HOPF BIFURCATION WITH TETRAHEDRAL AND OCTAHEDRAL SYMMETRY

ISABEL SALGADO LABOURIAU, ADRIAN C. MURZA

Abstract. In the study of the periodic solutions of a Γ-equivariant dynamical system, the H mod K theorem gives all possible periodic solutions, based on the group-theoretical aspects. By contrast, the equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for each **C**-axial subgroup of $\Gamma \times \mathbb{S}^1$. In this paper while characterizing the Hopf bifurcation, we identify which periodic solution types, whose existence is guaranteed by the H mod K theorem, are obtainable by Hopf bifurcation from the origin, when the group Γ is either tetrahedral or octahedral. The two groups are isomorphic, but their representations in $\mathbb{R}^3$ and in $\mathbb{R}^6$ are not, and this changes the possible symmetries of bifurcating solutions. We also discuss the periodic solutions for the full group of symmetries of the cube.

Keywords: equivariant dynamical system ; tetrahedral symmetry; octahedral symmetry; periodic solutions ; Hopf bifurcation.

[2010] 37C80 ; 37G40 ; 34C15 ; 34D06 ; 34C15

1. Introduction

The formalism of Γ-equivariant differential equations, ie. those equations whose associated vector field commutes with the action of a finite group Γ has been developed by Golubitsky, Stewart and Schaeffer in [4], [7] and [6]. Within this formalism, two methods for obtaining periodic solutions have been described: the H mod K theorem [2, 6, Ch.3] and the equivariant Hopf theorem [6, Ch.4]. The equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all **C**-axial subgroups of $\Gamma \times \mathbb{S}^1$, under some generic conditions. The H mod K theorem offers the complete set of possible periodic solutions based exclusively on the structure of the group Γ acting on the differential equation. It also guarantees the existence of a model with this symmetry having these periodic solutions, but it is not an existence result for any specific equation.

In this article, we discuss the cases when Γ is the tetrahedral group $\langle \mathbb{T}, \kappa \rangle$ of symmetries of the tetrahedron and when Γ is the octahedral group $\mathbb{O}$ of rotational symmetries of the cube. As abstract groups, $\langle \mathbb{T}, \kappa \rangle$ and $\mathbb{O}$ are isomorphic to the alternating group A_4, but they correspond to nonequivalent representations both in $\mathbb{R}^3$ and $\mathbb{C}^3$. We also discuss the full octahedral group $\langle \mathbb{O}, -Id \rangle$ of symmetries of the cube.

Steady-state bifurcation problems with octahedral symmetry are analysed by Melbourne [9] using results from singularity theory. For non-degenerate bifurcation problems equivariant with respect to the standard action on $\mathbb{R}^3$ of the octahedral group he finds three branches of symmetry-breaking steady-state bifurcations corresponding to the three maximal isotropy subgroups with one-dimensional fixed-point subspaces. Hopf bifurcation with the rotational symmetry of the tetrahedron is studied by Swift and Barany [12],

Date: February 3, 2015.

motivated by problems in fluid dynamics. They find periodic branches created at Hopf bifurcation, as well as evidence of chaotic dynamics, bifurcating directly from the equilibrium — instant chaos. Generic Hopf bifurcation with the rotational symmetries of the cube is studied by Ashwin and Podvigina [1], also with the motivation of fluid dynamics. They also find evidence of chaotic dynamics, this time arising from secondary bifurcations from periodic branches created at Hopf bifurcation.

In this article, we pose a more specific question: which periodic solutions predicted by the H mod K theorem are obtainable by Hopf bifurcation from the trivial steady-state when Γ is one of the groups discussed above?

Solutions predicted by the H mod K theorem cannot always be obtained by a generic Hopf bifurcation from the trivial equilibrium. When the group is finite abelian, the periodic solutions whose existence is allowed by the H mod K theorem that are realizable from the equivariant Hopf theorem are described in [3].

The relevant action for dealing with Γ-equivariant Hopf bifurcation is that of $\Gamma \times \mathbb{S}^1$. The representations of $A_4 \times \mathbb{S}^1$ on $\mathbb{C}^3$ induced by $\langle \mathbb{T}, \kappa \rangle$ and by $\mathbb{O}$ are equivalent, even though the representations $\langle \mathbb{T}, \kappa \rangle$ and $\mathbb{O}$ are not. Moreover, the action of the group $\langle \mathbb{O}, -Id \rangle \times \mathbb{S}^1$ is equivalent to that of $\mathbb{O} \times \mathbb{S}^1$. Thus it is interesting to compare results for these groups, and we find that indeed our question has different answers for them.

We will answer this question by finding for each group that not all periodic solutions predicted by the H mod K theorem occur as primary Hopf bifurcations from the trivial equilibrium. For this we analyse bifurcations taking place in four-dimensional invariant subspaces and giving rise to periodic solutions with very small symmetry groups. In particular, we find that some solutions predicted by the H mod K theorem for $\langle \mathbb{T}, \kappa \rangle$-equivariant vector fields are not compatible with the $\Gamma \times \mathbb{S}^1$ action, and if they bifurcate from an equilibrium, they either have more symmetry or they arise at a resonant Hopf bifurcation. This is not the case for $\mathbb{O}$, nor for $\langle \mathbb{O}, -Id \rangle$.

Framework of the article. The relevant actions of the groups $\Gamma = \langle \mathbb{T}, \kappa \rangle$ of symmetries of the tetrahedron, $\Gamma = \mathbb{O}$ of rotational symmetries of the cube, $\Gamma = \langle \mathbb{O}, -Id \rangle$ of all symmetries of the cube and of $\Gamma \times \mathbb{S}^1$ on $\mathbb{R}^6 \sim \mathbb{C}^3$ are described in Section 3, after stating some preliminary results and definitions in Section 2. Hopf bifurcation is treated in Section 4, where we summarise some of the results of Ashwin and Podvigina [1] on $\mathbb{O} \times \mathbb{S}^1$, together with the formulation of the same results when the group is interpreted as $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ and as $\langle \mathbb{O}, -Id \rangle \times \mathbb{S}^1$. This includes the analysis of Hopf bifurcation inside fixed-point subspaces for submaximal isotropy subgroups, one of which we perform in more detail than in [1], giving a geometric proof of the existence of up to three branches of submaximal periodic solutions, for some values of the parameters in a degree three normal form. Finally, we apply the H mod K theorem in Section 5 where we compare the bifurcations for the three group actions.

2. Preliminary results and definitions

Before stating the theorem we give some definitions from [6]. The reader is referred to this book for results on bifurcation with symmetry.

Let Γ be a compact Lie group. A representation of Γ on a vector space W is Γ-*simple* if either:

(a) $W \sim V \oplus V$ where V is absolutely irreducible for Γ, or
(b) the action of Γ on W is irreducible but not absolutely irreducible.

Let W be a Γ-simple representation and let f be a Γ-equivariant vector field in W. Then it follows [7, Ch. XVI, Lemma1.5] that if f is a Γ-equivariant vector field, and if Jacobian matrix $(df)_0$ of f evaluated at the origin has purely imaginary eigenvalues $\pm\omega i$, then in suitable coordinates $(df)_0$ has the form:

$$(df)_0 = \omega J = \omega \begin{bmatrix} 0 & -Id \\ Id & 0 \end{bmatrix}$$

where Id is the identity matrix. Consider the action of $\mathbb{S}^1$ on W given by $\theta x = e^{i\theta J}x$. A subgroup $\Sigma \subseteq \Gamma \times \mathbb{S}^1$ is $\mathbb{C}$-*axial* if Σ is an isotropy subgroup and $\dim \mathrm{Fix}(\Sigma) = 2$.

Let $\dot{x} = f(x)$ be a Γ-equivariant differential equation with a T-periodic solution $x(t)$. We call $(\gamma, \theta) \in \Gamma \times \mathbb{S}^1$ a spatio-temporal symmetry of the solution $x(t)$ if $\gamma \cdot x(t+\theta) = x(t)$ for all t. A spatio-temporal symmetry of the solution $x(t)$ for which $\theta = 0$ is called a spatial symmetry, since it fixes the point $x(t)$ at every moment of time.

The main tool here will be the following theorem.

Theorem 1 (Equivariant Hopf Theorem [6])**.** *Let a compact Lie group Γ act Γ-simply, orthogonally and nontrivially on $\mathbb{R}^{2m}$. Assume that*

(a) $f : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}^{2m}$ *is Γ-equivariant. Then $f(0, \lambda) = 0$ and $(df)_{0,\lambda}$ has eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$ each of multiplicity m;*
(b) $\sigma(0) = 0$ *and* $\rho(0) = 1$;
(c) $\sigma'(0) \neq 0$ *the eigenvalue crossing condition;*
(d) $\Sigma \subseteq \Gamma \times \mathbb{S}^1$ *is a $\mathbb{C}$-axial subgroup.*

Then there exists a unique branch of periodic solutions with period $\approx 2\pi$ emanating from the origin, with spatio-temporal symmetries Σ.

The group of all spatio-temporal symmetries of $x(t)$ is denoted $\Sigma_{x(t)} \subseteq \Gamma \times \mathbb{S}^1$. The symmetry group $\Sigma_{x(t)}$ can be identified with a pair of subgroups H and K of Γ and a homomorphism $\Phi : H \to \mathbb{S}^1$ with kernel K. We define

$$H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\} \qquad K = \{\gamma \in \Gamma : \gamma x(t) = x(t)\ \forall t\}$$

where $K \subseteq \Sigma_{x(t)}$ is the subgroup of spatial symmetries of $x(t)$ and the subgroup H of Γ consists of symmetries preserving the trajectory $x(t)$ but not necessarily the points in the trajectory. We abuse notation saying that H is a group of spatio-temporal symmetries of $x(t)$. This makes sense because the groups $H \subseteq \Gamma$ and $\Sigma_{x(t)} \subseteq \Gamma \times \mathbb{S}^1$ are isomorphic; the isomorphism being the restriction to $\Sigma_{x(t)}$ of the projection of $\Gamma \times \mathbb{S}^1$ onto Γ.

Given an isotropy subgroup $\Sigma \subset \Gamma$, denote by $N(\Sigma)$ the normaliser of Σ in Γ, satisfying $N(\Sigma) = \{\gamma \in \Gamma : \gamma\Sigma = \Sigma\gamma\}$, and by L_Σ the variety $L_\Sigma = \bigcup_{\gamma \notin \Sigma} \mathrm{Fix}(\gamma) \cap \mathrm{Fix}(\Sigma)$.

The second important tool in this article is the following result.

Theorem 2. *(H mod K Theorem [2, 6]) Let Γ be a finite group acting on $\mathbb{R}^n$. There is a periodic solution to some Γ-equivariant system of ODEs on $\mathbb{R}^n$ with spatial symmetries K and spatio-temporal symmetries H if and only if the following conditions hold:*

(a) *H/K is cyclic;*
(b) *K is an isotropy subgroup;*
(c) *$\dim Fix(K) \geqslant 2$. If $\dim Fix(K) = 2$, then either $H = K$ or $H = N(K)$;*
(d) *H fixes a connected component of* $\mathrm{Fix(K)} \backslash \mathrm{L_K}$.

Moreover, if (a)−(d) hold, the system can be chosen so that the periodic solution is stable.

When $H/K \sim \mathbb{Z}_m$, the periodic solution $x(t)$ is called either a standing wave or (usually for $m \geqslant 3$) a discrete rotating wave; and when $H/K \sim \mathbb{S}^1$ it is called a rotating wave [6, page 64]. Here all rotating waves are discrete.

3. Group actions

Our aim in this article is to compare the bifurcation of periodic solutions for generic differential equations equivariant under different representations of the same abstract group. In this section we describe the representations used.

3.1. **Symmetries of the tetrahedron.** The group $\mathbb{T}$ of rotational symmetries of the tetrahedron [12] has order 12. Its action on $\mathbb{R}^3$ is generated by two rotations R and C of orders 2 and 3, respectively, and given by

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Next we want to augment the group $\mathbb{T}$ with a reflection, given by

$$\kappa = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

to form $\langle \mathbb{T}, \kappa \rangle$, the full group of symmetries of the thetrahedron, that has order 24. We obtain an action on $\mathbb{R}^6$ by identifying $\mathbb{R}^6 \equiv \mathbb{C}^3$ and taking the same matrices as generators.

The representation of $\langle \mathbb{T}, \kappa \rangle$ on $\mathbb{C}^3$ is $\langle \mathbb{T}, \kappa \rangle$-simple. The isotropy lattice of $\langle \mathbb{T}, \kappa \rangle$ is shown in Figure 1.

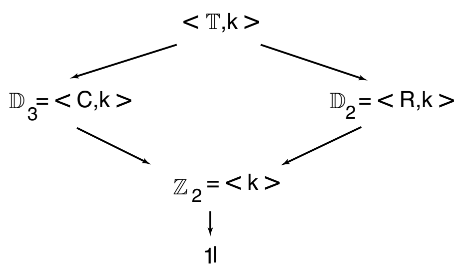


Figure 1. Isotropy lattice for the group $\langle \mathbb{T}, \kappa \rangle$ of symmetries of the tetrahedron.

3.2. **Symmetries of the cube.** The action on $\mathbb{R}^3$ of the group $\mathbb{O}$ of rotational symmetries of the cube is generated by the rotation C of order 3, with the matrix above, and by the rotation T of order 4

$$T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As in the case of the symmetries of the tetrahedron, we obtain an action of $\mathbb{O}$ on $\mathbb{R}^6$ by identifying $\mathbb{R}^6 \equiv \mathbb{C}^3$ and using the same matrices.

As abstract groups, $\langle \mathbb{T}, \kappa \rangle$ and $\mathbb{O}$ are isomorphic, the isomorphism maps C into itself and the rotation T of order 4 in $\mathbb{O}$ into the rotation-reflection $C^2 R\kappa$ in $\langle \mathbb{T}, \kappa \rangle$. However, the two representations are not equivalent, since $\mathbb{O}$ is a subgroup of $SO(3)$ and $\langle \mathbb{T}, \kappa \rangle$ is not. The isotropy lattice of $\mathbb{O}$ is shown in Figure 2.

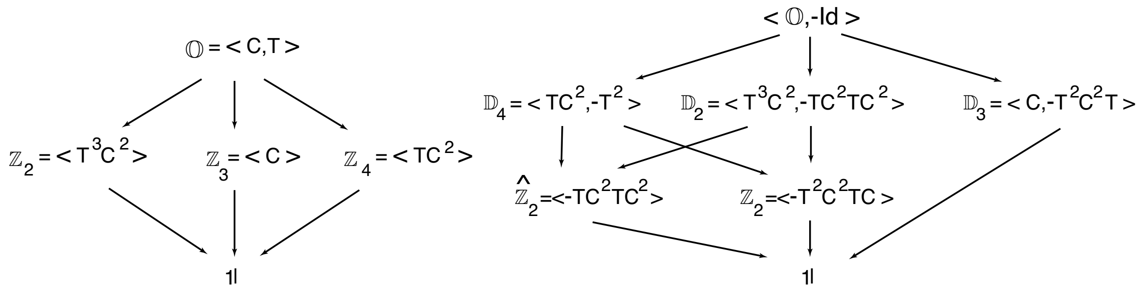


FIGURE 2. Isotropy lattices for the group $\mathbb{O}$ of rotational symmetries of the cube (left) and for the full group of symmetries of the cube $\langle \mathbb{O}, -Id \rangle$ (right).

Adding the rotation-reflection $-Id$ to $\mathbb{O}$ yields the full symmetry group of the cube, that will be denoted $\langle \mathbb{O}, -Id \rangle$. It has order 48 and its isotropy lattice is shown in Figure 2.

3.3. **Adding $\mathbb{S}^1$.** The corresponding actions of $\Gamma \times \mathbb{S}^1$ on $\mathbb{C}^3$, where Γ is one of the groups $\langle \mathbb{T}, \kappa \rangle$, $\mathbb{O}$ and $\langle \mathbb{O}, -Id \rangle$, are obtained by adding the elements $e^{i\theta} \cdot \mathrm{Id}$, $\theta \in (0, 2\pi)$ to the group Γ. Note that with this action the elements of $\mathbb{S}^1$ commute with those of Γ. Since $e^{i\pi} Id = -Id \in \langle \mathbb{O}, -Id \rangle$, then the groups $\langle \mathbb{O}, -Id \rangle \times \mathbb{S}^1$ and $\mathbb{O} \times \mathbb{S}^1$ coincide as subgroups of $\mathbf{O}(6)$. The product $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ also yields the same subgroups of $\mathbf{O}(6)$, because $T = e^{i\pi} C^2 R \kappa$. We will denote this group by $\Gamma \times \mathbb{S}^1$, its isotropy lattice is shown in Figure 3. Note that in the case $\Gamma = \langle \mathbb{O}, -Id \rangle$ the trivial subgroup $\mathbb{1}$ also contains the element $(-Id, \pi)$ that acts trivialy.

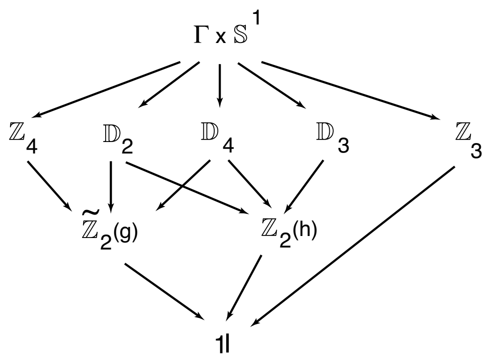


FIGURE 3. Isotropy lattice for the group $\Gamma \times \mathbb{S}^1$, where Γ is any of the three groups $\langle \mathbb{T}, \kappa \rangle$, $\mathbb{O}$ or $\langle \mathbb{O}, -Id \rangle$. The labels (g) and (h) on the two subgroups isomorphic to $\mathbb{Z}_2$ correspond to rows in Tables 1 and 2.

4. HOPF BIFURCATION

The first step in studying Γ-equivariant Hopf bifurcation is to obtain the $\mathbb{C}$-axial subgroups of $\Gamma \times \mathbb{S}^1$. The fixed-point subspaces for the isotropy subgroups are listed in Table

Table 1. Solution types, isotropy subgroups and fixed-point subspaces for the action of $\Gamma \times \mathbb{S}^1$ on $\mathbb{C}^3$ where Γ is either $\langle \mathbb{T}, \kappa \rangle$ or $\mathbb{O}$ or $\langle \mathbb{O}, -Id \rangle$. Here ω stands for $e^{2\pi i/3}$.

Index	Name	Isotropy subgroup	Fixed-point subspace	dim
(a)	Origin	$\Gamma \times \mathbb{S}^1$	$\{(0,0,0)\}$	0
(b)	Pure mode	$\mathbb{D}_4$	$\{(z,0,0)\}$	2
(c)	Standing wave	$\mathbb{D}_3$	$\{(z,z,z)\}$	2
(d)	Rotating wave	$\mathbb{Z}_3$	$\{(z,\omega z,\omega^2 z)\}$	2
(e)	Standing wave	$\mathbb{D}_2$	$\{(0,z,z)\}$	2
(f)	Rotating wave	$\mathbb{Z}_4$	$\{(z,iz,0)\}$	2
(g)	2-Sphere solutions	$\widetilde{\mathbb{Z}}_2$	$\{(0,z_1,z_2)\}$	4
(h)	2-Sphere solutions	$\mathbb{Z}_2$	$\{(z_1,z_2,z_2)\}$	4
(i)	General solutions	$\mathbb{1}$	$\{(z_1,z_2,z_3)\}$	6

Table 2. Generators of isotropy subgroups for the action of $\Gamma \times \mathbb{S}^1$ on $\mathbb{C}^3$ for $\Gamma = \langle \mathbb{T}, \kappa \rangle$, $\Gamma = \mathbb{O}$ and $\Gamma = \langle \mathbb{O}, -Id \rangle$, with $\omega = e^{2\pi i/3}$. The index (a) to (i) refers to the rows in Table 1.

Index	Isotropy subgroup	Generators in $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$	Generators in $\mathbb{O} \times \mathbb{S}^1$	Generators in $\langle \mathbb{O}, -Id \rangle \times \mathbb{S}^1$
(a)	$\Gamma \times \mathbb{S}^1$	$\{C,R,\kappa,e^{i\theta}\}$	$\{C,T,e^{i\theta}\}$	$\{C,T,e^{i\theta}\}$
(b)	$\mathbb{D}_4$	$\{e^{\pi i}C^2RC,\kappa\}$	$\{TC^2,e^{\pi i}T^2\}$	$\{TC^2,-T^2\}$
(c)	$\mathbb{D}_3$	$\{C,\kappa\}$	$\{C,e^{\pi i}T^2C^2T\}$	$\{C,-T^2C^2T\}$
(d)	$\mathbb{Z}_3$	$\{\bar{\omega}C\}$	$\{\bar{\omega}C\}$	$\{\bar{\omega}C\}$
(e)	$\mathbb{D}_2$	$\{e^{\pi i}R,\kappa\}$	$\{e^{\pi i}TC^2TC^2,T^3C^2\}$	$\{-TC^2TC^2,T^3C^2\}$
(f)	$\mathbb{Z}_4$	$\{e^{\pi i/2}C^2\mathrm{R}\kappa\}$	$\{e^{-\pi i/2}T\}$	$\{e^{-\pi i/2}T\}$
(g)	$\widetilde{\mathbb{Z}}_2$	$\{e^{\pi i}R\}$	$\{e^{\pi i}TC^2TC^2\}$	$\{-TC^2TC^2\}$
(h)	$\mathbb{Z}_2$	$\{\kappa\}$	$\{e^{\pi i}T^2C^2TC\}$	$\{-T^2C^2TC\}$
(i)	$\mathbb{1}$	$\{Id\}$	$\{Id\}$	$\{Id\}$

1. Generators of the isotropy subgroups are given in Table 2, in terms of the generators of $\langle \mathbb{T}, \kappa \rangle$, of $\mathbb{O}$ and of $\langle \mathbb{O}, -Id \rangle$.

The normal form for a $\Gamma \times \mathbb{S}^1$-equivariant vector field truncated to the cubic order is

$$(1)\qquad \begin{cases} \dot{z}_1 = z_1\left(\lambda + \gamma\left(|z_1|^2+|z_2|^2+|z_3|^2\right) + \alpha\left(|z_2|^2+|z_3|^2\right)\right) + \bar{z}_1\beta\left(z_2^2+z_3^2\right) \\ \dot{z}_2 = z_2\left(\lambda + \gamma\left(|z_1|^2+|z_2|^2+|z_3|^2\right) + \alpha\left(|z_1|^2+|z_3|^2\right)\right) + \bar{z}_2\beta\left(z_1^2+z_3^2\right) \\ \dot{z}_3 = z_3\left(\lambda + \gamma\left(|z_1|^2+|z_2|^2+|z_3|^2\right) + \alpha\left(|z_1|^2+|z_2|^2\right)\right) + \bar{z}_3\beta\left(z_1^2+z_2^2\right) \end{cases}$$

where $\alpha,\beta,\gamma,\lambda$ are all complex coefficients. This normal form is the same used in [12] for the $\mathbb{T} \times \mathbb{S}^1$ action, except that the extra symmetry κ forces some of the coefficients to be equal. The normal form (1) is slightly different, but equivalent, to the one given in [1] for the group $\mathbb{O} \times \mathbb{S}^1$. As in [1, 12], the origin is always an equilibrium of (1) and it undergoes a Hopf bifurcation when λ crosses the imaginary axis. By the Equivariant

Hopf Theorem, this generates several branches of periodic solutions, corresponding to the $\mathbb{C}$-axial subgroups of $\Gamma \times \mathbb{S}^1$.

Under additional conditions on the parameters in (1) there may be other periodic solution branches arising through Hopf bifurcation outside the fixed-point subspaces for $\mathbb{C}$-axial subgroups. These have been analysed in [1], we proceed to describe them briefly, with some additional information from [10].

4.1. **Submaximal branches in $\{(z_1, z_2, 0)\}$.** As a fixed-point subspace for the $\langle\mathbb{T}, \kappa\rangle \times \mathbb{S}^1$ action, this subspace is $\mathrm{Fix}\,(\mathbb{Z}_2\,(e^{\pi i}CRC^2))$ conjugated to $\mathrm{Fix}\,(\mathbb{Z}_2\,(e^{\pi i}R))$, (see Tables 1 and 2). The same subspace is described as $\mathrm{Fix}\,(\mathbb{Z}_2\,(e^{\pi i}T^2))$, under either the $\mathbb{O} \times \mathbb{S}^1$ or the $\langle\mathbb{O}, -Id\rangle \times \mathbb{S}^1$ actions, and is conjugated to the subspaces $\mathrm{Fix}\,(\mathbb{Z}_2\,(e^{\pi i}TC^2TC^2))$, and $\mathrm{Fix}\,(\mathbb{Z}_2\,(-TC^2TC^2))$, that appear in Table 2. It contains the fixed-point subspaces $\{(z, 0, 0)\}$, $\{(z, z, 0)\}$, $\{(iz, z, 0)\}$, corresponding to $\mathbb{C}$-axial subgroups of $\Gamma \times \mathbb{S}^1$, as well as a conjugate copy of each one of them.

Solution branches of (1) in the fixed-point subspace $\{(z_1, z_2, 0)\}$ have been analysed in [10, 1, 11]. The analysis in [11] is quite extensive, and is much more complete than the analysis in either [1] or the present article. We give an outline of their findings here, so results can be used in Section 5.

Restricting the normal form (1) to the subspace $\{(z_1, z_2, 0)\}$ and eliminating solutions that lie in the two-dimensional fixed-point subspaces, one finds that for $\beta \neq 0$ there is a solution branch with no additional symmetries, if and only if both $|\alpha/\beta| > 1$ and $-1 < \mathcal{R}e\,(\alpha/\beta) < 1$ hold. These solutions have the form $\{(\xi z, z, 0)\}$, with $\xi = re^{i\phi}$, where

$$\text{(2)}\quad \cos(2\phi) = -\mathcal{R}e\,(\alpha/\beta) \quad \sin(2\phi) = \pm\sqrt{1 - (\mathcal{R}e\,(\alpha/\beta))^2} \quad r^2 = \frac{\mathcal{I}m\,(\alpha/\beta) + \sin(2\phi)}{\mathcal{I}m\,(\alpha/\beta) - \sin(2\phi)}.$$

The submaximal branch of periodic solutions connects all the maximal branches that lie in the subspace $\{(z_1, z_2, 0)\}$. This can be deduced from the expressions (2), as we proceed to explain, and as illustrated in Figure 4.

When $\mathcal{R}e\,(\alpha/\beta) = +1$ we get $\xi = \pm i$ and the submaximal branches lie in the subspaces $\{(\pm iz, z, 0)\}$. The submaximal branches only exist for $\mathcal{R}e\,(\alpha/\beta) \leq 1$ and, when $\mathcal{R}e\,(\alpha/\beta)$ increases to $+1$, pairs of submaximal branches coalesce into the subspaces $\{(\pm iz, z, 0)\}$, as in Figure 4.

Similarly, for $\mathcal{R}e\,(\alpha/\beta) = -1$ we have $\xi = \pm 1$. As $\mathcal{R}e\,(\alpha/\beta)$ decreases to -1, the submaximal branches coalesce into the subspace$\{(\pm z, z, 0)\}$, at a pitchfork.

To see what happens at $|\alpha/\beta| = 1$, we start with $|\alpha/\beta| > 1$. The submaximal branch exists when $|\mathcal{R}e\,(\alpha/\beta)| < 1$ and hence $|\mathcal{I}m\,(\alpha/\beta)| > 1$. As $|\alpha/\beta|$ decreases, when we reach the value 1 we have $|\alpha/\beta|^2 = (\mathcal{R}e\,(\alpha/\beta))^2 + (\mathcal{I}m\,(\alpha/\beta))^2 = 1$ hence $\sin(2\phi)$ tends to $\pm\mathcal{I}m\,(\alpha/\beta)$. The expression for r^2 in (2) shows that in this case either r tends to 0 or r tends to ∞ and the submaximal branches bifurcate from the conjugate subspaces $\{(0, z, 0)\}$ and $\{(z, 0, 0)\}$.

From the expression for r^2 in (2) it follows that the condition $r = 1$ at the submaximal branch only occurs when $\sin(2\phi) = 0$ and the conditions $r = 0$, $r = \infty$ occur when $|\alpha/\beta| = 1$. These conditions are satisfied at the values of α and β for which the submaximal branch bifurcates from one of the $\mathbb{C}$-axial subspaces, as we have already seen. This implies that the submaximal solution branches have no additional symmetry, because, for any γ in $\Gamma \times \mathbb{S}^1$, the norm of the first coordinate of $\gamma \cdot (\xi z, z, 0)$ is either $|z|$, or $r|z|$, or zero. If r

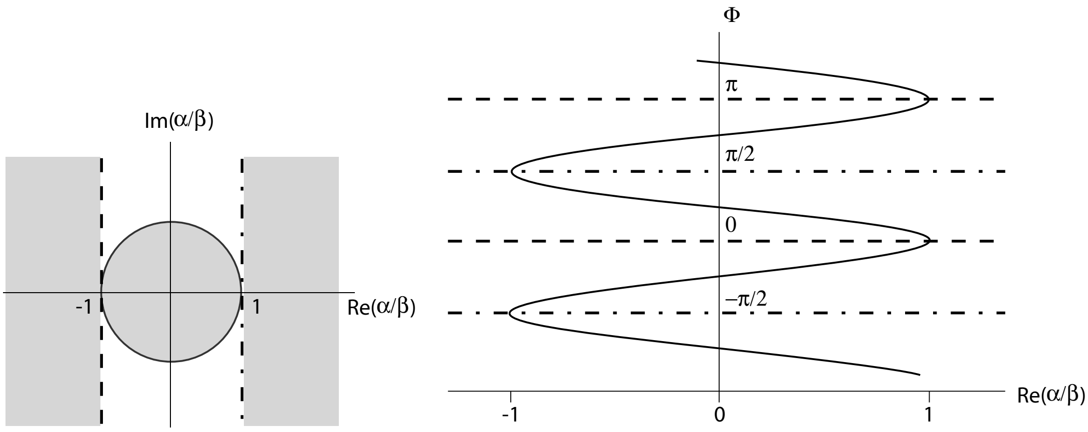


FIGURE 4. Submaximal solutions in the subspace $\{(z_1, z_2, 0)\}$ and their connection to the maximal branches. Left: submaximal branches exist in the white area of the α/β-plane. Bifurcation into the branches that lie in $\{(0, z, 0)\}$ and $\{(z, 0, 0)\}$, occurs at the circle $|\alpha/\beta| = 1$, for any value of the phase-shift ϕ. Right: dashed lines stand for the branches in $\{(\pm z, z, 0)\}$ and dot-dash for the branches in $\{(\pm iz, z, 0)\}$. Pairs of submaximal solutions bifurcate from these branches at pitchforks at specific values of the phase-shift ϕ when $\mathcal{R}e\,(\alpha/\beta) = \pm 1$.

is neither 0, 1 or ∞, then the only possible symmetries are those that fix the subspace $\{(z_1, z_2, 0)\}$.

4.2. **Submaximal branches in** $\{(z_1, z_1, z_2)\}$**.** As a fixed-point subspace for the $\langle \mathbb{T}, \kappa\rangle \times \mathbb{S}^1$ action, this subspace is conjugated to $\mathrm{Fix}\,(\mathbb{Z}_2\,(\kappa))$. Under the $\mathbb{O}\times\mathbb{S}^1$ action, it is conjugated to $\mathrm{Fix}\,(\mathbb{Z}_2\,(e^{\pi i}T^2C^2TC))$ and, under the $\langle \mathbb{O}, -Id\rangle \times \mathbb{S}^1$ action, to $\mathrm{Fix}\,(\mathbb{Z}_2\,(-T^2C^2TC))$ (see Tables 1 and 2), the conjugacy in all cases being realised by C^2. It contains the fixed-point subspaces $\{(z, z, 0)\}$, $\{(0, 0, z)\}$, $\{(z, z, z)\}$, corresponding to $\mathbb{C}$-axial subgroups, as well as a conjugate copy of each one of them. In what follows, we give a geometric construction for finding solution branches of (1) in the fixed-point subspace $\{(z_1, z_1, z_2)\}$, completing the description given in [1]. In particular, we show that the existence of these branches only depends on the value of α/β in the normal form, and that there are parameter values for which three submaximal solution branches coexist.

Restricting the normal form (1) to this subspace and eliminating some solutions that lie in the two-dimensional fixed-point subspaces, one finds that for $\beta \neq 0$ there is a solution branch through the point $(z, z, \xi z)$ with $\xi = re^{-i\psi}$, if and only if

$$\beta - \alpha + r^2\alpha + \beta\left(r^2 e^{-2i\psi} - 2e^{2i\psi}\right) = 0 \qquad \text{hence} \qquad r^2 = \frac{\alpha - \beta + 2\beta e^{2i\psi}}{\alpha + \beta e^{-2i\psi}}. \tag{3}$$

The expression for r^2 must be real and positive. For $(x, y) = (\cos 2\psi, \sin 2\psi)$, this is equivalent to the conditions $\mathcal{R}(x, y) > 0$ and $\mathcal{I}(x, y) = 0$ where:

$$\mathcal{R}(x, y) = (x - x_0)^2 - (y - y_0)^2 + K_r > 0 \qquad K_r = \left[(3\,\mathcal{I}m\,(\alpha/\beta))^2 - (\mathcal{R}e\,(\alpha/\beta) + 1)^2\right]/16$$

and

$$\mathcal{I}(x, y) = (x - x_0)(y - y_0) + K_i = 0 \qquad K_i = 3\,\mathcal{I}m\,(\alpha/\beta)\,[\mathcal{R}e\,(\alpha/\beta) + 1]\,/16$$

with

$$x_0 = (1 - 3\mathcal{R}e\,(\alpha/\beta))\,/4 \qquad y_0 = \mathcal{I}m\,(\alpha/\beta)\,/4 \quad \text{and} \quad r^2 = 2\mathcal{R}(x,y)/\left|e^{-2i\psi} + \alpha/\beta\right|^2.$$

Thus, solution branches will correspond to points (x, y) where the (possibly degenerate) hyperbola $\mathcal{I}(x, y) = 0$ intersects the unit circle, subject to the condition $\mathcal{R}(x, y) > 0$. There may be up to four intersection points. By inspection we find that $(x, y) = (1, 0)$ is always an intersection point, where either $\psi = 0$ or $\psi = \pi$. Substituting into (3), it follows that $r = 1$. Moreover, $\mathcal{R}(1, 0) = \left[(\mathcal{R}e(\alpha/\beta) + 1)^2 + (\mathcal{I}m(\alpha/\beta))^2\right]/2 \geq 0$. These solutions correspond to the two-dimensional fixed point subspace $\{(z, z, z)\}$ and its conjugate $\{(z, z, -z)\}$.

Solutions that lie in the fixed-point subspace $\{(z, z, 0)\}$ correspond to $r = 0$ and hence to $\mathcal{R}(x, y) = 0$. The subspace $\{(0, 0, z)\}$ corresponds to $r \to \infty$ in (3), i.e. to $e^{-2i\psi} = -\alpha/\beta$. At these points both $\mathcal{R}(x, y)$ and $\mathcal{I}(x, y)$ are zero.

Generically, the fact that $(1, 0)$ is an intersection point implies that $\mathcal{I}(x, y) = 0$ meets the circle on at least one more point. The discussion above shows that all the other intersections that satisfy $\mathcal{R}(x, y) > 0$ correspond to submaximal branches. We proceed to describe some of the situations that may arise.

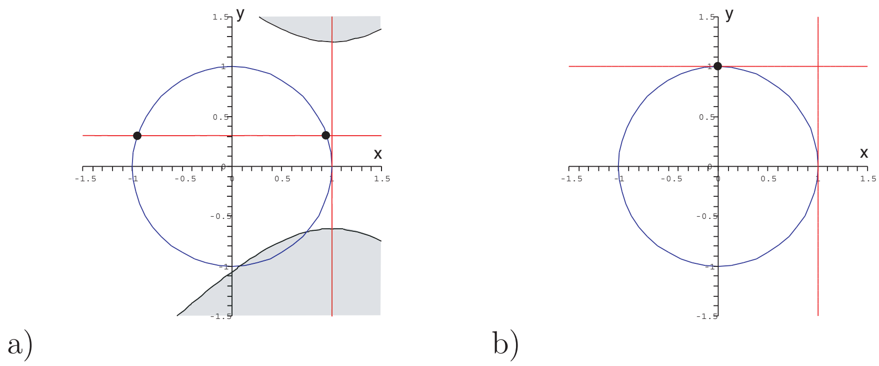


Figure 5. Graphs of $\mathcal{I}(x, y) = 0$ (degenerate hyperbola, two red lines, $x = 1$ and $y = y_0$) in the (x, y)-plane, for location of submaximal solutions (black dots), where the degenerate hyperbola intersects the unit circle (blue) with $\mathcal{R}(x, y) > 0$ (outside grey area). The intersection at $(x, y) = (1, 0)$ corresponds to a solution with maximal isotropy subgroup, the other intersections represent submaximal branches. For $\alpha/\beta = -1 + 5i/4$ there are two submaximal solutions, shown in a). The solutions come together when $\alpha/\beta = -1 + 4i$, shown in b).

When $\mathcal{R}e\,(\alpha/\beta) = -1$, we have $K_i = 0$ and hence $\mathcal{I}(x, y) = 0$ consists of the two lines $x = 1$ and $y = y_0$, shown in Figure 5. The line $x = 1$ is tangent to the unit circle, hence, when $0 < |\mathcal{I}m\,(\alpha/\beta)| \leq 4$, the lines $\mathcal{I}(x, y) = 0$ intersect the circle at three points, except in the limit cases $\mathcal{I}m\,(\alpha/\beta) = \pm 4$, when two solutions come together at a saddle-node. The other case when $\mathcal{I}(x, y) = 0$ consists of two lines occurs when $\mathcal{I}m\,(\alpha/\beta) = 0$, where $\mathcal{I}(x, y) = 0$ at $x = x_0$ or $y = y_0 = 0$, shown in Figure 6. Then $\mathcal{R}(x_0, y) < 0$ and hence the only possible submaximal branch corresponds to $(x, y) = (-1, 0)$. Therefore, when the hyperbola $\mathcal{I}(x, y) = 0$ degenerates to two lines, there may be 0, 1 or 2 submaximal solution branches.

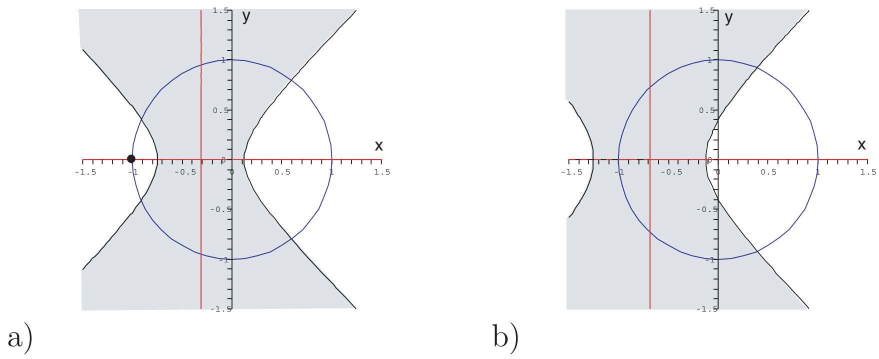


FIGURE 6. Graphs of $\mathcal{I}(x,y)=0$ (degenerate hyperbola, the $y=0$ axis and the red line $x=x_0$) in the (x,y)-plane, for location of submaximal solutions (black dots), where the degenerate hyperbola intersects the unit circle (blue) with $\mathcal{R}(x,y)>0$ (outside grey area). The intersection at $(x,y)=(1,0)$ has maximal isotropy. When $\mathcal{I}m\,(\alpha/\beta)=0$ there is at most one intersection with $\mathcal{R}(x,y)>0$ corresponding to a submaximal solution. Parameter values: a) $\alpha/\beta=3/4$ (one submaximal solution $(x,y)=(-1,0)$), b) $\alpha/\beta=5/4$ (no submaximal solution).

In the general case, when $\mathcal{R}e\,(\alpha/\beta)\neq -1$ and $\mathcal{I}m\,(\alpha/\beta)\neq 0$ (Figures 7 and 8), the curve $\mathcal{I}(x,y)=0$ is a hyperbola intersecting the unit circle at $(x,y)=(1,0)$ and at up to three other points. Again, these other intersections may correspond to submaximal branches or not, depending on the sign of $\mathcal{R}(x,y)$. Figure 7 shows examples with one branch and with no submaximal branches. Examples with one, two and three submaximal branches are shown in Figure 8. These branches bifurcate from the fixed-point subspace $\{(z,z,0)\}$, when the intersection meets the line $\mathcal{R}(x,y)=0$, or from the subspaces $\{(z,z,\pm z)\}$. A pair of branches may also terminate at a saddle-node bifurcation when the hyperbola is tangent to the unit circle.

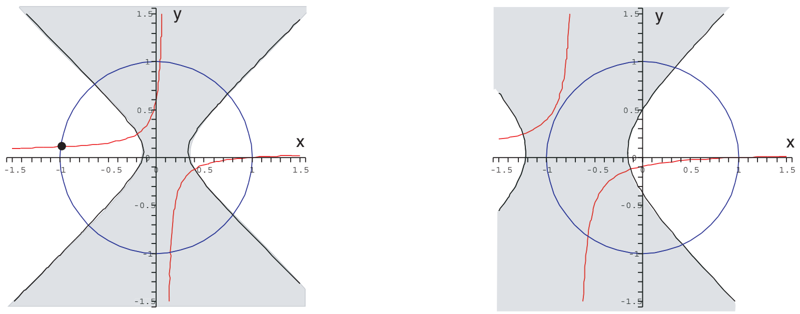
5. SPATIO-TEMPORAL SYMMETRIES

The H mod K theorem [2, 6] states necessary and sufficient conditions for the existence of a Γ-equivariant differential equation having a periodic solution with specified spatial symmetries $K\subset\Gamma$ and spatio-temporal symmetries $H\subset\Gamma$, as explained in Section 2.

For a given Γ-equivariant differential equation, the H mod K Theorem gives necessary conditions on the symmetries of periodic solutions. Not all these solutions arise by a Hopf bifurcation from the trivial equilibrium — we call this a *primary Hopf bifurcation*. In this section we address the question of determining which periodic solution types, whose existence is guaranteed by the H mod K theorem, are obtainable at primary Hopf bifurcations, when the symmetry group is either $\langle\mathbb{T},\kappa\rangle$ or $\mathbb{O}$, or $\langle\mathbb{O},-Id\rangle$.

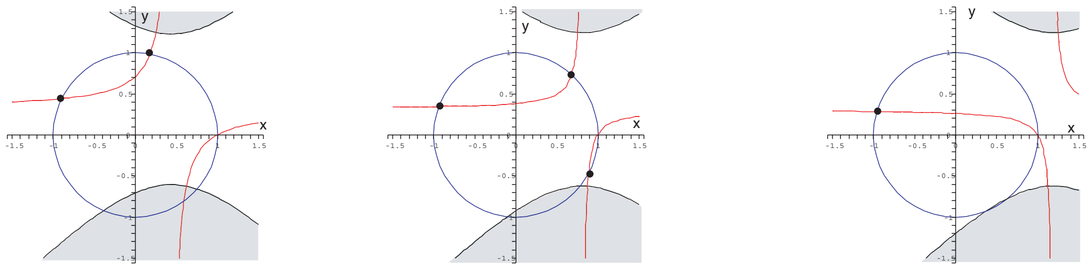
The first step in answering this question is the next lemma:

Lemma 1. *Pairs of subgroups H, K of symmetries of periodic solutions arising through a primary Hopf bifurcation for $\Gamma=\langle\mathbb{T},\kappa\rangle$ are given in Table 3, for $\Gamma=\mathbb{O}$ in Table 4, and for $\Gamma=\langle\mathbb{O},-Id\rangle$ in Table 5.*



a) b)

Figure 7. Graphs of $\mathcal{I}(x,y)=0$ (red hyperbola) in the (x,y)-plane, for location of submaximal solutions (black dots), where the hyperbola intersects the unit circle (blue) with $\mathcal{R}(x,y)>0$ (outside grey area). Parameter values: a) $\alpha/\beta = 1/5+i/4$ (one submaximal solution) and b) $\alpha/\beta = 5/4+i/4$ (no submaximal solutions). The intersection at $(x,y)=(1,0)$ corresponds to a solution with maximal isotropy subgroup.



a) b) c)

Figure 8. Graphs of $\mathcal{I}(x,y)=0$ (red hyperbola) in the (x,y)-plane, for location of submaximal solutions (black dots), where the hyperbola intersects the unit circle (blue) with $\mathcal{R}(x,y)>0$ (outside grey area). Parameter values: a) $\alpha/\beta = -1/4+5i/4$ (two submaximal solutions), b) $\alpha/\beta = -3/4+5i/4$ (three submaximal solutions), and c) $\alpha/\beta = -5/4+5i/4$ (no submaximal solutions). The intersection at $(x,y)=(1,0)$ corresponds to a solution with maximal isotropy subgroup.

Proof. The symmetries corresponding to the $\mathbb{C}$-axial subgroups of $\Gamma\times\mathbb{S}^1$ provide the first five rows of Tables 3, 4 and 5. The last two rows correspond to the submaximal branches found in 4.1 and 4.2 above. □

The next step is to identify the subgroups corresponding to Theorem 2.

Lemma 2. *Pairs of subgroups H, K satisfying conditions $(a)-(d)$ of Theorem 2 are given in Table 6 for $\Gamma=\langle\mathbb{T},\kappa\rangle$, in Table 7 for $\Gamma=\mathbb{O}$ and in Table 8 for $\Gamma=\langle\mathbb{O},-Id\rangle$.*

Proof. Conditions $(a)-(d)$ of Theorem 2 are immediate for the isotropy subgroups with two-dimensional fixed-point subspaces. For $\Gamma=\mathbb{O}$, these are all the non-trivial subgroups.

Table 3. Spatio-temporal symmetries of solutions arising through primary Hopf bifurcation from the trivial equilibrium and number of branches, for the action of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ on $\mathbb{C}^3$. The index refers to Table 1. Subgroups of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ below the dividing line are not $\mathbb{C}$-axial.

index	subgroup of $\Sigma \subset \langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ generators	Spatio-temporal symmetries H generators	Spatial symmetries K generators	number of branches
(b)	$\{e^{\pi i}C^2RC, \kappa\}$	$\{C^2RC, \kappa\}$	$\{R, \kappa\}$	3
(c)	$\{C, \kappa\}$	$\{C, \kappa\}$	$\{C, \kappa\}$	4
(d)	$\{e^{-2\pi i/3}C\}$	$\{C\}$	$\{Id\}$	8
(e)	$\{e^{\pi i}R, \kappa\}$	$\{R, \kappa\}$	$\{\kappa\}$	6
(f)	$\{e^{\pi i/2}C^2R\kappa\}$	$\{C^2R\kappa\}$	$\{Id\}$	6
(g)	$\{e^{\pi i}R\}$	$\{R\}$	$\{Id\}$	12
(h)	$\{\kappa\}$	$\{\kappa\}$	$\{\kappa\}$	12

Table 4. Spatio-temporal symmetries of solutions arising through primary Hopf bifurcation from the trivial equilibrium and number of branches, for the action of $\mathbb{O} \times \mathbb{S}^1$ on $\mathbb{C}^3$. The index refers to Table 1. Subgroups of $\mathbb{O} \times \mathbb{S}^1$ below the dividing line are not $\mathbb{C}$-axial.

index	subgroup of $\Sigma \subset \mathbb{O} \times \mathbb{S}^1$ generators	Spatio-temporal symmetries H generators	Spatial symmetries K generators	number of branches
(b)	$\{TC^2, e^{\pi i}T^2\}$	$\{TC^2, T^2\}$	$\{TC^2\}$	3
(c)	$\{C, e^{\pi i}T^2C^2T\}$	$\{C, T^2C^2T\}$	$\{C\}$	4
(d)	$\{e^{-2\pi i/3}C\}$	$\{C\}$	$\{Id\}$	8
(e)	$\{e^{\pi i}TC^2TC^2, T^3C^2\}$	$\{TC^2TC^2, T^3C^2\}$	$\{T^3C^2\}$	6
(f)	$\{e^{-\pi i/2}T\}$	$\{T\}$	$\{Id\}$	6
(g)	$\{e^{\pi i}TC^2TC^2\}$	$\{TC^2TC^2\}$	$\{Id\}$	12
(h)	$\{e^{\pi i}T^2C^2TC\}$	$\{T^2C^2TC\}$	$\{Id\}$	12

Condition (d) has to be verified for the isotropy subgroups K of $\langle \mathbb{T}, \kappa \rangle$ and $\langle \mathbb{O}, -Id \rangle$ that have a four-dimensional fixed-point subspace, as well as for $K = \mathbb{1}$ for the three groups Γ. Since, for each $\gamma \in \Gamma$, we have that $\dim \mathrm{Fix}(\gamma)$ is even, then L_K is always the union of a finite number of subspaces with even dimension. Therefore, L_K has even codimension in $\mathrm{Fix}(K)$ and hence $\mathrm{Fix}(K) \backslash L_K$ is connected and condition (d) follows in all cases.

For $\Gamma = \langle \mathbb{O}, -Id \rangle$, condition (a) has to be verified for the two subgroups with four-dimensional fixed-point subspaces. For $K = \mathbb{Z}_2 = \langle -T^2C^2TC \rangle$ we have $N(K) = \langle T^2C^2TC, T^3C^2, -I \rangle$ a subgroup of order 8. All elements of $N(K)$, except for Id, have order 2, hence $N(K)/K$ is not cyclic. This gives rise to 3 non-conjugate possibilities for H, all isomorphic to $\mathbb{D}_2$.

For $K = \widehat{\mathbb{Z}}_2 = \langle -TC^2TC^2 \rangle$ we have $N(\widehat{\mathbb{Z}}_2) = \langle TC^2, T^2, -I \rangle$, a subgroup of order 16, where TC^2 has order 4, T^2 and $-Id$ have order 2, hence $N(\widehat{\mathbb{Z}}_2)/\widehat{\mathbb{Z}}_2$ is not cyclic. This gives

TABLE 5. Spatio-temporal symmetries of solutions arising through primary Hopf bifurcation from the trivial equilibrium and number of branches, for the action of $\langle \mathbb{O}, -Id\rangle \times \mathbb{S}^1$ on $\mathbb{C}^3$. The index refers to Table 1. Subgroups of $\langle \mathbb{O}, -Id\rangle \times \mathbb{S}^1$ below the dividing line are not $\mathbb{C}$-axial.

index	subgroup of $\Sigma \subset \langle \mathbb{O}, -Id\rangle \times \mathbb{S}^1$ generators	Spatio-temporal symmetries H generators	Spatial symmetries K generators	number of branches
(b)	$\{TC^2, -T^2\}$	$\{TC^2, -T^2\}$	$\{TC^2, -T^2\}$	3
(c)	$\{C, -T^2C^2T\}$	$\{C, -T^2C^2T\}$	$\{C, -T^2C^2T\}$	4
(d)	$\{e^{-2\pi i/3}C\}$	$\{C\}$	$\{Id\}$	8
(e)	$\{-TC^2TC^2, T^3C^2\}$	$\{-TC^2TC^2, T^3C^2\}$	$\{-TC^2TC^2, T^3C^2\}$	6
(f)	$\{e^{-\pi i/2}T\}$	$\{T\}$	$\{Id\}$	6
(g)	$\{-TC^2TC^2\}$	$\{-TC^2TC^2\}$	$\{-TC^2TC^2\}$	24
(h)	$\{-T^2C^2TC\}$	$\{-T^2C^2TC\}$	$\{-T^2C^2TC\}$	24

TABLE 6. Possible pairs H, K for Theorem 2 in the action of $\langle \mathbb{T}, \kappa\rangle$ on $\mathbb{C}^3$.

K	Generators of K	H	Generators of H	Fix(K)	dim
$\mathbb{D}_2$	$\{R, \kappa\}$	K	$\{R, \kappa\}$	$\{(z, 0, 0)\}$	2
		$N(K) = \mathbb{D}_4$	$\{C^2RC, \kappa\}$		
$\mathbb{D}_3$	$\{C, \kappa\}$	$N(K) = K$	$\{C, \kappa\}$	$\{(z, z, z)\}$	2
$\mathbb{Z}_2$	$\{\kappa\}$	K	$\{\kappa\}$	$\{(z_1, z_2, z_2)\}$	4
		$N(K) = \mathbb{D}_2$	$\{R, \kappa\}$		
$\mathbb{1}$	$\{Id\}$	$\mathbb{Z}_4$	$\{C^2R\kappa\}$	$\mathbb{C}^3$	6
		$\mathbb{Z}_3$	$\{C\}$		
		$\mathbb{Z}_2$	$\{R\}$		
		$\mathbb{Z}_2$	$\{\kappa\}$		
		K	$\{Id\}$		

rise to 3 non-conjugate possibilities for H, one isomorphic to $\mathbb{D}_4$, the others isomorphic to $\mathbb{D}_2$.

Condition (a) also has to be used for the subgroup $K = \mathbb{1}$ in all cases, where the possibilities for H are then all the cyclic subgroups of Γ. We omit this information in Table 8, since there are too many subgroups in this case. □

Of all the subgroups of $\langle \mathbb{T}, \kappa\rangle$ of order two that appear as H in a pair $H \sim \mathbb{Z}_n, K = \mathbb{1}$ in Table 6, only $H \sim \mathbb{Z}_3(C)$ is an isotropy subgroup, as can be seen in Figure 1. The two subgroups of $\langle \mathbb{T}, \kappa\rangle$ of order two, $H = \mathbb{Z}_2(\kappa)$ and $H = \mathbb{Z}_2(R)$ are not conjugated, since κ fixes a four-dimensional subspace, whereas R fixes a subspace of dimension two. In contrast, all but one of the cyclic subgroups of $\mathbb{O}$ are isotropy subgroups, as can be seen comparing Table 7 to Figure 1, and noting that $\mathbb{Z}_4(T)$ is conjugated to $\mathbb{Z}_4(TC^2)$ and that $\mathbb{Z}_2(T^2C^2TC)$ is conjugated to $\mathbb{Z}_2(T^3C^2)$. This will have a marked effect on the primary Hopf bifurcations.

TABLE 7. Possible pairs H, K for Theorem 2 in the action of $\mathbb{O}$ on $\mathbb{C}^3$.

K	Generators of K	H	Generators of H	Fix(K)	dim
$\mathbb{Z}_4$	$\{TC^2\}$	K $N(K)=\mathbb{D}_4$	$\{TC^2\}$ $\{T^2, TC^2\}$	$\{(z,0,0)\}$	2
$\mathbb{Z}_3$	$\{C\}$	K $N(K)=\mathbb{D}_3$	$\{C\}$ $\{C, T^2C^2T\}$	$\{(z,z,z)\}$	2
$\mathbb{Z}_2$	$\{T^3C^2\}$	K $N(K)=\mathbb{D}_2$	$\{T^3C^2\}$ $\{TC^2TC^2, T^3C^2\}$	$\{(0,z,z)\}$	2
$\mathbb{1}$	$\{Id\}$	$\mathbb{Z}_4$ $\mathbb{Z}_3$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ K	$\{T\}$ $\{C\}$ $\{TC^2TC^2\}$ $\{T^2C^2TC\}$ $\{Id\}$	$\mathbb{C}^3$	6

TABLE 8. Possible pairs H, K, with $K \neq \mathbb{1}$ for Theorem 2 in the action of $\langle \mathbb{O}, -Id\rangle$ on $\mathbb{C}^3$.

K	Generators of K	H	Generators of H	Fix(K)	dim
$\mathbb{D}_4$	$\{TC^2, -T^2\}$	K $N(K)=\langle K, -Id\rangle$	$\{TC^2, -T^2\}$ $\{TC^2, T^2, -Id\}$	$\{(z,0,0)\}$	2
$\mathbb{D}_3$	$\{C, -T^2C^2T\}$	K $N(K)=\langle K, -Id\rangle$	$\{C, -T^2C^2T\}$ $\{C, T^2C^2T, -Id\}$	$\{(z,z,z)\}$	2
$\mathbb{D}_2$	$\{T^3C^2, -T^2C^2TC^2\}$	K $N(K)=\langle K, -Id\rangle$	$\{T^3C^2, -T^2C^2TC^2\}$ $\{T^3C^2, T^2C^2TC^2, -Id\}$	$\{(0,z,z)\}$	2
$\mathbb{Z}_2$	$\{-T^2C^2TC\}$	K $\mathbb{Z}_2 \oplus \mathbb{Z}_2(-Id)$ $\mathbb{Z}_2 \oplus \widehat{\mathbb{Z}}_2$	$\{-T^2C^2TC\}$ $\{T^2C^2TC, -Id\}$ $\{-T^2C^2TC, TC^2TC^2\}$	$\{(z_1,z_2,z_2)\}$	4
$\widehat{\mathbb{Z}}_2$	$\{-TC^2TC^2\}$	K $\mathbb{Z}_4(TC^2) \oplus \mathbb{Z}_2(-Id)$ $\widehat{\mathbb{Z}}_2 \oplus \mathbb{Z}_2(-Id)$ $\widehat{\mathbb{Z}}_2 \oplus \mathbb{Z}_2(T^2)$ $\widehat{\mathbb{Z}}_2 \oplus \mathbb{Z}_2(-T^2)$ $\widehat{\mathbb{Z}}_2 \oplus \mathbb{Z}_2(T^2C^2TC)$ $\mathbb{Z}_2 \oplus \widehat{\mathbb{Z}}_2$	$\{-TC^2TC^2\}$ $\{-Id, TC^2\}$ $\{-Id, TC^2TC^2\}$ $\{-TC^2TC^2, T^2\}$ $\{-TC^2TC^2, -T^2\}$ $\{-TC^2TC^2, T^2C^2TC\}$ $\{-TC^2TC^2, -T^2C^2TC\}$	$\{(0,z_1,z_2)\}$	4

Proposition 1. *For the representation of the group $\langle \mathbb{T}, \kappa\rangle$ on $\mathbb{R}^6 \sim \mathbb{C}^3$, all pairs of subgroups H, K satisfying the conditions of the H mod K Theorem, with $H \neq \mathbb{1}$, occur as spatio-temporal symmetries of periodic solutions arising through a primary Hopf bifurcation, except for:*

(1) *the pair $H = K = \mathbb{D}_2$, generated by $\{R, \kappa\}$;*
(2) *the pair $H = \mathbb{Z}_2(\kappa)$, $K = \mathbb{1}$.*

Proof. The result follows by inspection of Tables 3 and 6, we discuss here why these pairs do not arise in a primary Hopf bifurcation. Case (1) refers to a non-trivial isotropy subgroup $K \subset \langle \mathbb{T}, \kappa\rangle$ for which $N(K) \neq K$. For the group $\langle \mathbb{T}, \kappa\rangle$, there are two non-trivial isotropy subgroups in this situation, as can be seen in Table 6. The first one, $K = \mathbb{D}_2$, is not an isotropy subgroup of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$, so the pair $H = K = \mathbb{D}_2$ does not occur in Table 3 as a Hopf bifurcation from the trivial solution in a normal form with symmetry $\langle \mathbb{T}, \kappa \rangle$. The second subgroup, $K = \mathbb{Z}_2(\kappa)$ occurs as an isotropy subgroup of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ with four-dimensional fixed-point subspace. As we have seen in 4.2, in this subspace there are periodic solutions with $K = H = \mathbb{Z}_2(\kappa)$ arising through a Hopf bifurcation with submaximal symmetry. On the other hand, the normaliser of $\mathbb{Z}_2(\kappa)$ corresponds to a $\mathbb{C}$-axial subgroup of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$, so there is a Hopf bifurcation from the trivial solution with $H = N(K)$.

Case (2) concerns the situation when $K = \mathbb{1}$. All the cyclic subgroups $H \subset \langle \mathbb{T}, \kappa \rangle$, with the exception of $\mathbb{Z}_2(\kappa)$, are the projection into $\langle \mathbb{T}, \kappa \rangle$ of cyclic isotropy subgroups of $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$, so they correspond to primary Hopf bifurcations. □

Another reason why the pair $H = K = \mathbb{D}_2$ does not occur at a primary Hopf bifurcation is the following: a non-trivial periodic solution in $\text{Fix}(\mathbb{D}_2)$ has the form $X(t) = (z(t), 0, 0)$. Then $Y(t) = C^2RCX(t) = -X(t)$ is also a solution contained in the same plane. If the origin is inside $X(t)$ then the curves $Y(t)$ and $X(t)$ must intersect, so they coincide as curves, and this means that C^2RC is a spatio-temporal symmetry of $X(t)$. Hence, if $H = K = \mathbb{D}_2$, then $X(t)$ cannot encircle the origin, and hence it cannot arise from a Hopf bifurcation from the trivial equilibrium. Of course it may originate at a Hopf bifurcation from another equilibrium. The argument does not apply to the other subgroup K with $N(K) \neq K$, because in this case $\dim \text{Fix}(K) = 4$ and indeed, for some values of the parameters in the normal form, there are primary Hopf bifurcations into solutions whith $H = K = \mathbb{Z}_2(\kappa)$.

The second case in Proposition 1 is more interesting: if $X(t) = (z_1(t), z_2(t), z_3(t))$ is a ρ-periodic solution of a $\langle \mathbb{T}, \kappa \rangle$-invariant differential equation, with $H = \mathbb{Z}_2(\kappa)$, $K = \mathbb{1}$, then for some $\theta \neq 0 \pmod{\rho}$ and for all t we have $z_1(t+\theta) = z_1(t)$ and $z_3(t+\theta) = z_2(t) = z_2(t+2\theta)$.

If $z_2(t)$ and $z_3(t)$ are identically zero, then $\kappa \in K$, contradicting our assumption. If $z_1(t) \equiv 0$ and $z_2(t)$ and $z_3(t)$ are both non-zero, then $X(t)$ cannot be obtained from the $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ action, because in this case it would be $X(t) \in \text{Fix}(\mathbb{Z}_2(e^{\pi i}R))$, hence $R \in H$. The only other possibility is that to have all coordinates of $X(t)$ not zero, $X(t) = (z_1(t), z_2(t), z_2(t+\theta))$ with $2\theta = \rho$, with a θ-periodic z_1 and $z_2(t)$ and $z_3(t)$ having twice the period of $z_1(t)$. In this case, if $X(t)$ bifurcates from an equilibrium, at the bifurcation point the eigenvalues will be $\pm 2\pi i/\theta$ and $\pm \pi i/\theta$, a 2-1 resonance. This last situation does not occur for the group $\mathbb{O}$, as can be seen in the next result.

Proposition 2. *For the representation of the group $\mathbb{O}$ on $\mathbb{R}^6 \sim \mathbb{C}^3$, all pairs of subgroups H, K satisfying the conditions of the H mod K Theorem, with $H \neq \mathbb{1}$, occur as spatio-temporal symmetries of periodic solutions arising through a primary Hopf bifurcation, except for the following pairs:*

(1) *$H = K = \mathbb{Z}_4$, generated by $\{TC^2\}$;*
(2) *$H = K = \mathbb{Z}_3$ generated by $\{C\}$;*
(3) *$H = K = \mathbb{Z}_2$ generated by $\{T^3C^2\}$.*

Proof. As in Proposition 1, the result follows comparing Tables 4 and 7.

All the cyclic subgroups $H \subset \mathbb{O}$ are the projection into $\mathbb{O}$ of cyclic isotropy subgroups Σ of $\mathbb{O} \times \mathbb{S}^1$, so the pairs $H, K = \mathbb{Z}_n, \mathbb{1}$ correspond to primary Hopf bifurcations: for $\mathbb{Z}_4$

and $\mathbb{Z}_3$ the subgroup $\Sigma \subset \mathbb{O} \times \mathbb{S}^1$ is $\mathbb{C}$-axial, whereas for the subgroups of $\mathbb{O}$ of order two, the subspace $\mathrm{Fix}(\Sigma)$ is four-dimensional and has been treated in 4.1 and 4.2 above.

All the non-trivial isotropy subgroups K of $\mathbb{O}$ satisfy $N(K) \neq K$, so they are candidates for cases where $H = K$ does not occur. This is indeed the case, since they are not isotropy subgroups of $\mathbb{O} \times \mathbb{S}^1$. □

The pairs H, K in Proposition 2, that are not symmetries of solutions arising through primary Hopf bifurcation, are of the form $H = K$, where there would be only spatial symmetries. Here, $\dim \mathrm{Fix}(K) = 2$ for all cases. As remarked after the proof of Proposition 1, the origin cannot lie inside a closed trajectory with these symmetries, hence they cannot arise at a primary Hopf bifurcation — a bifurcation from the trivial equilibrium in the normal form (1). In a $\mathbb{O}$-equivariant differential equation they may only bifurcate from a non-trivial equilibrium.

For the larger group $\langle \mathbb{O}, -Id \rangle$ there are even more possibilities for the H mod K Theorem, and hence, even more of them will not occur as primary Hopf bifurcations:

Proposition 3. *For the representation of the group $\langle \mathbb{O}, -Id \rangle$ on $\mathbb{R}^6 \sim \mathbb{C}^3$, there are 11 pairs H, K with $K \neq \mathbb{1}$ satisfying the conditions of the H mod K Theorem, that do not occur as spatio-temporal symmetries of periodic solutions arising through a primary Hopf bifurcation.*

Proof. As in Proposition 1, the result follows comparing Tables 5 and 8. Since $-Id$ is in the normaliser of every subgroup of $\langle \mathbb{O}, -Id \rangle$ and does not belong to any isotropy subgroup, this increases dramatically the number of possible pairs H, K. Here, all the pairs of subgroups H, K that occur as spatio-temporal symmetries of periodic solutions arising through a primary Hopf bifurcation, also satisfy $H = K$, except for the pairs $H = \langle C \rangle$, $K = \mathbb{1}$ and $H = \langle T \rangle$, $K = \mathbb{1}$. □

When a subgroup of Γ occurs as H paired with $K \neq \mathbb{1}$ for the H mod K Theorem, then if H is cyclic it may also occur paired with the trivial subgroup $\mathbb{1}$. This is the case of $H = \langle \kappa \rangle$, for $\Gamma = \langle \mathbb{T}, \kappa \rangle$; of $H = \langle TC^2 \rangle$ (conjugated to $\langle T \rangle$), $H = \langle C \rangle$, and $H = \langle T^3C^2 \rangle$ (conjugated to $\langle TC^2TC^2 \rangle$), for $\Gamma = \mathbb{O}$; and of $H = \mathbb{Z}_2(-T^2C^2TC)$, and $H = \widehat{\mathbb{Z}}_2$ for $\Gamma = \langle \mathbb{O}, -Id \rangle$. In all these cases, there may be periodic solutions with the same spatio-temporal symmetry and less spatial symmetry. In all these cases, only one of the two pairs occurs at a primary Hopf bifurcation.

For the group $\langle \mathbb{O}, -Id \rangle$ there is also a non-trivial case where the same subgroup H is paired with different subgroups H, as can be seen in Table 8: the subgroup $H = \mathbb{Z}_2(-T^2C^2TC^2) \oplus \widehat{\mathbb{Z}}_2$ occurs both with $K = \mathbb{Z}_2(-T^2C^2TC^2)$ and with $K = \widehat{\mathbb{Z}}_2$. In this case, none of the possibilities occurs at a primary Hopf bifurcation.

Acknowledgements. The research of both authors at Centro de Matemática da Universidade do Porto (CMUP) had financial support from the European Regional Development Fund through the programme COMPETE and from the Portuguese Government through the Fundação para a Ciência e a Tecnologia (FCT) under the project PEst-C/MAT/UI0144/2011. A.C. Murza was also supported by the grant SFRH/ BD/ 64374/ 2009 of FCT.

References

[1] P. Ashwin, O. Podvigina, *Hopf bifurcation with cubic symmetry and instability of ABC flow*, Proc. Royal Soc. A **459**, (1991), 1801–1827

[2] P.L. Buono, M. Golubitsky, *Models of central pattern generators for quadruped locomotion I. Primary gaits*, J. Math. Biol. **42**, (2001), 291–326.

[3] N. Filipski, M. Golubitsky, *The Abelian Hopf H* mod *K Theorem*, SIAM J. Appl. Dynam. Sys. **9**, (2010), 283–291.

[4] M. Golubitsky, D.G. Schaeffer, *Singularities and groups in bifurcation theory I*, Applied mathematical sciences **51**, Springer-Verlag, (1985).

[5] M. Golubitsky, I.N. Stewart *Hopf bifurcation with dihedral group symmetry: coupled nonlinear oscillators*, In: Multiparameter Bifurcation Theory, Contemporary Mathematics **56**, (1986), 131–173.

[6] M. Golubitsky, I. Stewart, *The symmetry perspective: from equilibrium to chaos in phase space and physical space*, Birkhäuser, (2003).

[7] M. Golubitsky, I. Stewart, D.G. Schaeffer, *Singularities and groups in bifurcation theory II*, Applied mathematical sciences **69**, Springer-Verlag, (1988).

[8] J. Guckenheimer, P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcation of vector fields*, J. Marsden, L. Sirovich, F. John, eds., Applied mathematical sciences **42**, Springer-Verlag, (1983), 152–156.

[9] I. Melbourne, *A singularity theory analysis of bifurcation problems with octahedral symmetry*, Dyn. Syst. **1**, (1986), 293–321.

[10] A.C. Murza, *Bifurcation of periodic solutions of differential equations with finite symmetry groups*, PhD thesis, Univeristy of Porto, Portugal, 2014

[11] J. W. Swift, *Hopf bifurcation with the symmetry of the square*, Nonlin. **23**, (1988), 333–377.

[12] J.W. Swift, E. Barany, *Chaos in the Hopf bifurcation with tetrahedral symmetry: Convection in a rotating fluid with low Prandtl number*, Eur. J. Mech. B/Fluids **10**, (1991), 99–104.

I.S. Labouriau—Centro de Matemática da Universidade do Porto., Rua do Campo Alegre 687, 4169-007 Porto, Portugal, A.C. Murza — Departamento de Matemática, Faculdade de Ciências, Universidade do Porto., Rua do Campo Alegre 687, 4169-007 Porto, Portugal