Minimal Coupled Cell Networks

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Abstract

Golubitsky, Stewart and Török (Patterns of Synchrony in Coupled Cell Networks with Multiple Arrows, *SIAM J. Appl. Dynam. Sys.* **4** (1) (2005) 78–100) pointed out that non-isomorphic coupled cell networks can correspond to the same space of admissible vector fields, that is, be ODE-equivalent. Dias and Stewart (Linear Equivalence and ODE-equivalence for Coupled Cell Networks, *Nonlinearity* **18** (2005) 1003–1020) reduced ODE-equivalence to linear equivalence. Here we characterize the canonical normal forms of an ODE-equivalence class – the subclass of networks with minimal number of edges. Moreover, we present an algorithm that computes the canonical normal forms for a given ODE-class. This goes through the calculation of vectors with minimum length contained in a cone of a lattice described in terms of the adjacency matrices of any network in the ODE-class.

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1 Introduction

Mathematically, a coupled cell network can be idealized as a directed graph representing schematically a set of dynamical systems (the cells) that are coupled together, and the couplings among them. The nodes of the graph represent the cells and edges represent couplings. Here we consider that each cell represents a system of ordinary differential equations (ODEs).

Networks appear in many areas of science. For example, many biological systems can be modelled by networks of nonlinear dynamical systems. See Stewart [13] and references therein.

The dynamics of a network of dynamical systems is constrained by the topology of the graph. In particular, new phenomena can be typical for a given architecture, see Golubitsky *et al.* [9]. It follows then that one important aspect in the study of networks of dynamical systems is the understanding of the role of the network architecture in the dynamics.

Following Stewart *et al.* [14] and Golubitsky *et al.* [11], we associate to each coupled cell network a class of ODEs compatible with the structure of the network – the class of *coupled cell systems*. More precisely, to each coupled cell we associate a choice of cell phase space which we assume is a finite dimensional real vector space, the overall phase space of the coupled cell system being the direct product of the cell phase spaces. Cells represented by the same symbol have the same phase space and internal dynamics; different couplings correspond to different edges.

To each cell c we can associate the set of edges with head cell c. Two cells in a network are said to be isomorphic if there is an edge-type preserving isomorphism between their sets of edges. The symmetry groupoid of a network is given by all those isomorphisms between any two isomorphic cells in the network. The structure of a network can thus be described in terms of the symmetry groupoid associated to the network. A vector field corresponding to the coupled cell system is called *admissible* and respects the topology of the network – it is equivariant by its symmetry groupoid. For a survey of this formalism see Golubitsky and Stewart [10]. See also an alternative approach of Field [8].

The number of coupled cell networks grows exponentially with the number of cells and the number of edges. See Aldosray and Stewart [1] for the enumeration of networks in which there is a single type of node and a single type of edge and all the cells are isomorphic.

We follow the definition of coupled cell network introduced by Golubitsky *et al.* [11] where self-coupling and multiarrows are permitted. As pointed in [11], it is possible for two non-isomorphic coupled cell networks to generate the same space of admissible vector fields, that is to be *ODE-equivalent*. See Dias and Stewart [7] for the definition of ODE-equivalent coupled cell networks.

In [7] it is shown that two coupled cell networks are ODE-equivalent if and only if they are linearly equivalent. Basically, the ODE-equivalence reduces to *linear equivalence*, where two networks (with suitably identified phase spaces) are linearly equivalent if they determine the same space of linear admissible vector fields.

As every coupled cell network in a given ODE-equivalence class determines the same dynamical behaviour we look for the set of coupled cell networks in the ODE-class that are more amenable to treat. We aim to find a kind of *canonical normal forms* – a set of networks such that the number of edges is minimal among all the networks of the ODE-equivalence class, which we call the *minimal subclass*.

Using the results of Dias and Stewart [7] on ODE and linear equivalence of networks this problem can be posed in terms of the network adjacency matrices. That is, the problem of finding all the minimal networks in a given ODE-equivalence class corresponds to find all the *minimal bases* (with nonnegative integer entries) of the real vector space generated by the adjacency matrices of the networks in the ODE-class. The minimal bases define the adjacency matrices of the minimal networks.

For the coupled cell network represented in Figure 1 it is an easy linear algebra exercise to prove that the coupled cell network in Figure 2 is the unique minimal network in its ODE-equivalence class.

In general, the problem of finding the minimal subclass gets much more complicated. For example, the minimal subclass of the coupled cell network in Figure 3 is given by the minimal networks represented in Figure 4. In this case, the network has more than one
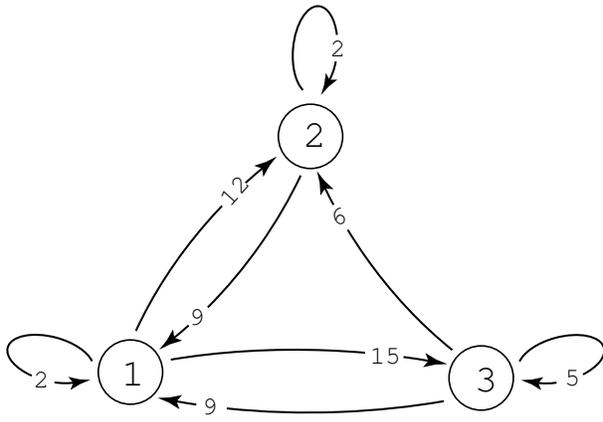
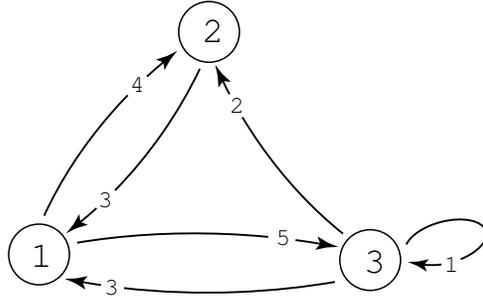Figure 1: Coupled cell network G_1 with one cell-type and one edge-type.

Figure 2: Minimal network G_2 of the ODE-equivalence class of the coupled cell network in Figure 1.

edge-type and the cells are all isomorphic. In general, an adjacency matrix of a network with isomorphic cells is a $\mathbb{R}$-linear combination of the adjacency matrices of any network in its ODE-class. Given a graph with isomorphic cells in an ODE-class, we consider the lattice consisting of the vectors with nonnegative integer entries in the real subspace generated by the adjacency matrices of the graph. The minimal networks are obtained by finding vectors (adjacency matrices) with shortest length in a cone of that lattice (whose rank grows with the number of edge-types).

In this paper, we describe the minimal subclass of a given coupled cell network and present an algorithm that computes it. The minimality of coupled cell networks basically reduces to the minimality of homogeneous networks, that is, networks whose cells are all isomorphic. We start by addressing the case of homogeneous networks (Theorem 8.7 and
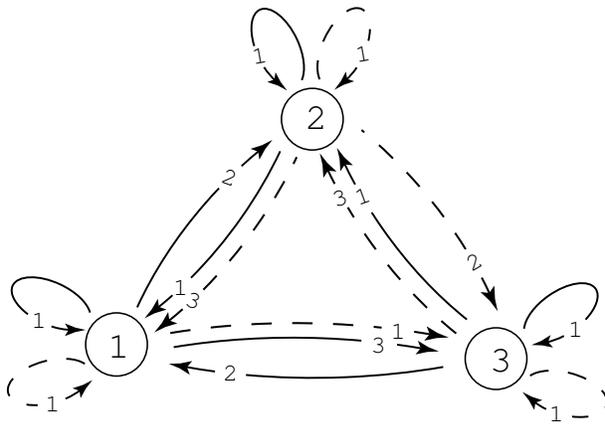
Figure 3: Coupled cell network with more than one edge type.
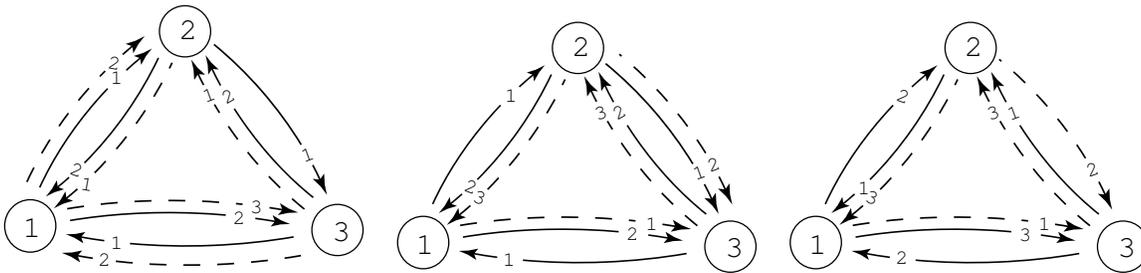
Figure 4: Minimal networks of the ODE-equivalence class of the coupled cell network of Figure 3.

Algorithm 8.8).

The rest of the paper is organized in the following way: Sections 2, 3 and 4 review the basic concepts on coupled cell networks. Section 5 relates ODE and linear equivalence of networks. The definition of minimality of networks appears in Section 6. The minimality of homogeneous networks is addressed in Section 7, answered in Section 8 and illustrated in Section 9. The special case of identical-edge homogeneous networks is treated in Section 10. Finally, in Section 11 we solve the general case using the results of Section 8.

2 Coupled Cell Networks

A coupled cell network can be represented schematically by a directed graph (see for e-xample Tutte [15]) whose nodes correspond to cells and whose edges represent couplings. See for example Figure 1 above. We employ the definition, introduced by Golubitsky *et al.* [11], which permits multiple arrows and self-coupling.

Definition 2.1 **[11]** A *coupled cell network* G consists of:

(a) A finite set $\mathcal{C} = \{1, \cdots, n\}$ of *nodes* or *cells.*

(b) An equivalence relation $\sim_C$ on cells in $\mathcal{C}$.

The *type* or *cell label* of cell c is the $\sim_C$-equivalence class $[c]_C$ of c.

(c) A finite set $\mathcal{E}$ of *edges* or *arrows.*

(d) An equivalence relation $\sim_E$ on edges in $\mathcal{E}$.

The *type* or *coupling label* of egde e is the $\sim_E$-equivalence class $[e]_E$ of e.

(e) Two maps $\mathcal{H}: \mathcal{E} \to \mathcal{C}$ and $\mathcal{T}: \mathcal{E} \to \mathcal{C}$.

For $e \in \mathcal{E}$ we call $\mathcal{H}(e)$ the *head* of e and $\mathcal{T}(e)$ the *tail* of e.

(f) Equivalent edges have equivalent tails and heads. That is, if $e_1, e_2 \in \mathcal{E}$ and $e_1 \sim_E e_2$, then

$$\mathcal{H}(e_1) \sim_C \mathcal{H}(e_2) \qquad \mathcal{T}(e_1) \sim_C \mathcal{T}(e_2)$$

We write $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$. $\diamond$

Remark 2.2 Observe that in this definition of coupled cell network, self-coupling is permitted since $\mathcal{H}(e) = \mathcal{T}(e)$ for $e \in \mathcal{E}$ is permitted. Also multiarrows are permitted since we can have $\mathcal{H}(e_1) = \mathcal{H}(e_2)$ and $\mathcal{T}(e_1) = \mathcal{T}(e_2)$ for $e_1 \neq e_2$. This definition of coupled cell network has several technical advantages over the original version introduced by Stewart *et al.* [14]. See [11] for details. $\diamond$

In a graphical representation of a coupled cell network, identical cells and identical edges are represented, respectively, by the same symbol. Multiple couplings of the same

type between two cells are represented by just one arrow with the number of couplings attached to it. Figures 1 and 2 show examples of coupled cell networks of three identical cells with self-coupling and multiarrows.

3   Symmetry Groupoid of a Coupled Cell Network

Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network as in Definition 2.1. We define now the 'symmetry groupoid' $\mathcal{B}_G$ of G.

Definition 3.1 [11] Given $c \in \mathcal{C}$, the *input set* of c is

$$I(c) = \{e \in \mathcal{E} : \ \mathcal{H}(e) = c\}$$

An element of $I(c)$ is called an *input edge* or *input arrow* of c.

The relation $\sim_I$ of *input-equivalence* on $\mathcal{C}$ is defined by $c \sim_I d$ if and only if there exists an arrow type preserving bijection

$$\beta : I(c) \rightarrow I(d)$$

That is, for every input edge $i \in I(c)$

$$i \sim_E \beta(i)$$

Any such bijection β is called an *input isomorphism* from cell c to cell d. Denote by $B(c, d)$ the set of all input isomorphisms from cell c to cell d, and define

$$\mathcal{B}_G = \dot{\bigcup_{c,d \, \in \, \mathcal{C}}} B(c, d)$$

where $\dot{\cup}$ indicates disjoint union. A natural product operation can be defined on $\mathcal{B}_G$ in the following way: $\beta_2 \in B(c, d)$ can be multiplied by $\beta_1 \in B(a, b)$ only when $b = c$, and in this case $\beta_2\beta_1 \in B(a, d)$ is the usual composition of functions. It follows that $\mathcal{B}_G$ is a groupoid whose objects are the nodes of G, and the $\mathcal{B}_G$-morphisms are the elements of the sets $B(c, d)$ with the product operation between the morphisms as defined above. Some references on groupoids are Brandt [2], Brown [3] and Higgins [12]. We call $\mathcal{B}_G$ the *symmetry groupoid* of the network G. Note that for any $c \in \mathcal{C}$, the subset $B(c, c)$ is always non-empty. Moreover, it is a group and it is called the *vertex group* corresponding to c.

$\diamondsuit$

Structure of $B(c, d)$

Let $B(c, d) \subseteq \mathcal{B}_G$. We can specify the structure of the set $B(c, d)$ in terms of the structure of G. We distinguish three cases:

1. If $c \not\sim_I d$ then $B(c, d) = \emptyset$.

2. If $c = d$ we can define an equivalence relation $\equiv_c$ on $\mathcal{T}(I(c))$ by

$$\mathcal{T}(j_1) \equiv_c \mathcal{T}(j_2) \iff j_1 \sim_E j_2 \tag{3.1}$$

   where $j_1, j_2 \in I(c)$. If $K_1, K_2, \ldots, K_{r(c)}$ are the $\equiv_c$-equivalence classes (on $\mathcal{T}(I(c))$), then

$$B(c, c) = \mathbf{S}_{K_1} \times \cdots \times \mathbf{S}_{K_{r(c)}} \tag{3.2}$$

   where each $\mathbf{S}_{K_i}$ comprises all permutations of the set K_i. If we extend by the identity on $\mathcal{T}(I(c)) \setminus K_i$, then there is a natural embedding of $\mathbf{S}_{K_1} \times \cdots \times \mathbf{S}_{K_{r(c)}}$, and hence of $B(c, c)$, in the group $\mathbf{S}_{n(c)}$, where $n(c) = |\mathcal{T}(I(c))|$ denotes the cardinality of the set $\mathcal{T}(I(c))$.

3. If $c \neq d$ and $c \sim_I d$ (so that $B(c, d) \neq \emptyset$), then for *any* $\beta \in B(c, d)$ we have

$$B(c, d) = \beta B(c, c) = B(d, d)\beta$$

See [14], end of Section 3, for details and proofs of the above facts.

Example 3.2 The symmetry groupoid of the coupled cell network G_1 in Figure 1 is given by $\mathcal{B}_{G_1} = \dot{\bigcup}_{c,d \in \{1,2,3\}} B(c, d)$ where

$$B(c, d) = \{\beta : \ \beta \text{ is a bijection from } I(c) \text{ to } I(d)\}$$

Note that there is only one edge-type. Moreover, we have $|\mathcal{T}(I(c))| = |\mathcal{T}(I(d))|$ and so $B(c, d) \neq \emptyset$ for every pair of cells c, d. This generalizes to any 'identical-edge homogeneous network' (see Definition 7.1). $\diamond$

4 Vector Fields on a Coupled Cell Network

We make now precise the connection between coupled cell systems and coupled cell networks. Again we follow Stewart *et al.* [14]. Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network with symmetry groupoid $\mathcal{B}_G$. We now define the class $\mathcal{F}_G^P$ of admissible vector fields corresponding to G, given a choice of 'total phase space' P. This class consists of all vector fields that are 'compatible' with the labeled graph structure. A *coupled cell system* associated with the coupled cell network G is a set of ordinary differential equations (ODEs) coupled together where the *couplings* correspond to the edges of the network and the nodes correspond to *cells*. To obtain these ODEs we must associate variables x_c with cells c, that is, we must choose a phase space for each cell.

For each cell $c \in \mathcal{C}$ define a *cell phase space* P_c, which for simplicity we assume is a nonzero finite-dimensional real vector space. If cells c and d are in the same $\sim_C$-equivalence class, we require that $P_c = P_d$ and we identify the two spaces canonically. Define the corresponding *total phase space* to be

$$P = \prod_{c \in \mathcal{C}} P_c$$

and employ the coordinate system

$$x = (x_c)_{c \in \mathcal{C}}$$

on P.

Let $\mathcal{D} = (d_1, \ldots, d_s)$ be any finite ordered subset of s cells in $\mathcal{C}$ where the same cell can appear more than once in $\mathcal{D}$. Define

$$P_{\mathcal{D}} = P_{d_1} \times \cdots \times P_{d_s}$$

and write

$$x_{\mathcal{D}} = (x_{d_1}, \ldots, x_{d_s})$$

where $x_{d_j} \in P_{d_j}$.

Given $c \in \mathcal{C}$, denote by $\mathcal{T}(I(c))$ the ordered set of cells $(\mathcal{T}(i_1), \ldots, \mathcal{T}(i_s))$ where the arrows i_k run through $I(c)$. Suppose that $c \sim_I d$. Note that given any bijection $\beta \in B(c, d)$

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then for all $i \in I(c)$ we have $i \sim_E \beta(i)$, and so $\mathcal{T}(i) \sim_C \mathcal{T}(\beta(i))$. Consider now the ordered sets $\mathcal{D}_1 = \mathcal{T}(I(c)) = (\mathcal{T}(i_1), \ldots, \mathcal{T}(i_s))$ and $\mathcal{D}_2 = \mathcal{T}(I(d)) = (\mathcal{T}(\beta(i_1)), \ldots, \mathcal{T}(\beta(i_s)))$ of $\mathcal{C}$. We can define the *pullback map*

$$\beta^* : \ P_{\mathcal{D}_2} \to P_{\mathcal{D}_1}$$

by

$$\left(\beta^*(z)\right)_{\mathcal{T}(j)} = z_{\mathcal{T}(\beta(j))}$$

for all $\mathcal{T}(j) \in \mathcal{D}_1$ and $z \in P_{\mathcal{D}_2}$. Thus $x_{\mathcal{T}(I(c))} = \left(x_{\mathcal{T}(i_1)}, \ldots, x_{\mathcal{T}(i_s)}\right)$ and $\beta^*\left(x_{\mathcal{T}(I(d))}\right) = \left(x_{\mathcal{T}(\beta(i_1))}, \ldots, x_{\mathcal{T}(\beta(i_s))}\right)$.

For a given cell c the *internal phase space* is P_c and the *coupling phase space* is

$$P_{\mathcal{T}(I(c))} = P_{\mathcal{T}(i_1)} \times \cdots \times P_{\mathcal{T}(i_s)}$$

where as before $\mathcal{T}(I(c))$ denotes the ordered set of cells $(\mathcal{T}(i_1), \cdots, \mathcal{T}(i_s))$.

Definition 4.1 [11] A vector field $f : \ P \to P$ is $\mathcal{B}_G$-*equivariant* or G-*admissible* if:

(a) For all $c \in \mathcal{C}$ the component $f_c(x)$ depends only on the internal phase space variables x_c and the coupling phase space variables $x_{\mathcal{T}(I(c))}$; that is, there exists a smooth function $\hat{f}_c : \ P_c \times P_{\mathcal{T}(I(c))} \to P_c$ such that

$$f_c(x) = \hat{f}_c(x_c, x_{\mathcal{T}(I(c))})$$

(b) For all $c, d \in \mathcal{C}$ and $\beta \in B(c, d)$

$$\hat{f}_d(x_d, x_{\mathcal{T}(I(d))}) = \hat{f}_c(x_d, \beta^*(x_{\mathcal{T}(I(d))}))$$

for all $x \in P$.

$\diamond$

Theorem 4.2 *Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network and $\mathcal{B}_G$ the corresponding symmetry groupoid. A vector field $f : \ P \to P$ for a given choice of the P_c is $\mathcal{B}_G$-equivariant if and only if for each $\sim_I$-equivalence class $\mathcal{Q}$ of $\mathcal{C}$, given (any) $c \in \mathcal{Q}$:*

(a) $\hat{f}_c$ is $B(c, c)$-invariant.

(b) For $d \in \mathcal{Q}$ such that $d \neq c$, given (any) $\beta \in B(c,d)$, we have

$$\hat{f}_d\big(x_d, x_{\mathcal{T}(I(d))}\big) = \hat{f}_c\big(x_d, \beta^*\big(x_{\mathcal{T}(I(d))}\big)\big)$$

Proof This is a generalization of [14, Lemma 4.5] and is proved the same way. □

Definition 4.3 Let G be a coupled cell network. For a given choice of the P_c, define $\mathcal{F}_G^P$ to consist of all smooth G-admissible vector fields on P which is a vector space over $\mathbb{R}$. Let $\mathcal{P}_G^P$ be the subspace of $\mathcal{F}_G^P$ consisting of the G-admissible polynomial vector fields on P, and let $\mathcal{L}_G^P$ be the subspace of $\mathcal{P}_G^P$ consisting of the G-admissible linear vector fields on P. ◇

Remark 4.4 Let G be a coupled cell network and P a given choice of the total phase space consistent with G. By Theorem 4.2, every smooth equivariant vector field $f \in \mathcal{F}_G^P$ is determined uniquely by its components f_c where c runs through a set of representatives for the $\sim_I$-equivalence classes (that is, the *connected components* of the groupoid $\mathcal{B}_G$). The only constraints on f_c are that it depends only on $x_c, x_{\mathcal{T}(I(c))}$ and is invariant under the vertex group $B(c,c)$. Thus every smooth equivariant vector field f is determined uniquely by a finite set of $B(c,c)$-invariant functions, where c runs through a set of representatives for the connected components of the groupoid. Moreover, if $d \sim_I c$ then f_d is related to f_c by a pullback map β^* for $\beta \in B(c,d)$. In particular, if there is only one $\sim_I$-equivalence class for $\mathcal{B}_G$ then each G-admissible vector field is uniquely determined by a single mapping f_c at some node c, which has to be invariant under the vertex group $B(c,c)$. ◇

5 ODE-equivalence and Linear Equivalence

As pointed by Golubitsky *et al.* [11], in the class of coupled cell networks that permits self-coupling and multiarrows, it is possible for two non-isomorphic coupled cell networks G_1 and G_2 to generate the same space of admissible vector fields. This comparison of admissible vector fields for two coupled cell networks involves identifying cells in the two networks, a step that it is formalised in general in terms of a bijection between the two

sets of cells. We follow the definition of ODE-equivalent coupled cell networks given by Dias and Stewart [7].

In what follows, given a coupled cell network G_i and a choice of total phase space P_i for G_i, we denote by $P_{i,c}$ the cell phase space corresponding to cell c of $\mathcal{C}_i$.

Definition 5.1 [7] Two coupled cell networks G_1 and G_2 are γ-*ODE-equivalent* if:

1. There is a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence, such that:

2. If we choose cell phase spaces $P_c \neq 0$ for G_1, and define the corresponding choice of cell phase spaces for G_2 by
$$P_{2,\gamma(c)} = P_{1,c}$$
so that the corresponding total phase spaces are
$$P_1 = \prod_{c \in \mathcal{C}_1} P_{1,c} \qquad P_2 = \prod_{c \in \mathcal{C}_1} P_{2,\gamma(c)}$$
then:

3. The condition
$$\mathcal{F}_{G_1}^{P_1} = \mathcal{F}_{G_2}^{P_2} \tag{5.3}$$
is satisfied.

Two coupled cell networks G_1 and G_2 are *ODE-equivalent* if they are γ-ODE-equivalent for some bijection γ. $\diamond$

Remark 5.2 The cells of G_2 can be renumbered so that $\gamma = $ id. In this case, we omit explicit reference to γ. $\diamond$

We define now the notion of 'linear equivalence' between two networks. In [7] it is shown that two coupled cell networks are ODE-equivalent if and only if they are linearly equivalent. Basically, the ODE-equivalence reduces to 'linear equivalence', where two networks (with suitably identified phase spaces) are linearly equivalent if they determine the same space of *linear* admissible vector fields. Moreover, when deciding linear equivalence,

it can without loss of generality be assumed that each cell phase space is one-dimensional since if (5.3) holds for some choice of nonzero cell phase spaces P_c, then it holds for all such choices (that are consistent with the structure of G).

Definition 5.3 Two coupled cell networks G_1 and G_2 are γ-*linearly equivalent* if there is a bijection $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ that preserves cell-equivalence and input-equivalence such that if item 2. of Definiton 5.1 is satisfied then:

3. The condition

$$\mathcal{L}_{G_1}^{P_1} = \mathcal{L}_{G_2}^{P_2} \tag{5.4}$$

is satisfied.

Two coupled cell networks G_1 and G_2 are *linearly equivalent* if they are γ-linearly equivalent for some γ. $\diamond$

Note that this definition is independent of the dimensions of the P_c. Also, we may renumber the cells to make γ the identity.

Throughout, we denote by $M_{n \times n}(\mathbb{Z}_0^+)$ the set of the square matrices of order n and nonnegative entries, and by Id_n the identity matrix in this set.

Example 5.4 The coupled cell network G_2 of Figure 2 is an example of a network linearly equivalent to the network G_1 of Figure 1. Both graphs have only one cell-equivalence class and one input-equivalence class. Trivially, the identity function on $\mathcal{C}_1 = \{1, 2, 3\}$ preserves both cell-equivalence and input-equivalence. Moreover, if we take $P_1 = P_2 = \mathbb{R}^3$ then $\mathcal{L}_{G_1}^{P_1} = \mathcal{L}_{G_2}^{P_2}$ since

$$\mathcal{L}_{G_1}^{P_1} = \mathbb{R}\left\{\mathrm{Id}_{\mathbb{R}^3}, h_1\right\} = \mathcal{L}_{G_2}^{P_2} = \mathbb{R}\left\{\mathrm{Id}_{\mathbb{R}^3}, h_2\right\} \tag{5.5}$$

where $\mathrm{Id}_{\mathbb{R}^3}$ denotes the identity function on $\mathbb{R}^3$,

$$h_1(x_1, x_2, x_3) = (2x_1 + 9x_2 + 9x_3, 12x_1 + 2x_2 + 6x_3, 15x_1 + 5x_3)$$

$$h_2(x_1, x_2, x_3) = (3x_2 + 3x_3, 4x_1 + 2x_3, 5x_1 + x_3)$$

and $(x_1, x_2, x_3) \in \mathbb{R}^3$. Note that (5.5) is equivalent as saying

$$
\mathbb{R}\left\{ \mathrm{Id}_3, \begin{pmatrix} 2 & 9 & 9 \\ 12 & 2 & 6 \\ 15 & 0 & 5 \end{pmatrix} \right\} = \mathbb{R}\left\{ \mathrm{Id}_3, \begin{pmatrix} 0 & 3 & 3 \\ 4 & 0 & 2 \\ 5 & 0 & 1 \end{pmatrix} \right\} \tag{5.6}
$$

$\diamond$

The next theorem reduces ODE-equivalence to linear equivalence, and says that the cell phase spaces may be assumed one-dimensional in that context.

Theorem 5.5 [7] *Let* $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ *be a bijection that preserves cell-equivalence and input-equivalence. Then the following conditions on two networks* G_1, G_2 *are equivalent:*

(a) G_1 *and* G_2 *are* γ*-ODE-equivalent.*

(b) G_1 *and* G_2 *are* γ*-linearly equivalent.*

(c) *With the identification* $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$*, the spaces* $\mathcal{L}_{G_1}^P$ *and* $\mathcal{L}_{G_2}^P$ *are equal when all cell phase spaces are taken to be* $\mathbb{R}$*.*

Proof   See Theorem 7.1 and Corollary 7.9 of [7].   □

Example 5.6 We return to Example 5.4. Recall (5.5). From the above theorem it follows that the two coupled cell networks of Figures 1 and 2 are ODE-equivalent. $\diamond$

6   Minimality

In this section we compare ODE-equivalent networks in terms of the number of edges.

Definition 6.1 Let G be an n-cell coupled cell network. We denote by $[G]$ the class of all coupled cell networks that are ODE-equivalent to G. By Theorem 5.5 we have that $[G]$ coincides with the class of all coupled cell networks that are linearly equivalent to G. $\diamond$

Note that for a given n-cell network $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$, the set

$$\{(\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1}) \in [G] : \text{card}(\mathcal{E}_1) \leq \text{card}(\mathcal{E})\} \subseteq [G],$$

where $\text{card}(\mathcal{E})$ denotes the cardinality of the set of edges $\mathcal{E}$, is finite. In particular, it follows that the set

$$\{\text{card}(\mathcal{E}_1) : (\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1}) \in [G]\}$$

has a minimum. We introduce now some notation:

Definition 6.2 Given an n-cell network G and the corresponding ODE-class $[G]$, let

$$m_{[G]} = \min \{\text{card}(\mathcal{E}_1) : (\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1}) \in [G]\}$$

and

$$\mathcal{M}in_{[G]} = \{(\mathcal{C}_1, \mathcal{E}_1, \sim_{C_1}, \sim_{E_1}) \in [G] : \text{card}(\mathcal{E}_1) = m_{[G]}\}$$

We call $\mathcal{M}in_{[G]}$ the *minimal subclass* of $[G]$. $\diamond$

The minimal subclass $\mathcal{M}in_{[G]}$ of $[G]$ is thus the subclass of networks of $[G]$ such that the number of edges is *minimal* among all the networks of $[G]$ – the *subclass of the canonical normal forms*. In this paper, we describe the minimal subclass $\mathcal{M}in_{[G]}$ of $[G]$, given an n-cell network G. More precisely, we give an algorithm that computes $\mathcal{M}in_{[G]}$. We start by adressing the question for networks with only one $\sim_I$-equivalence class (Sections 7-10). It follows then as a corollary an algorithm that computes $\mathcal{M}in_{[G]}$ for any ODE-class $[G]$ (Section 11).

Example 6.3 Consider the graph G_1 of Figure 1. For this example we have $m_{[G_1]} = 18$ and $\mathcal{M}in_{[G_1]} = \{G_2\}$, where G_2 is the coupled cell network of Figure 2. See Section 10 for details. $\diamond$

7 Homogeneous Networks

In this section we adress the minimality of networks in terms of the number of edges for ODE-classes associated with networks that have only one $\sim_I$-equivalence class.

Definition 7.1 An *homogeneous* network is a coupled cell network with only one input-equivalence class. An *identical-edge homogeneous* cell network is an homogeneous network in which all edges in $\mathcal{E}$ are equivalent. A *nonidentical-edge homogeneous* cell network is an homogeneous network with more than one edge-equivalence class. ◇

Note that homogeneous networks have only one cell-equivalence class since these have only one $\sim_I$-equivalence class and by definition the equivalence relation $\sim_I$ refines $\sim_C$.

The graphs of Figures 1 and 2 are examples of identical-edge homogeneous coupled cell networks. Figure 5 shows an example of a nonidentical-edge homogeneous network with 2 edge-types.
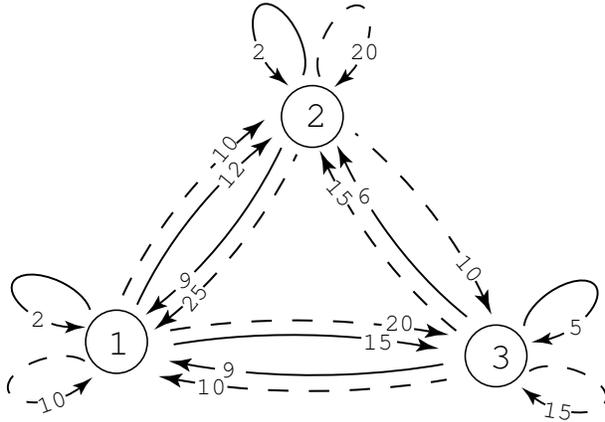

Figure 5: An example of a nonidentical-edge homogeneous network G_1.

Definition 7.2 Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be an homogeneous network with n cells, say $\mathcal{C} = \{1, \ldots, n\}$, and m edge-types, with $[e_1]_E, \cdots, [e_m]_E$, the $\sim_E$-equivalence classes. We define the *adjacency matrix* of G with respect to $[e_l]_E$, for $l = 1, \cdots, m$, to be the $n \times n$ matrix $M_{(G,l)}$ with rows and columns indexed by the cells of G in the following way: the (i,j)-entry of $M_{(G,l)}$ corresponds to the number of edges of type $[e_l]_E$ from cell j to cell i. Thus the sum of the jth entries of row i of the matrices $M_{(G,l)}$, for $l = 1, \cdots, m$, gives the number of input edges of cell i with tail cell j. ◇

Example 7.3 The adjacency matrices of the coupled cell network G_1 of Figure 5 with

respect to the two edge-types are

$$M_{(G_1,1)} = \begin{pmatrix} 2 & 9 & 9 \\ 12 & 2 & 6 \\ 15 & 0 & 5 \end{pmatrix} \text{ and } M_{(G_1,2)} = \begin{pmatrix} 10 & 25 & 10 \\ 10 & 20 & 15 \\ 20 & 10 & 15 \end{pmatrix}$$

$\diamond$

7.1 Linear Equivalence of Adjacency Matrices

We start by relating the adjacency matrices corresponding to linearly equivalent homogeneous networks. Definition 5.3 of linear equivalence is given by:

Proposition 7.4 *Let G_1 be an homogeneous network with n cells and m_1 edge-types, where $\mathcal{C}_1 = \{1, \ldots, n\}$. Let G_2 be an homogeneous network with n cells and m_2 edge-types. Denote by $M_{(G_1,l)} = [a_{ij}^l]_{1 \leq i,j \leq n}$ for $l = 1, \cdots, m_1$, and $M_{(G_2,k)} = [b_{ij}^k]_{1 \leq i,j \leq n}$ for $k = 1, \cdots, m_2$, the adjacency matrices of G_1 and G_2 with respect to $[e_l]_{E_1}$ and $[e_k]_{E_2}$, respectively. We have that G_1 and G_2 are linearly equivalent if and only if there is a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ such that:*

$$\mathbb{R}\left\{\mathrm{Id}_n, [a_{ij}^1]_{1 \leq i,j \leq n}, \cdots, [a_{ij}^{m_1}]_{1 \leq i,j \leq n}\right\} = \mathbb{R}\left\{\mathrm{Id}_n, [b_{\gamma(i)\gamma(j)}^1]_{1 \leq i,j \leq n}, \cdots, [b_{\gamma(i)\gamma(j)}^{m_2}]_{1 \leq i,j \leq n}\right\} \tag{7.7}$$

Proof Since both networks are homogeneous, trivially any bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ preserves cell-equivalence and input-equivalence. Thus Definition 5.3 of linear equivalence translates to equation (7.7). $\qquad\square$

If necessary, we can relabel the cells of G_2 so that $\mathcal{C}_1 = \mathcal{C}_2$ and G_2 is γ-linearly equivalent to G_1 where γ is the identity on $\mathcal{C}_1 = \{1, \ldots, n\}$. Moreover, since G_2 is linearly equivalent to G_1, then by Proposition 7.4 it follows that

$$\mathbb{R}\left\{\mathrm{Id}_n, [a_{ij}^1]_{1 \leq i,j \leq n}, \cdots, [a_{ij}^{m_1}]_{1 \leq i,j \leq n}\right\} = \mathbb{R}\left\{\mathrm{Id}_n, [b_{ij}^1]_{1 \leq i,j \leq n}, \cdots, [b_{ij}^{m_2}]_{1 \leq i,j \leq n}\right\} \tag{7.8}$$

Example 7.5 Using Proposition 7.4, the network G_1 of Figure 5 is linearly equivalent to the network G_2 of Figure 6, where γ is identity on $\{1, 2, 3\}$. $\diamond$
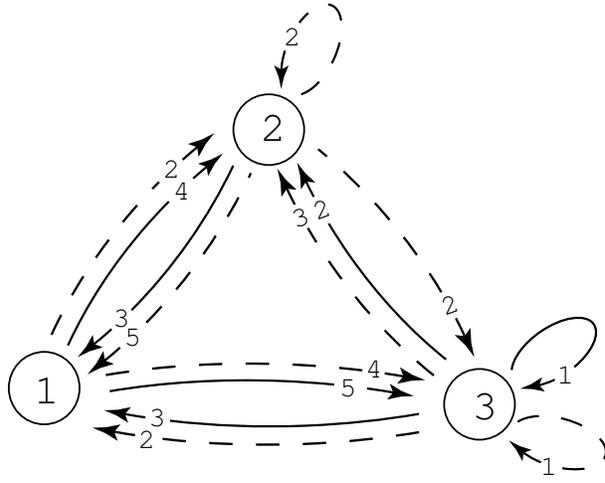
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Figure 6: An example of a nonidentical-edge homogeneous network G_2 linearly equivalent to the graph of Figure 5.

Remark 7.6 Let G_1 be an homogeneous network with n cells. Let $M_{(G,1)}, \ldots, M_{(G,m_1)}$ be the corresponding adjacency matrices. Then the set of matrices

$$\mathcal{L} = M_{n \times n}(\mathbb{Z}) \cap \mathbb{R} \left\{ \mathrm{Id}_n, M_{(G,1)}, \cdots, M_{(G,m_1)} \right\}$$

with the usual sum of matrices is a lattice [4, Chapter 3]: trivially, the set $\mathcal{L}$ is a discrete subgroup of $M_{n \times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$. $\diamond$

The next lemma shows that the networks of the ODE-class $[G_1]$ have adjacency matrices lying in the cone $\mathcal{L} \cap M_{n \times n}(\mathbb{Z}_0^+)$.

Lemma 7.7 *Let G_1 be an homogeneous network with n cells, say $\mathcal{C}_1 = \{1, \ldots, n\}$, and m_1 edge-types. Let $M_{(G_1,l)} = [a_{ij}^l]_{1 \leq i,j \leq n}$ for $l = 1, \cdots, m_1$ be the corresponding adjacency matrices. Then $M = [m_{ij}]_{1 \leq i,j \leq n} \in M_{n \times n}(\mathbb{Z}_0^+)$ is an adjacency matrix of a network G (of n cells) linearly equivalent to G_1 if and only if*

$$M \in \mathcal{L} \ \cap \ M_{n \times n}(\mathbb{Z}_0^+) \tag{7.9}$$

Proof Note that each $M_{(G_1,l)}$ has the following properties:

$$M_{(G_1,l)} \in M_{n \times n}(\mathbb{Z}_0^+); \tag{7.10}$$

$$\sum_{j=1}^{n} a_{ij}^l = \sum_{j=1}^{n} a_{kj}^l \text{ for all } i, k \tag{7.11}$$

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By Proposition 7.4 we have that if $M = [m_{ij}]_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{Z}_0^+)$ is an adjacency matrix of a network G (of n cells) linearly equivalent to G_1 then condition (7.9) follows.

Suppose now that $M = [m_{ij}]_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{Z}_0^+)$ satisfies (7.9). In particular, since G is homogeneous, it follows that

$$\sum_{j=1}^{n} m_{ij} = \sum_{j=1}^{n} m_{kj} \text{ for all } i, k$$

$\square$

7.2   Minimality

Given an homogeneous network G, we aim to describe the minimal subclass $\mathcal{M}in_{[G]}$ of $[G]$. Recall Definition 6.2. Using the above discussion on ODE-equivalence and linear equivalence, we can describe the graphs of $\mathcal{M}in_{[G]}$ by computing the corresponding adjacency matrices. Moreover, we have:

Lemma 7.8 *Let G_1 be an homogeneous network with n cells. Let $[G_1]$ the corresponding ODE-class and $M_{(G_1,l)}$, for $l = 1,\ldots,m_1$, the adjacency matrices of G_1. Let*

$$m = \dim \mathbb{R}\left\{\text{Id}_n, M_{(G_1,1)},\ldots,M_{(G_1,m_1)}\right\} - 1 \qquad (7.12)$$

Then the homogeneous networks of the minimal class $\mathcal{M}in_{[G_1]}$ have m edge types.

Proof   Let $G_2 \in \mathcal{M}in_{[G_1]}$ and suppose that G_2 has m_2 edge types. Trivially, $m_2 \geq m$ since

$$\mathbb{R}\left\{\text{Id}_n, M_{(G_1,1)},\ldots,M_{(G_1,m_1)}\right\} = \mathbb{R}\left\{\text{Id}_n, M_{(G_2,1)},\ldots,M_{(G_2,m_2)}\right\}$$

and so

$$m_2+1 \geq \dim\mathbb{R}\left\{\text{Id}_n, M_{(G_2,1)},\ldots,M_{(G_2,m_2)}\right\} = \dim\mathbb{R}\left\{\text{Id}_n, M_{(G_1,1)},\ldots,M_{(G_1,m_1)}\right\} = m+1$$

If $m_2 > m$ then

$$\mathbb{R}\left\{\text{Id}_n, M_{(G_2,1)},\ldots,M_{(G_2,m_2)}\right\} = \mathbb{R}\left\{\text{Id}_n, M_{(G_2,i_1)},\ldots,M_{(G_2,i_k)}\right\}$$

for some set $\{i_1,\ldots,i_k\} \subset \{1,\ldots,m_2\}$ and $G_2 \notin \mathcal{M}in_{[G_1]}$, a contradiction. Thus $m_2 = m$.

$\square$

Remark 7.9 Let m as defined in (7.12). Since $\dim \mathbb{R} \left\{ \mathrm{Id}_n, M_{(G,1)}, \cdots, M_{(G,m_1)} \right\}$ equals $m + 1$ we have that $\mathcal{L}$ is a lattice of rank $m + 1$. ◇

Definition 7.10 Given $M = [m_{ij}]_{1 \leq i,j \leq n} \in M_{n \times n}(\mathbb{Z}_0^+)$ denote by

$$l(M) = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}$$

More generally, for $u = (u_1, \ldots, u_k) \in (\mathbb{Z}_0^+)^k$ we call the *length* of u the nonnegative integer $l(u)$ defined by

$$l(u) = \sum_{i=1}^{k} u_i$$

Note that if $w_1, \ldots, w_m \in (\mathbb{Z}_0^+)^k$ then

$$l \left(\sum_{i=1}^{m} w_i \right) = \sum_{i=1}^{m} l(w_i)$$

Given a set of vectors $\{w_1, \ldots, w_r\} \subset (\mathbb{Z}_0^+)^k$ we denote

$$l (\{w_1, \ldots, w_r\}) = \sum_{j=1}^{r} l(w_j)$$

◇

We find the minimal subclass $\mathcal{M}in_{[G]}$ in the following way:

Proposition 7.11 *Let G be an homogeneous network of n cells and $[G]$ the corresponding ODE-class. Suppose that $M_{(G,1)}, \ldots, M_{(G,m_1)}$ are the adjacency matrices of G. Let*

$$m = \dim \mathbb{R} \left\{ \mathrm{Id}_n, M_{(G,1)}, \ldots, M_{(G,m_1)} \right\} - 1$$

An homogeneous network of the minimal class $\mathcal{M}in_{[G]}$ has adjacency matrices $M_1, \ldots, M_m$ defined by:

(i) $\{\mathrm{Id}_n, M_1, \ldots, M_m\} \subset M_{n \times n}(\mathbb{Z}_0^+)$;

(ii) $\{\mathrm{Id}_n, M_1, \ldots, M_m\}$ *is a basis of the real vector space* $\mathbb{R} \left\{ \mathrm{Id}_n, M_{(G,1)}, \cdots, M_{(G,m_1)} \right\}$;

(iii) $\sum_{k=1}^{m} l(M_k) = m_{[G]}$.

Proof   By Lemma 7.8 any graph $G_2 \in \mathcal{M}in_{[G]}$ has m edge types. Item (i) follows by the definition of adjacency matrix of a network. Moreover, by Lemma 7.7 and Proposition 7.4 we have (ii). Condition (iii) follows from Definition 6.2 of $\mathcal{M}in_{[G]}$.   □

Remark 7.12 Note that if $M \in M_{n \times n}(\mathbb{Z}_0^+)$ is an adjacency matrix of an homogeneous network, then $l(M) \geq n$ and so $l(M) \geq l(\mathrm{Id}_n)$. Denote by $M_0 = b_0 = \mathrm{Id}_n$. Assume the conditions of Proposition 7.11 and denote $S = \mathbb{R}\left\{\mathrm{Id}_n, M_{(G,1)}, \ldots, M_{(G,m_1)}\right\}$. A set $\{M_1, \ldots, M_m\}$ satisfies (i)-(iii) if and only if it satisfies

(i) $\{M_0, M_1, \ldots, M_m\} \subset M_{n \times n}(\mathbb{Z}_0^+)$;

(ii) $\{M_0, M_1, \ldots, M_m\}$ is a basis of S;

(iii) $\sum_{k=0}^m l(M_k) = \min\left\{\sum_{k=0}^m l(b_k) : \{b_0, b_1, \ldots, b_m\} \subset M_{n \times n}(\mathbb{Z}_0^+) \text{ is a basis of } S\right\}$.

                    ◇

8   Minimal Bases

By Proposition 7.11 and Remark 7.12 we have that the minimal subclass $\mathcal{M}in_{[G]}$ of $[G]$ where G is an homogeneous coupled cell network can be computed by describing the set of all possible adjacency matrices satisfying a minimality condition in terms of the length of those matrices. We abstract the problem in the following way.

Given a set of vectors $w_1, \ldots, w_p \in (\mathbb{Z}_0^+)^k$, denote by

$$S = \mathbb{R}\{w_1, \ldots, w_p\}$$

the real vector subspace of $\mathbb{R}^k$ generated by the vectors $w_1, \ldots, w_p$. Suppose that

$$\dim S = r,$$

with $r \leq p$. We choose any r linearly independent vectors from $w_1, \ldots, w_p$, say $w_1, \ldots, w_r$. Thus $w_1, \ldots, w_r$ form a basis of S. Denote by

$$s = l(\{w_1, \ldots, w_r\})$$

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Let

$$\mathbf{B} = \left\{ b : \ b = \{b_1, \ldots, b_r\} \subset (\mathbb{Z}_0^+)^k \text{ is a basis of } S \right\}$$

Lemma 8.1 *The set* $\{l(b) : \ b \in \mathbf{B}\}$ *has a minimum.*

Proof As $\{w_1, \ldots, w_r\} \subset (\mathbb{Z}_0^+)^k$ is a basis of S and $l\left(\{w_1, \ldots, w_r\}\right) = s$, it follows that the minimum of $\{l(b) : \ b \in \mathbf{B}\}$ is at most s. Thus,

$$r \leq \min \{l(b) : \ b \in \mathbf{B}\} \leq s.$$

Moreover,

$$\min \{l(b) : \ b \in \mathbf{B}\} = \min \{l(b) : \ b \in \mathbf{B}, \ r \leq l(b) \leq s\},$$

and $\{b \in \mathbf{B} : \ r \leq l(b) \leq s\}$ is a finite set. □

Definition 8.2 We call a basis $b \in \mathbf{B}$ of S such that

$$l(b) = \min \left\{ l\left(\bar{b}\right) : \ \bar{b} \in \mathbf{B} \right\}$$

a *minimal basis*. ◇

In this section we describe the set of all minimal bases $b \in \mathbf{B}$ of S. Moreover, we present an algorithm that computes this set.

Sketch of the Procedure

The method we follow for finding the minimal bases of S relies mostly in two steps:

1. Find the set C defined by

$$C = \left\{ w \in S \cap (\mathbb{Z}_0^+)^k : \ 0 < l(w) \leq s - r + 1 \right\}$$

2. Find all the minimal bases $b \in \mathbf{B}_C$ of S where

$$\mathbf{B}_C = \{b \in \mathbf{B} : \ b \subseteq C\}$$

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Remarks 8.3 (a) If $b = \{b_1, \ldots, b_r\} \subset (\mathbb{Z}_0^+)^k$ is a basis of S and $r \leq l(b) \leq s$ then $b \subseteq C$. Moreover, $\mathbf{B}_C \subseteq \mathbf{B}$ and

$$\min \{l(b) : \ b \in \mathbf{B}\} = \min \{l(b) : \ b \in \mathbf{B}_C\}$$

Thus the minimal bases $b = \{b_1, \ldots, b_r\}$ of S are contained in C.

(b) Note that the set C depends on the chosen basis $\{w_1, \ldots, w_r\}$ of S which determines the length $s = l(\{w_1, \ldots, w_r\})$. We show in Theorem 8.7 that the minimal bases do not depend on the chosen set of the vectors w_i that form a basis of S, neither on other choice of vectors (with nonnegative integers coordinates) generating S.

$\diamond$

Finding C

Let W be the $k \times r$ matrix with nonnegative integer entries with columns given by the vectors $w_1, \ldots, w_r$ that we assume linearly independent and generating S. In notation

$$W = (w_1| \cdots |w_r)$$

Denote by M_W the set of all $r \times r$ matrices formed by r rows of W and with nonzero determinant and take

$$d = \gcd \{\det(M) : \ M \in M_W\}$$

Lemma 8.4 *If $w \in S \cap (\mathbb{Z}_0^+)^k$ then there are rational numbers $x_1, \ldots, x_r$ whose denominators divide d such that*

$$w = x_1 w_1 + \cdots + x_r w_r$$

Proof By hypothesis $\{w_1, \ldots, w_r\}$ is a basis of S. Thus given $w \in S \cap (\mathbb{Z}_0^+)^k$, there are real numbers $x_1, \ldots, x_r \in \mathbb{R}$ such that $w = x_1 w_1 + \cdots + x_r w_r$. Moreover, we have that w has integer components. Given $M \in M_W$, if we consider the linear system in the variables $x_1, \ldots, x_r$ obtained from $x_1 w_1 + \cdots + x_r w_r = w$ by keeping the rows corresponding to M, by the Cramer's Rule we have that the unique $x_1, \ldots, x_r$ are rational whose denominator divide the (nonzero) determinant of M. $\square$

By the above lemma, we have that C is the set of linear combinations of $w_1, \ldots, w_r$, say $x_1 w_1 + \cdots + x_r w_r \in (\mathbb{Z}_0^+)^k$, with rational coefficients $x_1, \ldots, x_r$ whose denominators divide d and such that

$$0 < l(x_1 w_1 + \cdots + x_r w_r) \le s - r + 1$$

Thus $x_i = p_i/q_i$ where $p_i, q_i \in \mathbb{Z}$, $q_i \ne 0$ and $d = n_i q_i$ for some $n_i \in \mathbb{Z}$. It follows then that

$$(n_1 p_1) w_1 + \cdots + (n_r p_r) w_r \equiv 0 \pmod{d}$$

We describe C in the following way:

$$C = \left\{ \frac{1}{d} \left(y_1 w_1 + \cdots + y_r w_r \right) \in (\mathbb{Z}_0^+)^k : \begin{cases} y_1 w_1 + \cdots + y_r w_r \equiv 0 \pmod{d} \\ 0 < \frac{1}{d} l \left(y_1 w_1 + \cdots + y_r w_r \right) \le s - r + 1 \end{cases} \right\}$$

Remarks 8.5 (a) Note that $w_1, \ldots, w_r \in C$ since $d w_i \equiv 0 \pmod{d}$ and for $i = 1, \ldots, r$ we have $0 < l(w_i) \le s - r + 1$.

(b) The set C is finite since it is a subset of the finite set formed by the vectors $w \in (\mathbb{Z}_0^+)^k$ such that $0 < l(w) \le s - r + 1$.

$\diamond$

Finding the Minimal Bases

In order to find the bases $b = \{b_1, \ldots, b_r\} \subseteq C$ of S such that

$$l(b) = \min \{ l(b) : b \in \mathbf{B}_C \}$$

we partition the set C by the length of its vectors:

$$C = C_1 \,\dot{\cup}\, C_2 \,\dot{\cup}\, \cdots \,\dot{\cup}\, C_t$$

where given $u, v \in C$

$$\begin{cases} l(u) = l(v) \iff u, v \in C_i \text{ for some } i \\ u \in C_i, \ v \in C_j, \ i < j \iff l(u) < l(v) \end{cases}$$

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As $\{w_1, \ldots, w_r\}$ is a basis of S and $w_1, \ldots, w_r \in C$, it follows that the real subspace of $\mathbb{R}^k$ generated by C is S.

Definition 8.6 Let S_1, S_2 be real subspaces of $\mathbb{R}^k$. We denote

$$S_1 + S_2 = \{u + v : u \in S_1, v \in S_2\}$$

$\diamond$

Denote by S_i, $i = 1, \ldots, t$, the subspace of S generated by C_i. Thus

$$S_1 + \cdots + S_t = S$$

Moreover, there is q such that $1 \le q \le t$ and

$$\dim(S_1 + \cdots + S_q) = r, \qquad \dim(S_1 + \cdots + S_{q-1}) < r \qquad (8.13)$$

Thus we have

$$S_1 + \cdots + S_q = S$$

We show below that the minimal bases of S are contained in

$$C_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} C_q$$

Denote by l_i the common length of any vector $u \in C_i$, for $i = 1, \ldots, q$. Let

$$d_1 = \dim S_1$$
$$d_i = \dim\left(\textstyle\sum_{j=1}^{i} S_j\right) - \dim\left(\textstyle\sum_{j=1}^{i-1} S_j\right), \quad 2 \le i \le q.$$

Let $U_1 \subseteq C_1$ be a set of d_1 linearly independent vectors. For $i = 2, \ldots, q$, let $U_i \subseteq C_i$ be a set of d_i vectors such that $U_1 \cup \cdots \cup U_q$ is free.

Theorem 8.7 *The minimum of the set*

$$\{l(b) : b \in \mathbf{B}\}$$

is

$$m = \sum_{j=1}^{q} l_j d_j \qquad (8.14)$$

The minimal bases $b \in \mathbf{B}$ of S are the bases obtained by taking all the possible unions $U_1 \cup \cdots \cup U_q$.

Proof   By Lemma 8.1 the set $\{l(b) : b \in \mathbf{B}\}$ has a minimum, say m.

Any set $U_1 \cup \cdots \cup U_q = \{b_1, \ldots, b_r\}$ obtained as above is free and has r vectors by construction. (Recall that by assumption $\dim(S_1 + \cdots + S_q) = r$.) Thus it is a basis of S and

$$l\left(\{b_1, \ldots, b_r\}\right) = \sum_{j=1}^{q} l_j d_j$$

We prove now that the minimum m of the set $\{l(b) : b \in \mathbf{B}\}$ is

$$\sum_{j=1}^{q} l_j d_j$$

and that any minimal basis $b \in \mathbf{B}$ is the union $U_1 \cup \cdots \cup U_q$ of sets U_i constructed as above. In particular, it follows that $b \subseteq C$.

Starting with a different set of basis vectors w_i of S, the set C varies if the total length of the chosen basis, say s^*, differs from s. However, in this case, the new set, say C^*, would contain C if $s^* > s$, and would be contained in C if $s^* < s$. (Recall that C^* would be formed by the nonnull vectors in $S \cap (\mathbb{Z}_0^+)^k$ with length lower or equal to $s^* - r + 1$.) In both cases, we have $(C_1 \cup \cdots \cup C_q) \subseteq C^*$ since $(C_1 \cup \cdots \cup C_q) \subseteq S$ and generates S. Note that in the second case, where C^* would be contained in C, if $C_1 \cup \cdots \cup C_q \not\subseteq C^*$, then we would have $C^* = C_1 \cup \cdots \cup C_s$ where $s < q$ and so $S_1 + \cdots + S_s = S$, which cannot happen by the definition of q (recall (8.13)).

Suppose by contradiction that there is a basis $b = \{b_1, \ldots, b_r\} \in \mathbf{B}_C$ such that b is not a union $U_1 \cup \cdots \cup U_q$ of sets U_i constructed as above and that

$$l(b) = m$$

We have two cases to consider:

(i) $b \not\subseteq C_1 \cup \cdots \cup C_q$. Then b does not contain r linearly independent vectors of $C_1 \cup \cdots \cup C_q$. But $C_1 \cup \cdots \cup C_q$ contains r linearly independent vectors since by hypothesis $\dim(S_1 + \cdots + S_q) = r$. Thus we can choose a vector $u \in C_1 \cup \cdots \cup C_q$ such that $u \notin b$ and such that u and the vectors of $b \cap (C_1 \cup \cdots \cup C_q)$ are linearly independent. But $u \in S$ and b is a basis of S. Then we can write u as a linear combination of the vectors b_i and such that at least one of the nonzero coefficients is of a vector in b that do not belong to

$C_1 \cup \cdots \cup C_q$. Say b_i. It follows then that $l(b_i) > l(u)$ and so the basis obtained from b by substituting b_i by u, $\{b_1, \ldots, b_{i-1}, u, b_{i+1}, \ldots, b_r\}$ is such that

$$l\left(\{b_1, \ldots, b_{i-1}, u, b_{i+1}, \ldots, b_r\}\right) < m$$

A contradiction.

(ii) $b \subseteq C_1 \cup \cdots \cup C_q$ and there is j between 1 and q such that the number of vectors of b with length lower or equal to l_j is (strictly) lower than $\dim(S_1 + \cdots + S_j)$. Then there is $u \in C_1 \cup \cdots \cup C_j$ such that u and the vectors of $b \cap (C_1 \cup \cdots \cup C_j)$ are linearly independent. Again, u is a linear combination of the vectors b_i, and at least one nonzero coefficient corresponds to a vector, say b_k, such that $l(b_k) > l_j$. The basis obtained from b by substituting b_k by u has total length strictly lower than m. Again a contradiction.

It follows then that any basis $b = \{b_1, \ldots, b_r\} \subseteq \left((\mathbb{Z}_0^+)^k \cap C\right)$ of the space S such that $l\left(\{b_1, \ldots, b_r\}\right) = m$ is contained in $C_1 \cup \cdots \cup C_q$ and the number of vectors of b with length lower or equal to l_j is equal to $\dim(S_1 + \cdots + S_j)$, for $j = 1, \ldots, q$. Thus $b = U_1 \cup \cdots \cup U_q$ where the U_j are any sets as constructed above. Moreover, we have the equality (8.14).

$\square$

Algorithm

By Theorem 8.7 the following algorithm finds all the minimal bases of S in $(\mathbb{Z}_0^+)^k$, where S is the real vector subspace of $\mathbb{R}^k$ generated by p vectors $w_1, \ldots, w_p$ of $(\mathbb{Z}_0^+)^k$.

If M_1 and M_2 are matrices of order $m \times n$ and $m \times p$, respectively, we denote by $(M_1|M_2)$ the matrix of order $m \times (n + p)$ whose first n columns are those of M_1 and the last p columns are those of M_2.

Algorithm 8.8 Given p vectors $w_1, \ldots, w_p$ of $(\mathbb{Z}_0^+)^k$ that generate a real vector subspace S of $\mathbb{R}^k$, this algorithm finds all the minimal bases of S in $(\mathbb{Z}_0^+)^k$.

    1 [Initial base] Let r be the rank of the matrix $(w_1|\ldots|w_p)$. For each w_i, $i = 1, \ldots, p$ do the following: set $s_i = \sum_{j=1}^{k}(w_i)_j$ and then sort the vectors w_i by ascending order of the s_i. Let $v_1, \ldots, v_r$ be the 'first' r linearly independent vectors. Set $s = \sum_{j=1}^{k}(v_r)_j$.

2 [Compute gcd d] Let W be the matrix $(v_1|\ldots|v_r)$. Compute the set M_W of all the $r \times r$ matrices formed by r rows of W and with nonzero determinant. Let d be the greatest common divisor of the determinants of the matrices in M_W.

3 [Solve linear system of congruences] Apply HNF Algorithm A.3 to the matrix $(W|d\mathrm{Id}_k)$, and let $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ be a unimodular matrix and H the HNF matrix such that $(W|d\,\mathrm{Id}_k)U = (0|H)$. Here U_1 is a $r \times r$ square matrix. We can discard U_2, U_3 and U_4. Let H_B be the HNF of the matrix $(U_1|d\mathrm{Id}_r)$ (applying Algorithm A.3). Note that H_B is invertible (because H_B has rank r since it is the HNF of the matrix $(U_1|d\,\mathrm{Id}_r)$ for $d \neq 0$ that has rank r).

4 [Compute set C] Let W^* be a matrix in M_W. Determine the set

$$A = \left\{ (W^*H_B)^{-1}(d\,c_1, d\,c_2, \ldots, d\,c_r)^t : \begin{cases} c_1, \ c_2, \ \ldots, \ c_r \in \mathbb{Z}_0^+ \\ 0 < c_1 + c_2 + \cdots + c_r \leq s \end{cases} \right\} \cap \mathbb{Z}^r$$

Determine

$$C = \left\{ \frac{1}{d} W H_B (x_1, \ldots, x_r)^t : (x_1, \ldots, x_r)^t \in A \right\} \cap \left\{ w \in (\mathbb{Z}/(s+1)\mathbb{Z})^k : 0 < l(w) \leq s \right\}$$

For $i = 1$ until s, set $C_i = \emptyset$. For each w in C do the following: set $C_{l(w)} = C_{l(w)} \cup \{w\}$.

5 [Compute minimum sum] Let j be the least i such that $C_i \neq \emptyset$. Set $m = 0$, $i = j$ and $d = 0$. While $i \leq s$ and $d < r$ do the following: if $C_i \neq \emptyset$, let d_i be the dimension of the real subspace generated by $C_j \cup \cdots \cup C_i$ minus the dimension of the real subspace generated by $C_j \cup \cdots \cup C_{i-1}$ and set $d = d + d_i$, $m = m + id_i$, else set $d_i = 0$; set $i = i + 1$.

6 [All subsets of C with minimum sum] Set $E = \emptyset$, $q = 0$ and $S_{j-1} = \{0\}$. For $n = j$ until $i - 1$ do the following: if $d_n \neq 0$, set $I_n = \{w \in C_n : w \notin S_j + \cdots + S_{n-1}\}$ and let E_n be the set of all the subsets of I_n with d_n linearly independent vectors and set $E = E \times E_n$ and $q = q + 1$.

7 [All minimal bases] Set $U = \emptyset$. For each $(U_1, \ldots, U_q)$ in E do the following: if the dimension of the subspace generated by $U_1 \cup \ldots \cup U_q$ is r (that is, $U_1 \cup \ldots \cup U_q$ is a minimal base of S in $(\mathbb{Z}_0^+)^k$) then set $U = U \cup \{\{U_1 \cup \ldots \cup U_q\}\}$.

The set U contains all the minimal bases of S in $(\mathbb{Z}_0^+)^k$. Output U and m and terminate algorithm.

$\diamond$

The item 3 of Algorithm 8.8 for solving homogeneous linear systems of congruences is based on Cohen [6, Algorithm 4.1.22] .

9 Examples

In this section we apply Algorithm 8.8 to two examples of homogeneous coupled cell networks G obtaining the adjacency matrices corresponding to the minimal networks of the subclasses $\mathcal{Min}_{[G]}$ of $[G]$.

Example 9.1 Consider the homogeneous network G of three cells of Figure 7. Thus the
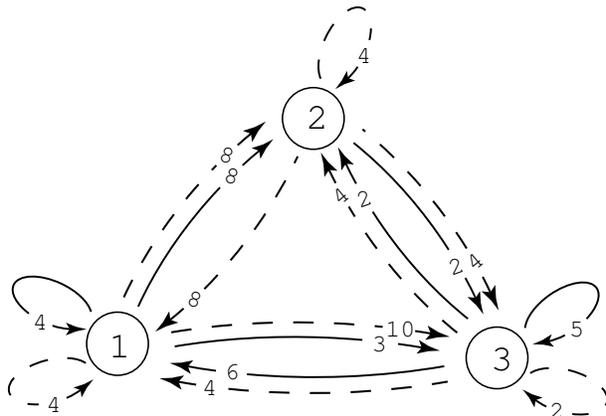

Figure 7: Homogeneous coupled cell network of Example 9.1.

adjacency matrices are

$$M_{(G,1)} = \begin{pmatrix} 4 & 0 & 6 \\ 8 & 0 & 2 \\ 3 & 2 & 5 \end{pmatrix}, \quad M_{(G,2)} = \begin{pmatrix} 4 & 8 & 4 \\ 8 & 4 & 4 \\ 10 & 4 & 2 \end{pmatrix}$$

Note that

$$\dim \mathbb{R}\left\{\mathrm{Id}_3, M_{(G,1)}, M_{(G,2)}\right\} = 3$$

and recall that the adjacency matrices of the networks in $\mathcal{Min}_{[G]}$ satisfy the three properties of Remark 7.12. Thus we aim to find all the possible matrices $M_1, M_2 \in M_{3\times3}(\mathbb{Z}_0^+)$ such that $\mathbb{R}\{M_0 = \mathrm{Id}_3, M_1, M_2\} = \mathbb{R}\{\mathrm{Id}_3, M_{(G,1)}, M_{(G,2)}\}$ and

$$\sum_{k=0}^{2} l(M_k) = \min\left\{\sum_{k=0}^{2} l(b_k) : \{b_0 = \mathrm{Id}_3, b_1, b_2\} \subset M_{3\times3}(\mathbb{Z}_0^+) \text{ is a basis of } \mathbb{R}\left\{\mathrm{Id}_3, M_{(G,1)}, M_{(G,2)}\right\}\right\}$$

For that we apply Algorithm 8.8. We consider

$$w_1 = (1,0,0,0,1,0,0,0,1)^t, \quad w_2 = (4,0,6,8,0,2,3,2,5)^t, \quad w_3 = (4,8,4,8,4,4,10,4,2)^t$$

in $(\mathbb{Z}_0^+)^9$ and S the real subspace of $\mathbb{R}^9$ generated by these linearly independent vectors:

$$S = \mathbb{R}\ \{w_1, w_2, w_3\}$$

Note that

$$s = \max\{l(w_1),\ l(w_2),\ l(w_3)\} = l(w_3) = 48$$

and the matrix

$$W = \begin{pmatrix} 1 & 4 & 4 \\ 0 & 0 & 8 \\ 0 & 6 & 4 \\ 0 & 8 & 8 \\ 1 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 3 & 10 \\ 0 & 2 & 4 \\ 1 & 5 & 2 \end{pmatrix}$$

has rank 3. The greatest common divisor of the determinants of all 3×3 matrices formed by 3 rows of W and with nonzero determinant is $d = 8$. Applying the HNF Algorithm A.3 to the matrix $(W|8\mathrm{Id}_9)$ we obtain matrices U and the HNF H of $(W|8\mathrm{Id}_9)$ such that

$(W|8\mathrm{Id}_9)U = (0|H)$:

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} = \begin{pmatrix}
-8 & 44 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & -5 & 1 \\
0 & -10 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\
0 & 3 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -3 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 6 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 7 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\
1 & -7 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

where

$$U_1 = \begin{pmatrix} -8 & 44 & -16 \\ 0 & -10 & 4 \\ 0 & 3 & -2 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix}
8 & 0 & 0 & 0 & 0 & 0 & 4 & 7 & 1 \\
0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 8 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 0 & 4 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 8 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Computing now H_B, the HNF of the matrix $(U_1|8\mathrm{Id}_3)$ (applying Algorithm A.3), we

obtain

$$
H_B = \begin{pmatrix} 8 & 0 & 4 \\ 0 & 8 & 2 \\ 0 & 0 & 1 \end{pmatrix}
\quad \text{and} \quad
WH_B = \begin{pmatrix} 8 & 32 & 16 \\ 0 & 0 & 8 \\ 0 & 48 & 16 \\ 0 & 64 & 24 \\ 8 & 0 & 8 \\ 0 & 16 & 8 \\ 0 & 24 & 16 \\ 0 & 16 & 8 \\ 8 & 40 & 16 \end{pmatrix}
$$

and so

$$
C = \left\{ \begin{pmatrix} x_1 + 4x_2 + 2x_3 \\ x_3 \\ 6x_2 + 2x_3 \\ 8x_2 + 3x_3 \\ x_1 + x_3 \\ 2x_2 + x_3 \\ 3x_2 + 2x_3 \\ 2x_2 + x_3 \\ x_1 + 5x_2 + 2x_3 \end{pmatrix} \in (\mathbb{Z}_0^+)^9 : \ x_1, x_2, x_3 \in \mathbb{Z}, \ 0 < x_1 + 10x_2 + 5x_3 \le 16 \right\}
$$

In particular, it follows that for

$$
w = (x_1 + 4x_2 + 2x_3, x_3, 6x_2 + 2x_3, 8x_2 + 3x_3, x_1 + x_3, 2x_2 + x_3, 3x_2 + 2x_3, 2x_2 + x_3, x_1 + 5x_2 + 2x_3)^t \in C
$$

we have $l(w) \geq 3x_3$ and $x_3 \geq 0$. Easy computations show that

$$C_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad C_6 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}, \quad C_9 = \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right\}, \quad C_{12} = \left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Moreover, if S_i denotes the real subspace of S generated by C_i, we have that

$$\dim S_3 = \dim(S_3 + S_6) = \dim(S_3 + S_6 + S_9) = 1, \quad \dim(S_3 + S_6 + S_9 + S_{12}) = \dim S = 3$$

By Theorem 8.7, there is a unique minimal basis of S which is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 2 \\ 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Thus, by Proposition 7.11 and Remark 7.12, $\mathcal{M}in_{[G]}$ contains only one minimal network with adjacency matrices given by

$$M_1 = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$
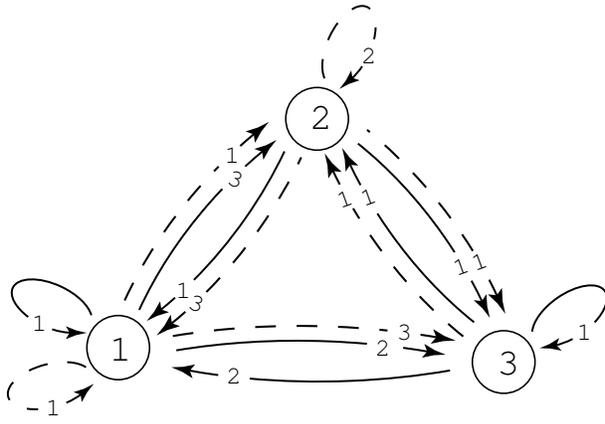
See Figure 8.

$\diamond$

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Figure 8: Minimal network of the ODE-equivalence class of the homogeneous coupled cell network of Example 9.1.

Example 9.2 Consider the homogeneous network G of Figure 3 in Section 1. The adjacency matrices of G are

$$
M_{(G,1)} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}, \quad M_{(G,2)} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \quad \text{and } \dim \mathbb{R}\left\{\mathrm{Id}_3, M_{(G,1)}, M_{(G,2)}\right\} = 3
$$

We aim to find all the possible matrices $M_1, M_2 \in M_{3\times3}(\mathbb{Z}_0^+)$ such that $\mathbb{R}\{M_0 = \mathrm{Id}_3, M_1, M_2\} = \mathbb{R}\{\mathrm{Id}_3, M_{(G,1)}, M_{(G,2)}\}$ and such that

$$
\sum_{k=0}^{2} l(M_k) = \min\left\{\sum_{k=0}^{2} l(b_k) : \ \{b_0 = \mathrm{Id}_3, b_1, b_2\} \subset M_{3\times3}(\mathbb{Z}_0^+) \text{ is a basis of } \mathbb{R}\left\{\mathrm{Id}_3, M_{(G,1)}, M_{(G,2)}\right\}\right\}
$$

Again, we apply Algorithm 8.8 to the vectors

$$
w_1 = (1,0,0,0,1,0,0,0,1)^t, \quad w_2 = (1,1,2,2,1,1,3,0,1)^t, \quad w_3 = (1,3,0,0,1,3,1,2,1)^t
$$

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in $(\mathbb{Z}_0^+)^9$. Note that $s = l(w_3) = 12$. Easy computations show that

$$C = \left\{ \begin{pmatrix} x_1 + x_2 + x_3 \\ x_2 + 2x_3 \\ 2x_2 + x_3 \\ 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_2 + 2x_3 \\ 3x_2 + 2x_3 \\ x_3 \\ x_1 + x_2 + x_3 \end{pmatrix} \in (\mathbb{Z}_0^+)^9 : \; x_1, x_2, x_3 \in \mathbb{Z}, \; 0 < x_1 + 4x_2 + 4x_3 \le 4 \right\}$$

Moreover,

$$C_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \; C_6 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}, \; C_9 = \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \\ 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \\ 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 3 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

If S_i denotes the real subspace of S generated by C_i, we have that

$$\dim S_3 = \dim(S_3 + S_6) = 1, \quad \dim(S_3 + S_6 + S_9) = \dim S = 3$$

Thus, by Proposition 7.11 and Remark 7.12, $\mathcal{Min}_{[G]}$ contains three minimal networks with adjacency matrices given, respectively, by

$$\left\{ M_1 = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \right\}, \quad \left\{ M_1 = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{pmatrix} \right\}$$

and

$$\left\{ M_1 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix}, \ M_2 = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{pmatrix} \right\}$$

See Figure 4 in Section 1.

$\diamond$

10 Identical-Edge Homogeneous Networks

In the special class of identical-edge homogeneous cell networks, the application of Algorithm 8.8 simplifies. Recall that an identical-edge homogeneous cell network is an homogeneous network in which all edges in $\mathcal{E}$ are equivalent. We prove below (Theorem 10.3) that, if G is an identical-edge homogeneous network with n cells, with adjacency matrix $M_{(G,1)} = [a_{ij}]_{1 \leq i,j \leq n}$, and $[G]$ the corresponding ODE-class, then $\mathcal{M}in_{[G]}$ has a unique graph with adjacency matrix M defined by

$$M = \frac{1}{d} \left(M_{(G,1)} - m \operatorname{Id}_n \right)$$

where $d = \gcd\{a_{ij} : i, j = 1, \ldots, n; \ j \neq i\}$ and $m = \min\{a_{ii} : i = 1, \ldots, n\}$. Direct application of Algorithm 8.8 gives the same result. See Remark 10.5. Here, we consider only networks G such that there is at least one edge $e \in \mathcal{E}$ satisfying $\mathcal{H}(e) \neq \mathcal{T}(e)$, and so $M_{(G,1)}$ is not a scalar multiple of the identity matrix.

Example 10.1 The graphs G_1 and G_2 of Figures 1 and 2 in Section 1, respectively, are examples of identical-edge homogeneous coupled cell networks. The corresponding adjacency matrices are

$$M_{(G_1,1)} = \begin{pmatrix} 2 & 9 & 9 \\ 12 & 2 & 6 \\ 15 & 0 & 5 \end{pmatrix}, \quad M_{(G_2,1)} = \begin{pmatrix} 0 & 3 & 3 \\ 4 & 0 & 2 \\ 5 & 0 & 1 \end{pmatrix}$$

We show below that the minimal subclass of $[G_1]$ is $\mathcal{M}in_{[G_1]} = \{G_2\}$. $\diamond$

Linear Equivalence and Adjacency Matrices

Definition 5.3 of linear equivalence for identical-edge homogeneous networks in terms of the adjacency matrices is now a special case of Proposition 7.4. More precisely, if G_1 and G_2 are two identical-edge homogeneous networks with n cells, where $\mathcal{C}_1 = \{1, \ldots, n\}$, and $M_{(G_1,1)} = [a_{ij}]_{1\leq i,j\leq n}$ and $M_{(G_2,2)} = [b_{ij}]_{1\leq i,j\leq n}$ are the corresponding adjacency matrices, then G_1 and G_2 are linearly equivalent if and only if there is a bijection $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$ such that:

$$\mathbb{R}\left\{\mathrm{Id}_n, [a_{ij}]_{1\leq i,j\leq n}\right\} = \mathbb{R}\left\{\mathrm{Id}_n, [b_{\gamma(i)\gamma(j)}]_{1\leq i,j\leq n}\right\} \tag{10.15}$$

Example 10.2 Consider the identical-edge homogeneous networks of Figures 1 and 2 in Section 1, and recall Example 5.4. Trivially, the identity function on $\{1, 2, 3\}$ is a bijection that preserves cell-equivalence and input-equivalence (like any bijection on $\{1, 2, 3\}$). Moreover, equation (5.6) of Example 5.4 on the corresponding adjacency matrices is equivalent to the linear equivalence of the networks. $\diamond$

Minimality

We prove now that the minimal subclass $\mathcal{M}in_{[G]}$ of an ODE-class $[G]$ where G is an identical-edge homogeneous network contains a unique graph. We then describe an algorithm that computes this minimal graph in $\mathcal{M}in_{[G]}$.

Theorem 10.3 *Let G be an identical-edge homogeneous network of n cells, say $\mathcal{C} = \{1, \ldots, n\}$, and $[G]$ the corresponding ODE-class. Let $M_{(G,1)} = [a_{ij}]_{1\leq i,j\leq n}$ be the adjacency matrix of G and assume that it is not a scalar multiple of the identity matrix. Let*

$$d \;=\; \gcd\{a_{ij} : \; i, j = 1, \ldots, n; \; j \neq i\}$$

$$m \;=\; \min\{a_{ii} : \; i = 1, \ldots, n\}$$

Then

$$\mathcal{M}in_{[G]} = \{G_2\}$$

where G_2 is the network with adjacency matrix

$$M_{(G_2,1)} = \frac{1}{d}\left(M_{(G,1)} - m\,\mathrm{Id}_n\right)$$

Proof The proof consists in showing that G_2 as described above is linearly equivalent to G, and that if G_1 is linearly equivalent to G_2 and $G_1 \neq G_2$ then $\text{card}(\mathcal{E}_2) < \text{card}(\mathcal{E}_1)$.

As $M_{(G_2,1)} = (1/d)(M_{(G,1)} - m\text{Id}_n)$ it follows that the identical-edge homogeneous network with adjacency matrix $M_{(G_2,1)}$ is linearly equivalent to G and the entries of $M_{(G_2,1)}$ are nonnegative integers as we show now. Let $M_{(G_2,1)} = [b_{ij}]_{1 \leq i,j \leq n}$, then

$$b_{ii} = \frac{a_{ii} - m}{d}, \quad i = 1, \ldots, n \tag{10.16}$$

$$b_{ij} = \frac{a_{ij}}{d}, \quad i, j = 1, \ldots, n; \ j \neq i. \tag{10.17}$$

Since G is an identical-edge homogeneous network it follows that $M_{(G,1)}$ satisfies for any $i, k \in \{1, \ldots, n\}$

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{kj}$$

and thus

$$a_{ii} - a_{kk} = \sum_{j \neq k} a_{kj} - \sum_{j \neq i} a_{ij}$$

Let k be such that $m = a_{kk}$. Then $a_{kk} \leq a_{ii}, \forall i$ and $\sum_{j \neq k} a_{kj} \geq \sum_{j \neq i} a_{ij}$. Moreover, d is a divisor of $\sum_{j \neq k} a_{kj} - \sum_{j \neq i} a_{ij}$ and thus of $a_{ii} - a_{kk}$.

Let G_1 be a network linearly equivalent to G_2 (and so to G) with adjacency matrix $M_{(G_1,1)}$. It follows then that

$$M_{(G_1,1)} = \frac{1}{\beta}\left(M_{(G_2,1)} - \alpha\text{Id}_n\right),$$

for some $\alpha \in \mathcal{Q}$ and $\beta \in \mathcal{Q}^+$. Since there is i such that $b_{ii} = 0$ we have that $\alpha \in \mathbb{Z}_0^-$. If $\text{card}(\mathcal{E}_1) \leq \text{card}(\mathcal{E}_2)$ then $\beta \geq 1$. Since the greatest common divisor of the entries b_{ij} is 1, then $\beta = 1$. If $\alpha = 0$ then $G_1 = G_2$. Otherwise, $\text{card}(\mathcal{E}_1) > \text{card}(\mathcal{E}_2)$. $\qquad \square$

Example 10.4 Recall the graphs G_1 and G_2 of Figures 1 and 2 in Section 1. By Theorem 10.3 it follows that $\mathcal{M}in_{G_1} = \{G_2\}$. Note that $d = 3$ and $m = 2$. $\qquad \diamond$

If we denote the rows of a $k \times r$ matrix M by $M_1, \ldots, M_k$, then M^* represents the column vector $(M_1, \ldots, M_k)^t$.

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Remark 10.5 Direct application of Algorithm 8.8 gives the result of Theorem 10.3 as we show now. Suppose G is an identical-edge homogeneous network with adjacency matrix $M_{(G,1)} = [a_{ij}]_{1 \leq i,j \leq n}$. Assume $M_{(G,1)}$ is not a scalar multiple of the identity matrix. Using the same procedure as in Section 8 we have the following: we take the $n^2 \times 2$ matrix $W = (w_1 | w_2)$ where $w_1 = \mathrm{Id}_n^*$ and $w_2 = M_{(G,1)}^*$; the greatest common divisor d of the set of all 2×2 matrices formed by 2 rows of W and with nonzero determinant is the greatest common divisor of the set $\{a_{ij} : i \neq j\} \cup \{a_{ii} - a_{kk} : i \neq k\}$. From the proof of Theorem 10.3, it follows that d is the greatest common divisor of the set $\{a_{ij} : i \neq j\}$. We aim to find the finite set C and so to find $y_1, y_2 \in \mathbb{Z}$ such that

$$y_1 w_1 + y_2 w_2 = dw$$

for some w with nonnegative integer entries and such that $0 < l(w) \leq l(w_2)$. Note that

$$l(w) = \frac{1}{d} (y_1 n + y_2 l(w_2))$$

and

$$dw_{(k-1)n+k} = y_1 + y_2 a_{kk} \quad (k = 1, \ldots, n)$$
$$dw_{(k-1)n+i} = y_2 a_{ki} \quad (i, k = 1, \ldots, n; \ i \neq k)$$

Thus $y_2 \geq 1$. Trivially, the minimum $l(w)$ is obtained by taking $y_2 = 1$ and $y_1 = -m$ where

$$m = \min\{a_{kk} : \ k = 1, \ldots, n\}$$

and so there is a unique minimal basis of $\mathbb{R}\{w_1, w_2\}$ corresponding to the minimal network with adjacency matrix

$$M_{(G_2,1)} = \frac{1}{d} \left(M_{(G,1)} - m\mathrm{Id}_n \right)$$

$\Diamond$

Algorithm 10.6 Given the adjacency matrix $M_{(G,1)} = [a_{ij}]_{1 \leq i,j \leq n}$, with $M_{(G,1)} \neq \alpha \mathrm{Id}_n$, of an identical-edge homogeneous network G, this algorithm computes the adjacency matrix $M_{(G_2,1)} = [b_{ij}]_{1 \leq i,j \leq n}$ corresponding to the unique network in $\mathcal{M}in_{[G]}$.

1 [Compute gcd d and minimum m] Set $m = \min\{a_{ii} : \ i = 1, \ldots, n\}$ and $d = \gcd\{a_{ij} : \ i, j = 1, \ldots, n; \ j \neq i\}$.

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2 [Compute $M_{(G_2,1)}$] For $i, j = 1, \cdots, n$ do the following: if $j \neq i$ then set $b_{ij} = a_{ij}/d$ else set $b_{ii} = (a_{ii} - m)/d$. Output $M_{(G_2,1)} = [b_{ij}]_{1 \leq i,j \leq n}$.

$\diamond$

11   Minimality for coupled cell networks

Let G be a coupled cell network. From Theorem 4.2 (recall also Remark 4.4) the space of $\mathcal{B}_G$-equivariant maps has a natural decomposition according to the connected components (the $\sim_I$-equivalence classes) of the groupoid $\mathcal{B}_G$, and this decomposition is inherited by the polynomial and linear vector fields:

Definition 11.1 Let $\mathcal{Q} \subseteq \mathcal{C}$ be an $\sim_I$-equivalence class. Define

$$\mathcal{F}_G^P(\mathcal{Q}) = \{f \in \mathcal{F}_G^P : f_s(x) = 0, \ \forall s \notin \mathcal{Q}\}$$
$$\mathcal{P}_G^P(\mathcal{Q}) = \{f \in \mathcal{P}_G^P : f_s(x) = 0, \ \forall s \notin \mathcal{Q}\}$$
$$\mathcal{L}_G^P(\mathcal{Q}) = \{f \in \mathcal{L}_G^P : f_s(x) = 0, \ \forall s \notin \mathcal{Q}\}$$

We say that vector fields in $\mathcal{F}_G^P(\mathcal{Q})$, $\mathcal{P}_G^P(\mathcal{Q})$, and $\mathcal{L}_G^P(\mathcal{Q})$ are *supported on* $\mathcal{Q}$. $\diamond$

Remark 11.2 From Theorem 4.2, there are direct sum decompositions

$$\mathcal{F}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{F}_G^P(\mathcal{Q}) \qquad \mathcal{P}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{P}_G^P(\mathcal{Q}) \qquad \mathcal{L}_G^P = \bigoplus_{\mathcal{Q}} \mathcal{L}_G^P(\mathcal{Q})$$

where $\mathcal{Q}$ runs over the $\sim_I$-equivalence classes of G. $\diamond$

For detailed proofs see [14], end of Section 4, especially Proposition 4.6.

By Theorem 5.5, two coupled cell networks G_1 and G_2 are ODE-equivalent if and only if they are linearly equivalent. That is, there exists some bijection γ between the corresponding sets of cells $\mathcal{C}_i$, preserving cell-equivalence and input-equivalence, such that with the identification $\gamma : \mathcal{C}_1 \to \mathcal{C}_2$, the vector spaces $\mathcal{L}_{G_1}^P$ and $\mathcal{L}_{G_2}^P$ are equal. By the above discussion, this is equivalent to have $\mathcal{L}_{G_1}^P(\mathcal{Q}) = \mathcal{L}_{G_2}^P(\mathcal{Q})$, for each connected component $\mathcal{Q}$. Moreover, all cell phase spaces may be assumed to be $\mathbb{R}$ in this context.

We look now for $\mathcal{M}in_{[G]}$ where G is an n-cell network. Suppose that G has connected components ($\sim_I$-equivalence classes) $\mathcal{Q}_1, \ldots, \mathcal{Q}_l$. For $i = 1, \ldots, l$ define the subnetworks G_i in the following way:

$$G_i = \{\mathcal{C}, \mathcal{E}_i, \sim_C, \sim_{E_i})$$

where

$$\mathcal{E}_i = \{e \in \mathcal{E} : \ \mathcal{H}(e) \in \mathcal{Q}_i\} \text{ and } \forall e_1, e_2 \in \mathcal{E}_i, e_1 \sim_{E_i} e_2 \Leftrightarrow e_1 \sim_E e_2$$

We can apply Algorithm 8.8 to the graphs G_i, for $i = 1, \ldots, l$, computing $\mathcal{M}in_{[G_1]}, \ldots,$ $\mathcal{M}in_{[G_l]}$. The minimal class $\mathcal{M}in_{[G]}$ is then obtained by considering all possible graphs that can be formed by junction of one graph of each class $\mathcal{M}in_{[G_1]}, \ldots, \mathcal{M}in_{[G_l]}$.

A   Algorithms Appendix

In this section we present an algorithm given by Cohen [5] to calculate the Hermite normal form of a matrix with integer coefficients. We use the following definition:

Definition A.1 ([5], Definition 2.4.2) We say that an $m \times n$ matrix $M = [m_{ij}]$ with integer coefficients is in *Hermite normal form* (abbreviated HNF) if there exists $r \leq n$ and a strictly increasing map f from $[r+1, n]$ to $[1, m]$ satisfying the following properties:
(1) For $r + 1 \leq j \leq n$, $m_{f(j)j} \geq 1$, $m_{ij} = 0$ if $i > f(j)$ and $0 \leq m_{f(k)j} < m_{f(k)k}$ if $k < j$.
(2) The first r columns of M are equal to 0. $\diamond$

We have the following result:

Theorem A.2 ([5], Theorem 2.4.3) *Let A be an $m \times n$ matrix with coefficients in $\mathbb{Z}$. Then there exists a unique $m \times n$ matrix $B = [b_{ij}]$ in HNF of the form $B = AU$ with $U \in GL_n(\mathbb{Z})$, where $GL_n(\mathbb{Z})$ is the group of matrices with integer coefficients which are invertible, that is, whose determinant is equal to ± 1.*

Although B is unique, the matrix U is not unique. The matrix W formed by the nonzero columns of B is called the Hermite normal form of the matrix A.

Algorithm A.3 ([5], Algorithm 2.4.4) (Hermite Normal Form). Given an $m \times n$ matrix A with integer coefficients $[a_{ij}]$ this algorithm finds the Hermite normal form H of A

and a $n \times n$ unimodular matrix U such that $AU = (0|H)$. We write h_{ij} for the coefficients of H, A_i (resp. H_i, U_i) for the columns of A (resp. H, U).

1. [Initialize] Set $i = m$, $k = n$ and $U = \mathrm{Id}_n$. If $m \leq n$ then set $l = 1$ else set $l = m - n + 1$.

2. [Row finished?] If all the a_{ij} with $j < k$ are zero, then if $a_{ik} < 0$ replace column A_k by $-A_k$, replace column U_k by $-U_k$ and go to step 5.

3. [Choose non-zero entry] Pick among the non-zero a_{ij}, with $j \leq k$, one with the smallest absolute value, say a_{ij_0}. Then if $j_0 < k$, exchange column A_k with column A_{j_0} and exchange column U_k with column U_{j_0}. In addition, if $a_{ik} < 0$ replace column A_k by $-A_k$ and replace column U_k by $-U_k$. Set $b = a_{ik}$.

4. [Reduce] For $j = 1, \ldots, k - 1$ do the following: set $q = \lfloor a_{ik}/b \rfloor$, and $A_j = A_j - qA_k$, $U_j = U_j - qU_k$. Then go to step 2.

5. [Final reductions] Set $b = a_{ik}$. If $b = 0$, set $k = k + 1$ and go to step 6. Otherwise, for $j > k$ do the following: set $q = \lfloor a_{ik}/b \rfloor$, and $A_j = A_j - qA_k$, $U_j = U_j - qU_k$.

6. [Finished?] If $i = l$ then for $j = 1, \ldots, n - k + 1$ set $H_j = A_{j+k-1}$ and terminate the algorithm, else set $i = i - 1, k = k - 1$ and go to step 2.

$\diamondsuit$

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