TOTALLY SYNCHRONIZING MONOTONIC GRAPHS

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Abstract. In this paper, we solve the problem of synchronizing acyclic and aperiodic digraphs as well as those for which the resulting automata preserve some order in the vertex set. We also find minimum rank words for the same classes of graphs.

1. Introduction

Forty five years ago, Černý [24] presented a family of synchronizing automata with n states whose shortest reset words have size $(n-1)^2$ and conjectured that for every automaton with n states, if there is a synchronizing word, then there is one with at most $(n-1)^2$ letters. Several advances have been made towards the proof of this conjecture [2, 3, 5–9, 12, 13, 16–23, 25, 26], but the general case remains open. The best known upper bound is $(n^3-n)/6$, due to Pin [15], using a combinatorial result from Frankl [10].

The following generalization of Černý's conjecture was suggested by Pin [14]: if for a given automaton with n states there is a word of rank r, then there is such a word of size at most $(n-r)^2$. However, Kari [11] gave a counterexample to this conjecture. A reformulated version of Pin's conjecture, which states that every automaton with n states and minimal rank r has a word of rank r of size at most $(n-r)^2$, still remains open. This is known as the Rank conjecture, or the Černý-Pin conjecture, and Černý's conjecture is the case where $r=1$.

In 2008, at the School on Algebraic Theory of Automata in Lisbon, Mikhail V. Volkov suggested several problems related to the conjectures mentioned above. Among those problems were the questions of characterizing totally synchronizing digraphs, that is, such that every automaton obtained from them by suitably labeling the edges is synchronizing, and finding universal reset words for such graphs. Relating this to the Černý-Pin conjecture, one obtains the rank problem for digraphs, that is, the problem of computing shortest minimal rank words for such graphs.

Following the definition of monotonic automata from [4], we solve the rank problem for monotonic graphs. We also present a solution of that problem for generalized monotonic graphs and aperiodic graphs, corresponding to the definitions of generalized monotonic automata [5] and aperiodic automata [23], respectively, by showing that those classes of graphs are equal to the class of acyclic digraphs.

2. Definitions

For a digraph G with n vertices and constant outdegree k, we assume that k is never greater than n, because in that case every vertex in G would have at least $k-n$ pairs of outgoing edges with the same end point, hence, we would be able to delete $k-n$ outgoing edges for each vertex in G and obtain a graph with the same relevant properties and constant outdegree equal to n. Actually, we assume

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the outdegree of a graph to be the number of distinct outgoing edges in a vertex for which that number is maximum.

Given a finite digraph G with constant outdegree k and an alphabet Σ with k letters, we say that a labeling of the edges in G using the letters from Σ is *suitable*, if it turns G into a deterministic finite automaton (without sets of initial and final states) $\mathcal{G} = (S, \Sigma, \delta)$, where S is the set of vertices of G (set of states of $\mathcal{G}$) and $\delta : S \times \Sigma \to S$ is the transition function that to $(s, x) \in S \times \Sigma$ associates the ingoing vertex of the only edge leaving s with label x. The function δ is extended on the second component to the set of all words in the alphabet Σ, Σ^*, in the obvious way.

We call a directed graph G *acyclic* if all the directed cycles in G are *trivial*, that is, they involve only one vertex. We say that a directed cycle is *nontrivial* if it involves more than one vertex of G.

A semigroup A is said to be *aperiodic* if all its subgroups are trivial, which is equivalent to the property that for every $a \in A$, there is some $m \in \mathbb{N}$ such that $a^m = a^{m+1}$. A deterministic finite automaton $\mathcal{G} = (S, \Sigma, \delta)$ whose transition semigroup is aperiodic is called *aperiodic* as well. We say that a finite digraph G with constant outdegree is *aperiodic* if every automaton obtained from G by a suitable labeling is aperiodic.

We say that a digraph G with constant outdegree is *monotonic* if for every transition function δ associated with a suitable labeling of G, there is a total order $\leq$ on the vertex set S, such that for all $p, q \in S$ and $x \in \Sigma$, $p \leq q \Rightarrow \delta(p, x) \leq \delta(q, x)$, we call such an order a *perfect order*.

Given a deterministic finite automaton $\mathcal{G} = (S, \Sigma, \delta)$, a *congruence* on the state set S is an equivalence relation ρ, such that for every $x \in \Sigma$ and every $p, q \in S$, $(p, q) \in \rho \Rightarrow (\delta(p, x), \delta(q, x)) \in \rho$. Denote by $[p]_\rho$ the ρ-class that contains the state $p \in S$. We define the *quotient automaton* $\mathcal{G}/\rho$ as the automaton $(S/\rho, \Sigma, \delta_\rho)$, where $S/\rho = \{[p]_\rho : p \in S\}$ and the transition function δ_ρ is such that for every ρ-class $[p]_\rho$, $\delta_\rho([p]_\rho, x) = [\delta(p, x)]_\rho$.

Consider a congruence ρ on the automaton $\mathcal{G} = (S, \Sigma, \delta)$. We say that $\mathcal{G}$ is *ρ-monotonic*, see [5], if there is a partial order $\leq$ on the state set S for which:

- the states p and q are $\leq$-comparable if and only if $(p, q) \in \rho$;
- for all $p, q \in S$ and $x \in \Sigma$, $p \leq q \Rightarrow \delta(p, x) \leq \delta(q, x)$.

An automaton $\mathcal{G}$ is said to be *generalized monotonic of level l* [5], if there is a sequence of congruences on $\mathcal{G}$ $\rho_0 \subsetneq \rho_1 \subsetneq \ldots \subsetneq \rho_l$, such that ρ_0 is the equality relation, ρ_l is the universal relation and $\mathcal{G}/\rho_{i-1}$ is ρ_i/ρ_{i-1}-monotonic for every $i \in \{1, 2, \ldots, l\}$. This way, generalized monotonic automata of level 1 are just monotonic automata. We say that the automaton $\mathcal{G}$ is *generalized monotonic* if it is generalized monotonic of level l for some l. The digraph G is called *generalized monotonic* if every automaton obtained from G by suitably labeling its edges is generalized monotonic.

We say that a digraph G with constant outdegree is *synchronizing* if there is a suitable labeling for which the resulting automaton $\mathcal{G} = (S, \Sigma, \delta)$ is synchronizing, that is, there is a word $w \in \Sigma^*$ such that the function δ restricted to $S \times \{w\}$ is constant. We say that G is *totally synchronizing* if each of its suitable labelings leads to a synchronizing automaton. Given such a graph, a *universal reset word* is a word $w \in \Sigma$, such that w is a reset word for every automaton obtained from G by suitably labeling its edges.

Given a digraph G with constant outdegree, an automaton $\mathcal{G} = (S, \Sigma, \delta)$ resulting from a suitable labeling of the edges in G and a word $w \in \Sigma^*$, we say that w has *rank* r with respect to $\mathcal{G}$, if $\delta(S, w)$ has exactly r elements. The *rank* of the automaton $\mathcal{G}$ is the minimum rank of all words in Σ^* with respect to $\mathcal{G}$. The *rank* of the graph G is the maximum rank of all the automata obtained from G by suitably labeling its edges.

Given a digraph G, a *fixed point* is a vertex v such that all edges leaving v are loops. A *sink* is a fixed point v such that for every vertex u in G there is a path connecting it to v. From these definitions we conclude that a digraph G cannot have more than one sink, but it can have several fixed points. We say that the subset S of vertices of G is *invariant* if every edge leaving a vertex in S has its endpoint in S.

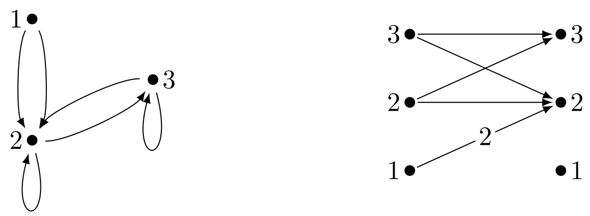

FIGURE 2.1. A graph and its function-like diagram.

As is shown in Figure 2.1, to represent graphs we will use function-like diagrams, in which we have two columns, each one with a copy of the vertices represented in the same order, and arrows going from the first to the second column. An arrow connecting i on the left column to j on the right column indicates that there is a directed edge in the graph going from the vertex i to the vertex j. We place labels on top of certain edges to represent their multiplicity and edges without label have multiplicity 1. To represent paths (instead of edges) we use dashed arrows.

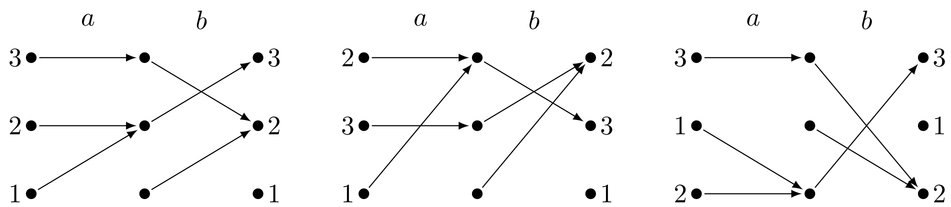

FIGURE 2.2. The function-like diagrams of a deterministic finite automaton resulting from the graph in Figure 2.1.

When considering deterministic finite automata, the function-like diagram will be drawn by dividing the edges between several columns, according to their label, which will be indicated at the top of its column. This way, if a digraph is monotonic, for every automaton resulting from it, there is an order on the state set, such that there are no crosses between arrows in the same column of the function-like diagram. As we can see in Figure 2.2, the graph from Figure 2.1 is not monotonic.

If for a digraph G with constant outdegree, there is an order on its vertex set, such that there are no crosses between the arrows in its function-like diagram, then G is monotonic. Examples of such graphs can be seen in Figure 2.3. We were not able to prove the converse statement nor to find a counterexample for it.

3. APERIODIC, GENERALIZED MONOTONIC AND ACYCLIC GRAPHS

In this section we establish important relations between the classes of graphs that are being considered.

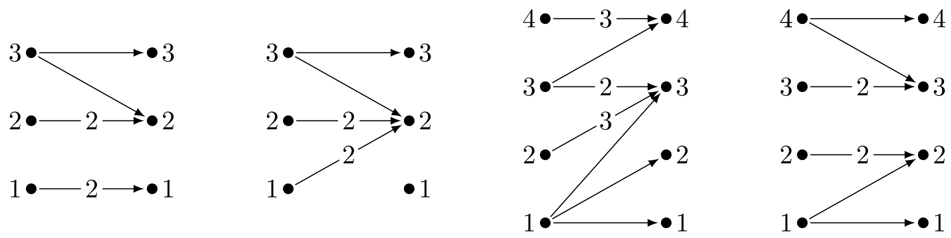

Figure 2.3. Examples of monotonic graphs.

Lemma 3.1. *Suppose that C is a nontrivial directed cycle of minimum length in a digraph G. Then we may consider a suitable labeling of the edges in G such that every edge in C has the same label x.*

Proof. Since C has minimum length among the nontrivial cycles, every vertex t in C has only one outgoing edge that belongs to C. Therefore, when building a suitable labeling of the edges in G. we may label all the edges in C with the letter x. □

Lemma 3.2. *Every monotonic graph is acyclic.*

Proof. Let G be a monotonic graph with n vertices ($n \geq 2$). If G is not acyclic, then it has some nontrivial cycle. Let C be such a cycle of minimum length and suppose that it has the form $t_1 \longrightarrow t_2 \longrightarrow \cdots \longrightarrow t_m \longrightarrow t_1$. According to Lemma 3.1, we may consider a suitable labeling of G with transition function δ, such every edge in C has label x. By definition of monotonic graph, we have $t_i \leq t_{i+1} \Rightarrow \delta(t_i, x) \leq \delta(t_{i+1}, x)$, hence $t_i \leq t_{i+1} \Rightarrow t_{i+1} \leq t_{i+2}$, for $i \in \{1, 2, \ldots, m-2\}$, and equally $t_{m-1} \leq t_m \Rightarrow t_m \leq t_1$. This way, $t_1 \leq t_2 \Rightarrow t_2 \leq t_3 \Rightarrow \cdots \Rightarrow t_{m-1} \leq t_m \Rightarrow t_m \leq t_1 \Rightarrow t_1 \leq t_2$, so $t_1 = t_2 = \cdots = t_m$, which is absurd. A similar argument can be used for the case $t_1 \geq t_2$. Therefore G is acyclic. □

A counterexample to the reverse implication of Lemma 3.2 can be found in Figure 3.1. The graph represented is acyclic and yet, the chosen automaton is not monotonic.

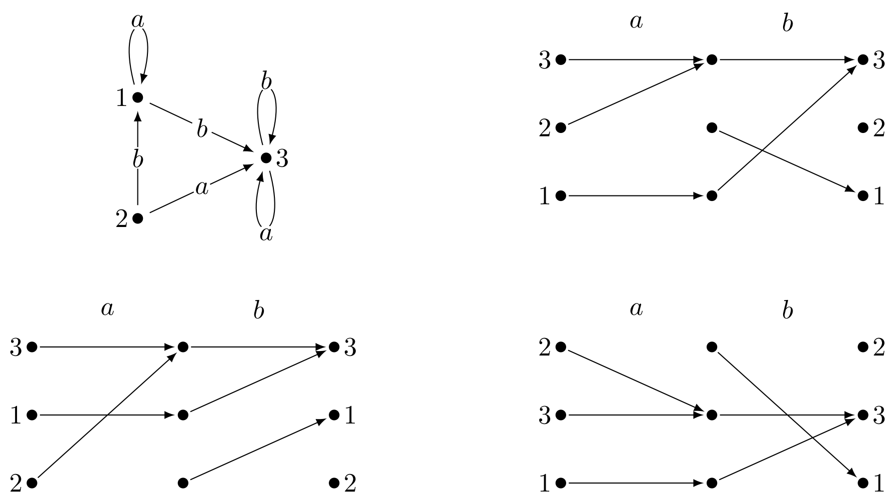

Figure 3.1. A graph with only trivial directed cycles that is not monotonic.

Proposition 3.3. *For a finite digraph G with constant outdegree, the following conditions are equivalent:*

(1) *G is aperiodic;*
(2) *G is generalized monotonic;*
(3) *G is acyclic.*

Proof. (1) $\Rightarrow$ (3). We prove the contrapositive. Suppose that G has some nontrivial directed cycle. Let C be such a cycle of minimum length. Using Lemma 3.1 consider a suitable labeling of the edges in G such that every edge in C has label $a \in \Sigma$. Let $\mathcal{G}$ be the resulting automaton. If $l > 1$ is the length of C, then a acts as a permutation of order l on the vertices of C, thus there can be no $m \in \mathbb{N}$ such that $a^m = a^{m+1}$ and G is not aperiodic.

(3) $\Rightarrow$ (1). We demonstrate the contrapositive. Suppose that G is not aperiodic, then some automaton $\mathcal{G} = (S, \Sigma, \delta)$ obtained from G by suitably labeling its edges is not aperiodic, that is, there is some element $a \neq 1$ of the transition semigroup A of $\mathcal{G}$ such that $a^m \neq a^{m+1}$ for every $m \in \mathbb{N}$. On the other hand, since A is finite, there are $l, j \in \mathbb{N}$ such that $j > l$ and $a^l = a^j$. But this means that a^{j-l} labels some nontrivial directed cycle in $\mathcal{G}$, because if all such cycles involved only one vertex, we would have $\delta(p, a^l) = \delta(p, a^{l+1}) = \cdots = \delta(p, a^j)$ for every $p \in S$, which is absurd, since $a^l \neq a^{l+1}$. Thus, G is not acyclic.

(2) $\Rightarrow$ (3). Once more, we prove the contrapositive. Suppose that C is a nontrivial directed cycle of G with minimum length $m > 1$. Using Lemma 3.1 consider a suitable labeling of the edges in G such that every edge in C has label $a \in \Sigma$ and let $\mathcal{G} = (S, \Sigma, \delta)$ be the resulting automaton. For every congruence ρ on S, if $(p, q) \in \rho$ with $p \neq q$ vertices in C, then either every vertex in C is in $[p]_\rho$ or the number of vertices of C that belong to $[p]_\rho$ is a proper divisor $d \neq 1$ of m. This way, when trying to build a chain of congruences that makes $\mathcal{G}$ generalized monotonic, either we collapse the entire cycle C at once or we do it in several steps. In the latter case, we start by joining d_1 vertices in the same class, where $d_1 \neq 1$ is a proper divisor of m, and obtain in the quotient automaton a cycle with m/d_1 elements. After i steps, we collapse d_{i+1} vertices of the remaining cycle, where $d_{i+1} \neq 1$ is a divisor of $m/(d_1 d_2 \dots d_i)$ and obtain in the quotient automaton a cycle with $m/(d_1 \dots d_i d_{i+1})$ elements. But at some point, since m is finite a congruence ρ_{j+1} will collapse an entire nontrivial cycle, that is, $d_{j+1} = m/(d_1 d_2 \dots d_j) \neq 1$.

Now, let $\mathcal{H}$ be a subautomaton of $\mathcal{G}/\rho_j$ consisting only of the nontrivial cycle that is collapsed by ρ_{j+1}. If $\mathcal{G}/\rho_j$ is ρ_{j+1}/ρ_j-monotonic, then $\mathcal{H}$ is monotonic and its underlying graph H is also monotonic (it has only one possible suitable labeling). But this is absurd, because Lemma 3.2 establishes that every monotonic graph is acyclic. Hence, $\mathcal{G}/\rho_j$ is not ρ_{j+1}/ρ_j-monotonic and G is not generalized monotonic.

(3) $\Rightarrow$ (2). Suppose that G has vertex set S and is acyclic. If G has no edges, then it is obviously monotonic, hence we may assume that G has constant outdegree greater than 0. We define inductively a sequence of subsets of S as follows. The graph G must have at least one fixed point, since it is finite and acyclic. Let $F_1 \subseteq S$ be the set of fixed points of G. If $F_1 = S$, then G is obviously generalized monotonic, actually it is monotonic. Suppose that $F_1 \subsetneq S$, that we have already defined the sets $F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_{i-1} \subsetneq S$ and they are such that $|F_{j+1} - F_j| = 1$, for every $j \in \{1, 2, \dots, i-1\}$. Since G is finite and acyclic, there is some vertex in $G - F_{i-1}$ such that all its outgoing edges are loops or end in F_{i-1}. Let p be such a vertex and let $F_i = F_{i-1} \cup \{p\}$. Since S is finite and the sequence $F_1, F_2, \dots, F_{i-1}, F_i, \dots$ is strictly increasing, there must be some l such that $F_l = S$.

Define for every $j \in \{1, 2, \dots, l\}$ the equivalence relation ρ_j on S such that $[s]_{\rho_j} = F_j$, for each $s \in F_j$ and $[s]_{\rho_j} = \{s\}$ for each $s \notin F_j$. Given an automaton $\mathcal{G}$ obtained from G by suitably labeling its edges, ρ_j is a congruence on $\mathcal{G}$, because the only ρ_j-class that is not a singleton is an invariant set. If we consider the equality relation ρ_0, we have $\rho_0 \subsetneq \rho_1 \subsetneq \ldots \subsetneq \rho_l$, with ρ_l the universal relation on S. Since the only ρ_1-class that is not a singleton is the set of fixed points of S, $\mathcal{G}$ is ρ_1-monotonic. Also, because $|F_{j+1} - F_j| = 1$, $\mathcal{G}/\rho_j$ is ρ_{j+1}/ρ_j-monotonic for every $j \in \{1, 2, \ldots, l\}$. We conclude that $\mathcal{G}$ is generalized monotonic of level l and, since $\mathcal{G}$ was any automaton obtained from G by suitably labeling its edges, G is generalized monotonic. □

Alternatively, one could prove directly that every generalized monotonic digraph is aperiodic as follows. If G is generalized monotonic, any automaton $\mathcal{G}$ obtained from G by suitably labeling its edges is generalized monotonic. But it was observed in [5] that, since the notion of generalized monotonic automaton is an automata-theoretic counterpart to the notion of transformation monoid preserving a chain of interval partitions, introduced and studied by Almeida and Higgins [1], and every such monoid is aperiodic, then generalized monotonic automata are aperiodic. Hence $\mathcal{G}$ is aperiodic and therefore G is aperiodic.

Using Proposition 3.3, in order to solve the rank problem for aperiodic graphs and generalized monotonic graphs, all we have to do is solve it for acyclic graphs, which is what is presented in the next section.

4. Synchronizing acyclic graphs

Throughout this section, G is an acyclic digraph with n vertices ($n \geq 2$) and constant outdegree k ($1 < k \leq n$), v is a word with k distinct letters, over the alphabet $\Sigma = \{x_1, x_2, \ldots, x_k\}$, and δ is the transition function associated with a suitable labeling of the edges in G with the letters in Σ.

Proposition 4.1. *The following conditions are equivalent:*

(1) *every deterministic finite automaton $\mathcal{G} = (S, \Sigma, \delta)$ obtained from G by suitably labeling its edges has rank r;*
(2) *G has rank r;*
(3) *G has precisely r fixed points.*

Proof. (1) $\Rightarrow$ (2). Obvious from the definitions.

(2) $\Rightarrow$ (3). Consider a suitable labeling of the edges in G and suppose that the resulting automaton $\mathcal{G} = (S, \Sigma, \delta)$ has rank r. Let $w \in \Sigma^*$ be a word of rank r with respect to $\mathcal{G}$ and let $s \in \delta(S, w)$. If $\delta(s, x) = t$ for $t \in S$, $x \in \Sigma$, then $\delta(s, xw) = \delta(t, w) \in \delta(S, w)$, hence xw acts as a permutation on the elements of $\delta(S, w)$. This means that xw must be the identity, otherwise it would label one or more directed cycles involving distinct vertices of G. But if xw is the identity, then it labels a directed cycle passing through s and t, thus $s = t$. This way, s is fixed by every letter in Σ, which means that it is a fixed point of G.

On the other hand, if s is a fixed point of G, then it is necessarily fixed by every letter in Σ and therefore it is fixed by the word w, which means that it belongs to $\delta(S, w)$.

(3) $\Rightarrow$ (1). Assume $\mathcal{G} = (S, \Sigma, \delta)$ is a deterministic finite automaton resulting from a suitable labeling of the edges in G. We already know from the proof of the previous implication that for a word w of minimum rank with respect to $\mathcal{G}$, the r fixed points from G belong to $\delta(S, w)$. We also know that any other element in this set would be a fixed point for G, hence w has rank r. □

Corollary 4.2. *The following conditions are equivalent:*

(1) *G is totally synchronizing;*
(2) *G is synchronizing;*
(3) *G has a sink.*

Lemma 4.3. *For every vertex t in G, either $\delta(t, v) \neq t$ or t is a fixed point.*

Proof. If t is not a fixed point, then for some $x_i \in \Sigma$, $\delta(t, x_i) \neq t$. Hence, since x_i is in v, $\delta(t, v) \neq t$, otherwise v would label a nontrivial directed cycle in G, which is absurd. □

Lemma 4.4. *Assume G is totally synchronizing. Let s be a sink in G and T be an invariant subset of vertices, such that $|T| = m$. Then, for every vertex q in T, $\delta(q, v^{m-1}) = s$.*

Proof. For $i \geq 0$, $\delta(q, v^i) \in T$, since T is invariant. This way, $Q = \{\delta(q, v^i) : 0 \leq i \leq m-1\} \subset T$. Using Lemma 4.3, we know that for $j > i$, $\delta(q, v^i) = \delta(q, v^j) \Rightarrow (\delta(q, v^i) = \delta(q, v^l) = s, \forall l > i)$. Hence, either Q has m distinct elements and $\delta(q, v^{m-1}) = s$ or Q has $i < m$ distinct elements and $\delta(q, v^i) = \delta(q, v^{m-1}) = s$. □

Lemma 4.5. *Let S be the vertex set of G, $t \in S$, $S_t = \{q \in S : \exists$ a directed path $[q, t]$ from q to t in $G\} \cup \{t\}$ and n_t be the size of S_t. Then $\delta(S_t, v^{n_t - 1}) \subseteq \{t\} \cup (S - S_t)$.*

Proof. If $q \in S_t$ is such that $q \neq t$, then q is not a fixed point. Hence, using Lemma 4.3, we know that $\delta(q, v^i) \neq q$ for every $i > 0$. Now, if $\delta(q, v^i) \notin S_t$, then $\delta(q, v^j)$ is also not in S_t for every $j > i$. On the other hand, if $\delta(q, v^i) \in S_t$ for every $i \in \{0, 1, \ldots, n_t - 1\}$, then $\{\delta(q, v^i) : i = 0, 1, \ldots, n_t - 1\} = S_t$. But, since G is acyclic, there can be no path connecting t to any other vertex in S_t, hence $\delta(S_t, v^{n_t - 1}) = t$. □

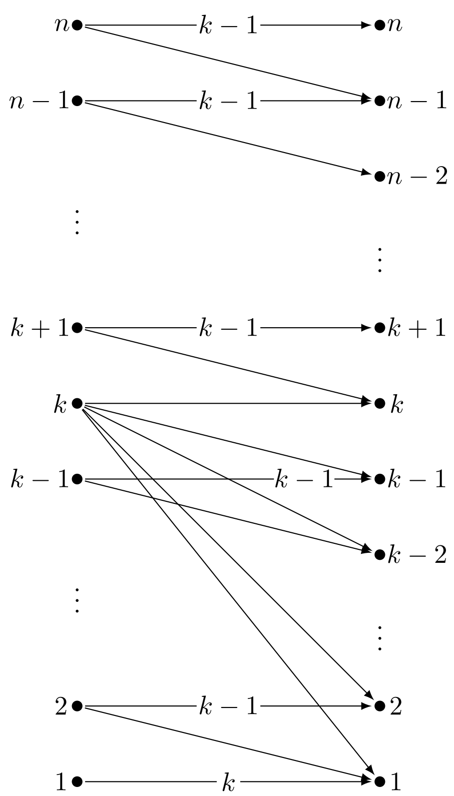

FIGURE 4.1. A totally synchronizing acyclic graph.

Theorem 4.6. *If G is totally synchronizing, then*

$$ {}^{n}_{k}\bar{w} = (x_1x_2\dots x_k)^{(n-2)}x_1x_2 $$

is a universal reset word for G.

Proof. Let S be the vertex set of G, s be its sink and $p \in S$ be such that the number of distinct edges leaving p is equal to k. Considering the set S_p as in Lemma 4.5 and the word $u = x_1x_2\dots x_k$, we know that $\delta(S_p, u^{n_p-1}) \subseteq \{p\} \cup (S - S_p)$, where $n_p = |S_p|$.

Now, since p has k distinct outgoing edges, $\delta(p, x_1x_2) \neq p$, because for each suitable labeling considered, only one letter may fix p. Thus, $\delta(S_p, u^{n_p-1}x_1x_2) \notin S_p$. Indeed, for $q \in S_p$, the word u^{n_p-1} labels a path leaving q and passing through p, if $\delta(q, u^{n_p-1}x_1x_2) = t \in S_p$, then by definition of S_p we would have a path from t to p and so the graph G would have a nontrivial directed cycle.

Let $\bar{u} = x_3x_4\dots x_kx_1x_2$ be a word in the alphabet Σ. For $q \in S - S_p$, there is no path connecting it to p, therefore there is no path connecting it to any vertex in S_p. Hence $S - S_p$ is an invariant subset of vertices. Using Lemma 4.4, we conclude that $\delta(q, \bar{u}^{n-n_p-1}) = s$.

Finally, let t be any vertex in G, then $\delta(t, u^{n_p-1}x_1x_2) \in S - S_p$, hence,

$$ \delta(t, u^{n_p-1}x_1x_2\bar{u}^{n-n_p-1}) = \delta(t, w) = s. \quad \square $$

For each $n > 1$ and $1 \leq k < n$, consider the graph ${}^{n}_{k}\bar{G}$ represented by the diagram in Figure 4.1. The word ${}^{n}_{k}\bar{w}$ in the theorem is a shortest universal reset word for this graph. Indeed, consider for each $i \in \{1, 2, \dots, k\}$, a suitable labeling of ${}^{n}_{k}\bar{G}$ with transition function δ_i, such that for $j \notin \{1, k\}$, $\delta_i(j, x_i) = j-1$ and $\delta_i(j, x_l) = j$ when $l \neq i$. Since $\delta_i(S, w') \subset \{1, 2, \dots, k\}$ implies that there are at least $n-k$ x_i in w', we conclude that a word that takes all the vertices in ${}^{n}_{k}\bar{G}$ to $\{1, 2, \dots, k\}$, independently of the suitable labeling considered, must have at least $n - k$ occurrences of each letter in Σ. After this, we need 2 distinct letters to make sure that the vertex k goes to some vertex in the subset $\{1, 2, \dots, k-1\}$, since the first letter will fix k in some labelings. Finally, using the same argument as above, we need a word with $k - 2$ occurrences of each letter in Σ to take $\{1, 2, \dots, k-1\}$ to $\{1\}$ independently of the labeling.

Theorem 4.7. *If G has rank r, then*

$$ {}^{n}_{k}\bar{w}_r = (x_1x_2\dots x_k)^{(n-r-1)}x_1x_2 $$

is a minimum rank word for G.

Proof. Using Proposition 4.1, we know that G has exactly r fixed points. Any vertex in G that is not a fixed point must have a path connecting it to at least one fixed point. Let us call a vertex *indecisive* if it is connected to more than one fixed point.

For now, assume that G has no indecisive vertices. In this case, it is the union of r disjoint acyclic totally synchronizing digraphs, all with no more than $n - r + 1$ vertices. Since, by the previous result, the word ${}^{n}_{k}\bar{w}_r$ synchronizes all these subgraphs, it is a word of rank r for G.

In the general case, we build r new graphs resulting from G, one for each fixed point. Every new graph contains one fixed point, all the vertices that are connected to it by some path in G and all the edges in G that have their endpoints in that new graph. Since all indecisive vertices belong to at least two new graphs, to maintain the constant outdegree k in those graphs, we add the necessary number of loops.

Every new graph is acyclic, because G is acyclic and the eventual addition of loops to some vertices will not create nontrivial cycles. Also, the new graphs are

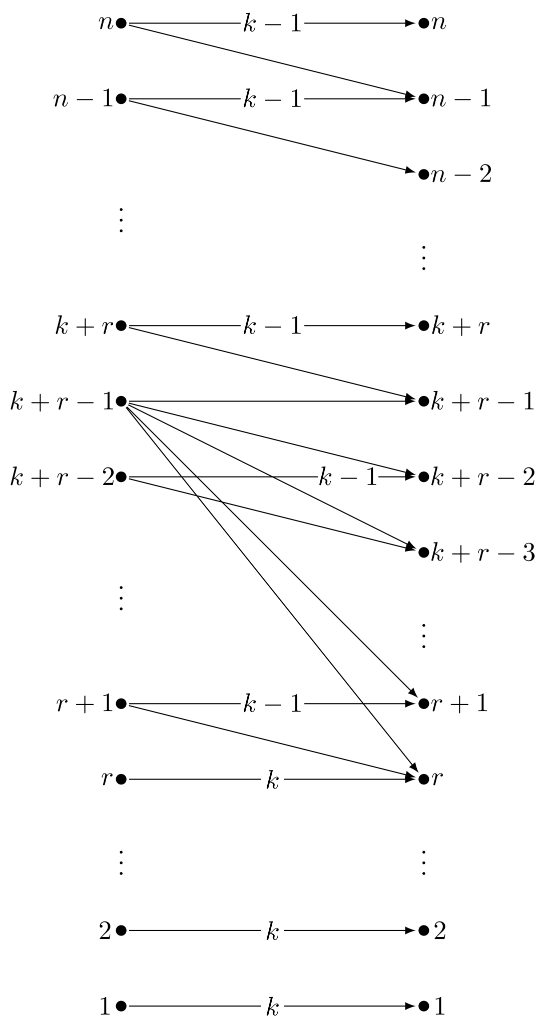

FIGURE 4.2. An acyclic graph with rank r.

totally synchronizing and cannot have more than $n-r+1$ vertices each, hence they can be synchronized by the word ${}^{n}_{k}\bar{w}_r$.

In the graph G, ${}^{n}_{k}\bar{w}_r$ sends each indecisive vertex to one of the fixed points to which it is connected. All the other vertices are sent to the fixed points as in the new graphs. Thus, ${}^{n}_{k}\bar{w}_r$ is a word of rank r for G. $\square$

Using the example from Figure 4.1, it is easy to see that ${}^{n}_{k}\bar{w}_r$ is a shortest maximum rank word for the graph ${}^{n}_{k}\bar{G}_r$ represented in Figure 4.2.

Corollary 4.8. *Suppose G has rank $r \geq 1$. Then G has a rank r word of size $n(n-r-1)+2$ and this bound is tight.*

Proof. We know that G has a rank r word of size $k(n-r-1)+2$ and that this bound is tight, according to Theorem 4.7 and Figure 4.2, respectively. Hence to finish the proof, it is enough to observe that $k(n-r-1)+2$ is maximum when $k=n$. $\square$

Corollary 4.9. *If G is totally synchronizing, then it has a universal reset word of size $n(n-2)+2$ and this bound is tight.*

5. Synchronizing Monotonic Graphs

Throughout this section, G is a monotonic graph with n vertices ($n \geq 2$) and constant outdegree k ($1 < k \leq n$), $\Sigma = \{x_1, x_2, \ldots, x_k\}$ is the alphabet used to form suitable labelings of the edges in G and δ is the transition function associated with one of those labelings.

The next result is a direct consequence of Lemma 3.2 and Proposition 4.1.

Corollary 5.1. *The following conditions are equivalent:*

(1) *every deterministic finite automaton $\mathcal{G} = (S, \Sigma, \delta)$ obtained from G by suitably labeling its edges has rank r;*
(2) *G has rank r;*
(3) *G has precisely r fixed points.*

Corollary 5.2. *The following conditions are equivalent:*

(1) *G is totally synchronizing;*
(2) *G is synchronizing;*
(3) *G has a sink.*

For example, among the monotonic graphs considered in Figure 2.3, the second and the third are totally synchronizing since they have sinks, and the first and fourth have rank 2, since that is the number of fixed points each one of them has.

Lemma 5.3. *Suppose that p and q are vertices and that there is a path from p to q (or q to p) in G. Consider a suitable labeling of the edges in G, such that all the edges in that path have the same label, and a perfect order for that labeling. Assume that $p < q$ for that order. Then every vertex t in the path from p to q (or q to p) is such that $p \leq t \leq q$.*

Proof. Let $[p, q]$ be a path from p to q and assume, without loss of generality, that it has no loops. We only consider this case, because the other one, in which there is a path from q to p, is the same as this one applied to the reverse order.

Aiming for a contradiction, assume there is a vertex t in $[p, q]$ such that $t < p$. Let $\bar{t}$ be the first vertex in those conditions and $\bar{p}$ be the vertex immediately before $\bar{t}$ in $[p, q]$. Let $\bar{q}$ be the first vertex in $[p, q]$ after $\bar{t}$ such that $\bar{q} > \bar{t}$ (it exists because $q > \bar{t}$) and let $\bar{s}$ be the vertex immediately before $\bar{q}$ in the path $[p, q]$. Then there is a cross between the edges $(\bar{p}, \bar{t})$ and $(\bar{s}, \bar{q})$, which is absurd because they have the same label and we considered a perfect order for this labeling. If we assume that $t > q$ for some vertex t in $[p, q]$, then similar arguments lead to a contradiction. □

Lemma 5.4. *Suppose that G is totally synchronizing. Let p be a vertex in G such that the number of distinct edges leaving p is equal to k and $T = \{t_1, t_2, \ldots, t_k\}$ be the set distinct vertices such that the edges $(p, t_1), (p, t_2), \ldots, (p, t_k)$ are in G. Then, there can only be loops at one vertex in $T - \{p\}$.*

Proof. Aiming towards a contradiction, suppose that $t_i, t_j \in T - \{p\}$ are such that the edges (t_i, t_i) and (t_j, t_j) are in G. Consider a suitable labeling of the edges in G such that $(p, t_i), (t_j, t_j)$ have label $a \in \Sigma$ and $(p, t_j), (t_i, t_i)$ have label $b \in \Sigma$. There are two possibilities:

(1) we have a path in G from one of the vertices with loops to the other and we assume, without loss of generality, that path is $[t_i, t_j]$;
(2) some other vertex $s \neq t_i, t_j$ is the sink of G, hence we have paths $[t_i, s]$, $[t_j, s]$ and there are no paths of the forms $[t_i, t_j], [t_j, t_i]$.

In case (1), consider the label a in the path $[t_i, t_j]$ (that path cannot contain p, since G is acyclic). So, we have the diagram in Figure 5.1. Assuming that $p > t_i$,

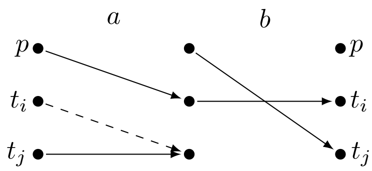

FIGURE 5.1

and using Lemma 5.3, we must have $t_i > t_j$. But this means we have a cross between the edges (p, t_j) and (t_i, t_i), both with label b and that is absurd.

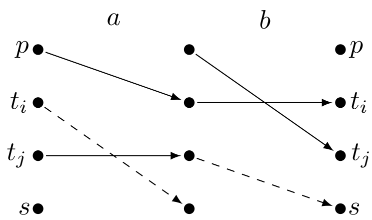

FIGURE 5.2

In case (2), consider the label a in the path $[t_i, s]$ (that path cannot contain the vertex p, since G is acyclic, and it also cannot contain the vertex t_j, because there is no path from t_i to t_j). For similar reasons we may label the path $[t_j, s]$ with b. This way, we have the diagram in Figure 5.2. Assuming that $p > t_i$ and using Lemma 5.3, we must have $t_i > s$. According to the same result $s < t_j < p$, because we have a path from p to s passing through t_j and with all edges having the same label. Now, if $t_i < t_j < p$ there is a cross between (p, t_i) and (t_j, t_j), hence $s < t_j < t_i$. But then (p, t_j) crosses (t_i, t_i), which is absurd. □

Lemma 5.5. *Let p and T be as in Lemma 5.4. Then, there is no path in G passing through more than two vertices in $T - \{p\}$.*

Proof. Aiming towards a contradiction, let $t_i, t_j, t_l \in T - \{p\}$ be three distinct vertices such that there is a path from t_i to t_l passing through t_j, that is, the paths $[t_i, t_j]$ and $[t_j, t_l]$ are in G. We consider a suitable labeling in the edges of G such that (p, t_i) has label $a \in \Sigma$, (p, t_j) has label $b \in \Sigma$ and (p, t_l) has label $c \in \Sigma$. There are two possibilities:

(1) there is an edge (t_j, t_l) in G;
(2) the path $[t_j, t_l]$ has at least two edges and, in this case we consider a vertex q such that the path $[t_j, q]$ and the edge (q, t_l) are in G.

In case (1), label (t_j, t_l) with c and $[t_i, t_j]$ with a (p is not in that path). Since G has outdegree $k \geq 3$, there must be some edge leaving t_i besides the one that belongs to the path $[t_i, t_j]$. Let (t_i, t) be that edge and label it c. Given a perfect order for this labeling, we may assume that $p < t_i$ and according to Lemma 5.3, $t_i < t_j$. If $t_l < t$, then (t_i, t) crosses (t_j, t_l), as we can observe in Figure 5.3. If $t < t_l$, then (t_i, t) crosses (p, t_l). Since all these edges have label c, we must have $t = t_l$.

Now, consider in (t_j, t_l) the label c, in (t_i, t_l) the label b and in the path $[t_i, t_j]$ the label c, this can be done, since p is not in that path. Considering a perfect order for this labeling, we may assume that $t_i < t_j$. Lemma 5.3 allows us to conclude

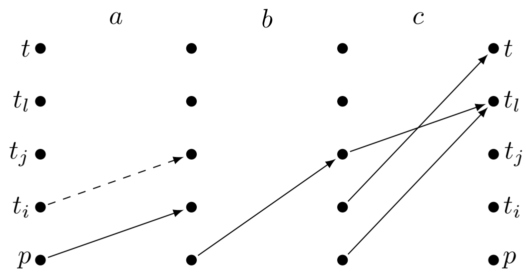

FIGURE 5.3

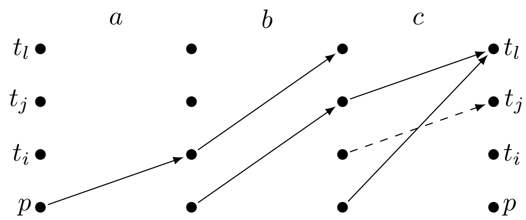

FIGURE 5.4

that $t_j < t_l$. Now, if $t_i < p$, (p, t_j) crosses (t_i, t_l) and they have the same label b, so this is absurd. Hence, $p < t_i$ and we are in the situation of Figure 5.4. But then, using Lemma 5.3, we conclude that the edges in the path $[t_i, t_j]$ cross (p, t_l), since they have the same label c, this is absurd. Therefore, case (1) is not possible.

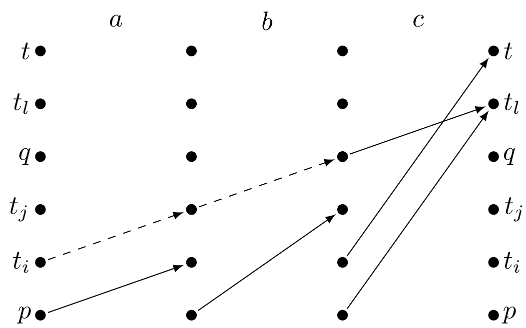

FIGURE 5.5

In case (2), let (q, t_l) have label c, let the path $[t_j, q]$ have label b and let the path $[t_i, t_j]$ have label a (p is not in those paths). Since G has outdegree $k \geq 3$, there must be some edge leaving t_i besides the one that belongs to the path $[t_i, t_j]$. Let (t_i, t) be that edge and label it c. Given a perfect order for this labeling, we may assume that $p < t_i$ and, according to Lemma 5.3, $t_i < t_j < q$. If $t_l < t$, then (t_i, t) and (t_j, t_l) cross each other, as we can see in Figure 5.5. If $t < t_l$, then (t_i, t) and (p, t_l) cross each other. Since all these edges have label c, we must have $t = t_l$.

Now, consider in (t_i, t_l) the label b, in the path $[t_i, t_j]$ the label c (p is not in that path), in the path $[t_j, q]$ the label c (p is not in that path, neither is any of the vertices in the path $[t_i, t_j]$) and in (q, t_l) label c also. Considering a perfect order for

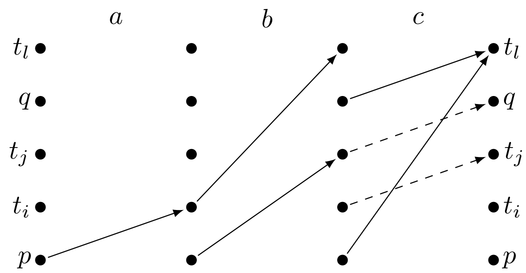

FIGURE 5.6

this labeling, we may assume that $t_i < t_j$. Lemma 5.3 allows us to conclude that $t_j < q < t_l$. Now, if $t_i < p$, (p, t_j) crosses (t_i, t_l) and they have the same label b, so this is absurd. Hence, $p < t_i$ and we are in the situation of Figure 5.6. But then, using Lemma 5.3, we conclude that the edges in the path $[t_i, t_j]$ cross (p, t_l), since they have the same label c, this is absurd. Therefore, case (2) is also impossible and we have reached the desired contradiction. ☐

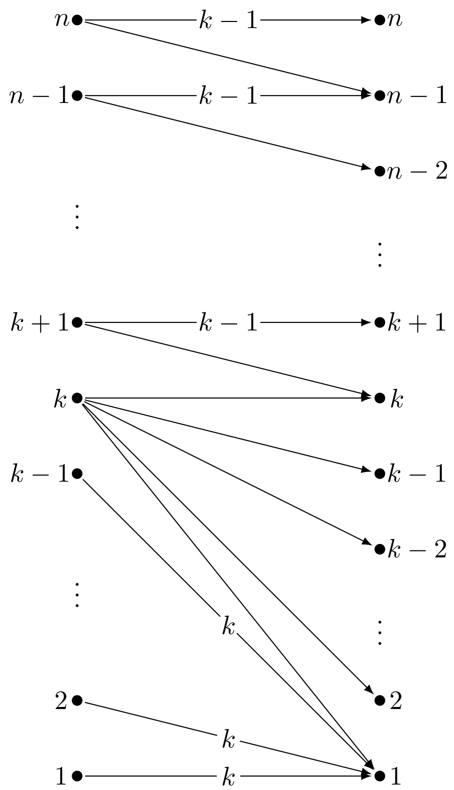

FIGURE 5.7. A totally synchronizing monotonic graph.

Theorem 5.6. *If G is totally synchronizing, then*

$$ {}^{n}_{k}w = (x_1 x_2 \dots x_k)^{(n-k)} x_1 \dots x_\alpha, $$

with $\alpha = \min\{3, k\}$, is a universal reset word for G.

Proof. Let S be the vertex set of G and s be its sink. Let p and T be as in Lemma 5.5. Let S_p and n_p be as in Lemma 4.5 and let $u = x_1x_2 \dots x_k \in \Sigma^*$.

Suppose for now that $k > 2$. Using Lemma 5.4, suppose that t_k is the only vertex in $T - \{p\}$ that may have loops. We know that $\delta(p, x_1x_2) \neq p$ and that $\delta(t_i, x_j) \neq t_i$, for all $i \in \{1, 2, \dots, k-1\}$ and $j \in \{1, 2, \dots, k\}$. We also know that, by Lemma 3.2, $\delta(q, u) = q \Rightarrow q = s$ for $q \in S$. Now, according to Lemma 5.5, if there is a path from q to s then one of the following situations must occur:

(1) the path passes first through p and then through t_i and t_j for some $i, j \in \{1, 2, \dots, k\}$ (possibly $t_i = p$ or $i = j$);
(2) the path passes first through t_i and then through t_j for some $i, j \in \{1, 2, \dots, k\}$ without passing through p (possibly $i = j$);
(3) the path does not pass through p nor any vertex in T.

In case (1), we know from Lemma 4.5, that $\delta(q, u^{n_p - 1}) \in \{p\} \cup (S - S_p)$, hence $\delta(q, u^{n_p-1}x_1x_2) \in S - S_p$. Now, from p we must go to some vertex $t_i \in T$, therefore if $i \neq k$, then $\delta(q, u^{n_p-1}x_1x_2x_3) \in S - S_{t_i}$. Let $\bar{u} = x_4x_5 \dots x_kx_1x_2x_3$. By Lemma 5.5, the path from q to s can only contain one more vertex in T, while it cannot contain any more vertices in S_p, hence it can only pass through $n - n_p - k + 2$ vertices more (we need to include the case where some $t_l = p$ and so when adding n_p and k we are counting p twice). This is enough to conclude that $\delta(q, u^{n_p-1}x_1x_2x_3\bar{u}^{n-n_p-k+1}) = \delta(q, {}^n_kw) = s$. If $i = k$, then $j \neq k$, hence the only difference is that we would have to use a few copies of $\tilde{u} = x_3x_4 \dots x_kx_1x_2$ to get from t_k to t_j, the letter x_3 to get out of t_j and then the number of necessary copies of $\bar{u}$ to reach s, also concluding that $\delta(q, {}^n_kw) = s$. For this reason, in the next case we will simply assume $i \neq k$, since everything works basically the same way.

In case (2), if $n_i = |S_{t_i}|$, then we know from Lemma 4.5 that $\delta(q, u^{n_i-1}) \in \{t_i\} \cup (S - S_{t_i})$, hence $\delta(q, u^{n_i-1}x_1) \in S - S_{t_i}$ and so $\delta(q, u^{n_i-1}x_1x_2x_3) \in S - S_{t_i}$, because this set is invariant. By Lemma 5.5, the path from q to s can only contain one more vertex in T while it cannot contain any more vertices in S_{t_i}, hence it can only pass through $n - n_i - k + 2$ vertices more. This way, we conclude that $\delta(q, u^{n_i-1}x_1x_2x_3\bar{u}^{n-n_i-k+1}) = \delta(q, {}^n_kw) = s$.

Finally, in case (3), we know that the path from q to s cannot contain any vertex in T hence it can only pass through $n - k$ vertices. Therefore $\delta(q, u^{n-k-1}) = s$ and so $\delta(q, u^{n-k}x_1x_2x_3) = \delta(q, {}^n_kw) = s$.

When $k = 2$, only two letters are necessary (there is no x_3), but everything else works the same and $\delta(q, {}^n_kw) = s$ for every $q \in S$. □

For each $n > 1$ and $1 \leq k < n$, consider the graph ${}^n_kG$ represented by the diagram in Figure 5.7. The word ${}^n_kw$ in the theorem is a shortest universal reset word for this graph. Indeed, consider for each $i \in \{1, 2, \dots, k\}$, a suitable labeling of ${}^n_kG$ with transition function δ_i, such that for $j > k$, $\delta_i(j, x_i) = j - 1$ and $\delta_i(j, x_l) = j$ when $l \neq i$. Since $\delta_i(S, w') \subset \{1, 2, \dots, k\}$ implies that there are at least $n - k$ occurrences of x_i in w', we conclude that a word that takes all the vertices in ${}^n_kG$ to $\{1, 2, \dots, k\}$, no matter what suitable labeling is considered, must have at least $n-k$ occurrences of the letter x_i, for each $i \in \{1, 2, \dots, k\}$. Finally, we need $\min\{3, k\}$ distinct letters to take $\{1, 2, \dots, k\}$ to $\{1\}$ independently of the labeling. To see this, observe that for some labeling the first letter fixes the vertex k and the second letter sends it to a vertex j such that $1 \leq j < k$; now, if $k > 2$ we can have $j > 1$ and the third letter is necessary to send j to 1.

Lemma 5.7. *If G has rank $r \geq 2$ and s_1, s_2 are two fixed points in G, then there is at most one vertex in G for which there are paths connecting it to s_1 and s_2.*

Proof. Suppose that p and q are distinct vertices such that $[p, s_1], [p, s_2], [q, s_1]$ and $[q, s_2]$ are paths in G. For any suitable labeling of the edges in G and any perfect order for that labeling, let $i, j \in \{1, 2\}$, with $i \neq j$. If s_i is between p and s_j, then there is a cross between some edge in the path $[p, s_j]$ and the edge (s_i, s_i) with the same label, which is absurd. When s_i is between q and s_j, a similar contradiction is reached. Hence, we must have $s_i < p, q < s_j$. We assume, without loss of generality that $s_1 < p, q < s_2$.

Now, if t is a vertex that belongs to both paths $[p, s_1]$ and $[q, s_2]$, then there are paths from p to s_2 and from q to s_1 going through t, but this is in contradiction with Lemma 5.3, because $t \neq p, s_1, s_2$, hence both the conditions $p < t < s_2$ and $s_1 < t < p$ would have to be satisfied and that is impossible.

This way, there are no common vertices between the paths $[p, s_1]$ and $[q, s_2]$, hence we may consider a suitable labeling such that all the edges in these two paths have label a. Also, there are no common vertices between the paths $[q, s_1]$ and $[p, s_2]$, hence we may consider that in the previous labeling all the edges in these two paths have label b.

Suppose that $s_1 < p < q < s_2$. Let t be the first vertex after p in the path $[p, s_2]$, we have $t \leq s_2$. There must be some vertex in the path $[q, s_1]$ that is between p and s_1 in the considered order. Let $\bar{t}$ be the first vertex in those conditions and let $\tilde{t}$ be the vertex immediately before that in the path $[q, s_1]$. Then there is a cross between the edges $(\tilde{t}, \bar{t})$ and (p, t), because $p < \tilde{t}$ and $\bar{t} < p < t$. Since they have the same label, this is absurd.

The case $s_1 < q < p < s_2$ is similar and also leads to a contradiction. Therefore, we cannot have two distinct vertices p and q in the initial conditions. □

Theorem 5.8. *If G has rank $r \geq 1$, such that $k + r < n$, then*

$$ {}^{n}_{k}w_r = (x_1 x_2 \dots x_k)^{(n-k-r+1)} x_1 \dots x_\alpha, $$

with $\alpha = \min\{3, k\}$, is a word of rank r for G.

Proof. Since G has rank r, according to Corollary 5.1 it has r fixed points. Any vertex in G that is not a fixed point must have a path connecting it to at least one fixed point and cannot be connected to more than two fixed points, because this would imply the existence of a cross between one of those connecting paths and the loops of a fixed point. Let us call the vertices which are connected to two distinct fixed points *indecisive.*

For now, assume that G has no indecisive vertices. In this case, it is the union of r disjoint monotonic totally synchronizing digraphs, all with no more than $n - r + 1$ vertices. Since, by Theorem 5.6, the word ${}^{n}_{k}w_r$ synchronizes all these subgraphs, it is a word of rank r for G.

Now, for the general case, we build r new graphs resulting from G, one for each fixed point. Besides the fixed point, every new graph contains all the vertices that are connected to that fixed point by some path in G and all the edges in G that have their endpoints in that new graph. Each indecisive vertex will belong to two new graphs, and for that reason, to maintain the constant outdegree k in the new graphs, we add the necessary number of loops.

According to Lemma 5.7, each new graph cannot contain more than two indecisive vertices. Let p be a vertex that is undecided between the fixed points s_1 and s_2 and let $\bar{p}, q, \bar{q}$ be vertices such that the edge $(p, \bar{p})$ is replaced by (p, p) in the new graph containing s_2 and $(q, \bar{q})$ is an edge in that graph that was already in the original one and such that $q \neq p$. Consider a suitable labeling of G such that all the edges in a path $[p, s_1]$ that contains $\bar{p}$ have label a. Suppose also that the edges in a path $[q, s_2]$ that contains $\bar{q}$ have the same label a (if $t \in [q, s_2]$ then $t \notin [p, s_1]$, according to Lemma 5.7). Regard a perfect order for that labeling and the corresponding labeling and order in the new graph. Without loss of generality,

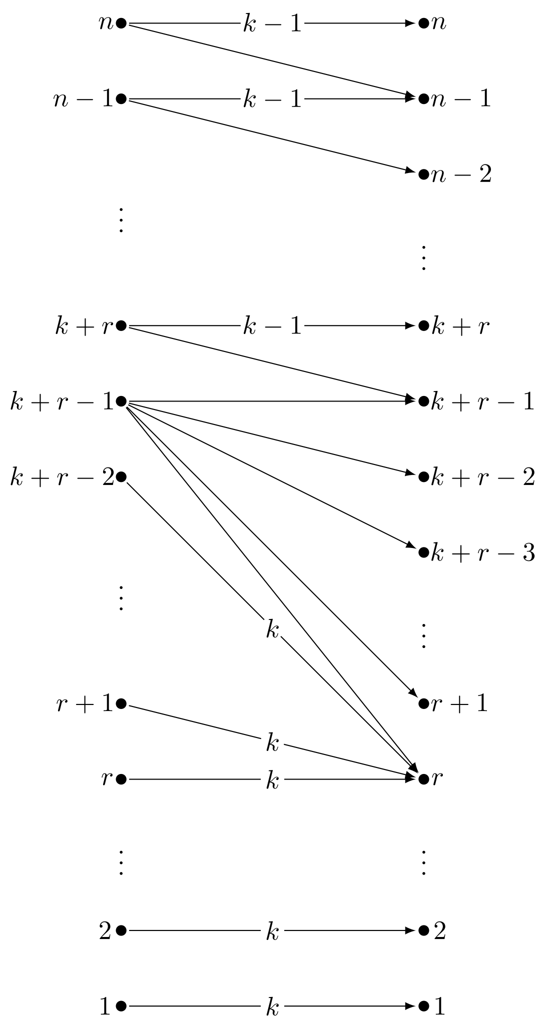

FIGURE 5.8. A monotonic graph with rank r.

it can be assumed that $s_1 > p > s_2$ for that order. Suppose that there is a cross between the edges (p, p) and $(q, \bar{q})$ in the new graph. Then p must be between q and $\bar{q}$. By Lemma 5.3, we cannot have $q < p < \bar{q}$, because $\bar{q}$ must be between q and s_2, hence $\bar{q} < p < q$ therefore there is a cross between $(p, \bar{p})$ and $(q, \bar{q})$ in G, which is absurd. This allows us to conclude that the new graphs are all monotonic.

Each new graph is also totally synchronizing, because it has only one fixed point, and cannot have more than $n - r + 1$ vertices, hence it can be synchronized by the word ${}^n_k w_r$.

In the graph G, ${}^n_k w_r$ sends each indecisive vertex to one of the two fixed points to which it is connected. All the other vertices are sent to the fixed points as in the new graphs. This way, ${}^n_k w_r$ is a word of rank r for G. $\square$

Using the example from Figure 5.7, it is easy to see that ${}^n_k w_r$ is a shortest maximum rank word for the graph ${}^n_k G_r$ represented in Figure 5.8.

Corollary 5.9. *Let G have rank $r \geq 1$. Then, for $n-r = 1, 2, 3$ there are rank r words of sizes $2, 4, 6$, respectively. If $n - r \geq 4$, there is a rank r word for G of size*

$$\left\lfloor \frac{n-r+1}{2} \right\rfloor \left\lfloor \frac{n-r+2}{2} \right\rfloor + 3.$$

All these bounds are tight.

Proof. Note that if $k = 1$, there is only one suitable labeling for G, we have a rank r word of length at most $n - r$ and this bound is tight. Hence we may assume $k > 1$, because this leads to bigger rank r words.

Let

$$\operatorname{len}(n, r, k) = k(n - k - r + 1) + \min\{3, k\}$$

be the function that associates with each triple (n, r, k) such that $n, k > 1$, $r > 1$ and $k + r < n$, the length of the word ${}^{n}_{k}w_r$ in Theorem 5.8. When $k + r \geq n$, let $\operatorname{len}(n, r, k)$ be the maximum length of a shortest minimum rank word for a monotonic digraph with n vertices, rank r and constant outdegree k. Also let

$$\operatorname{len}(n, r) = \max_k \{\operatorname{len}(n, r, k)\}.$$

To finish the proof of the result, all we have to do is compute $\operatorname{len}(n, r)$. The conclusions come from Theorem 5.8 and the graph ${}^{n}_{k}G_r$ considered above, which is used to show that the bounds are tight.

If $k = 2$, then $\operatorname{len}(n, r, 2) = 2(n - r)$. Otherwise, $\min\{3, k\} = 3$, $\operatorname{len}(n, r, k) = k(n - k - r + 1) + 3$ and the maximum is obtained when $k = \lfloor (n - r + 1)/2 \rfloor$. If $n - r < 5$, $\lfloor (n - r + 1)/2 \rfloor < 3$, hence we need to study these cases separately. But when $n - r \geq 5$, $\lfloor (n - r + 1)/2 \rfloor \geq 3$ and

$$\operatorname{len}(n, r) = \left\lfloor \frac{n-r+1}{2} \right\rfloor \left\lfloor \frac{n-r+2}{2} \right\rfloor + 3,$$

because

$$\left\lfloor \frac{n-r+1}{2} \right\rfloor \left\lfloor \frac{n-r+2}{2} \right\rfloor + 3 > 2(n - r),$$

for all n, r as above.

Finally, observing the following table

$n-r$	k	$\operatorname{len}(n,r,k)$
1	2	2
2	2	4
2	3	2
3	2	6
3	3	6
3	4	4
4	2	8
4	3	9
4	4	7
4	5	5

allows us to conclude that

$n-r$	$\operatorname{len}(n,r)$
1	2
2	4
3	6
4	9

Observe that for $n - r = 4$, $\operatorname{len}(n, r) = 9 = \lfloor (4 + 1)/2 \rfloor \lfloor (4 + 2)/2 \rfloor + 3$. □

Corollary 5.10. *Let G be totally synchronizing. Then, for $n = 2, 3, 4$ there are universal reset words of sizes $2, 4, 6$, respectively. If $n \geq 5$, there is a universal reset word for G of size*

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor + 3.$$

All these bounds are tight.

6. Acknowledgments

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