# POLYNOMIAL SEQUENCES ASSOCIATED WITH CLASSICAL LINEAR FUNCTIONALS 

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#### Abstract

This work in mainly devoted to the study of polynomial sequences, not necessarily orthogonal, defined by integral powers of certain first order differential operators in deep connection to the classical polynomials of Hermite, Laguerre, Bessel and Jacobi. This connection is streamed from the canonical element of their dual sequences. Meanwhile new Rodrigues-type formulas for the Hermite and Bessel polynomials are achieved.


## 1. Introduction and preliminary results

Throughout the text, $\mathbb{N}$ will denote the set of all positive integers, $\mathbb{N}_{0}=\mathbb{N} \cup\{0\}$, whereas $\mathbb{R}$ and $\mathbb{C}$ the field of the real and complex numbers, respectively. The notation $\mathbb{R}_{+}$corresponds to the set of all positive real numbers. The present investigation is primarily targeted at analysis of sequences of polynomials whose degrees equal its order, which will be shortly called as PS. Whenever the leading coefficient of each of its polynomials equals 1 , the PS is said to be a MPS (monic polynomial sequence). A PS or a MPS forms a basis of the vector space of polynomials with coefficients in $\mathbb{C}$, here denoted as $\mathscr{P}$. Further notations are introduced as needed.

The dual sequence $\left\{u_{n}\right\}_{n \geqslant 0}$ of a given MPS $\left\{P_{n}(x)\right\}_{n \geqslant 0}$, whose elements are called forms (or linear functionals) belong to the dual space $\mathscr{P}^{\prime}$ of $\mathscr{P}$ and are defined according to

$$
\left\langle u_{n}, P_{k}\right\rangle:=\delta_{n, k}, n, k \geqslant 0,
$$

where $\delta_{n, k}$ represents the Kronecker delta function. Its first element, $u_{0}$, earns the special name of canonical form of the MPS. Here, by $\langle u, f\rangle$ we mean the action of $u \in \mathscr{P}^{\prime}$ over $f \in \mathscr{P}$, but a special notation is given to the action over the elements of the canonical sequence $\left\{x^{n}\right\}_{n \geqslant 0}$ - the moments of $u \in \mathscr{P}^{\prime}:(u)_{n}:=$ $\left\langle u, x^{n}\right\rangle, n \in \mathbb{N}_{0}$. Any element $u$ of $\mathscr{P}^{\prime}$ can be written in a series of any dual sequence $\left\{\mathbf{v}_{n}\right\}_{n \geqslant 0}$ of a MPS $\left\{P_{n}\right\}_{n \geqslant 0}[10]:$

$$
\begin{equation*}
u=\sum_{n \geqslant 0}\left\langle u, P_{n}\right\rangle u_{n} . \tag{1.1}
\end{equation*}
$$

Concerning the recursive relation of any MPS, we have [10]:

$$
\begin{equation*}
P_{n+2}(x)=\left(x-\beta_{n+1}\right) P_{n+1}(x)-\sum_{v=0}^{n} \chi_{n, v} P_{v}(x), n \in \mathbb{N}_{0} \tag{1.2}
\end{equation*}
$$

where

$$
\begin{align*}
& \beta_{n}=\left\langle u_{n}, x P_{n}\right\rangle, n \in \mathbb{N}_{0},  \tag{1.3}\\
& \chi_{n, v}=\left\langle u_{v}, x P_{n+1}\right\rangle, 0 \leqslant v \leqslant n, n \in \mathbb{N}_{0} . \tag{1.4}
\end{align*}
$$

[^0]Differential equations or other kind of linear relations realized by the elements of the dual sequence can be deduced by transposition of those relations fulfilled by the elements of the corresponding MPS, insofar as a linear operator $T: \mathscr{P} \rightarrow \mathscr{P}$ has a transpose ${ }^{t} T: \mathscr{P}^{\prime} \rightarrow \mathscr{P}^{\prime}$ defined by

$$
\begin{equation*}
\left\langle{ }^{t} T(u), f\right\rangle=\langle u, T(f)\rangle, \quad u \in \mathscr{P}^{\prime}, f \in \mathscr{P} . \tag{1.5}
\end{equation*}
$$

For example, for any form $u$ and any polynomial $g$, let $D u=u^{\prime}$ and $g u$ be the forms defined as usual by $\left\langle u^{\prime}, f\right\rangle:=-\left\langle u, f^{\prime}\right\rangle,\langle g u, f\rangle:=\langle u, g f\rangle$, where $D$ is the differential operator [10]. Thus, $D$ on forms is minus the transpose of the differential operator $D$ on polynomials.

The investigation about the orthogonality of a MPS can be performed in a purely algebraic point of view. Precisely, a form $v \in \mathscr{P}^{\prime}$ is said to be regular if we can associate a PS $\left\{Q_{n}\right\}_{n \geqslant 0}$ such that $\left\langle v, Q_{n} Q_{m}\right\rangle=k_{n} \delta_{n, m}$ with $k_{n} \neq 0$ for all $n, m \in \mathbb{N}_{0}[10,11]$. The $\operatorname{PS}\left\{Q_{n}\right\}_{n \geqslant 0}$ is then said to be orthogonal with respect to $v$ and we can assume the system (of orthogonal polynomials) to be monic. Therefore, we can set $v=v_{0}$ and the remaining elements of the corresponding dual sequence $\left\{v_{n}\right\}_{n \geqslant 0}$ are represented by

$$
\begin{equation*}
v_{n+1}=\left(\left\langle v_{0}, Q_{n+1}^{2}(\cdot)\right\rangle\right)^{-1} Q_{n+1}(x) v_{0}, n \in \mathbb{N}_{0} \tag{1.6}
\end{equation*}
$$

When $v \in \mathscr{P}^{\prime}$ is regular, let $\Phi$ be a polynomial such that $\Phi v=0$, then $\Phi=0$ [12].
This unique MOPS $\left\{Q_{n}(x)\right\}_{n \geqslant 0}$ with respect to the regular form $v_{0}$ can be characterized by the popular second order recurrence relation

$$
\left\{\begin{array}{l}
Q_{0}(x)=1 \quad ; \quad Q_{1}(x)=x-\beta_{0}  \tag{1.7}\\
Q_{n+2}(x)=\left(x-\beta_{n+1}\right) Q_{n+1}(x)-\gamma_{n+1} Q_{n}(x), \quad n \in \mathbb{N}_{0}
\end{array}\right.
$$

The MPS $\left\{\mathrm{e}^{x} \mathscr{A}^{n} \mathrm{e}^{-x}\right\}_{n \in \mathbb{N}_{0}}$ where $\mathscr{A}=x^{2}-x \frac{d}{d x} x \frac{d}{d x}$, was recently investigated in [19]. It triggered the study of a wider class of polynomial sequences $\left\{\mathrm{e}^{x} x^{-\alpha} \mathscr{A}^{n} x^{\alpha} \mathrm{e}^{-x}\right\}_{n \in \mathbb{N}_{0}}$ which, despite not being orthogonal (in the usual sense), the canonical element of their corresponding dual form is regular as long as $\alpha>0$, and the existence of a MOPS with respect to this canonical form is ensured. The characterization of such MOPS is undoubtedly an issue, that we could not settle, mainly because of the inherent difficulties of dealing with regular forms fulfilling second order differential equations.

On the other hand, this raised the problem of characterizing polynomial sequences generated by integral composite powers of a first order differential operator, whose canonical form we are able to fully characterize, like the classical forms, that is, regular forms $u_{0}$ fulfilling

$$
\begin{equation*}
\left(\phi u_{0}\right)^{\prime}+\psi u_{0}=0 \tag{1.8}
\end{equation*}
$$

with $\operatorname{deg} \phi \leqslant 2$, $\operatorname{deg} \psi=1$ and $\frac{n}{2} \phi^{\prime \prime}(0)-\psi^{\prime}(0) \neq 0, n \in \mathbb{N}_{0}$. There are essentially four different equivalence classes depending on the nature of the polynomial $\phi[11,12]$, whose representatives are summarized in the next table.

TABLE 1. Expressions for the polynomials $\phi$ and $\psi$ for each classical family.

| $:$ Hermite |  | Laguerre | Bessel | Jacobi |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Regularity <br> conditions <br> $n \in \mathbb{N}_{0}$ |  | $\alpha \neq-(n+1)$ | $\alpha \neq-\frac{n}{2}$ | $\alpha, \beta \neq-(n+1)$ <br> $\alpha+\beta \neq-(n+2)$ |  |
| $\phi(x)$ | $:$ | 1 | $x$ | $x^{2}$ | $x(x-1)$ |
| $\psi(x)$ | $:$ | $2 x$ | $x-\alpha-1$ | $-2(\alpha x+1)$ | $-(\alpha+\beta+2) x+(\alpha+1)$ |

After setting all the required properties of these polynomial sequences generated by integral composite powers of a first order differential operator on $\S 2$, we seek those possessing orthogonality, where, as it will be shown in Proposition 2.1 the solution is reduced to the Hermite polynomials. Afterwards, the characterization of all the polynomial sequences whose canonical form is classical will be handled on $\S 3$, where we will unravel the aforementioned MPSs, for each of the four arisen possibilities, either by determining the coefficients or by seeking the connection with the well known classical polynomial sequences. Meanwhile, the procedure will permit to infer new Rodrigues type formulas for the classical polynomials of Hermite and Bessel. At last, some problems are left open in the case of Jacobi form.

## 2. Polynomials generated by integral powers of a first order differential operator.

Lemma 2.1. Let $E$ be a complex-valued smooth function not identically equal to zero and let $\phi$ and $\psi$ be two polynomials. The sequence of functions

$$
\begin{equation*}
p_{n}(x)=\frac{1}{E(x)} \mathscr{A}^{n} E(x), n \in \mathbb{N}_{0} \tag{2.1}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathscr{A}=-\phi(x) \frac{d}{d x}+\psi(x)+\phi(x) \frac{E^{\prime}(x)}{E(x)} \tag{2.2}
\end{equation*}
$$

represent a polynomial sequence whose elements have degree $n$ (in short, PS) if and only if

$$
\begin{equation*}
\operatorname{deg} \phi \leqslant 2, \operatorname{deg} \psi=1 \text { and } \frac{n}{2} \phi^{\prime \prime}(0)-\psi^{\prime}(0) \neq 0 \quad, \quad n \in \mathbb{N}_{0} \tag{2.3}
\end{equation*}
$$

Moreover, such PS $\left\{p_{n}\right\}_{n \geqslant 0}$ can be equivalently represented by

$$
\begin{array}{ll}
p_{0}(x) & =1  \tag{2.4}\\
p_{n+1}(x) & =-\phi(x) p_{n}^{\prime}(x)+\psi(x) p_{n}(x), n \in \mathbb{N}_{0}
\end{array}
$$

Proof. Under the assumptions evoked for the function $E(x)$, consider the sequence of functions $\left\{p_{n}\right\}_{n \geqslant 0}$ defined by (2.1). The first element of this sequence is the constant function $p_{0}(x)=1$. Insofar as the identity

$$
\begin{equation*}
\frac{1}{E(x)} \mathscr{A}(E(x) g(x))=-\phi(x) g^{\prime}(x)+\psi(x) g(x) \tag{2.5}
\end{equation*}
$$

holds for any analytic function $g(x)$, it readily follows from (2.1) that the sequence of functions $\left\{p_{n}\right\}_{n \geqslant 0}$ can be represented by (2.4), because we successively have

$$
p_{n+1}(x)=\frac{1}{E(x)} \mathscr{A}\left(E(x) \frac{1}{E(x)} \mathscr{A}^{n}(E(x))\right)=\frac{1}{E(x)} \mathscr{A}\left(E(x) p_{n}(x)\right)=-\phi(x) p_{n}^{\prime}(x)+\psi(x) p_{n}(x),, n \in \mathbb{N}_{0}
$$

Conversely, if a sequence of functions $\left\{p_{n}\right\}_{n \geqslant 0}$ is defined by (2.4), we then have $p_{1}(x)=\frac{1}{E(x)} \mathscr{A}(E(x))$ and performing analogous steps as the ones made above, by induction, we conclude that necessarily $\left\{p_{n}\right\}_{n \geqslant 0}$ is also given by (2.1).

Now, it remains to show that $\left\{p_{n}\right\}_{n \geqslant 0}$ is actually a sequence of polynomials of exactly degree $n$ if and only if (2.3) hold. Indeed if these constraints for the pair of polynomials $(\phi, \psi)$ are realized, then (2.4) (as well as (2.1)) ensures that $p_{1}(x)=\psi(x)$ is a one degree polynomial. Through a finite induction procedure, the relation (2.4) enables the result.

Conversely, if each $p_{n}(x)$ is a polynomial of exactly degree $n$, the condition (2.4) implies the claimed constraints for $(\phi, \psi)$, because they do not depend on $n$.

Corollary 2.2. The polynomial sequence $\left\{p_{n}\right\}_{n \geqslant 0}$ given by (2.1) under the constraints (2.3) does not depend on the choice of the smooth function $E$.

Proof. The result is a mere consequence of the equivalent shown between (2.1) and (2.4), as long as $(\phi, \psi)$ satisfy (2.3).

This paper aims to describe the objects given by (2.1), which under certain choices for the pair $(\phi, \psi)$ represent polynomials of degree $n$. From this point forth the pair $(\phi, \psi)$ will be considered to be a pair of polynomials fulfilling (2.3).

For a question of normalization, we are primarily interested in dealing with monic polynomial sequences. For this reason, we will deal instead with the MPS $\left\{P_{n}\right\}_{n \geqslant 0}$ obtained from $\left\{p_{n}\right\}_{n \geqslant 0}$ after the division by its leading coefficient, say $\lambda_{n}$. Without loss of generality, we will as well consider $\phi$ to be a monic polynomial.

To set things more in concrete, we consider $\left\{P_{n}\right\}_{n \geqslant 0}$ such that

$$
\begin{equation*}
\lambda_{n} P_{n}(x)=\frac{1}{E(x)} \mathscr{A}^{n} E(x), n \in \mathbb{N}_{0} \tag{2.6}
\end{equation*}
$$

where $\lambda_{n}$ is a nonzero constant compelling $P_{n}$ to be monic. According to (2.4), it naturally follows that

$$
\begin{equation*}
P_{n+1}(x)=\frac{\lambda_{n}}{\lambda_{n+1}}\left(-\phi(x) P_{n}^{\prime}(x)+\psi(x) P_{n}(x)\right), n \in \mathbb{N}_{0} \tag{2.7}
\end{equation*}
$$

which, in turn, provides the equalities

$$
\lambda_{n}=\left\{\begin{array}{ll}
\left(\psi^{\prime}(0)\right)^{n} & , \quad \operatorname{deg} \phi \leqslant 1  \tag{2.8}\\
(-2)^{n}\left(-\psi^{\prime}(0) / 2\right)_{n} & , \quad \operatorname{deg} \phi=2
\end{array}, n \in \mathbb{N}_{0}\right.
$$

where $(x)_{n}$ denotes the Pochhammer symbol defined by $(x)_{0}:=1,(x)_{n}=x(x+1) \ldots(x+n-1)$ for $n \in \mathbb{N}$.
Now, we redirect the problem of characterizing this MPS to the study of the corresponding dual sequence, to which we will refer to as $\left\{u_{n}\right\}_{n \geqslant 0}$.

Lemma 2.3. The dual sequence of the MPS $\left\{P_{n}\right\}_{n \geqslant 0}$ above defined in (2.6) (or by (2.7)) fulfills

$$
\begin{align*}
& \left(\phi u_{0}\right)^{\prime}+\psi u_{0}=0  \tag{2.9}\\
& \left(\phi u_{n+1}\right)^{\prime}+\psi u_{n+1}=\frac{\lambda_{n+1}}{\lambda_{n}} u_{n}, n \in \mathbb{N}_{0} \tag{2.10}
\end{align*}
$$

Proof. The action of $u_{0}$ over (2.7) is given by

$$
\left\langle u_{0},-\phi P_{n}^{\prime}+\psi P_{n}\right\rangle=0, n \in \mathbb{N}_{0}
$$

which, by transposition, on account of (1.5), is equivalent to

$$
\left\langle\left(\phi u_{0}\right)^{\prime}+\psi u_{0}, P_{n}\right\rangle=0, n \in \mathbb{N}_{0}
$$

providing (2.9). Likewise, the action of $u_{k+1}$ over (2.7) yields

$$
\left\langle u_{k+1},-\phi P_{n}^{\prime}+\psi P_{n}\right\rangle=\frac{\lambda_{n+1}}{\lambda_{n}} \delta_{k, n}, n, k \in \mathbb{N}_{0}
$$

and again, due to (1.5), we may write this latter as

$$
\left\langle\left(\phi u_{k+1}\right)^{\prime}+\psi u_{k+1}, P_{n}\right\rangle=\frac{\lambda_{n+1}}{\lambda_{n}} \delta_{k, n}, n, k \in \mathbb{N}_{0}
$$

Considering (1.1), the relation (2.10) is then a consequence of this latter equality.

Additionally, the moments of the dual sequence fulfill

$$
\begin{align*}
& \left(\psi^{\prime}(0)-\frac{k}{2} \phi^{\prime \prime}(0)\right)\left(u_{n+1}\right)_{k+1}+\left(\psi(0)-k \phi^{\prime}(0)\right)\left(u_{n+1}\right)_{k}-k \phi(0)\left(u_{n+1}\right)_{k-1}=\frac{\lambda_{n+1}}{\lambda_{n}}\left(u_{n}\right)_{k}, n \in \mathbb{N}_{0}  \tag{2.11}\\
& \left(\psi^{\prime}(0)-\frac{k}{2} \phi^{\prime \prime}(0)\right)\left(u_{0}\right)_{k+1}+\left(\psi(0)-k \phi^{\prime}(0)\right)\left(u_{0}\right)_{k}-k \phi(0)\left(u_{0}\right)_{k-1}=0, k \in \mathbb{N}_{0}
\end{align*}
$$

with the initial conditions that $\left(u_{n}\right)_{k}=\delta_{n, k}$ whenever $0 \leqslant k \leqslant n$.
It is worth to notice indeed that

$$
\begin{equation*}
x^{n}=\sum_{v=0}^{n}\left(u_{v}\right)_{n} P_{v}(x), n \in \mathbb{N}_{0} \tag{2.12}
\end{equation*}
$$

All the regular forms fulfilling equations like in (2.9) under the constraints (2.3) have been deeply explored and they are essentially classical forms - see Table 1

Proposition 2.1. If the $\operatorname{MPS}\left\{P_{n}\right\}_{n \geqslant 0}$ defined in (2.6) is orthogonal, then $\left\{P_{n}\right\}_{n \geqslant 0}$ is the Hermite MOPS.
Proof. The orthogonality assumption over the MPS $\left\{P_{n}\right\}_{n \geqslant 0}$ ensures the elements of the corresponding dual sequence $\left\{u_{n}\right\}_{n \geqslant 0}$ to be given by (1.6). The combination of this information with (2.10) leads to

$$
P_{n+1}\left(\left(\phi u_{0}\right)^{\prime}+\psi u_{0}\right)+P_{n+1}^{\prime} \phi u_{0}=\frac{\left\langle u_{0}, P_{n+1}^{2}\right\rangle}{\left\langle u_{0}, P_{n}^{2}\right\rangle} P_{n} u_{0}, n \geqslant 0
$$

which, on account of (2.9) together with the regularity of the canonical form $u_{0}$, enables

$$
\phi(x) P_{n+1}^{\prime}(x)=\frac{\left\langle u_{0}, P_{n+1}^{2}\right\rangle}{\left\langle u_{0}, P_{n}^{2}\right\rangle} P_{n}(x), n \geqslant 0
$$

A mere comparison of the leading coefficients shows that $\operatorname{deg} \phi=0$ and, because $\phi$ was assumed to be monic, $\phi(x)=1$ and, concomitantly, $\frac{\left\langle u_{0}, P_{n+1}^{2}\right\rangle}{\left\langle u_{0}, P_{n}^{2}\right\rangle}=(n+1)$. Thus, it follows $P_{n+1}^{\prime}(x)=(n+1) P_{n}(x)$, for $n \in \mathbb{N}_{0}$, and consequently, $\left\{P_{n}\right\}_{n \geqslant 0}$ coincides with the Hermite polynomial sequence.

Since the $\operatorname{deg} \phi \leqslant 2$ and $\operatorname{deg} \psi=1$ there are only four possible arising cases, better to say, the analysis shall then be split into four different classes. On the other hand, looking at the equation (2.9) fulfilled by the form $u_{0}$ we readily come to the conclusion that $u_{0}$ is necessarily a regular form. Needless to say that this does not imply $\left\{P_{n}\right\}_{n \geqslant 0}$ to be orthogonal (ergo, classical). Actually, Proposition 2.1 ensures the nonorthogonal (in the usual sense) character of the MPS $\left\{P_{n}\right\}_{n \geqslant 0}$ when $\operatorname{deg} \phi \geqslant 1$.

## 3. ANALYSIS OF THE FOUR POSSIBLE SITUATIONS

The analysis taken throughout this section will be drawn according to the nature of the polynomial $\phi$ and therefore split into four different cases. Representatives $\phi$ and $\psi$ for each one of the four possible situations will be chosen from Table 1. Other choices can be considered as long as they realize the required conditions (2.3), guaranteeing the admissibility of regular solutions $u_{0}$ of (2.9) [11].

Among the characterization properties are: an explicit expansion in terms of the monomials, a generating function and a recursive relation (1.2). This latter is important not only for computational reasons but also because it permits to know whether a MPS can be or not $d$-orthogonal [9, 18], which, roughly speaking, means that the elements of the MPS would then fulfill a recursive relation of order $d+1$ (a constant value, independent of the order of the element). In this case more specific expressions for the $\beta$ and $\chi$ coefficients
presented in (1.2) and defined in (1.3)-(1.4) can be straightforwardly obtained from (2.7)-(2.10):

$$
\begin{align*}
& \beta_{n+1}=\beta_{n}+\frac{\lambda_{n}}{\lambda_{n+1}}\left\langle u_{n+1}, \phi P_{n}\right\rangle, n \in \mathbb{N}_{0}  \tag{3.1}\\
& \chi_{n+1, v+1}=\frac{\lambda_{v+1} \lambda_{n+1}}{\lambda_{v} \lambda_{n+2}} \chi_{n, v}+\frac{\lambda_{n+1}}{\lambda_{n+2}}\left\langle u_{v+1}, \phi P_{n+1}\right\rangle, 0 \leqslant v \leqslant n, n \in \mathbb{N}_{0} \tag{3.2}
\end{align*}
$$

with the initial conditions $\beta_{0}=\left(u_{0}\right)_{1}$, determined from the analysis of the moments of order 0 of (2.9), so that

$$
\begin{equation*}
\beta_{0}=-\frac{\psi(0)}{\psi^{\prime}(0)} \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\chi_{n, 0}=\frac{\lambda_{n}}{\lambda_{n+1}}\left\langle u_{0}, x\left(-\phi P_{n}^{\prime}+\psi P_{n}\right)\right\rangle=\frac{\lambda_{n}}{\lambda_{n+1}}\left\langle x\left(\left(\phi u_{0}\right)^{\prime}+\psi u_{0}\right)+\phi u_{0}, P_{n}\right\rangle=\frac{\lambda_{n}}{\lambda_{n+1}}\left\langle u_{0}, \phi P_{n}\right\rangle, n \in \mathbb{N}_{0} \tag{3.4}
\end{equation*}
$$

From this point forth we need to split the analysis into the four possible cases.
3.1. Hermite Case. So far, we have seen that when the orthogonality of the MPS $\left\{P_{n}\right\}_{n \geqslant 0}$ is assumed, we are necessarily handling with the Hermite polynomial sequence, ergo the Hermite form. Now, we are willing to find all polynomial sequences $\left\{P_{n}\right\}_{n \geqslant 0}$ defined in (2.6) such that $u_{0}$ is the regular form of Hermite. As representatives for this case we consider $\phi(x)=1$ and $\psi(x)=2 x$ (see Table 1).

As a matter of fact, (3.1) together with (3.3) implies $\beta_{n}=0$ for all $n \in \mathbb{N}_{0}$, while (3.2) provides

$$
\begin{align*}
& \chi_{n+1, v+1}=\chi_{n, v}, 0 \leqslant v \leqslant n-1  \tag{3.5}\\
& \chi_{n+1, n+1}=\chi_{n, n}+\frac{1}{2}, n \in \mathbb{N}_{0} \tag{3.6}
\end{align*}
$$

According to (3.4) it follows $\chi_{n, 0}=\frac{1}{2} \delta_{n, 0}, n \in \mathbb{N}_{0}$. Thus, $\chi_{n, n}=\frac{n+1}{2}$ and $\chi_{n, v}=0$ for $0 \leqslant v \leqslant n-1$. We achieve therefore the conclusion that the canonical form of an MPS $\left\{P_{n}\right\}_{n \geqslant 0}$ defined by (2.6) with $\operatorname{deg} \phi=0$ is regular if and only if $\left\{P_{n}\right\}_{n \geqslant 0}$ is the Hermite polynomial sequence.

The latter result permits to obtain many Rodrigues type formulas for the Hermite polynomials, since they are represented by

$$
\begin{equation*}
P_{n}(x)=2^{-n} \frac{1}{E(x)}\left(-\frac{d}{d x}+2 x+\frac{E^{\prime}(x)}{E(x)}\right)^{n} E(x), n \in \mathbb{N}_{0} \tag{3.7}
\end{equation*}
$$

rather than the well known one, which could be recovered from upon the choice $E(x)=\mathrm{e}^{-x^{2} / 2}$. Other possible choices for $E(x)$ could be, for instance, $E(x)=\mathrm{e}^{\mathrm{e}^{x}}$.
3.2. Laguerre case. Here we consider the case where $\operatorname{deg} \phi=1$, we make use of the Laguerre form, which is the unique regular form, up to an affine transformation, that is solution of (2.9) under the assumed conditions. As representative of this class, we set $\phi(x)=x$ and $\psi(x)=x-\alpha-1$, and therefore (2.7) becomes

$$
P_{n+1}(x ; \alpha)=-x P_{n}^{\prime}(x ; \alpha)+(x-\alpha-1) P_{n}(x ; \alpha)
$$

## Expressing

$$
\begin{equation*}
P_{n}(x ; \alpha)=\sum_{v=0}^{n}(-1)^{n+v} c_{n, v}(\alpha) x^{v}, n \in \mathbb{N}_{0} \tag{3.8}
\end{equation*}
$$

from (2.7) we derive

$$
\sum_{v=0}^{n+1}(-1)^{n+v+1} c_{n+1, v} x^{v}=-\sum_{\tau=0}^{n} v(-1)^{n+v} c_{n, v} x^{v}+\sum_{v=0}^{n}(-1)^{n+v} c_{n, v}\left(x^{v+1}-(\alpha+1) x^{v}\right), n \in \mathbb{N}_{0}
$$

(under the notation $c_{n, v}:=c_{n, v}(\alpha)$ ) and therefore

$$
\left\{\begin{array}{l}
c_{n, n}(\alpha)=1 \quad, \quad c_{n, 0}(\alpha)=(\alpha+1)^{n}, \quad c_{n, n+v+1}(\alpha)=0, n, v \in \mathbb{N}_{0}  \tag{3.9}\\
c_{n+1, v}(\alpha)=c_{n, v-1}(\alpha)+(v+\alpha+1) c_{n, v}(\alpha), 0 \leqslant v \leqslant n, n \in \mathbb{N}_{0}
\end{array}\right.
$$

under the convention $c_{n,-1}(\alpha)=0$.
These correspond to the non-central Stirling numbers of second kind (or simply, the generalized Stirling numbers) treated in [6], where it was also pointed out the denomination of non-central Lah numbers proposed in [5]. Without entering into further considerations, their explicit formula is

$$
\begin{equation*}
c_{n, v}(\alpha)=\frac{1}{v!} \sum_{i=0}^{v}\binom{v}{i}(-1)^{v-i}(i+\alpha+1)^{n}=\left.\frac{1}{v!}\left(\Delta_{\alpha+1}^{v} x^{n}\right)\right|_{x=0}, n \in \mathbb{N}_{0} \tag{3.10}
\end{equation*}
$$

where $\Delta_{\alpha+1} f(x)=f(x+\alpha+1)-f(x)$. Moreover, these coefficients are the bridge to connect the canonical MPS with the (factorial) polynomial sequences $\left\{(-1)^{n}(-x+\alpha+1)_{n}\right\}_{n \geqslant 0}$ because for $x \neq 0$

$$
\begin{equation*}
x^{n}=\sum_{k=0}^{n} c_{n, k}(\alpha)(-1)^{k}(-x+\alpha+1)_{k} \quad \text { or } \quad(x+\alpha+1)^{n}=\sum_{k=0}^{n} c_{n, k}(\alpha)(-1)^{k}(-x)_{k}, n \in \mathbb{N}_{0} \tag{3.11}
\end{equation*}
$$

and

$$
\sum_{k=0}^{n} c_{n, k}(\alpha)(-1)^{n+k}(\alpha+1)_{k}=1, n \in \mathbb{N}_{0}
$$

Conversely, regarding the moments of the dual sequence, we can consider the inverse relation of (3.8). Thus, from (2.11), we have

$$
\left\{\begin{array}{l}
\left(u_{n+1}\right)_{k+1}=\left(u_{n}\right)_{k}+(\alpha+1+k)\left(u_{n+1}\right)_{k}, n \in \mathbb{N}_{0}  \tag{3.12}\\
\left(u_{0}\right)_{k}=(\alpha+1)_{k}, k \in \mathbb{N}_{0}
\end{array}\right.
$$

with the initial conditions that $\left(u_{n}\right)_{k}=\delta_{n, k}$ whenever $0 \leqslant k \leqslant n$. Thus, the set $\left\{\left(u_{n}\right)_{k}\right\}$ mimics the set of Stirling numbers of first kind, and differ from it by the decentralizing factor $(\alpha+1)$. They are actually the non-central Stirling numbers of first kind, pointed in [6]. Finally, we have (2.12), where $\left(u_{n}\right)_{k}$ satisfies (3.12).

Lemma 3.1. The $\operatorname{MPS}\left\{P_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$ have the following generating function

$$
G(x, t)=\mathrm{e}^{-(\alpha+1) t-x\left(\mathrm{e}^{-t}-1\right)}=\sum_{n \geqslant 0} P_{n}(x ; \alpha) \frac{t^{n}}{n!}
$$

Proof. Indeed, a generating function $G(x, t)=\sum_{n \geqslant 0} P_{n}(x) \frac{t^{n}}{n!}$ must be a solution of the partial differential equation

$$
\frac{\partial}{\partial t} G(x, t)=-x \frac{\partial}{\partial x} G(x, t)+(x-\alpha-1) G(x, t)
$$

satisfying the boundary conditions $\lim _{t \rightarrow 0} G(x, t)=1$ and $\lim _{x \rightarrow 0} G(x, t)=\mathrm{e}^{-(\alpha+1) t}$. Thus, the function $\Phi(x, t)=\mathrm{e}^{-(\alpha+1) t-x\left(\mathrm{e}^{-t}-1\right)}$ is a solution of the problem.

The latter result brings the status of Sheffer-type for the underlying MPS $\left\{P_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$ [14].
Meanwhile, if we bring into discussion the generating function for the (monic) Laguerre polynomials, $\left\{Q_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$,

$$
H(x, t)=(t+1)^{-(\alpha+1)} \mathrm{e}^{x^{\frac{t}{t+1}}}=\sum_{n \geqslant 0} Q_{n}(x ; \alpha) \frac{t^{n}}{n!}
$$

we readily observe that the generating function of the aforementioned MPS $\left\{P_{n}\right\}_{n \geqslant 0}$ can be expressed as

$$
G(x, t)=H\left(x, \mathrm{e}^{t}-1\right)
$$

Hence, by recalling [2, p.51]

$$
\frac{\left(\mathrm{e}^{t}-1\right)^{k}}{k!}=\sum_{n \geqslant k} S(n, k) \frac{t^{n}}{n!}, n \in \mathbb{N}_{0}
$$

we deduce the identity

$$
P_{n}(x ; \alpha)=\frac{1}{n!} \sum_{v=0}^{n} S(n, v) Q_{v}(x ; \alpha), n \in \mathbb{N}_{0}
$$

Conversely, the identity [2, p.51]

$$
\frac{(\log (t+1))^{k}}{k!}=\sum_{n \geqslant k} s(n, k) \frac{t^{n}}{n!}, n \in \mathbb{N}_{0}
$$

provides the reciprocal

$$
Q_{n}(x ; \alpha)=\frac{1}{n!} \sum_{v=0}^{n} s(n, v) P_{v}(x ; \alpha), n \in \mathbb{N}_{0}
$$

An alternative way to obtain the latter identities, but rather less intuitive, would be via Faa di Bruno's formula.

Lemma 3.2. The structure relation of the $\operatorname{MPS}\left\{P_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$ is

$$
P_{n+2}(x ; \alpha)=(x-n-\alpha-2) P_{n+1}(x ; \alpha)-\sum_{v=0}^{n}\binom{n+1}{v}\left(\alpha+\frac{n+2}{n+1-v}\right) P_{v}(x ; \alpha), n \geqslant 0 .
$$

Proof. In this case $\beta_{0}=\alpha+1$ and therefore the remaining ones are

$$
\beta_{n}=n+\alpha+1, n \in \mathbb{N}_{0}
$$

(which match the $\beta \mathrm{s}$ of the second order recursive relation of the Laguerre polynomials). Besides, according to (3.4)

$$
\chi_{n, 0}=\left\langle u_{0}, x P_{n}\right\rangle=\chi_{n-1,0}=\chi_{0,0}=\alpha+1
$$

because $\chi_{0,0}=\alpha+1$, which, in particular, guarantees this sequence not to be $d$-orthogonal. The remaining coefficients satisfy the recurrence relation

$$
\left\{\begin{array}{l}
\chi_{n+1, n+1}=\chi_{n, n}+\beta_{n+1}, n \in \mathbb{N}_{0} \\
\chi_{n+1, v+1}=\chi_{n, v}+\chi_{n, v+1}, 0 \leqslant v \leqslant n-1, n \in \mathbb{N} \\
\chi_{n, 0}=(\alpha+1), n \in \mathbb{N}_{0} \\
\chi_{0, n}=(\alpha+1) \delta_{0, n}, n \in \mathbb{N}_{0}
\end{array}\right.
$$

whence $\chi_{n, v}=\binom{n+2}{v}+\binom{n+1}{v} \alpha, 0 \leqslant v \leqslant n, n \in \mathbb{N}_{0}$.

The fact that $\chi_{n, 0} \neq 0$ for all $n \in \mathbb{N}_{0}$ discards the possibility of $\left\{P_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$ to be orthogonal or $d$ orthogonal.

The example obtained under the choices of $E(x)=x^{\alpha} \mathrm{e}^{-x}, \phi(x)=x$ and $\psi(x)=x-\alpha$, has received a special attention, as we may read in [4, pp.254-255] (or in the references therein) after the work taken in [16, 17].
3.3. Bessel case. The choice of $\phi(x)=x^{2}$ and $\psi(x)=-2(\alpha x+1)$, launch the polynomial sequence

$$
\begin{equation*}
P_{n}(x ; \alpha)=\frac{1}{(2 \alpha)_{n}} \frac{1}{E(x)} \mathscr{A}^{n} E(x) \tag{3.13}
\end{equation*}
$$

equivalently defined by the differential relation

$$
\begin{equation*}
P_{n+1}(x ; \alpha)=\frac{1}{2 \alpha+n}\left(x^{2} P_{n}^{\prime}(x ; \alpha)+2(\alpha x+1) P_{n}(x ; \alpha)\right), n \in \mathbb{N}_{0} \tag{3.14}
\end{equation*}
$$

Lemma 3.3. The MPS $\left\{P_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$ and the canonical sequence $\left\{x^{n}\right\}_{n \geqslant 0}$ realize the following inverse relations:

$$
\begin{equation*}
P_{n}(x ; \alpha)=\sum_{v=0}^{n}\binom{n}{v} \frac{2^{n-v}}{(2 \alpha+v)_{n-v}} x^{v}, n \in \mathbb{N}_{0} \tag{3.15}
\end{equation*}
$$

whereas

$$
\begin{equation*}
x^{k}=\frac{(-1)^{k} 2^{k}}{(2 \alpha)_{k}} P_{0}(x)+\sum_{v=0}^{k-1}\left(\frac{(-1)^{k-v-1} 2^{k-v-1}(2 \alpha)_{v+1}}{(2 \alpha)_{k}} \sum_{\mu=v}^{k-1}(-1)^{v+\mu-1}\binom{\mu}{v}\right) P_{v+1}(x), k \in \mathbb{N}_{0} \tag{3.16}
\end{equation*}
$$

Proof. From (3.14), we deduce

$$
P_{n}(x ; \alpha)=\sum_{v=0}^{n} \frac{2^{n-v}(2 \alpha)_{v}}{(2 \alpha)_{n}} \widehat{c}_{n, v}(\alpha) x^{v}, n \in \mathbb{N}_{0}
$$

where $\widehat{c}_{n, v}(\alpha)$ fulfills the triangular relation

$$
\begin{aligned}
& \widehat{c}_{n+1, v}(\alpha)=\widehat{c}_{n, v-1}(\alpha)+\widehat{c}_{n, v}(\alpha), 0 \leqslant v \leqslant n ; n, v \in \mathbb{N}_{0} \\
& \widehat{c}_{n, 0}(\alpha)=1, n \in \mathbb{N}_{0}
\end{aligned}
$$

which yields $\widehat{c}_{n, v}=\binom{n}{v}, 0 \leqslant v \leqslant n, n, v \in \mathbb{N}_{0}$, whence (3.15).
Regarding the reciprocal relation of (3.15), i.e., to write the monomials in terms of the polynomials $P_{n}(\cdot ; \alpha)$, within the framework of (2.12), we seek an expression for the moments of the dual sequence $\left\{u_{n}(\alpha)\right\}_{n \geqslant 0}$ given in (2.9)-(2.10) also realizing (2.11). In this case,

$$
\left(u_{n}\right)_{k}=\frac{(-1)^{k-n} 2^{k-n}(2 \alpha)_{n}}{(2 \alpha)_{k}} d_{k, n}
$$

where $d_{n, k}$ fulfills the triangular relation

$$
d_{k+1, n+1}=d_{k, n}-d_{k, n+1} \quad ; \quad d_{n, 0}=1 \quad ; \quad d_{0, n}=\delta_{0, n}, n \in \mathbb{N}_{0}
$$

which provides

$$
d_{k+1, n+1}=\sum_{\mu=n}^{k}(-1)^{n+\mu-1}\binom{\mu}{n}, 0 \leqslant n \leqslant k, n, k \in \mathbb{N}_{0}
$$

For other considerations regarding the set of numbers $\left\{\left|d_{k, n}\right|\right\}_{k, n}$ we refer to the entry $\underline{\text { A059260 }}$ of [15]. Consequently,

$$
\begin{equation*}
\left(u_{0}\right)_{k}=\frac{(-1)^{k} 2^{k}}{(2 \alpha)_{k}} \quad, \quad\left(u_{v+1}\right)_{k+1}=\frac{(-1)^{k-v} 2^{k-v}(2 \alpha)_{v+1}}{(2 \alpha)_{k+1}} \sum_{\mu=v}^{k}(-1)^{v+\mu-1}\binom{\mu}{v} \tag{3.17}
\end{equation*}
$$

and, (2.12) becomes (3.16).
Recalling the expression for the (classical) monic Bessel polynomials

$$
\begin{equation*}
B_{n}(x ; \alpha)=\sum_{v=0}^{n}\binom{n}{v} \frac{2^{n-v}}{(2 \alpha+n-1+v)_{n-v}} x^{v}, n \in \mathbb{N}_{0} \tag{3.18}
\end{equation*}
$$

a simple relation between the nonorthogonal sequence $\left\{P_{n}(x ; \alpha)\right\}_{n \geqslant 0}$ and the orthogonal sequence of Bessel polynomials $\left\{B_{n}(x ; \alpha)\right\}_{n \geqslant 0}$ comes out:

$$
\begin{equation*}
P_{n}(x ; \alpha)=B_{n}\left(x ; \alpha-\frac{n-1}{2}\right), n \in \mathbb{N}_{0} \tag{3.19}
\end{equation*}
$$

This fact, actually has the consequence of providing new Rodrigues type formulas for the Bessel polynomials rather than the one already known [3]

$$
B_{n}(x ; \alpha)=\frac{(-1)^{n}}{(-2 n-2 \alpha+2)_{n} x^{2 \alpha-2} \mathrm{e}^{-2 / x}} \frac{d^{n}}{d x^{n}}\left(x^{2 \alpha-2+2 n} \mathrm{e}^{-2 / x}\right), n \in \mathbb{N}_{0}
$$

Lemma 3.4. The Bessel polynomials $\left\{B_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$ with $\alpha \neq-\frac{n}{2}, n \in \mathbb{N}_{0}$, can be generated by the Rodrigues type formula

$$
B_{n}(x ; \alpha)=\frac{1}{(2 \alpha)_{n}} \frac{1}{E(x)}\left(-x^{2} \frac{d}{d x}-2\left(\alpha+\frac{n+1}{2}\right) x-2+x^{2} \frac{E^{\prime}(x)}{E(x)}\right)^{n} E(x), n \geqslant 0
$$

Proof. The result is a mere consequence of the definition of the polynomials $\left\{P_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$ and (3.19) written in the reverse way: $B_{n}(x ; \alpha)=P_{n}\left(x ; \alpha+\frac{n+1}{2}\right), n \in \mathbb{N}_{0}$.

Besides, from (3.15) (or by simply considering the identity $B_{n+1}^{\prime}(x ; \alpha)=(n+1) B_{n}(x ; \alpha+1)$ ) we deduce

$$
P_{n+1}^{\prime}(x ; \alpha)=(n+1) P_{n}\left(x ; \alpha+\frac{1}{2}\right), n \in \mathbb{N}_{0}
$$

Combining this latter with (3.14), another structure relation for these polynomials comes out:

$$
\frac{n}{2 \alpha+n} x^{2} P_{n}\left(x ; \alpha+\frac{1}{2}\right)=P_{n+1}(x ; \alpha)-\frac{2}{2 \alpha+n}(\alpha x+1) P_{n}(x ; \alpha), n \in \mathbb{N}_{0}
$$

Concerning the generating function, $G(x, t)$, for the PS $\left\{(2 \alpha)_{n} P_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$, its expression can be deduced based on the differential relation (3.14), which implies $G(x, t)$ to be solution of the partial differential equation

$$
\frac{\partial}{\partial t} G(x, t)=-x^{2} \frac{\partial}{\partial x} G(x, t)-2(\alpha x+1) G(x, t)
$$

satisfying the boundary condition $G(x, 0)=1$. As a consequence, we have

$$
G(x, t)=\mathrm{e}^{2 t}(t x-1)^{-2 \alpha}=\sum_{n \geqslant 0}(2 \alpha)_{n} P_{n}(x ; \alpha) \frac{t^{n}}{n!}
$$

Despite the non-orthogonality of the MPS $\left\{P_{n}(\cdot):=P_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$, we may envisage whether the $d$ orthogonality can fit in this MPS, for some positive integer $d \geqslant 2$. However, we have:

Lemma 3.5. The MPS $\left\{P_{n}(\cdot ; \alpha)\right\}_{n \geqslant 0}$ cannot be $d$-orthogonal because the order of its recursive relation depends on the order of their elements, precisely

$$
P_{n+2}(x)=\left(x-\beta_{n+1}\right) P_{n+1}(x)-\sum_{v=0}^{n} \chi_{n, v} P_{v}(x)
$$

with

$$
\beta_{n}=-\frac{2(2 \alpha-1)}{(n+2 \alpha-1)(n+2 \alpha)} \quad \text { and } \quad \chi_{n, 0}=\frac{-2^{n+2}(n+1)!}{(2 \alpha)_{n+1}(2 \alpha)_{n+2}} \neq 0, n \in \mathbb{N}_{0}
$$

Proof. From (3.1) with $\lambda_{n+1}=-(2 \alpha+n) \lambda_{n}$ we have in this case

$$
\beta_{n+1}=\beta_{n}-\frac{1}{2 \alpha+n}\left\langle u_{n+1}, x^{2} P_{n}(x)\right\rangle, n \in \mathbb{N}_{0}
$$

However, due to (1.2) with $n \rightarrow n-1$ we deduce

$$
\left\langle u_{n+1}, x^{2} P_{n}(x)\right\rangle=\beta_{n+1}+\beta_{n}, n \in \mathbb{N}_{0}
$$

whence,

$$
\beta_{n}=\beta_{0} \frac{(2 \alpha-1)_{n}}{(2 \alpha+1)_{n}}=\beta_{0} \frac{2 \alpha(2 \alpha-1)}{(2 \alpha-1+n)(2 \alpha+n)}, n \in \mathbb{N}_{0}
$$

and finally by appealing to (3.3), we obtain the desired expression for $\beta_{n}$.
Instead of following a similar procedure as the one taken to determine the coefficients $\chi_{n, v}$ in the Laguerre case, we will use the relations (3.15) together with (3.17) to write

$$
\chi_{n, v}=\left\langle u_{v}, x P_{n+1}\right\rangle=\sum_{\sigma=0}^{n+1}\binom{n+1}{\sigma} \frac{2^{n+1-\sigma}(2 \alpha)_{\sigma}}{(2 \alpha)_{n+1}}\left(u_{v}\right)_{\sigma+1}
$$

The particular choice of $v=0$ becomes

$$
\chi_{n, 0}=\sum_{\sigma=0}^{n+1}\binom{n+1}{\sigma} \frac{2^{n+2}(-1)^{\sigma+1}}{(2 \alpha)_{n+1}(2 \alpha+\sigma)}=-\frac{2^{n+2}(n+1)!}{(2 \alpha)_{n+1}(2 \alpha)_{n+2}} \neq 0, n \in \mathbb{N}_{0}
$$

whereas

$$
\chi_{n, v+1}=\frac{2^{n+1-v}(2 \alpha)_{v+1}}{(2 \alpha)_{n+1}} \sum_{\sigma=v}^{n+1}\binom{n+1}{\sigma} \frac{(-1)^{\sigma}}{(2 \alpha+\sigma)} \sum_{\mu=v}^{\sigma}(-1)^{\mu-1}\binom{\mu}{v}, 0 \leqslant v \leqslant n-1, n \in \mathbb{N} .
$$

The condition $\chi_{n, 0} \neq 0$ refute the $d$-orthogonality of $\left\{P_{n}\right\}_{n \geqslant 0}$.
3.4. Jacobi case. Proceeding in a similar way as in the precedent cases, we consider $\phi(x)=x(x-1)$ and $\psi(x)=-(\alpha+\beta+2) x+(\alpha+1)$, so that the MPS $\left\{P_{n}\right\}_{n \geqslant 0}$ fulfill

$$
\begin{equation*}
P_{n+1}(x, \alpha, \beta)=\frac{1}{n+\alpha+\beta+2}\left(x(x-1) P_{n}^{\prime}(x, \alpha, \beta)+((\alpha+\beta+2) x-(\alpha+1)) P_{n}(x, \alpha, \beta)\right), n \in \mathbb{N}_{0} \tag{3.20}
\end{equation*}
$$

and the corresponding canonical form, which coincides with the Jacobi form, satisfies

$$
\begin{equation*}
\left(x(x-1) u_{0}\right)^{\prime}-((\alpha+\beta+2) x-(\alpha+1)) u_{0}=0 \tag{3.21}
\end{equation*}
$$

Lemma 3.6. The elements of the MPS $\left\{P_{n}(\cdot ; \alpha, \beta)\right\}_{n \geqslant 0}$ are explicitly given by

$$
\begin{equation*}
P_{n}(x, \alpha, \beta)=\sum_{v=0}^{n} \frac{(\alpha+\beta+2)_{v}}{(\alpha+\beta+2)_{n}}(-1)^{n+v} c_{n, v}(\alpha) x^{v}, n \in \mathbb{N}_{0} \tag{3.22}
\end{equation*}
$$

where the set of numbers $\left\{c_{n, v}\right\}_{0 \leqslant v \leqslant n}$ is defined in (3.9)-(3.10).

Proof. Based on this differential-recursive relation (3.20) fulfilled by the MPS $\left\{P_{n}\right\}_{n \geqslant 0}$, we derive the explicit expression of their elements. Indeed, by setting

$$
\begin{equation*}
P_{n}(x, \alpha, \beta)=\sum_{v=0}^{n} \frac{\widetilde{c}_{n, v}(\alpha, \beta)}{(\alpha+\beta+2)_{n}} x^{v} \tag{3.23}
\end{equation*}
$$

then, regarding (3.20), the set of coefficients $\widetilde{c}_{n, v}$ fulfill the triangular relation

$$
\begin{aligned}
& \widetilde{c}_{n+1, v}(\alpha, \beta)=(v+\alpha+1) \widetilde{c}_{n, v}(\alpha, \beta)-(v+\alpha+\beta+1) \widetilde{c}_{n, v-1}(\alpha, \beta) \\
& \widetilde{c}_{n, 0}(\alpha, \beta)=(-1)^{n}(\alpha+1)^{n}
\end{aligned}
$$

which can be shrunk to the same coefficients $c_{n, v}$, given in (3.9)-(3.10) on the aforementioned Laguerre case, if we consider

$$
\widetilde{c}_{n, v}(\alpha, \beta)=(-1)^{n+v}(\alpha+\beta+2)_{v} c_{n, v}(\alpha)
$$

The new set of coefficients $\left\{c_{n, v}(\alpha)\right\}$ no longer depends on $\beta$. Consequently, we obtain (3.22).
Notwithstanding the generating function for the $\operatorname{MPS}\left\{P_{n}(x ; \alpha, \beta)\right\}_{n \geqslant 0}$ seems to be complicate to obtain, we succeeded in determining the following:

Lemma 3.7. The $P S\left\{p_{n}(x ; \alpha, \beta)\right\}_{n \geqslant 0}$ where $p_{n}(x ; \alpha, \beta)=(\alpha+\beta+2)_{n} P_{n}(x ; \alpha, \beta)$ have the following generating function

$$
\begin{equation*}
G(x, t)=\frac{\mathrm{e}^{(\alpha-\beta) t}}{(\cosh (t)+x \sinh (t))^{\alpha+\beta+2}}=\sum_{n \geqslant 0} p_{n}(x ; \alpha, \beta) \frac{t^{n}}{n!} \tag{3.24}
\end{equation*}
$$

Proof. The differential-recursive relation

$$
p_{n+1}(x ; \alpha, \beta)=-x^{2} p_{n}^{\prime}(x ; \alpha, \beta)+(-(\alpha+\beta+2) x+\alpha-\beta) p_{n}(x ; \alpha, \beta), n \in \mathbb{N}_{0}
$$

fulfilled by the PS $\left\{p_{n}(x ; \alpha, \beta)\right\}_{n \geqslant 0}$, implies the generating function to be solution of the differential equation

$$
\frac{\partial}{\partial t} G(x, t)=-x^{2} \frac{\partial}{\partial x} G(x, t)+(-(\alpha+\beta+2) x+\alpha-\beta) G(x, t)
$$

satisfying the boundary condition $G(x, 0)=1$. The desired solution is (3.24).
Regarding the structure relation fulfilled by $\left\{P_{n}(\cdot ; \alpha, \beta)\right\}_{n \geqslant 0}$ we have the following result:
Lemma 3.8. The MPS $\left\{P_{n}(\cdot):=P_{n}(\cdot ; \alpha, \beta)\right\}_{n \geqslant 0}$ fulfills the recursive relation

$$
P_{n+2}(x)=\left(x-\beta_{n+1}\right) P_{n+1}(x)-\sum_{v=0}^{n} \chi_{n, v} P_{v}(x)
$$

with

$$
\beta_{n}=\frac{2(\alpha+\beta+1)(\alpha+1)+n(n+3+2 \alpha+2 \beta)}{2(n+\alpha+\beta+2)(n+\alpha+\beta+1)}, n \in \mathbb{N}_{0}
$$

and

$$
\chi_{n, 0}=\frac{1}{(\alpha+\beta+2)_{n+1}} \sum_{v=0}^{n+1} \frac{(-1)^{n+v+1} c_{n+1, v}(\alpha)(\alpha+1)_{v+1}}{(\alpha+\beta+2+v)}, n \in \mathbb{N}_{0}
$$

Proof. The procedure is very similar to the one taken in the Bessel case. Precisely, from (3.1), where $\lambda_{n+1}=(\alpha+\beta+n+2) \lambda_{n}$, and by taking into consideration (1.2) and (3.3) it follows

$$
(n+\alpha+\beta+3) \beta_{n+1}=(n+\alpha+\beta+1) \beta_{n}+1, n \in \mathbb{N}_{0}
$$

and $\beta_{0}=\frac{\alpha+1}{\alpha+\beta+2}$, which amounts to the same as

$$
(n+\alpha+\beta+2)(n+\alpha+\beta+1) \beta_{n}=(\alpha+\beta+1)(\alpha+1)+\frac{n}{2}(n+3+2 \alpha+2 \beta), n \in \mathbb{N}_{0}
$$

and finally by appealing to (3.3), we obtain the desired expression for $\beta_{n}$.
Taking into account (3.22) and the moments of $u_{0}$, precisely

$$
\left(u_{0}\right)_{n}=\frac{(\alpha+1)_{n}}{(\alpha+\beta+2)_{n}}, n \in \mathbb{N}_{0}
$$

we conclude

$$
\begin{aligned}
\chi_{n, 0} & =\left\langle u_{0}, x P_{n+1}\right\rangle=\sum_{v=0}^{n+1} \frac{(\alpha+\beta+2)_{v}}{(\alpha+\beta+2)_{n+1}}(-1)^{n+1+v} c_{n+1, v}(\alpha)\left(u_{0}\right)_{v+1} \\
& =\frac{1}{(\alpha+\beta+2)_{n+1}} \sum_{v=0}^{n+1} \frac{(-1)^{n+v+1} c_{n+1, v}(\alpha)(\alpha+1)_{v+1}}{(\alpha+\beta+2+v)}, n \in \mathbb{N}_{0}
\end{aligned}
$$

Actually, as far as $\alpha+\beta>-1, \chi_{n, 0} \neq 0$ for $n \in \mathbb{N}_{0}$. Indeed, bearing in mind (3.11) with $n \rightarrow n+1$, i.e.,

$$
\sum_{v=0}^{n+1}(-1)^{n+v+1} c_{n+1, v}(\alpha)(\alpha+1)_{v+1}=(\alpha+1)(-1)^{n+1} \sum_{v=0}^{n+1}(-1)^{v} c_{n+1, v}(\alpha)(\alpha+2)_{v}=\alpha+1
$$

it readily follows

$$
\left|\chi_{n, 0}\right|>\left|\frac{1}{(\alpha+\beta+2)_{n+1}}\right|\left|\sum_{v=0}^{n+1} \frac{(-1)^{n+v+1} c_{n+1, v}(\alpha)(\alpha+1)_{v+1}}{\alpha+\beta+2+v}\right|=\left|\frac{(\alpha+1)}{(\alpha+\beta+2)_{n+2}}\right|>0, n \in \mathbb{N}_{0}
$$

Therefore, if $\alpha+\beta>-1$, the MPS $\left\{P_{n}(\cdot ; \alpha, \beta)\right\}_{n \geqslant 0}$ cannot be $d$-orthogonal.
Otherwise, if $\alpha+\beta<-1$ fulfilling the regularity conditions pointed out in Table 1 , we conjecture that $\chi_{n, 0} \neq 0$.

The determination of the remaining coefficients $\chi_{n, v}$ seems to require more laborious computations, that are deferred for a further work. Besides, regarding the nature of this MPS $\left\{P_{n}(\cdot ; \alpha, \boldsymbol{\beta})\right\}_{n \geqslant 0}$, the connection with the well known Jacobi classical polynomial sequences is not as simple to establish as in the precedent cases of Hermite, Laguerre or Bessel. For this reason, we leave this issue as an open problem.

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