SYSTEMS OF PROPORTIONALLY MODULAR DIOPHANTINE INEQUALITIES

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1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* S is a submonoid of $\mathbb{N}$ (that is, it is closed under addition and $0 \in S$) such that $\mathbb{N} \setminus S$ is finite. Given two integers a and b with $b \neq 0$, we denote by $a \bmod b$ the remainder of the division of a by b. Following the notation of [5], a proportionally modular Diophantine inequality is an expression of the form $ax \bmod b \leq cx$, where a, b and c are positive integers. The set $\mathrm{S}(a, b, c)$ of all integer solutions to this inequality is a numerical semigroup. A numerical semigroup S is *proportionally modular* if $S = \mathrm{S}(a, b, c)$ for some positive integers a, b and c, that is, it is the set of integer solutions of the inequality $ax \bmod b \leq cx$. In this setting, we say that $\mathrm{S}(a, b, c)$ is a proportionally modular representation of S.

Let $a_1, \ldots, a_r, b_1, \ldots, b_r, c_1, \ldots, c_r$ be positive integers. Then the set of solutions of the system of inequalities

$$\begin{cases} a_1 x \bmod b_1 \leq c_1 x, \\ \qquad \cdots \\ a_r x \bmod b_r \leq c_r x, \end{cases}$$

is $\mathrm{S}(a_1, b_1, c_1) \cap \cdots \cap \mathrm{S}(a_r, b_r, c_r)$, and thus it is a numerical semigroup. A numerical semigroup is *system proportionally modular* if it is the set of solutions of a system of proportionally modular Diophantine inequalities. If $S = \mathrm{S}(a_1, b_1, c_1) \cap \cdots \cap \mathrm{S}(a_r, b_r, c_r)$, then we say that $\mathrm{S}(a_1, b_1, c_1) \cap \cdots \cap \mathrm{S}(a_r, b_r, c_r)$ is a system proportionally modular representation of S.

The aim of this paper is the study of system proportionally modular numerical semigroups. We will mainly focus on the following aspects.

- We will see that every subset A of $\mathbb{N}$ with $\gcd(A) = 1$ (gcd stands for greatest common divisor) uniquely determines a system proportionally modular numerical semigroup.
- We give an algorithmic procedure to recurrently construct the set of all system proportionally modular numerical semigroups.
- We give a procedure to determine whether or not a numerical semigroup is system proportionally modular.

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- For a system proportionally modular numerical semigroup, we show how to obtain a system proportionally modular representation. We will also give a method to compute a minimal representation, that is, if S is a system proportionally modular numerical semigroup, then we find the least possible number of proportionally modular Diophantine inequalities determining S.
- Finally, we explain the relationship of system proportionally modular numerical semigroups with numerical semigroups having a Toms' decomposition. These semigroups appear as positive cones of the K_0-group of certain C^*-algebras (see [9]).

2. $\mathcal{SPM}$-SYSTEMS OF GENERATORS

Let A be a set of nonnegative integers. We denote by $\langle A\rangle$ the submonoid of $(\mathbb{N},+)$ generated by A, that is, the set of elements of the form $\lambda_1 a_1 + \cdots + \lambda_n a_n$ with $n \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_n \in \mathbb{N}$ and $a_1,\ldots,a_n \in A$. If $M = \langle A\rangle$, then we say that A is a *system of generators* of M. This system of generators is *minimal* if no proper subset of A generates M. It is easy to show that every submonoid of $\mathbb{N}$ admits a unique minimal system of generators (see for instance [3]), which turns out to have finitely many elements. The cardinality of the minimal system of generators of a submonoid M of $\mathbb{N}$ is known as the *embedding dimension* of M, and it is denoted by $\mathrm{e}(M)$. It is also well known (see for instance [3]) that if $A \subseteq \mathbb{N}$, then $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$. The following result can be deduced from [5].

Lemma 1. *Every numerical semigroup of embedding dimension two is proportionally modular.*

The intersection of proportionally modular numerical semigroups is a submonoid of $\mathbb{N}$, but in general it is not a numerical semigroup since it does not need to be co-finite: clearly, $\bigcap_{n\in\mathbb{N}\setminus\{0\}}\langle d, dn+1\rangle = \langle d\rangle$ for every positive integer d. If the family of proportionally modular numerical semigroups that we intersect is finite, then the resulting semigroup is a numerical semigroup, since the intersection of finitely many numerical semigroups is again a numerical semigroup.

A submonoid M of $\mathbb{N}$ is a $\mathcal{SPM}$*-semigroup* if it can be expressed as the intersection of proportionally modular numerical semigroups. Thus the intersection of $\mathcal{SPM}$-semigroups is again a $\mathcal{SPM}$-semigroup. Note that a co-finite $\mathcal{SPM}$-semigroup is a system proportionally numerical semigroup. If $A \subseteq \mathbb{N}$, it makes sense to talk about the $\mathcal{SPM}$-semigroup generated by A, which is the intersection of all $\mathcal{SPM}$-semigroups containing A, that is, the least (with respect to set inclusion) $\mathcal{SPM}$-semigroup containing A. We will denote this submonoid of $\mathbb{N}$ as $\mathcal{SPM}(A)$ and call it the $\mathcal{SPM}$*-closure* of A. If $M = \mathcal{SPM}(A)$, then we say that A is a $\mathcal{SPM}$*-system of generators* of M, and, as usual, we say that A is *minimal* if not proper subset of A is a $\mathcal{SPM}$-system of generators of M. The following result is easy to prove.

Lemma 2. *Let A and M be a subset and a submonoid of $\mathbb{N}$, respectively. The following conditions are equivalent.*

1) $M = \mathcal{SPM}(A)$.
2) *M is the intersection of all proportionally modular numerical semigroups containing A.*
3) *M is the intersection of all proportionally modular numerical semigroups containing $\langle A \rangle$.*

Proposition 3. *Let A be a subset of $\mathbb{N}$. Then $\mathcal{SPM}(A)$ is a numerical semigroup if and only if* $\gcd(A) = 1$.

Proof. Necessity. Assume that $\gcd(A) = d \neq 1$. Then $A \subseteq \langle d \rangle$, and we already know that $\langle d \rangle$ is a $\mathcal{SPM}$-semigroup. Thus $\mathcal{SPM}(A) \subseteq \langle d \rangle$. As $\mathbb{N} \setminus \mathcal{SPM}(A)$ has infinitely many elements, $\mathcal{SPM}(A)$ is not a numerical semigroup.

Sufficiency. If $\gcd(A) = 1$, then $\langle A \rangle$ is a numerical semigroup. Hence $\mathbb{N} \setminus \langle A \rangle$ has finitely many elements, whence there are only finitely many numerical semigroups containing $\langle A \rangle$. Thus, only finitely many proportionally modular numerical semigroups contain $\langle A \rangle$. In view of Lemma 2, $\mathcal{SPM}(A)$ is the intersection of all proportionally modular numerical semigroups containing $\langle A \rangle$. As there are only finitely many of them, this intersection is a numerical semigroup. □

With this, we can state the following result.

Theorem 4. *Let A be a subset of $\mathbb{N}$ with* $\gcd(A) = 1$. *Then $\mathcal{SPM}(A)$ is a system proportionally modular numerical semigroup. Moreover, every system proportionally modular numerical semigroup is of this form.*

3. Uniqueness of minimal $\mathcal{SPM}$-systems of generators

Our goal in this section is to prove that every $\mathcal{SPM}$-semigroup admits a unique minimal $\mathcal{SPM}$-system of generators. In order to achieve this, we need to recall some known results.

Lemma 5. [5, Corollary 9] *Let $c < a < b$ be positive integers. Then* $\mathrm{S}(a, b, c) = T \cap \mathbb{N}$*, where T is the submonoid of $\mathbb{R}_0^+$ generated by the closed interval $[\frac{b}{a}, \frac{b}{a-c}]$. Conversely, given positive integers a_1, a_2, b_1, b_2 with $\frac{a_1}{b_1} < \frac{a_2}{b_2}$, if T is the submonoid of $\mathbb{R}_0^+$ generated by $[\frac{a_1}{b_1}, \frac{a_2}{b_2}]$, then* $T \cap \mathbb{N} = \mathrm{S}(a_2 b_1, a_1 a_2, a_2 b_1 - a_1 b_2)$.

Note that the inequality $ax \bmod b \leq cx$ has the same solutions as the inequality $(a \bmod b)x \bmod b \leq cx$, and if $c \geq a$, then $\mathrm{S}(a, b, c) = \mathbb{N}$. Thus, the condition $c < a < b$ imposed in Lemma 5 is not restrictive.

If T is the submonoid of $\mathbb{R}_0^+$ generated by $[\frac{a_1}{b_1}, \frac{a_2}{b_2}]$, we will refer to $T \cap \mathbb{N}$ as the proportionally modular numerical semigroup associated to $[\frac{a_1}{b_1}, \frac{a_2}{b_2}]$, and we will denote it by $\mathrm{S}([\frac{a_1}{b_1}, \frac{a_2}{b_2}])$.

More generally, let I be any interval of $\mathbb{R}_0^+$ with more than one element. If T is the submonoid of $\mathbb{R}_0^+$ generated by I, then $\mathrm{S}(I) = T \cap \mathbb{N}$ turns out to be a proportionally modular numerical semigroup as explained in [7] (see also [11]). The following result, which can be deduced from [7, Lemma 2] (or [11, Lemma 6.2]), solves the membership problem for $\mathrm{S}(I)$.

Lemma 6. *Let I be an interval of nonnegative real numbers with more than one element. Then a positive integer x belongs to* $\mathrm{S}(I)$ *if and only if there exists a positive integer y such that $\frac{x}{y} \in I$.*

Let S be a numerical semigroup. The largest integer not belonging to S is known as the Frobenius number of S, and it is denoted by $\mathrm{F}(S)$. The following result renders proportionally modular numerical semigroups with given a Frobenius number. Its proof is easy and is part of [7, Lemma 6] (it can also be deduced from [11, Lemma 6.6]; open intervals with ends α and β will be denoted by $]\alpha,\beta[$).

Lemma 7. *Let a and b be positive integers with $a < b$. Then* $\mathrm{S}(]\frac{b}{a}, \frac{b}{a-1}[)$ *is a proportionally modular numerical semigroup with Frobenius number b.*

Remark 8. We are going to use the convenient notation $\frac{x}{0} = \infty$, with x a positive integer. Moreover, we convey that $r < \infty$ for every rational number r. For $a = 1$ in the preceding result, the interval is $]\frac{b}{a}, \infty[$, and the result still holds.

Remark 9. Note that

1) if M is a $\mathcal{SPM}$-semigroup, then $\mathcal{SPM}(M) = M$,
2) if $A \subseteq B \subseteq \mathbb{N}$, then $\mathcal{SPM}(A) \subseteq \mathcal{SPM}(B)$.

Lemma 10. *Let $A \subseteq \mathbb{N}$ and let $M = \mathcal{SPM}(A)$. Then A is a minimal $\mathcal{SPM}$-system of generators of M if and only if $a \notin \mathcal{SPM}(A \setminus \{a\})$ for all $a \in A$.*

Proof. Assume that $a \in \mathcal{SPM}(A \setminus \{a\})$. Then $A \subseteq \mathcal{SPM}(A \setminus \{a\})$ and $M = \mathcal{SPM}(A) \subseteq \mathcal{SPM}(\mathcal{SPM}(A \setminus \{a\})) = \mathcal{SPM}(A \setminus \{a\})$, contradicting that A is a minimal $\mathcal{SPM}$-system of generators.

If A is not a minimal $\mathcal{SPM}$-system of generators, then $M = \mathcal{SPM}(B)$ for some $B \subsetneq A$. Let $a \in A \setminus B$. Then $a \in M = \mathcal{SPM}(B) \subseteq \mathcal{SPM}(A \setminus \{a\})$. □

The following result is the key to prove the uniqueness of minimal $\mathcal{SPM}$-systems of generators.

Lemma 11. *Let $A = \{a_1 < a_2 < \cdots\}$ be a set of positive integers. If x is in $\mathcal{SPM}(A)$ and $x < a_k$, then $x \in \mathcal{SPM}(\{a_1, \ldots, a_{k-1}\})$.*

Proof. Assume to the contrary that $x \notin \mathcal{SPM}(\{a_1, \ldots, a_{k-1}\})$. Then there exists a proportionally modular numerical semigroup T such that $\{a_1, \ldots, a_{k-1}\} \subset T$ and $x \notin T$. In view of Lemma 5, there exist positive rational numbers $1 < \alpha < \beta$ such that $T = \mathrm{S}([\alpha,\beta])$. As $x \notin T$, from Lemma 6, there exists a positive integer d such that

$$\frac{x}{d+1} < \alpha < \beta < \frac{x}{d}.$$

Hence $\{a_1, \ldots, a_{k-1}\} \subset T \subseteq \mathrm{S}(]\frac{x}{d+1}, \frac{x}{d}[)$. Moreover, from Lemma 7, we deduce that for every $a \in A$ such that $a > x$, we have that $a \in \mathrm{S}(]\frac{x}{d+1}, \frac{x}{d}[)$. Thus $x \notin \mathrm{S}(]\frac{x}{d+1}, \frac{x}{d}[)$ and $A \subseteq \mathrm{S}(]\frac{x}{d+1}, \frac{x}{d}[)$, contradicting that $x \in \mathcal{SPM}(A)$ (Lemma 2). □

In the above lemma, if $k = 1$, then $x < a_1$. This implies that $x = 0$ because $\{0, a_1, \rightarrow\}$ is proportionally modular (this fact is easy to prove, but can be seen as a particular case of [5, Theorem 16]; the arrow means that every integer greater than

a_1 is in the set). And 0 is in $\mathcal{SPM}(\emptyset) = \{0\}$, the intersection of all proportionally modular numerical semigroups.

Theorem 12. *If A and B are minimal $\mathcal{SPM}$-systems of generators of the submonoid M of $\mathbb{N}$, then $A = B$.*

Proof. Assume that $A \neq B$, and that $A = \{a_1 < a_2 < \cdots\}$ and $B = \{b_1 < b_2 < \cdots\}$. Let $i = \min\{k \mid a_k \neq b_k\}$ and suppose without loss of generality that $a_i < b_i$ (this minimum exists, because $A \neq B$). Since $a_i \in M = \mathcal{SPM}(A) = \mathcal{SPM}(B)$, by Lemma 11, $a_i \in \mathcal{SPM}(\{b_1, \ldots, b_{i-1}\})$. However, $\{b_1, \ldots, b_{i-1}\} = \{a_1, \ldots, a_{i-1}\}$, which implies that $a_i \in \mathcal{SPM}(\{a_1, \ldots, a_{i-1}\})$, contradicting that A was a minimal $\mathcal{SPM}$-system of generators of S (Lemma 10). □

If M is a $\mathcal{SPM}$-semigroup, then from this theorem we know that M admits a unique minimal $\mathcal{SPM}$-system of generators. We will refer to the cardinality of this set as the $\mathcal{SPM}$-*dimension* of M. Note that if $\{n_1, \ldots, n_p\}$ is the minimal system of generators of M, then $\mathcal{SPM}(\{n_1, \ldots, n_p\}) = M$. Thus the embedding dimension is greater than or equal to the $\mathcal{SPM}$-dimension. This implies that describing $\mathcal{SPM}$-semigroups by their minimal $\mathcal{SPM}$-system of generators is cheaper than by their minimal system of generators.

4. The tree of all system proportionally modular numerical semigroups

In this section we are bound to construct the tree of all system proportionally modular numerical semigroups. The underlying idea is the following. If S is a numerical semigroup other than $\mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is again a numerical semigroup. If this numerical semigroup is not $\mathbb{N}$, then we can repeat the same process. After a finite number of steps we reach $\mathbb{N}$. More precisely, there exists a chain of numerical semigroups

$$S = S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{t-1} \subseteq S_t = \mathbb{N},$$

where $S_{i+1} = S_i \cup \{\mathrm{F}(S_i)\}$ for all $i \in \{1, \ldots, t-1\}$. We first prove that if S is system proportionally modular, then this chain consists of system proportionally modular numerical semigroups. This would allow us to move downwards the tree from every node to the root $\mathbb{N}$. If we want to move upwards, that is, constructing sons instead of parents, then we must check whether $S \setminus \{n\}$ is a system proportionally modular numerical semigroup, with $n \in S$ and S a system proportionally modular numerical semigroup, and so that this operation is the dual of adding the Frobenius number. We show that $S \setminus \{n\}$ is system proportionally modular provided that n is in the minimal $\mathcal{SPM}$-system of generators of S. Thus for constructing the sons of a node S in the tree, we only have to remove from the node those elements greater than $\mathrm{F}(S)$ belonging to the minimal $\mathcal{SPM}$-system of generators of S (we impose the condition of being greater than the Frobenius number, so that the resulting semigroups are indeed sons of the starting node).

Lemma 13. *Let S be a proportionally modular numerical semigroup with Frobenius number $g \neq -1$, that is, $S \neq \mathbb{N}$. Then $S \cup \{g\}$ is system proportionally modular.*

Proof. Assume that S is generated by $\{n_1, \ldots, n_p\}$. Since S is proportionally modular, $S = \mathrm{S}([\alpha,\beta])$ for some real numbers α,β with $1 < \alpha < \beta$ (Lemma 5). By Lemma 6, for every $i \in \{1, \ldots, p\}$, there exists $d_i \in \{1, \ldots, n_i - 1\}$ such that $\frac{n_i}{d_i} \in [\alpha,\beta]$. Assume that after rearranging the generators (if needed), we have that

$$\frac{n_1}{d_1} < \cdots < \frac{n_p}{d_p}.$$

Then $\mathrm{S}([\frac{n_1}{d_1}, \frac{n_p}{d_p}]) \subseteq \mathrm{S}([\alpha,\beta]) = S$, and as $n_1, \ldots, n_p \in \mathrm{S}([\frac{n_1}{d_1}, \frac{n_p}{d_p}])$ (Lemma 6 again), we deduce that $S = \mathrm{S}([\frac{n_1}{d_1}, \frac{n_p}{d_p}])$. Since $g \notin S$, by using Lemma 6 once more, there exists $d \in \mathbb{N}$ such that

$$\frac{g}{d+1} < \frac{n_1}{d_1} < \cdots < \frac{n_p}{d_p} < \frac{g}{d}.$$

If $d = 0$, then $g < \frac{n_i}{d_i}$ and thus $g < n_i$ for all $i \in \{1, \ldots, p\}$. Hence $g < s$ for all $s \in S \setminus \{0\}$, which means that $S = \{0, g+1, \rightarrow\}$. In this setting $S \cup \{g\} = \{0, g, \rightarrow\}$, which is proportionally modular and thus system proportionally modular.

Now assume that $d \neq 0$. We prove that $S \cup \{g\} = \mathrm{S}([\frac{g}{d+1}, \frac{n_p}{d_p}]) \cap \mathrm{S}([\frac{n_1}{d_1}, \frac{g}{d}])$. The inclusion $S \cup \{g\} \subseteq \mathrm{S}([\frac{g}{d+1}, \frac{n_p}{d_p}]) \cap \mathrm{S}([\frac{n_1}{d_1}, \frac{g}{d}])$ is clear. Now, assume that there exists $x \in \mathrm{S}([\frac{g}{d+1}, \frac{n_p}{d_p}]) \cap \mathrm{S}([\frac{n_1}{d_1}, \frac{g}{d}])$, with $x \notin S \cup \{g\}$. Thus $x < g$ and there exist n and m positive integers such that $\frac{g}{d+1} \leq \frac{x}{n} \leq \frac{n_p}{d_p}$ and $\frac{n_1}{d_1} \leq \frac{x}{m} \leq \frac{g}{d}$ (Lemma 6). As $\frac{x}{n}, \frac{x}{m} \notin [\frac{n_1}{d_1}, \frac{n_p}{d_p}]$ (this would imply that $x \in S$), we have that

$$\frac{g}{d+1} \leq \frac{x}{n} < \frac{n_1}{d_1} < \frac{n_p}{d_p} < \frac{x}{m} \leq \frac{g}{d},$$

which in particular implies that $m < n$. Thus

$$gn \leq xd + x \leq gm + x < gm + g = g(m+1),$$

and this leads to $n < m + 1$. We conclude that $m < n < m + 1$, which is impossible. □

Theorem 14. *Let S be a system proportionally modular numerical semigroup. If $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is also a system proportionally modular numerical semigroup.*

Proof. As S is system proportionally modular, there exist $S_1, \ldots, S_t$ proportionally modular numerical semigroups such that $S = S_1 \cap \cdots \cap S_t$. Let $g = \mathrm{F}(S)$. Then $S \cup \{g\} = (S_1 \cup \{g\}) \cap \cdots \cap (S_t \cup \{g\})$. For $i \in \{1, \ldots, t\}$, if $g \in S_i$, then $S_i \cup \{g\} = S_i$, which is trivially system proportionally modular. If $g \notin S_i$, as $S \subseteq S_i$, this implies that $g = \mathrm{F}(S_i)$. In view of Lemma 13, $S_i \cup \{g\}$ is system proportionally modular. Thus $S \cup \{g\}$ is the intersection of (finitely many) system proportionally modular numerical semigroups, which means that $S \cup \{g\}$ is a system proportionally modular numerical semigroup. □

The dual operation of adding the Frobenius number to a numerical semigroup is that of removing a minimal generator greater than the Frobenius number of the semigroup. If S is a numerical semigroup and $0 \neq n \in S$, then $S \setminus \{n\}$ is a numerical semigroup if and only if n is a minimal generator of S. If in addition S is system proportionally modular, then, in general, $S \setminus \{n\}$ does not have to be system proportionally modular, as explained in the following example. However, there is a natural restriction we can impose on n in order to ensure that $S \setminus \{n\}$ is system proportionally modular.

Example 15. Let $S = \langle 4, 6, 7, 9 \rangle$. This numerical semigroup is system proportionally modular; it is the intersection of $\langle 3, 4 \rangle$ and $\langle 4, 5, 6, 7 \rangle$, which are proportionally modular (the first is generated by two elements, the latter is $\{0, 4, \rightarrow\}$, which is a particular example of [5, Theorem 16]). The semigroup $S \setminus \{6\} = \langle 4, 7, 9, 10 \rangle = \mathrm{S}([\frac{9}{7}, \frac{10}{7}])$ is proportionally modular and thus system proportionally modular. However, $S \setminus \{9\} = \langle 4, 6, 7 \rangle$ is not system proportionally modular (see [5, Example 28]). □

Theorem 16. *Let M be a $\mathcal{SPM}$-semigroup, and let n be one of its minimal generators. The following conditions are equivalent.*

i) *$M \setminus \{n\}$ is a $\mathcal{SPM}$-semigroup.*
ii) *n belongs to the minimal $\mathcal{SPM}$-system of generators of M.*

Proof. Assume that $M \setminus \{n\}$ is a $\mathcal{SPM}$-semigroup and let $A \subseteq M$ be the minimal $\mathcal{SPM}$-system of generators of M. If $n \notin A$, then $A \subseteq M \setminus \{n\}$, which is a $\mathcal{SPM}$-semigroup. Thus $\mathcal{SPM}(A) \subseteq M \setminus \{n\}$, contradicting that $\mathcal{SPM}(A) = M$.

Now assume that n is in the minimal $\mathcal{SPM}$-system of generators of M. We always have the following chain of numerical semigroups

$$M \setminus \{n\} \subseteq \mathcal{SPM}(M \setminus \{n\}) \subseteq M.$$

From the uniqueness of the minimal $\mathcal{SPM}$-system of generators (Theorem 12), $M \neq \mathcal{SPM}(M \setminus \{n\})$. Hence $\mathcal{SPM}(M \setminus \{n\}) = M \setminus \{n\}$. Thus $M \setminus \{n\}$ is a $\mathcal{SPM}$-semigroup. □

Corollary 17. *Let S be a numerical semigroup. The following conditions are equivalent.*

i) *$S = S' \cup \{\mathrm{F}(S')\}$ for some system proportionally modular numerical semigroup S'.*
ii) *S is system proportionally modular and has an element greater that* $\mathrm{F}(S)$ *in its minimal $\mathcal{SPM}$-system of generators.*

Proof. If $S = S' \cup \{\mathrm{F}(S')\}$ with S' a system proportionally modular numerical semigroup, then by Theorem 14, S is system proportionally modular. Moreover, $S \setminus \{\mathrm{F}(S')\} = S'$, which implies that $\mathrm{F}(S')$ is a minimal generator of S and, in view of Theorem 16, $\mathrm{F}(S')$ belongs to the minimal $\mathcal{SPM}$-system of generators of S. Observe that $\mathrm{F}(S') > \mathrm{F}(S)$.

For the converse, let $n > \mathrm{F}(S)$ be in the minimal $\mathcal{SPM}$-system of generators of S. Then by Theorem 16, $S' = S \setminus \{n\}$ is system proportionally modular. Finally, note that as $n > \mathrm{F}(S)$, we have that $\mathrm{F}(S') = n$. Clearly, $S = S' \cup \{\mathrm{F}(S')\}$. □

Corollary 17 together with Theorem 14 allow us to construct recursively the set of all system proportionally modular numerical semigroups, starting from $\mathbb{N}$. This construction arranges this set in a tree rooted in $\mathbb{N}$, and the sons of a node S are $S \setminus \{n_1\}, \ldots, S \setminus \{n_r\}$, with $\{n_1, \ldots, n_r\}$ the set of elements in the minimal $\mathcal{SPM}$-system of generators of S greater than $\mathrm{F}(S)$. This node is a leaf if $\{n_1, \ldots, n_r\}$ is empty.

This construction requires the computation of minimal $\mathcal{SPM}$-system of generators, which will be the subject of next section. An explicit construction will be given in Example 25.

5. Computing minimal $\mathcal{SPM}$-systems of generators

In this section we give an easy procedure to compute the minimal $\mathcal{SPM}$-system of generators for a $\mathcal{SPM}$-semigroup and present some of its properties. If $n_1 < \cdots < n_p$ are positive integers, then in view of Lemma 11, $\{n_1 < n_2 < \cdots < n_p\}$ is a minimal $\mathcal{SPM}$-system of generators of S if and only if $n_k \notin \mathcal{SPM}(\{n_1, \ldots, n_{k-1}\})$ for all $k \in \{2, \ldots, p\}$. Thus the following algorithm computes the minimal $\mathcal{SPM}$-system of generators for $\mathcal{SPM}(\{n_1, \ldots, n_p\})$.

Algorithm 18.

Input: *Positive integers $n_1 < \cdots < n_p$.*
Output: *The minimal $\mathcal{SPM}$-system of generators of $\mathcal{SPM}(\{n_1, \cdots, n_p\})$.*

> $A = \{n_1\}$.
> *For k from* 1 *to p do*
>   *if $n_k \notin \mathcal{SPM}(A)$, then $A := A \cup \{n_k\}$.*
> *Return A.*

Thus the problem reduces to determine a procedure for deciding whether or not a positive integer is in the $\mathcal{SPM}(A)$ for a given finite set A. The key to this procedure is given in the following result.

Proposition 19. *Let $x, n_1, \ldots, n_p$ be positive integers. The following conditions are equivalent:*

1) $x \notin \mathcal{SPM}(\{n_1, \ldots, n_p\})$,
2) there exists $k \in \{0, \ldots, x-1\}$ and $d_1, \ldots, d_r$ positive integers such that $d_i \in \{1, \ldots, n_i - 1\}$ and

$$\frac{x}{k+1} < \frac{n_1}{d_1}, \ldots, \frac{n_p}{d_p} < \frac{x}{k}.$$

Proof. Let $M = \mathcal{SPM}(\{n_1, \ldots, n_p\})$.

1) implies 2). As $x \notin M$, in view of Lemma 2, there exists a proportionally modular numerical semigroup T such that $x \notin T$ and $\{n_1, \ldots, n_p\} \subset T$. Let α, β be rational numbers such that $T = \mathrm{S}([\alpha, \beta])$ and $1 < \alpha < \beta$ (Lemma 5). We use a similar argument to the one appearing in the proof of Lemma 13.

- Since $x \notin T$, there exists $k \in \mathbb{N}$ with $\frac{x}{k+1} < \alpha < \beta < \frac{x}{k}$. This in particular implies that $0 \leq k < x$.

- From $\{n_1, \ldots, n_p\} \subset \mathrm{S}([\alpha,\beta])$ and Lemma 6, we deduce that there exist $d_1, \ldots, d_p \in \mathbb{N} \setminus \{0\}$ with $\alpha \leq \frac{n_i}{d_i} \leq \beta$ for all $i \in \{1, \ldots, p\}$. From these equalities we deduce that $d_i \in \{1, \ldots, n_i - 1\}$ for all $i \in \{1, \ldots, p\}$.

Thus
$$\frac{x}{k+1} < \alpha \leq \frac{n_1}{d_1}, \ldots, \frac{n_p}{d_p} \leq \beta < \frac{x}{k}.$$

2) implies 1). Let $T = \mathrm{S}(]\frac{x}{k+1}, \frac{x}{k}[)$. By Lemma 7, T is a proportionally modular numerical semigroup such that $x \notin T$. From Lemma 6, we deduce that $\{n_1, \ldots, n_p\} \subset T$. Thus $x \notin M$. □

From this result and taking into account that for n, x, d positive integers and k a nonnegative integer
$$\frac{x}{k+1} < \frac{n}{d} < \frac{x}{k} \quad \text{if and only if} \quad k\frac{n}{x} < d < (k+1)\frac{n}{x},$$
the correctness of the next algorithm follows easily.

Algorithm 20.

Input: *Positive integers x and $n_1 < \cdots < n_p$.*

Output: *Returns true if $x \in \mathcal{SPM}(\{n_1, \ldots, n_p\})$; false otherwise.*

> *If $x < n_1$, return false.*
> $Q := \{\frac{n_i}{x} \mid i \in \{1, \ldots, p\}\}$.
> *For k from 1 to $x - 1$ do*
> > *if for all $q \in Q$ there exists an integer in $]kq, (k+1)q[$, then return false.*
>
> *Return true.*

Let S be a numerical semigroup. The set of gaps of S is $\mathrm{H}(S) = \mathbb{N} \setminus S$. The $\mathcal{SPM}$-closure of S, $\mathcal{SPM}(S)$, is a numerical semigroup containing S. Assume that $\{n_1, \ldots, n_p\}$ is a system of generators of S. Then $\mathcal{SPM}(S) = \mathcal{SPM}(\{n_1, \ldots, n_p\})$. Thus we can use Algorithm 20 in order to check which elements of $\mathrm{H}(S)$ are in $\mathcal{SPM}(S)$. In this way we know the set $\mathrm{H}(\mathcal{SPM}(S))$, and thus $\mathcal{SPM}(S)$.

Example 21. Let $S = \langle 4, 6, 7 \rangle$. Then $\mathrm{H}(S) = \{1, 2, 3, 5, 9\}$. Among these gaps, only $9 \in \mathcal{SPM}(S)$. Hence $\mathrm{H}(\mathcal{SPM}(S)) = \{1, 2, 3, 5\}$ and thus $\mathcal{SPM}(S) = S \cup \{9\} = \langle 4, 6, 7, 9 \rangle$.

Next we will show that for proportionally modular numerical semigroups, the concepts of minimal $\mathcal{SPM}$-systems of generators and minimal systems of generators coincide. First we need a lemma, which can be deduced from [6, Corollary 26].

Lemma 22. *Let S be a proportionally modular numerical semigroup minimally generated by $\{n_1 < \cdots < n_p\}$. Then $\langle n_1, \ldots, n_k \rangle$ is a proportionally modular numerical semigroup for all $k \in \{2, \ldots, p\}$.*

Proposition 23. *For proportionally modular numerical semigroups, the concepts of minimal $\mathcal{SPM}$-system and minimal system of generators are the same.*

Proof. Let S be a proportionally modular numerical semigroup minimally generated by $\{n_1 < \cdots < n_p\}$. By applying Algorithm 18 to $\{n_1, \ldots, n_p\}$, we obtain that a minimal $\mathcal{SPM}$-system of generators of $\mathcal{SPM}(\{n_1, \ldots, n_p\})$ is $\{n_1, \ldots, n_p\}$, since $\mathcal{SPM}(\{n_1, \ldots, n_k\}) = \langle n_1, \ldots, n_k \rangle$ by Lemma 22 for all $k \in \{2, \ldots, p\}$, and for $k = 1$ we already know that the equality also holds. $\square$

As a consequence of Theorem 16 and Proposition 23, since $S \setminus \{n_{i_1}, \ldots, n_{i_r}\} = S \setminus \{n_{i_1}\} \cap \cdots \cap S \setminus \{n_{i_r}\}$, we obtain the following.

Corollary 24. *Let S be a proportionally modular numerical semigroup with minimal system of generators $\{n_1, \ldots, n_p\}$. Then $S \setminus \{n_{i_1}, \ldots, n_{i_r}\}$ is a system proportionally modular numerical semigroup for all $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, p\}$.*

Example 25. Now that we know how to compute minimal $\mathcal{SPM}$-systems of generators, we illustrate how to construct the tree $\mathcal{T}$ of all system proportionally modular numerical semigroups.

Starting from $\mathbb{N} = \langle 1 \rangle$ and removing 1 from it, we obtain $\langle 2, 3 \rangle$. And this yields $\langle 2, 3 \rangle \setminus \{2\} = \langle 3, 4, 5 \rangle$ and $\langle 2, 3 \rangle \setminus \{3\} = \langle 2, 5 \rangle$. Now, if we focus on $\langle 3, 4, 5 \rangle$, we obtain three new system proportionally numerical semigroups (every minimal generator is greater than 2, the Frobenius number): $\langle 4, 5, 6, 7 \rangle$, $\langle 3, 5, 7 \rangle$ and $\langle 3, 4 \rangle$. Note that $\langle 3, 4 \rangle$ is a "leaf" of $\mathcal{T}$, since it has no generator greater than 5 (its Frobenius number).

We can keep repeating this process, and so for instance,

- $\langle 2, 5 \rangle$ yields $\langle 2, 7 \rangle$, which produces $\langle 2, 9 \rangle$ and so on;
- $\langle 4, 5, 6, 7 \rangle$ has "sons" $\langle 5, 6, 7, 8, 9 \rangle$, $\langle 4, 6, 7, 9 \rangle$, $\langle 4, 5, 7 \rangle$ and the leaf $\langle 4, 5, 6 \rangle$;
- from $\langle 3, 5, 7 \rangle$, we obtain $\langle 3, 7, 8 \rangle$ and $\langle 3, 5 \rangle$ (which is also a leaf of $\mathcal{T}$).

$$\begin{tikzcd}
\mathbb{N} \arrow[r] & \langle 2,3 \rangle \arrow[r] \arrow[dr] & \langle 2,5 \rangle \arrow[r] & \langle 2,7 \rangle \arrow[r] & \langle 2,9 \rangle \arrow[r] & \cdots \\
 & & \langle 3,4,5 \rangle \arrow[r] \arrow[ddddr] \arrow[ddddddr] & \langle 4,5,6,7 \rangle \arrow[r] \arrow[dr] \arrow[ddr] \arrow[dddr] & \langle 5,6,7,8,9 \rangle \arrow[r] & \cdots \\
 & & & & \langle 4,6,7,9 \rangle \arrow[r] & \cdots \\
 & & & & \langle 4,5,7 \rangle \arrow[r] & \cdots \\
 & & & & \langle 4,5,6 \rangle & \\
 & & & \langle 3,5,7 \rangle \arrow[r] \arrow[dr] & \langle 3,7,8 \rangle \arrow[r] & \cdots \\
 & & & & \langle 3,5 \rangle & \\
 & & & \langle 3,4 \rangle & &
\end{tikzcd}$$

If S is a system proportionally modular numerical semigroup with $\mathrm{e}(S) = 3$, then its minimal $\mathcal{SPM}$-system of generators agrees with its minimal system of generators (see Remark 28 below). Numerical semigroups generated by two elements and, more generally, by arithmetic progressions are proportionally modular (see [5, Theorem 16]). This is why for example $\langle 4,5,6,7\rangle$ has four sons.

Note that $S = \langle 4,6,7,9\rangle$ has only three sons, namely $S\setminus\{4\} = \langle 6,7,8,9,10,11\rangle$, $S\setminus\{6\} = \langle 4,7,9,10\rangle$ and $S\setminus\{7\} = \langle 4,6,9,11\rangle$, since the minimal $\mathcal{SPM}$-system of generators of S is $\{4,6,7\}$. As expected, $\langle 4,6,7\rangle = \langle 4,6,7,9\rangle\setminus\{9\}$ is not system proportionally modular numerical semigroup. $\square$

6. $\mathcal{SPM}$-closure of a numerical semigroup

Let A be a subset of $\mathbb{N}$, we prove next that $\mathcal{SPM}(A) = \gcd(A)\mathcal{SPM}(A/\gcd(A))$ (where by rX we mean $\{rx \mid x\in X\}$, for r any rational number and X any subset of $\mathbb{N}$). That is, for computing $\mathcal{SPM}$-closures, it suffices to compute the $\mathcal{SPM}$-closure of a numerical semigroup.

Proposition 26. *Let A be a subset of $\mathbb{N}$ with $\gcd(A) = 1$ and let d be an integer greater than one. Then $\mathcal{SPM}(dA) = d\mathcal{SPM}(A)$.*

Proof. Let T be a proportionally modular numerical semigroup containing dA. From Lemma 5, we know that there exist rational numbers $1 < \alpha < \beta$ such that $T = \mathrm{S}([\alpha,\beta])$. Then for all $da \in dA$, there exists a positive integer k such that $\alpha \le \frac{da}{k} \le \beta$, or equivalently, $\frac{\alpha}{d} \le \frac{a}{k} \le \frac{\beta}{d}$. Thus $A \subseteq \mathrm{S}([\frac{\alpha}{d},\frac{\beta}{d}])$. Hence $\mathcal{SPM}(A) \subseteq \mathrm{S}([\frac{\alpha}{d},\frac{\beta}{d}])$. Arguing as above, and unwinding the process, we deduce that $d\mathcal{SPM}(A) \subseteq \mathrm{S}([\alpha,\beta]) = T$. This proves that $d\mathcal{SPM}(A) \subseteq \mathcal{SPM}(dA)$. For the other inclusion, observe that in view of the proof of Proposition 3, $\mathcal{SPM}(dA) \subseteq \langle d\rangle$. Thus every element in $\mathcal{SPM}(dA)$ is of the form dx for some positive integer x. Take $dx \in \mathcal{SPM}(dA)$. We show that $x \in \mathcal{SPM}(A)$. Let $T = \mathrm{S}([\alpha,\beta])$, with $1 < \alpha < \beta$ be a proportionally modular numerical semigroup containing A. Then $dA \subseteq \mathrm{S}([d\alpha, d\beta])$, whence $\mathcal{SPM}(dA) \subseteq \mathrm{S}([d\alpha, d\beta])$, and thus $dx \in \mathrm{S}([d\alpha, d\beta])$. From this it is easy to deduce that $x \in \mathrm{S}([\alpha,\beta]) = T$, and this occurs for every T proportionally modular numerical semigroup containing A. We conclude that $x \in \mathcal{SPM}(A)$. $\square$

Remark 27. From this result, as we know that every numerical semigroup of embedding dimension two is proportionally modular, we deduce that every submonoid of $\mathbb{N}$ with embedding dimension two is a $\mathcal{SPM}$-semigroup. Let n_1 and n_2 be two positive integers with $\gcd\{n_1,n_2\} = d$. Then

$$\mathcal{SPM}(\{n_1,n_2\}) = \mathcal{SPM}(d\{\frac{n_1}{d},\frac{n_2}{d}\}) = d\mathcal{SPM}(\langle\frac{n_1}{d},\frac{n_2}{d}\rangle) = d\langle\frac{n_1}{d},\frac{n_2}{d}\rangle = \langle n_1,n_2\rangle.$$

Remark 28. Let M be a $\mathcal{SPM}$-semigroup minimally generated by $\{n_1 < \cdots < n_p\}$, with $p \ge 3$. Since $\mathcal{SPM}(\{n_1\}) = \langle n_1\rangle$, $\mathcal{SPM}(\{n_1,n_2\}) = \langle n_1,n_2\rangle$, and neither $n_2 \in \langle n_1\rangle$ nor $n_3 \in \langle n_1,n_2\rangle$, in view of Algorithm 18, we deduce that n_1, n_2 and n_3 are always in the minimal $\mathcal{SPM}$-system of generators of M.

Next we are concerned with the problem of computing $S_1,\ldots,S_r$ proportionally modular numerical semigroups such that $\mathcal{SPM}(S) = S_1\cap\cdots\cap S_r$.

We first need to recall some definitions. Let

$$\mathrm{EH}(S) = \{x \in \mathrm{H}(S) \mid S\cup\{x\} \text{ is a numerical semigroup}\}.$$

It is easy to prove that

$$\mathrm{EH}(S) = \{x \in \mathrm{H}(S) \mid 2x \in S,\ x + s \in S \text{ for all } s \in S \setminus \{0\}\}.$$

Assume that S is minimally generated by $\{n_1, \dots, n_p\}$. Let B_S be as in [5, Algorithm 27], that is,

$$B_S = \{(x, y) \mid x \in (\mathrm{EH}(S) \setminus \{1\}) \cup \{n_1, \dots, n_p\},\ 1 \leq y < x\}.$$

Consider B_S as a list ordered in the following way: $(x, y) < (x', y')$ if $\frac{x}{y} < \frac{x'}{y'}$ or $\frac{x}{y} = \frac{x'}{y'}$ and $x < x'$. Thus, B_S can be expressed in the form

$$B_S = ((x_1, y_1), \dots, (x_m, y_m)).$$

We say that a segment $\ell = ((x_q, y_q), \dots, (x_r, y_r))$ of B is *closed* if

- $q = 1$ or $\frac{x_{q-1}}{y_{q-1}} \neq \frac{x_q}{y_q}$, and
- $r = m$ or $\frac{x_r}{y_r} \neq \frac{x_{r+1}}{y_{r+1}}$.

For the segment $\ell = ((x_q, y_q), \dots, (x_r, y_r))$, write $\pi_1(\ell) = \{x_q, \dots, x_r\}$.

If $\ell = ((x_q, y_q), \dots, (x_r, y_r))$ is a closed segment of B_S, set $S_\ell = \mathrm{S}([\frac{x_q}{y_q}, \frac{x_r}{y_r}])$, which is a proportionally modular numerical semigroup. Clearly, in view of Lemma 6, $\pi_1(\ell) \subset S_\ell$.

Lemma 29. *Under the standing hypothesis, if T is a proportionally modular numerical semigroup containing S, then there exists a closed segment ℓ of B_S such that $\{n_1, \dots, n_p\} \subseteq \pi_1(\ell)$ and $S_\ell \subseteq T$.*

Proof. We use an argument similar to the one used in the proof of Proposition 19. From Lemma 5, there exist rational numbers α and β with $1 < \alpha < \beta$ such that $T = \mathrm{S}([\alpha, \beta])$. As $S \subseteq T$, by Lemma 6, for every $i \in \{1, \dots, p\}$ there exists a positive integer d_i such that $\alpha \leq \frac{n_i}{d_i} \leq \beta$. Note that this forces $1 \leq d_i < n_i$, whence the pair $(n_i, d_i) \in B_S$. Let $\ell = ((x_q, y_q), \dots, (x_r, y_r))$ be the least closed segment containing $\{(n_1, d_1), \dots, (n_p, d_p)\}$. Then $\alpha \leq \frac{x_q}{y_q} = \frac{n_i}{d_i}$ and $\frac{x_r}{y_r} = \frac{n_j}{d_j} \leq \beta$ for some $i, j \in \{1, \dots, p\}$. Hence $S_\ell \subseteq T$, and $\{n_1, \dots, n_p\} \subset \pi_1(\ell)$ follows from the way we have chosen ℓ. □

Thus we obtain the following consequence.

Corollary 30. *If T is a minimal (with respect to set inclusion) proportionally modular numerical semigroup containing the numerical semigroup S, then there exists a closed segment of B_S such that $T = S_\ell$.*

With this, it is straightforward to prove the next result, which gives an easy method to compute an expression of $\mathcal{SPM}(S)$ as intersection of finitely many proportionally modular numerical semigroups.

Proposition 31. *Let S be a numerical semigroup minimally generated by $\{n_1, \dots, n_p\}$. Let $\ell_1, \dots, \ell_r$ be those closed segments of B_S such that $\{n_1, \dots, n_p\} \subseteq \pi_1(\ell_i)$. Then*

$$\mathcal{SPM}(S) = S_{\ell_1} \cap \cdots \cap S_{\ell_r}.$$

Remark 32. Among those closed segments referred in the proposition above, we only need to consider those that are minimal with respect to set inclusion.

Example 33. Let $S = \langle 4, 6, 7\rangle$. Then $\mathrm{EH}(S) = \{9\}$ and the list B_S is

$$((9,8),(7,6),(6,5),(9,7),(4,3),(7,5),(6,4),(9,6),(7,4),(9,5),(4,2),$$
$$(6,3),(9,4),(7,3),(6,2),(9,3),(7,2),(4,1),(9,2),(6,1),(7,1),(9,1)).$$

Using the above remark, we obtain

(1) $\ell_1 = ((7,6),\dots,(4,3))$, and $S_{\ell_1} = \mathrm{S}([\frac{7}{6},\frac{4}{3}])$;
(2) $\ell_2 = ((6,5),\dots,(7,5))$, and $S_{\ell_2} = \mathrm{S}([\frac{6}{5},\frac{7}{5}])$;
(3) $\ell_3 = ((4,3),\dots,(9,6))$, and $S_{\ell_3} = \mathrm{S}([\frac{4}{3},\frac{9}{6}])$;
(4) $\ell_4 = ((7,4),\dots,(6,3))$, and $S_{\ell_4} = \mathrm{S}([\frac{7}{4},\frac{6}{3}])$;
(5) $\ell_5 = ((4,2),\dots,(7,3))$, and $S_{\ell_5} = \mathrm{S}([\frac{4}{2},\frac{7}{3}])$;
(6) $\ell_6 = ((6,2),\dots,(4,1))$, and $S_{\ell_6} = \mathrm{S}([\frac{6}{2},\frac{4}{1}])$;
(7) $\ell_7 = ((7,2),\dots,(6,1))$, and $S_{\ell_7} = \mathrm{S}([\frac{7}{2},\frac{6}{1}])$;
(8) $\ell_8 = ((4,1),\dots,(7,1))$, and $S_{\ell_8} = \mathrm{S}([\frac{4}{1},\frac{7}{1}])$.

Then

$$S_{\ell_1} = S_{\ell_2} = S_{\ell_7} = S_{\ell_8} = \langle 4,5,6,7\rangle,$$
$$S_{\ell_3} = S_{\ell_6} = \langle 3,4\rangle,$$

and

$$S_{\ell_4} = S_{\ell_5} = \langle 2,7\rangle.$$

For these semigroups it is not too difficult to obtain a minimal system of generators (one can just use Lemma 6). For more complicated ones, one can use the algorithm described in [6]. Even for this case, we have used the implementation of this algorithm done in the GAP [10] package NumericalSgps [1].

Thus

$$\mathcal{SPM}(S) = \mathrm{S}([4,7]) \cap \mathrm{S}([3,4]) \cap \mathrm{S}([2,\frac{7}{3}]) = \langle 4,6,7,9\rangle.$$

This decomposition is not minimal since

$$\mathcal{SPM}(S) = \mathrm{S}([4,7]) \cap \mathrm{S}([3,4]) = \mathrm{S}([4,7]) \cap \mathrm{S}([2,\frac{7}{3}])$$
$$=< 4,5,6,7 > \cap < 3,4 >=< 4,5,6,7 > \cap < 2,7 > .$$

This example also stresses out that minimal decompositions do not have to be unique. □

Encoded in B_S there is still more information that can be used for instance to compute the minimal $\mathcal{SPM}$-system of generators of $\mathcal{SPM}(S)$, and thus for $\mathcal{SPM}(\{n_1,\dots,n_p\})$. Next we show how to achieve this, giving in this way an alternative procedure to the one explained in Section 5.

Lemma 34. *Let S be a numerical semigroup and let T be a submonoid of S. Then* $\min(S \setminus T)$ *is a minimal generator of S.*

Proof. Let $n = \min(S \setminus T)$. Assume that $n = s_1 + s_2$ with $s_1, s_2 \in S \setminus \{0\}$. Then s_1 and s_2 are smaller than n, and the minimality of n implies that both s_1 and s_2 are in T. But this is impossible, since T is a monoid and thus $s_1 + s_2 = s \in T$. □

Proposition 35. *Let S be a system proportionally modular numerical semigroup minimally generated by $\{n_1, \ldots, n_p\}$. Let $A \subseteq \{n_1, \ldots, n_p\}$. The following conditions are equivalent.*

1) $\mathcal{SPM}(A) = S$.

2) For every closed segment ℓ of B_S such that $A \subseteq \pi_1(\ell)$, we have that $\{n_1, \ldots, n_p\} \subseteq \pi_1(\ell)$.

Proof. 1) implies 2). Let $\ell = ((x_q, y_q), \ldots, (x_r, y_r))$ be a closed segment of B_S such that $A \subseteq \{x_q, \ldots, x_r\}$. Then by Lemma 6, $\mathcal{SPM}(A) \subseteq S_\ell$. As $\mathcal{SPM}(A) = S$, this implies that $S \subseteq S_\ell$. But then there are $d_i \in \{1, \ldots, n_i - 1\}$ with $\frac{n_i}{d_i} \in [\frac{x_q}{y_q}, \frac{x_r}{y_r}]$. From the way B_S is constructed, this leads to $(n_i, d_i) \in \ell$ for all $i \in \{1, \ldots, p\}$.

2) implies 1). Clearly, $\mathcal{SPM}(A) \subseteq S$. Assume that $\mathcal{SPM}(A) \neq S$, and let $n = \min(S \setminus \mathcal{SPM}(A))$. Then by the preceding lemma $n \in \{n_1, \ldots, n_p\}$. Hence, there exists a proportionally modular numerical semigroup T with $A \subset T$ and $n \notin T$. In view of Lemma 5, there exist two rational numbers α and β with $1 < \alpha < \beta$ and $T = \mathrm{S}([\alpha, \beta])$. From Lemma 6, we deduce that if $A = \{n_{i_1}, \ldots, n_{i_k}\}$, then for all $j \in \{1, \ldots, k\}$, there exists $d_{i_j} \in \{1, \ldots, n_{i_j} - 1\}$ such that $\alpha \leq \frac{n_{i_j}}{d_{i_j}} \leq \beta$. Take $\ell = ((x_q, y_q), \ldots, (x_r, y_r))$ to be a closed segment of B_S containing $\{(n_{i_1}, d_{i_1}), \ldots, (n_{i_k}, d_{i_k})\}$ and with $\frac{x_q}{y_q}, \frac{x_r}{y_r} \in \{\frac{n_{i_1}}{d_{i_1}}, \ldots, \frac{n_{i_k}}{d_{i_k}}\}$. Then $A \subseteq \pi_1(\ell)$ and from the hypothesis we deduce that $n \in \pi_1(\ell)$. If $\gamma = \min\{\frac{n_{i_1}}{d_{i_1}}, \ldots, \frac{n_{i_k}}{d_{i_k}}\}$ and $\delta = \max\{\frac{n_{i_1}}{d_{i_1}}, \ldots, \frac{n_{i_k}}{d_{i_k}}\}$, then $n \in \pi_1(\ell) \subset \mathrm{S}([\gamma, \delta]) \subseteq T$, a contradiction. □

Example 36. Let $S = \langle 4, 6, 7, 9 \rangle$. The list B_S is

$$\begin{aligned}((9,8),(7,6),(6,5),(5,4),(9,7),(4,3),(7,5),(3,2),(6,4),(9,6),(5,3),(7,4),\\(9,5),(2,1),(4,2),(6,3),(9,4),(7,3),(5,2),(3,1),(6,2),(9,3),(7,2),(4,1),\\(9,2),(5,1),(6,1),(7,1),(9,1)).\end{aligned}$$

Observe that $A = \{4, 6, 7\}$ is such that for every closed segment ℓ of B_S, if $A \subseteq \pi_1(\ell)$, then $9 \in \pi_1(\ell)$. Thus $\mathcal{SPM}(\{4, 6, 7\}) = \langle 4, 6, 7, 9 \rangle$. Observe also that for each proper subset A' of $\{4, 6, 7\}$ there exists a closed segment ℓ of B_S such that $A' \subseteq \pi_1(\ell)$ but $\{4, 6, 7, 9\}$ is not contained in $\pi_1(\ell)$. Thus $\{4, 6, 7\}$ is a $\mathcal{SPM}$-minimal system of generators of S. □

7. A minimal decomposition and representation

As we have seen, the method described above does not produce in general a minimal decomposition of $\mathcal{SPM}(S)$ in terms of proportionally modular numerical semigroups. The concept of minimality can be thought in two different ways: the first as a decomposition with the least possible number of factors, and the second, as a decomposition in which no factor is redundant.

Let S be a numerical semigroup. Denote by $\mathcal{P}(S)$ the set of all proportionally modular numerical semigroups containing S. Every decomposition of a system proportionally modular numerical semigroup S can be transformed into another in which any factor belongs to $\mathrm{Minimals}_{\subseteq}(\mathcal{P}(S))$.

Proposition 37. *Let S be a numerical semigroup. If $\mathcal{SPM}(S) = S_1 \cap \cdots \cap S_n$ with $S_1, \ldots, S_n \in \mathcal{P}(S)$, then there exist $S'_1, \ldots, S'_n \in \mathrm{Minimals}_{\subseteq}(\mathcal{P}(S))$ such that $\mathcal{SPM}(S) = S'_1 \cap \cdots \cap S'_n$.*

Proof. For every $i \in \{1, \ldots, n\}$, if S_i is not minimal, then choose S'_i minimal with $S'_i \subseteq S_i$; otherwise set $S_i = S'_i$. Clearly $S'_1 \cap \cdots \cap S'_n \subseteq S_1 \cap \cdots \cap S_n$, and in view of Lemma 2, $\mathcal{SPM}(S) \subseteq S'_1 \cap \cdots \cap S'_n$, which concludes the proof. □

Recall that from Corollary 30,

$$\mathrm{Minimals}_{\subseteq}(\mathcal{P}(S)) \subseteq \{S_\ell \mid \ell \text{ is a closed segment of } B_S\}.$$

The trick to find a minimal decomposition relies in the following result.

Lemma 38. [4, Proposition 25] *Let S be a numerical semigroup and let $S_1, \ldots, S_n$ be oversemigroups of S. The following conditions are equivalent.*

1) $S = S_1 \cap \cdots \cap S_n$.
2) For all $h \in \mathrm{EH}(S)$, there exists $i \in \{1, \ldots, n\}$ such that $h \notin S_i$.

Let S be a numerical semigroup. We already know that $\mathcal{SPM}(S) = S_{\ell_1} \cap \cdots \cap S_{\ell_r}$ for some closed segments ℓ_i of B_S. For every $i \in \{1, \ldots, r\}$ set

$$C_i = \{h \in \mathrm{EH}(\mathcal{SPM}(S)) \mid h \notin S_{\ell_i}\}.$$

Then for $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, r\}$,

$$\mathcal{SPM}(S) = S_{\ell_{i_1}} \cap \cdots \cap S_{\ell_{i_n}}$$

if and only if

$$C_{i_1} \cup \cdots \cup C_{i_n} = \mathrm{EH}(\mathcal{SPM}(S)).$$

Thus a minimal decomposition can be found by choosing $\{i_1, \ldots, i_n\}$ minimal fulfilling this condition, and if we are looking for a decomposition with the least number of factors, then we must choose n minimal.

Example 39. Let $S = \langle 4, 6, 7 \rangle$. Then B_S is

$$((9,8), (7,6), (6,5), (9,7), (4,3), (7,5), (6,4), (9,6), (7,4), (9,5), (4,2), (6,3), (9,4), (7,3), (6,2), (9,3), (7,2), (4,1), (9,2), (6,1), (7,1), (9,1))$$

and $B_{\mathcal{SPM}(S)}$ is

$$((9,8), (7,6), (6,5), (5,4), (9,7), (4,3), (7,5), (3,2), (6,4), (9,6), (5,3), (7,4), (9,5), (2,1), (4,2), (6,3), (9,4), (7,3), (5,2), (3,1), (6,2), (9,3), (7,2), (4,1), (9,2), (5,1), (6,1), (7,1), (9,1)).$$

$\mathrm{EH}(\mathcal{SPM}(S)) = \mathrm{EH}(\langle 4, 6, 7, 9 \rangle) = \{2, 3, 5\}$, whence $C_1 = \{2, 3\}$, $C_2 = \{2, 3\}$ and $C_3 = \{2, 5\}$, which suffices to ensure that

$$\mathcal{SPM}(S) = S_{\ell_1} \cap S_{\ell_3}$$

is a minimal decomposition. □

From the results we have obtained so far, we can also find the least possible number of inequalities describing a system proportionally modular numerical semigroup. We can also decide whether an inequality is or not redundant.

Observe that in the way a decomposition is obtained, in view of Lemma 5, it is easy to find a system proportionally modular representation for the closure of a numerical semigroup. Let S be a numerical semigroup, and let $S_{\ell_1}, \dots, S_{\ell_r}$ be the proportionally modular numerical semigroups such that

$$\mathcal{SPM}(S) = S_{\ell_1} \cap \cdots \cap S_{\ell_r}$$

is a minimal decomposition, computed as shown above. For every $i \in \{1, \dots, r\}$, we know that $S_{\ell_i} = \mathrm{S}([\frac{x_{q_i}}{y_{q_i}}, \frac{x_{r_i}}{y_{r_i}}])$. Set

- $a_i = x_{r_i} y_{q_i}$,
- $b_i = x_{q_i} x_{r_i}$,
- $c_i = x_{r_i} y_{q_i} - x_{q_i} y_{r_i}$.

Then by Lemma 5, S_{ℓ_i} is the set of integer solutions x of the inequality $a_i x \bmod b_i \leq c_i x$. Hence

$$\mathcal{SPM}(S) \equiv \begin{cases} a_1 x \bmod b_1 \leq c_1 x, \\ \quad \dots \\ a_r x \bmod b_r \leq c_r x. \end{cases}$$

Example 40. Let us go back to $S = \langle 4, 6, 7 \rangle$. We already know that $\mathcal{SPM}(S) = \langle 4, 6, 7, 9 \rangle$ and that, for instance, $\mathcal{SPM}(S) = \mathrm{S}([4, 7]) \cap \mathrm{S}([3, 4])$ is a minimal decomposition. Hence $\mathcal{SPM}(S)$ is the set of integer solutions of

$$\begin{cases} 7x \bmod 28 \leq 3x, \\ 4x \bmod 12 \leq x. \end{cases}$$

□

8. Toms' decompositions

Let M be a submonoid of $\mathbb{N}$ and let d be a positive integer. Then

$$\frac{M}{d} = \{n \in \mathbb{N} \mid dn \in M\}$$

is a submonoid of $\mathbb{N}$, called the *quotient of M by d*. Proportionally modular numerical semigroups can be characterized as those that are quotients of numerical semigroups generated by arithmetic progressions [5, Theorem 16]. This characterization is sharpened in [8, Theorem 5], where it is shown that S is a proportionally modular numerical semigroup if and only if S is the quotient of a numerical semigroup with embedding dimension two, that is, there exist n_1, n_2 and d positive integers such that $S = \frac{\langle n_1, n_2 \rangle}{d}$ and $\gcd(\{n_1, n_2\}) = 1$. Thus in view of Lemma 2 we obtain the following result.

Proposition 41. *Let S be a numerical semigroup. The following conditions are equivalent.*

1) S is a system proportionally modular numerical semigroup.

2) S is the intersection of finitely many quotients of numerical semigroups of embedding dimension two.

Let S be a numerical semigroup. According to [2], we say that S has a Toms' decomposition if there exist $q_1, \ldots, q_n$, $m_1, \ldots, m_n$ and L such that

1) $\gcd(\{q_i, m_i\}) = \gcd(\{L, q_i\}) = \gcd(\{L, m_i\}) = 1$ for all $i \in \{1, \ldots, n\}$,
2) $S = \frac{1}{L} \bigcap_{i=1}^{n} \langle q_i, m_i \rangle$.

The importance of numerical semigroups with a Toms' decomposition relies on the following realization property.

Proposition 42. [9, Theorem 1.1] *If S has a Toms' decomposition, then there exists a simple, separable, amenable and unital C^*-algebra with ordered K_0-group isomorphic to $\mathbb{Z}$ with positive cone S.*

Toms in his paper [9] wondered whether or not every numerical semigroup has a Toms' decomposition. Observe that

$$\frac{1}{L} \bigcap_{i=1}^{n} \langle q_i, m_i \rangle = \bigcap_{i=1}^{n} \frac{\langle q_i, m_i \rangle}{L}$$

(this is easy to deduce and was already observed in [2, Lemma 2.2]). Hence from Proposition 41 we deduce the following.

Corollary 43. *Every numerical semigroup having a Toms' decomposition is system proportionally modular.*

We have given in this paper numerical semigroups that are not system proportionally modular, and thus the answer to Toms' question is negative (actually $\langle 4, 6, 7 \rangle$ already appears in [5, Example 28]). In [2] several families of numerical semigroups having Toms' decomposition are given.

In [8] it is shown that if S is proportionally modular, then $S = \mathrm{S}([\frac{a_1}{b_1}, \frac{a_2}{b_2}])$ for some positive integers a_1, b_1, a_2, b_2 with $1 < \frac{a_1}{b_1} < \frac{a_2}{b_2}$ and $\gcd(\{a_1, a_2\}) = 1$ (this condition on the gcd is the main difference with Lemma 5). Moreover, a_1, b_1, a_2, b_2 can be derived from a representation of the semigroup as the set of solutions of the inequality $ax \bmod b \leq cx$. Theorem 5 in [8] then states that $S = \frac{\langle a_1, a_2 \rangle}{d}$, with $d = a_2 b_1 - a_1 b_2$. In some sense, d measures the "size" of the interval $[\frac{a_1}{b_1}, \frac{a_2}{b_2}]$. Thus we could think in Toms' decomposition, as a decomposition in which all intervals are taken to have the same size.

Example 44. Observe that

$$\begin{aligned} \langle 4, 6, 7, 9 \rangle &= \langle 4, 5, 6, 7 \rangle \cap \langle 3, 4 \rangle = \tfrac{\langle 4,7 \rangle}{3} \cap \tfrac{\langle 3,4 \rangle}{1} \\ &= \langle 4, 5, 6, 7 \rangle \cap \langle 2, 7 \rangle = \tfrac{\langle 4,7 \rangle}{3} \cap \tfrac{\langle 2,7 \rangle}{1} \\ &= \langle 2, 7 \rangle \cap \langle 3, 4 \rangle = \tfrac{\langle 2,7 \rangle}{1} \cap \tfrac{\langle 3,4 \rangle}{1} \end{aligned}$$

This last decomposition is a Toms' decomposition.

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