$ with $i_1 < \cdots < i_l$. Then $\nu(w_i) = (x_{i_1}^\omega y^\omega \cdots x_{i_l}^\omega y^\omega)^\omega = (x_{i_1} \cdots x_{i_l} y)^\omega$ applying the reduction rule (rr2). Note that $i_1 < \cdots < i_l < y$ (assuming that the new letter y is larger than any of the others) and, therefore, the last presented factorization of $\nu(w_i)$ is in canonical form. Therefore, an idempotent of $F_n \mathcal{V}$ has as image an idempotent of $F_{n+1} \mathcal{V}$ and a word of length k has as image a product of $2k$ idempotents of $F_{n+1} \mathcal{V}$, in canonical form.

Consider now the product $w_i w_j$ with $j = i + 1$. Note that, by definition of canonical form, w_i and w_j are not both words. Suppose that w_i is a word and w_j is an idempotent. Then

$$\nu(w_i w_j) = x_{i_1}^\omega y^\omega \cdots x_{i_k}^\omega y^\omega \cdot (x_{j_1} \cdots x_{j_l} y)^\omega = x_{i_1}^\omega y^\omega \cdots x_{i_k}^\omega (x_{j_1} \cdots x_{j_l} y)^\omega$$

applying the reduction rule (rr3). By hypothesis $x_{i_k} \notin c(w_j)$ and we conclude that the last factorization of $\nu(w_i w_j)$ is in canonical form. If w_i is an idempotent and w_j is a word, or if both w_i and w_j are idempotents, then we have, respectively, the following canonical forms for $w_i w_j$:

$$\nu(w_i w_j) = (x_{i_1} \cdots x_{i_k} y)^\omega x_{j_1}^\omega y^\omega \cdots x_{j_l}^\omega y^\omega$$

and

$$\nu(w_i w_j) = (x_{i_1} \cdots x_{i_k} y)^\omega (x_{j_1} \cdots x_{j_l} y)^\omega.$$

Let $\nu(w) = \bar{w}_1 \cdots \bar{w}_{m'}$ and $\nu(z) = \bar{z}_1 \cdots \bar{z}_{n'}$ be the factorizations in canonical form of $\nu(w)$ and $\nu(z)$, respectively. Since $\nu(w) = \nu(z)$, by [1, Theorem 8.2.8] we have $m' = n'$ and $\bar{w}_i = \bar{z}_i$, for all i . Three cases can occur for each factor \bar{w}_i : $\bar{w}_i = x_j^\omega$, $\bar{w}_i = y^\omega$ or $\bar{w}_i = (x_{j_1} \cdots x_{j_l} y)^\omega$, for some j, l . Note that the content of the idempotent in the last case has cardinal bigger than 2, whether in the other cases is 1. We recover w as follows. In the first two cases, we substitute \bar{w}_i by x_j and by y , respectively. In the last case, we substitute \bar{w}_i by $(x_{j_1} \cdots x_{j_l})^\omega$. It is easy to see that the canonical forms of w and z are recovered and they are equal. It follows that $w = z$ and ν is injective. \square

Finally, we treat the case of DA using the representation of implicit operations over DA by means of labeled orderings obtained by Moura [10], which is similar to the case of the pseudovariety R. So, we omit most details and we refer the reader to [10] as needed. In that paper, we proved that there is a bijection between the free profinite semigroup over DA, $\bar{\Omega}_A \text{DA}$, and the set of all reduced A -labeled $*$ -linear orderings, $\mathbf{rLO}^*(A)$.

Proposition 5.5. *The function $\mu_{\text{DA}} : \bar{\Omega}_n \text{DA} \rightarrow \bar{\Omega}_{n+1} \text{DA}$ is an embedding.*

Proof. Since $\bar{\Omega}_A \text{DA}$ and $\mathbf{rLO}^*(A)$ are isomorphic, it suffices to prove that the following mapping is injective:

$$\begin{aligned} \nu : \mathbf{rLO}^*(A) &\rightarrow \mathbf{rLO}^*(B) \\ (o, l) &\mapsto ((\omega + \omega^*)2o, l'). \end{aligned}$$

By [10], $\nu(o, l)$ is constructed from (o, l) in the following way: each position of o is replaced by the ordering $(\omega + \omega^*)2$ and, if this position is labeled $a \in A$, the label of each position on the resulting ordering is a or b , depending on whether the position is in the first or second term of the form $\omega + \omega^*$. Thus, given two consecutive positions $p < q$ from $\nu(o, l)$, one and only one of the following cases can occur: $l'(p) = l'(q) = a$, $l'(p) = l'(q) = b$, $l'(p) = a$ and $l'(q) = b$, or $l'(p) = b$ and

$l'(q) = a$, for some $a \in A$. In the first three cases, both positions are in the same interval $(\omega + \omega^*)2$ of o , resulting from the replacement of a position of o labeled a , for some $a \in A$. In the fourth case, the positions are in consecutive intervals corresponding to the replacement of consecutive positions of o . We split $(\omega + \omega^*)2o$ in intervals I_p , $p \in o$, of the form $(\omega + \omega^*)2$ and that are maximal for the following condition: $I_p = I_{p,1} \cup I_{p,2}$ where $I_{p,1}$ is an interval whose elements are labeled with the same letter of A and $I_{p,2}$ is an interval whose elements are labeled with b . It follows that $(\omega + \omega^*)2o = \bigcup_{p \in o} I_p$ and $l'(\bar{p}) = l(p)$, for all $\bar{p} \in I_{p,1}$, with $I_p = I_{p,1} \cup I_{p,2}$ satisfying the above condition. Thus, we may recover (o, l) from $((\omega + \omega^*)2o, l')$ considering the ordering of such intervals with the labeling function that labels each interval by a if the set of labels of its positions is $\{a, b\}$. \square

Combining Propositions 5.2, 5.3, 5.4, and 5.5 with Theorem 5.1, we obtain the following result, of which only the case of DA was not already proved by the alternative approach referred in Section 3.

Corollary 5.6. *The pseudovarieties R, L, J and DA satisfy the equation $\mathbf{V}_E = \mathbf{V}$.*

It remains an open problem whether the pseudovariety DS satisfies the equality $\mathbf{V}_E = \mathbf{V}$. This motivates the study of the free profinite semigroup over DS, for which no representation result is currently known.

6. RANKS

For a pseudovariety \mathbf{V} we consider the following associated parameters:

- $\text{rank } \mathbf{V}$ is the least positive integer n such that \mathbf{V} is defined by a set of pseudoidentities on at most n variables, unless there is no such n , in which case we let $\text{rank } \mathbf{V} = \infty$;
- for a positive integer n , $\mathbf{V}(n)$ denotes the pseudovariety generated by all n -generated members of \mathbf{V} , that is the class of all finite continuous homomorphic images of $\bar{\Omega}_n \mathbf{V}$;
- the *generator rank* of \mathbf{V} , denoted $\text{grank } \mathbf{V}$, is the least positive integer n such that $\mathbf{V} = \mathbf{V}(n)$, unless there is no such n , in which case we set $\text{grank } \mathbf{V} = \infty$;
- the *idempotent generator rank* of \mathbf{V} , denoted $\text{idgrank } \mathbf{V}$, is the least positive integer n such that \mathbf{V} is generated by its members which are generated by at most n idempotents, unless there is no such n , in which case we put $\text{idgrank } \mathbf{V} = \infty$.

The following are simple observations concerning these parameters.

Lemma 6.1. *Let \mathbf{V} be a pseudovariety and n a positive integer. Denote by Σ_n the set of all pseudoidentities in at most n variables which are valid in \mathbf{V} . Then the following hold:*

- (1) $\text{rank } \mathbf{V} \leq n$ if and only if $\mathbf{V} = \llbracket \Sigma_n \rrbracket$;
- (2) $\text{grank } \mathbf{V} \leq n$ if and only if $\mathbf{V} = \mathbf{V}(n)$;
- (3) $\text{idgrank } \mathbf{V} \leq n$ implies $\mathbf{V} = \mathbf{V}_E(n)$;
- (4) $\text{rank } \mathbf{V} \leq \text{grank } \mathbf{V} \leq \text{idgrank } \mathbf{V}$.

Proof. Everything but the first inequality in (4) follows immediately from the definitions. A standard proof of Reiterman's Theorem (see, for instance, [2]) shows that every element of $\llbracket \Sigma_n \rrbracket$ is a continuous homomorphic image of $\bar{\Omega}_n \mathbf{V}$. Hence $\llbracket \Sigma_n \rrbracket \subseteq \mathbf{V}(n)$, which implies the first inequality in (4) in view of (1) and (2). \square

Lemma 6.2. *Let x_1, \dots, x_n be n distinct variables and consider the word $u_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$. Then the pseudoidentity*

$$(6.1) \quad (u_1^\omega \cdots u_n^\omega)^{\omega+1} = u_1^\omega \cdots u_n^\omega$$

holds in $\text{DS}(n-1)$ but not in $\text{J}(n)$.

Proof. Ordering the variables by increasing order of the indices, the canonical form of the implicit operation over J determined by the left side of the pseudoidentity (6.1) is $(x_1 \cdots x_n)^\omega$ while the right side is already in canonical form. By [1, Theorem 8.2.7] it follows that $\overline{\Omega}_n \text{J}$ fails (6.1), whence so does $\text{J}(n)$.

Let $\varphi : \overline{\Omega}_n \text{S} \rightarrow \overline{\Omega}_{n-1} \text{DS}$ be any continuous homomorphism. We need to show that φ identifies the two sides of (6.1), that is that $\varphi(u_1^\omega \cdots u_n^\omega)$ is regular. Now, by [1, Theorems 8.1.10 and 8.2.7], $\varphi(u_1^\omega \cdots u_n^\omega)$ is regular if and only if it has the same content as some $\varphi(u_i^\omega)$. Since $c(\varphi(u_1^\omega \cdots u_n^\omega))$ contains at most $n-1$ variables, there is an index $i \in \{1, \dots, n\}$ such that $c(\varphi(u_1^\omega \cdots u_n^\omega)) = \bigcup_{j \in \{1, \dots, n\} \setminus \{i\}} c(\varphi(x_j))$, whence $c(\varphi(u_1^\omega \cdots u_n^\omega)) = c(\varphi(u_i^\omega))$. \square

The following result is an immediate application of Lemma 6.2.

Proposition 6.3. *Every pseudovariety in the interval $[\text{J}, \text{DS}]$ has infinite grank.* \square

In contrast, the pseudoidentity definitions of the pseudovarieties $\text{J}, \text{R}, \text{L}, \text{DA}, \text{DG}, \text{DO}, \text{DS}$ given at the end of Section 2 show that the rank of each of them is two. Indeed, the smallest pseudovarieties defined by one-variable pseudoidentities containing the first four, respectively the last three, of these are respectively A and S .

For the idgrank, we can prove the following results.

Lemma 6.4. *We define, recursively, two sequences of implicit operations as follows: for $n \geq 3$, we put*

$$\begin{aligned} v_3 &= (x_1 x_2)^\omega x_3 \cdot x_1^2 \cdot x_3 (x_1 x_2)^\omega \\ w_3 &= (x_1 x_2)^\omega x_3 \cdot x_1 \cdot x_3 (x_1 x_2)^\omega \\ v_{n+1} &= (x_1 \cdots x_n)^\omega x_{n+1} \cdot v_n \cdot x_{n+1} (x_1 \cdots x_n)^\omega \\ w_{n+1} &= (x_1 \cdots x_n)^\omega x_{n+1} \cdot w_n \cdot x_{n+1} (x_1 \cdots x_n)^\omega. \end{aligned}$$

(a) *If S is a semigroup from DA generated by $n \geq 2$ idempotents, then S satisfies the pseudoidentity $v_{n+1} = w_{n+1}$.*

(b) *The pseudovariety $\text{J}(n)$ fails the pseudoidentity $v_{n+1} = w_{n+1}$.*

Proof. (a) By hypothesis, there exists some continuous homomorphism $\pi : \overline{\Omega}_n \text{DA} \rightarrow S$ that maps the free generators x_i to idempotents.

We proceed by induction on n . Given $\varphi : \overline{\Omega}_{n+1} \text{S} \rightarrow S$, we must show that $\varphi(v_{n+1}) = \varphi(w_{n+1})$. Since π is onto and $\overline{\Omega}_n \text{S}$ is free profinite, φ factors through π , say as $\varphi = \pi \circ \psi$ for some continuous homomorphism $\psi : \overline{\Omega}_{n+1} \text{S} \rightarrow \overline{\Omega}_n \text{DA}$.

At the basis of the induction, let us consider first the case $n = 2$. If $\psi(x_1 x_2)$ has full content, then $\varphi((x_1 x_2)^\omega)$ belongs to the minimum ideal of S . Since this ideal is a rectangular band and both $\varphi(v_3)$ and $\varphi(w_3)$ are \mathcal{L} and \mathcal{R} -below $\varphi((x_1 x_2)^\omega)$, they are both equal to this idempotent. Otherwise, $\psi(x_1 x_2)$ only involves one of the free generators of $\overline{\Omega}_n \text{DA}$ and so $\varphi(x_1)$ is an idempotent, in which case the equality $\varphi(v_3) = \varphi(w_3)$ is trivially verified.

For the general case $n > 2$, similarly, if $\overline{\psi(x_1 \cdots x_n)}$ has full content, then $\varphi(v_{n+1}) = \varphi(w_{n+1})$. Otherwise, let $T = \overline{\langle x_1, \dots, x_n \rangle}$ so that $\varphi(T)$ is a semigroup of DA generated by at most $n - 1$ idempotents. By induction hypothesis, it satisfies the pseudoidentity $v_n = w_n$, whence $\varphi(v_n) = \varphi(w_n)$. Taking into account the definition of v_{n+1} and w_{n+1} , we conclude that $\varphi(v_{n+1}) = \varphi(w_{n+1})$.

(b) Let $\varphi = \overline{\Omega_{n+1}\mathbf{S}} \rightarrow \overline{\Omega_n\mathbf{J}}$ be the continuous homomorphism that fixes x_1 and sends each other x_i to x_{i-1} . Consider the factorizations of v_{n+1} and w_{n+1} in idempotents and maximal explicit factors between them which results from the recursive definition of these implicit operations. Then a straightforward induction shows that both these factorizations and the result of applying φ to each factor (and eliminating the repetition of x_1 within each ω -power) are in canonical form. Hence $\varphi(v_{n+1}) \neq \varphi(w_{n+1})$ by [1, Theorem 8.2.7]. Since $\overline{\Omega_n\mathbf{J}}$ is residually finite, this shows that there is some member of $\mathbf{J}(n)$ that fails $v_{n+1} = w_{n+1}$. \square

In view of the definitions, Lemma 6.4 yields the following result.

Proposition 6.5. *The inequality $\text{idrank } \mathbf{V}(n) > n$ holds for every pseudovariety \mathbf{V} in the interval $[\mathbf{J}, \text{DA}]$.* \square

Combining Propositions 6.5, 5.4, 5.2, 5.3, and 5.5 yields the following result.

Corollary 6.6. *The equality $\text{idrank } \mathbf{V}(n) = n + 1$ holds for every pseudovariety \mathbf{V} in $\{\mathbf{J}, \mathbf{R}, \mathbf{L}, \text{DA}\}$.* \square

We finish the paper with a brief comparison of the results obtained here for the equality $\mathbf{V} = \mathbf{V}_E$, with $\mathbf{V} \in \{\mathbf{J}, \mathbf{R}, \mathbf{L}\}$, and the results that follow from the work of Straubing [18] and Howie [7], and of Pin [15] and Laradji and Umar [8], respectively.

Straubing showed that an n -element \mathcal{J} -trivial monoid divides \mathcal{C}_{n+1} , and Pin proved that a finite \mathcal{R} -trivial monoid with cardinal n embeds into \mathcal{E}_n .

On the other hand, several works deal with the ranks and idempotent ranks of various finite transformation semigroups. Recall that the *rank* of a finite semigroup is the minimum number of generators, and the *idempotent rank* of an idempotent-generated finite semigroup is the minimum number of idempotent generators. Gomes and Howie [5] showed that the rank and idempotent rank of the subsemigroup of \mathcal{T}_n consisting of all full transformations with range less than n are both equal to $n(n - 1)/2$. The rank and idempotent rank of the subsemigroup of all contractive finite full transformations are both equal to $n + 1$, as showed by Umar [19]. Finally, Laradji and Umar [8] proved that the rank and idempotent rank of the subsemigroup of all contractive and order-preserving finite full transformations are both equal to n . We refer the reader to [20, 8] for detailed information on this topic.

Therefore, in the results quoted in Section 3 that follow from the works of these authors, the idempotent rank of the idempotent-generated semigroup is related with the cardinality of the embedded semigroup. In this section together with Section 5, we proved that any finite semigroup of \mathbf{J} , \mathbf{R} , \mathbf{L} or DA with rank n divides an idempotent-generated semigroup of the same type with idempotent rank at most $n + 1$ (cf. Corollary 6.6). So, here the control on the number of generators is related with the number of generators of the embedded semigroup, which may be much smaller than its cardinality.

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