CENTRAL FACTORIALS UNDER THE KONTOROVICH-LEBEDEV TRANSFORM OF POLYNOMIALS

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ABSTRACT. We show that slight modifications of the Kontorovich-Lebedev transform lead to an automorphism of the vector space of polynomials. This circumstance along with the Mellin transformation property of the modified Bessel functions perform the passage of monomials to central factorial polynomials. A special attention is driven to the polynomial sequences whose KL-transform is the canonical sequence, which will be fully characterized. Finally, new identities between the central factorials and the Euler polynomials are found.

1. INTRODUCTION AND PRELIMINARY RESULTS

Throughout the text, $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, whereas $\mathbb{R}$ and $\mathbb{C}$ the field of the real and complex numbers, respectively. The notation $\mathbb{R}_+$ corresponds to the set of all positive real numbers. The present investigation is primarily targeted at analysis of sequences of polynomials whose degrees equal its order, which will be shortly called as PS. Whenever the leading coefficient of each of its polynomials equals 1, the PS is said to be a MPS (*monic polynomial sequence*). A PS or a MPS forms a basis of the vector space of polynomials with coefficients in $\mathbb{C}$, here denoted as $\mathscr{P}$. The convention $\prod_{\sigma=0}^{-1} := 1$ is assumed. Further notations are introduced as needed.

We will show that, upon slight modifications on the *Kontorovich-Lebedev transform* (hereafter, we will shortly call *KL-transform*), introduced in [19], permit to transform the canonical polynomial sequence $\{x^n\}_{n\geqslant 0}$ into the so called central factorials of even or odd order [33]:

$$\left(x - \frac{n}{2} + \frac{1}{2}\right)_n = \begin{cases} (-1)^k (1-x)_k (1+x)_k & \text{if} \quad n = 2k \\ (-1)^k \left(\frac{1}{2} - x\right)_k \left(\frac{1}{2} + x\right)_k & \text{if} \quad n = 2k+1 \end{cases}$$

where the $(x)_n$ represents the *Pochammer symbol*: $(x)_n := \prod_{\sigma=0}^{n-1} (x+\sigma)$ when $n \geqslant 1$ and $(x)_0 = 1$. Indeed the set $\{\left(x - \frac{n}{2} + \frac{1}{2}\right)_n\}_{n\geqslant 0}$ is an Appell sequence with respect to the central difference operator δ, defined by $(\delta f)(x) = f(x+\frac{1}{2}) - f(x-\frac{1}{2})$, for any $f \in \mathscr{P}$ [6, 33], since $\delta \left(x - \frac{n}{2} - \frac{1}{2}\right)_{n+1} = (n+1)\left(x - \frac{n}{2} + \frac{1}{2}\right)_n$.

Precisely, we define the two following modifications of the *Kontorovich-Lebedev transform*, which figure out to be our main tools [20, 36, 43, 45]

$$KL_s[f](\tau) = \frac{2\sinh(\pi\sqrt{\tau})}{\pi\sqrt{\tau}} \int_0^\infty K_{2i\sqrt{\tau}}(2\sqrt{x}) f(x)dx \, , \tag{1.1}$$

and

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