Never Minimal Automata and the rainbow bipartite subgraph problem

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Abstract. Never minimal automata, introduced in [4], are strongly connected automata which are not minimal for any choice of their final states. In [4] the authors raise the question whether recognizing such automata is a polynomial time task or not. In this paper, we show that the complement of this problem is equivalent to the problem of checking whether or not in an edge-colored graph there is a bipartite subgraph whose edges are colored using all the colors. We prove that this graph theoretic problem is **NP**-complete, showing that checking the property of never-minimality is unlikely a polynomial time task.

1 Introduction

Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a deterministic (not necessarily complete) finite-state automaton (DFA). The action of the transition function δ can naturally be extended to the free monoid Σ^*. This extension will still be denoted by δ. For convenience for each $v \in \Sigma^*$ and $q \in Q$ we will write $q \,.\, v = \delta(q, v)$ and put $S \,.\, v = \{q \,.\, v \mid q \in S\}$ for any $S \subseteq Q$. A congruence σ of the automaton $\mathscr{A}$ is an equivalence relation on Q such that if $q \sigma q'$ then $(q \,.\, u)\sigma(q' \,.\, u)$ for all $u \in \Sigma^*$. By $\mathcal{M}(\mathscr{A})$ we denote the set of minimal congruences of $\mathscr{A}$. In [4] the authors introduce some classes of automata with extremal conditions. The class of uniformly minimal automata is formed by strongly connected automata which are minimal for any choice of the final states. In this paper they also provide a characterization in term of the state-pair graph which leads to a polynomial time algorithm to decide whether a given DFA is uniformly minimal. This class has also interesting connections with *multi-entry automata* and *symbolic dynamics*. The other interesting case introduced in [4] is the opposite extremal case of *never-minimal automata* which is considered in our paper.

Definition 1. *Except for the last section, we restrict our attention to strongly connected automata. We say that a DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is never-minimal if and only if for any $F \subseteq Q$ and $i \in Q$ the automaton $\mathscr{A}_{i,F} = \langle Q, \Sigma, \delta, i, F \rangle$ is not minimal.*

In [4] the authors exhibit an infinite sequence of never-minimal automata and raise the problem of characterizing such property in order to give a polynomial

time algorithm for recognizing such automata. Formally NEVER-MINIMAL is the problem that given in input a strongly connected DFA $\mathscr{A}$ checks whether or not $\mathscr{A}$ is never-minimal. In this paper we prove that co-NEVER-MINIMAL is an **NP**-complete showing that NEVER-MINIMAL is unlikely in **P**.
The paper is organized as follows. In Section 2 we introduce the concept of syntactic graph which is useful in characterizing never-minimal automata. In Section 3 we introduce some graph theoretic problems which are proved to be **NP**-complete. This graph theoretic problems turn out to be equivalent to the DISJUNCTIVE SET problem already considered in [1] in which the authors show the **NP**-completeness. However our reduction gives the **NP**-completeness for a smaller class. In Section 4 we prove that co-NEVER-MINIMAL is equivalent to the DISJUNCTIVE SET problem showing that this problem is also **NP**-complete. Finally in Section 5 we explore some connections with the SYNTACTIC MONOID problem (cf. [1]).

2 The syntactic graphs

In this paper we deal with graphs which are simple undirected and without loops. Given a symmetric, reflexive relation $R \subseteq V \times V$, there is a natural way to associate to R a graph $\mathcal{G}(R) = (V, E)$. Namely for each pair of distinct elements x, y we say that $\{x, y\} \in E$ if $(x, y) \in R$. Conversely a graph G gives rise to a symmetric reflexive relation $\mathcal{R}(G)$ in the obvious way. We say that a family $\mathfrak{R}$ of equivalence relations on a set V is *orthogonal* (or *pairwise separating* in [1]) if for any pair $R, R' \in \mathfrak{R}$ of distinct relations, $R \cap R' = 1_V$, where 1_V is the identity relation on V. In [4] the authors introduce the state-pair graphs as a tool to characterize uniformly minimal automata. We introduce an analogous tool which is a slight generalization of these graphs. This will be useful to characterize never-minimal automata. For any pair x, y of distinct states we associate an undirected graph $\mathfrak{G}_{x,y}$ called the *syntactic graph* generated by the pair x, y.

Definition 2 (syntactic graph of the pair $\{x, y\}$). *Let $x \neq y$ be two states of the automaton $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$. The syntactic graph of the pair $\{x, y\}$ is the undirected graph $\mathfrak{G}_{x,y} = (Q, E_{x,y})$ having as set of vertices Q and the set of edges $E_{x,y}$ formed by the pair $\{\alpha, \beta\}$ with $\alpha \neq \beta$ such that there is some $u \in \Sigma^*$ with $\{x, y\} . u = \{\alpha, \beta\}$. We denote by $\Gamma^i_{x,y}$, for $i = 1, \ldots, C(x, y)$, the connected components of $\mathfrak{G}_{x,y}$.*

Given a set $F \subseteq Q$, the *syntactic congruence* σ_F generated by F is the largest congruence saturating F and it is defined by $a \sigma_F b$ if $\forall w \in \Sigma^*$ $a . w \in F \Leftrightarrow b . w \in F$. The following proposition characterizes the syntactic congruences σ_F, $F \subseteq Q$ with $x \sigma_F y$ in term of the connected components of the syntactic graph $\mathfrak{G}_{x,y}$. Using the notation of Definition 2 we have the following proposition.

Proposition 1. *Let $x \neq y$ be two states of the automaton $\mathscr{A}$ and let $F \subseteq Q$. Then σ_F is a syntactic congruence with $x \sigma_F y$ if and only if*

$$F = \bigcup_{i \in I} \Gamma^i_{x,y}$$

for some $I \subseteq \{1, \ldots, C(x,y)\}$*. Moreover the number of sets* $F \subseteq Q$ *such that* $x\sigma_F y$ *is* $2^{|C(x,y)|} - 1$.

Proof. This is a consequence of Definition 2. □

Therefore, given x, y, the connected components of $\mathfrak{G}_{x,y}$ describe all the possible subsets $F \subseteq Q$ such that x, y are identified via the syntactic congruence σ_F.
We have a first characterization of never-minimal automata given in term of their syntactic graphs.

Theorem 1. *A strongly connected DFA* $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ *is never-minimal if and only if for all* $F \subseteq Q$ *there is a pair of distinct elements* x, y *and a subset* $I \subseteq \{1, \ldots, C(x,y)\}$ *such that*

$$F = \bigcup_{i \in I} \Gamma^i_{x,y}$$

Proof. Clearly $\mathscr{A}$ is never-minimal if and only if for all $F \subseteq Q$ there is a pair of distinct elements x, y such that $x\sigma_F y$. By Proposition 1 this is equivalent to say

$$F = \bigcup_{i \in I} \Gamma^i_{x,y}$$

for some $I \subseteq \{1, \ldots, C(x,y)\}$. □

We have the following fact.

Proposition 2. *The equivalence relation* $\sigma_{x,y} \subseteq Q \times Q$ *defined by* $\alpha\sigma_{x,y}\beta$ *if* $\alpha, \beta \in \Gamma^i_{x,y}$ *for some* $i \in \{1, \ldots, C(x,y)\}$ *is a congruence. Moreover it is the smallest congruence which identifies* x *with* y.

Proof. Suppose that for some $a \in \Sigma$, $\alpha \,.\, a \in \Gamma^j_{x,y}$ for some $j \in \{1, \ldots, C(x,y)\}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a path in $\Gamma^i_{x,y}$ connecting $\alpha = \alpha_1$ with $\beta = \alpha_n$. Since $\{\alpha_i, \alpha_{i+1}\} \in E_{x,y}$ then the image of this path through the action $.\,a$ is also a path contained in some connected component which contains also α thus this path is contained in $\Gamma^j_{x,y}$, whence $\beta \,.\, a \in \Gamma^j_{x,y}$. The last statement is also an easy consequence of the definition of syntactic graph. □

Definition 3. *Let* $\mathscr{G}$ *be the set of all syntactic graphs of the automaton* $\mathscr{A}$*. We define a preorder in* $\mathscr{G}$ *by* $\mathfrak{G}_{x,y} \preceq \mathfrak{G}_{x',y'}$ *if for every* $i \in \{1, \ldots, C(x,y)\}$ *there is a* $j \in \{1, \ldots, C(x',y')\}$ *such that* $\Gamma^i_{x,y} \subseteq \Gamma^j_{x',y'}$*. Equivalently* $\mathfrak{G}_{x,y} \preceq \mathfrak{G}_{x',y'}$ *if and only if the partition induced by the connected components of* $\mathfrak{G}_{x,y}$ *is a refinement of the partition induced by the connected components of* $\mathfrak{G}_{x',y'}$*, i.e.* $\sigma_{x,y} \subseteq \sigma_{x',y'}$.

We have the following lemma.

Lemma 1. *Let* x, y *be two distinct states of the automaton* $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ *and let* $\Gamma^i_{x,y}$ *be a connected component of* $\mathfrak{G}_{x,y}$*. For any pair* $x', y' \in \Gamma^i_{x,y}$ *of distinct vertices we have* $\mathfrak{G}_{x',y'} \preceq \mathfrak{G}_{x,y}$.

Proof. We remark that the statement of the lemma is equivalent to the following: if x', y' are two distinct pair of states with $(x', y') \in \sigma_{x,y}$ then $\sigma_{x',y'} \subseteq \sigma_{x,y}$. By Proposition 2, $\sigma_{x,y}$ is a congruence, moreover by the definition $x'\sigma_{x,y}y'$. By the minimality of $\sigma_{x',y'}$ it is clear that $\sigma_{x',y'} \subseteq \sigma_{x,y}$, i.e. $\mathfrak{G}_{x',y'} \preceq \mathfrak{G}_{x,y}$. $\square$

We remark that the relation $\sim$ defined on $(\mathscr{G}, \preceq)$ by $\mathfrak{G}_{x',y'} \sim \mathfrak{G}_{x,y}$ if $\mathfrak{G}_{x',y'} \preceq \mathfrak{G}_{x,y}$ and $\mathfrak{G}_{x,y} \preceq \mathfrak{G}_{x',y'}$ is an equivalence relation such that $\mathscr{G}/\sim$ is endowed with an obvious partial order which is isomorphic to the partial order $(\{\sigma_{x,y} : x \neq y\}, \subseteq)$. In view of Proposition 2 it is not difficult to see that $\mathcal{M}(\mathscr{A})$ coincides with the set of minimal elements of $(\{\sigma_{x,y} : x \neq y\}, \subseteq))$. We have the following property.

Proposition 3. *The family of equivalence relations $\mathcal{M}(\mathscr{A})$ is orthogonal. Moreover the automaton $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is never-minimal if and only if for any subset $F \subseteq Q$ there is a $\sigma \in \mathcal{M}(\mathscr{A})$ such that F is union of equivalence classes of σ.*

Proof. Let $\sigma_{x,y}, \sigma_{x',y'} \in \mathcal{M}(\mathscr{A})$. Suppose that $\sigma_{x,y} \cap \sigma_{x',y'} \neq 1_Q$ and so let $(z, t) \in \sigma_{x,y} \cap \sigma_{x',y'}$ for some pair of distinct states $z, t \in Q$. By Lemma 1 we get $\mathfrak{G}_{z,t} \preceq \mathfrak{G}_{x,y}$ and $\mathfrak{G}_{z,t} \preceq \mathfrak{G}_{x',y'}$, thus $\sigma_{z,t} \subseteq \sigma_{x,y}$ and $\sigma_{z,t} \subseteq \sigma_{x',y'}$. Hence by the minimality of $\sigma_{x,y}, \sigma_{x',y'}$ we have $\sigma_{x,y} = \sigma_{z,t} = \sigma_{x',y'}$. The last statement is a consequence of Theorem 1 and the definition of $\mathcal{M}(\mathscr{A})$. $\square$

3 The rainbow bipartite subgraph problem

In this section we introduce some graph theoretic problems and we study their computational complexity class. In this paper a colored graph is a pair (G, φ) where $G = (V, E)$ is a graph and φ is a function, called coloring, from the set of edges E to a set $C = \{1, \ldots, N\}$ of colors. This definition can be extended to the case of list colored graphs (G, φ), where the list coloring is a function $\varphi : E \to 2^C$. For each $i \in C$ by $G(i)$ we denote the maximal subgraph of G formed by the edges whose lists contain i. We call the subgraphs $G(i) = (V, E_{G(i)}), i = 1, \ldots, n$, the *maximal monochromatic components* of (G, φ). It is clear that a list coloring φ is completely described by all the maximal monochromatic components $\{G(i)\}_{i\in C}$. Namely $\varphi(\{\alpha, \beta\}) = \{i \in C : \{\alpha, \beta\} \in E_{G(i)}\}$.
We say that a coloring φ is *splittable* if there is a partition of the set of vertices of V into two sets V_1, V_2 such that for any $i \in \{1, \ldots, n\}$ there is an edge $\{v_1, v_2\}$ with $v_1 \in V_1, v_2 \in V_2$ such that $\varphi(\{v_1, v_2\}) = i$. In this case we say that the partition V_1, V_2 splits φ. A more graph-theoretic way to see this property is via the concept of rainbow subgraphs (cf. [2]), i.e. subgraphs (in the weakest sense) such that all the edges are colored differently. Indeed a coloring φ is splittable if there is a bipartite rainbow subgraph of G colored using all the colors $\{1, \ldots, n\}$. Everything extends naturally to the case of the list coloring. Indeed we say that a list coloring φ is *list splittable* (for short splittable) if there is a partition of V into two sets V_1, V_2 such that for any $i \in \{1, \ldots, N\}$ there is an edge $\{v_1, v_2\}$ with $v_1 \in V_1, v_2 \in V_2$ such that $i \in \varphi(\{v_1, v_2\})$. Also in this case a similar characterization holds in term of bipartite subgraphs. Indeed a list coloring φ is splittable if there is a bipartite subgraph such that each color is contained in

some color list of some edge. Extending the definition of n-bounded coloring (cf. [2]) from colored graphs to list colored graphs, we say that a list coloring φ of a graph G is n-bounded if each color appears at most n times in the lists of colors associated to the edges.

With the previous definitions it makes sense defining the problem `SPLITTABLE`. This is the problem of determining, given a colored graph (G,φ), whether φ is splittable. Analogously `LIST-SPLITTABLE` is the problem of determining, given a list colored graph (G,φ) whether φ is list splittable. The n-`SPLITTABLE` problem is the sub-problem of checking, given a n-bounded colored graph (G,φ), whether φ is splittable. n-`LIST-SPLITTABLE` is defined analogously. We say that the (list) coloring φ on a graph $G = (V,E)$ is *anti-incidence* if for all pairs of incident edges $\{v, v_1\}, \{v, v_2\}$, $\varphi(\{v, v_1\}) \cap \varphi(\{v, v_2\}) = \emptyset$. Although `LIST-SPLITTABLE` may appear a more difficult problem with respect to `SPLITTABLE`, the following proposition shows that this is not the case.

Proposition 4. *There is a reduction η from `LIST-SPLITTABLE` to `SPLITTABLE` bringing a n-bounded list colored graph (G,φ) into a n-bounded colored graph (G',φ'). Moreover we can find a reduction η' which brings a n-bounded colored graph (G,φ) into a n-bounded colored graph (G'',φ'') such that φ'' is anti-incidence.*

Proof. Given an instance (G,φ) with $G = (V,E)$ of `LIST-SPLITTABLE`, where φ is a list coloring on the set of colors $C = \{1,\ldots,N\}$, we reduce it to an instance (G',φ') with $G' = (V',E')$ of `SPLITTABLE`. Starting from (G,φ) we iteratively apply the following construction. For each edge $\{v, v'\}$ of a list colored graph (H,ψ) with $H = (Y,T)$ such that $\psi(\{v,v'\}) = \{i_0,\ldots,i_k\}$ with $k \geq 1$, we add $k+1$ new vertices $\overline{v}, v_1,\ldots,v_k$, $2k+1$ new edges and $k+1$ new colors $c_0,\ldots,c_k$. For each $j \in \{1,\ldots,k\}$ we add an edge $\{v', v_j\}$ colored by $\psi'(\{v', v_j\}) = i_j$ and take $\psi'(\{v', v\}) = i_0$. Putting $v_0 = v$ we also add for each $j \in \{0,\ldots,k\}$ edges $\{v_j, \overline{v}\}$ colored by $\psi'(\{v_j, \overline{v}\}) = c_j$ (see Fig. 1).

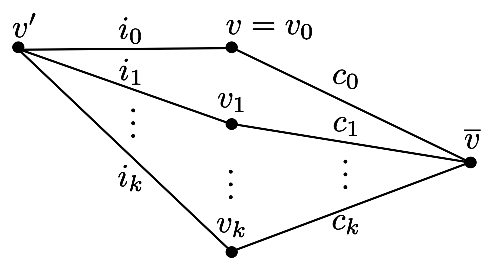

Fig. 1. One step of the iterated construction.

Leaving $\psi = \psi'$ for the other edges, we obtain a new list colored graph (H',ψ')

with $H' = (Y', T')$ such that $|T'| - |T| \leq 2N - 1$ and the number of added colors is upper bounded by N. It is clear that after at most $|E|$ iterations we get a colored graph (G', φ') with $|E'| \leq 2N|E|$ whose number of colors is upper bounded by $N|E| + N$. Thus to prove that η is a reduction we have to show that φ is list splittable iff φ' is splittable. To prove it, it is enough to show that the splittability property is preserved in each step of the previous iteration. Indeed suppose that ψ' is list splittable in H' and let Y_1', Y_2' be the associated partition. We observe that, since the edges $\{v_j, \overline{v}\}$ for $j \in \{0, \ldots, k\}$ are the only edges in H' colored by $c_0, \ldots, c_k$, then $\overline{v} \in Y_l'$ for some $l \in \{1, 2\}$ iff $v_0, \ldots, v_k \in Y_{3-l}'$. We claim that $Y \cap Y_1', Y \cap Y_2'$ is a partition that splits ψ. Indeed let $c \in \{1, \ldots, N\}$. Since ψ' is splittable there is and edge $\{w, z\}$ with $c \in \psi'(\{w, z\})$ and $w \in Y_l', z \in Y_{3-l}'$. If $w, z \in Y$ we are done, hence we may assume without loss of generality that $w = v' \in Y$ and z belongs to the added vertices $\{v_1, \ldots, v_k\}$, say $z = v_j$. Since $w = v' \in Y_l' \cap Y$ and $z = v_j \in Y_{3-l}'$, it follows from the previous remark that $\overline{v} \in Y_l'$ and $v \in Y_{3-l}' \cap Y$. Since $c \in \psi(\{v, v'\})$, we get that $Y \cap Y_1', Y \cap Y_2'$ splits ψ.
Conversely suppose that Y_1, Y_2 splits the list coloring ψ. Suppose that $\{v, v'\} \in T$ with $v \in Y_l, v' \in Y_k$ for some $k, l \in \{1, 2\}$. We put all the added vertices $\{v_1, \ldots, v_k\}$ in Y_l' and $\overline{v}$ in Y_{3-l}'. In this way all the added colors are splitted by Y_l', Y_{3-l}'. Thus, by the construction of the graph H', we get that Y_l', Y_{3-l}' splits ψ'.
It is easy to check that if φ is a n-bounded coloring then also φ' is n-bounded.
The last statement can be obtained in a similar way. Indeed starting from the previous colored graph (G', φ'), to obtain (G'', φ'') it is enough to apply the following construction iteratively. Suppose that the colored graph (H, ψ) has a vertex v such that there are two incident edges $\{v, v_1\}, \{v, v_2\}$ with $\varphi(\{v, v_1\}) = \varphi(\{v, v_2\}) = i$. Then build a new colored graph (H', ψ') erasing from H the edge $\{v, v_2\}$ and adding two new vertices $\overline{v}, v'$ and the following three new edges: $\{v', v_2\}$, colored by $\psi'(\{v', v_2\}) = i$, and the edges $\{\overline{v}, v\}, \{\overline{v}, v'\}$ colored by two new colors $\psi'(\{\overline{v}, v\}) = c$, $\psi'(\{\overline{v}, v'\}) = c'$. Since c, c' are new colors it is easy to see that in a splitting, v, v' belong to the same component of the partition. Therefore ψ' is spittable iff ψ is splittable. Since in each iteration we reduce the number of incident edges having the same colors, the number of iterations is upper bounded by $N(|V'| - 2)$ where N is the number of colors of (G', φ') and $G' = (V', E')$. Thus $|E''| \leq 2N(|V'| - 2)$ and so η'' is a reduction. $\square$

Since `SPLITTABLE` is a sub-problem of `LIST-SPLITTABLE`, from Proposition 4 we have that actually `SPLITTABLE` and `LIST-SPLITTABLE` are equivalent problems. It is easy to see that both 1-`LIST-SPLITTABLE` and 1-`SPLITTABLE` coincide with the problem of checking if a graph is bipartite and so they belong to the computational class **P**. The following theorem shows that things change radically when we consider 2-`SPLITTABLE`, indeed we have the following:

Theorem 2. *2-SPLITTABLE is **NP**-complete.*

Proof. The problem is clearly in **NP**. To prove the completeness we reduce `NAESAT` to 2-`SPLITTABLE`. `NAESAT` is the problem of checking, given a boolean formula in CNF

$$\mathcal{F} = \bigwedge_{i=1}^{m} \mathcal{C}_i$$

where in each clause C_i there are three literals, whether there is a truth assignment such that in no clause all three literals are equal in truth value (neither all true nor all false). This is a well know **NP**-complete problem [3].
Suppose that the boolean formula $\mathcal{F}$ is on the set of variables $X = \{x_1, \ldots, x_n\}$. We build a graph $G = (V, E)$ and a 2-bounded list coloring φ in the following way. The set of vertices is $V = X \cup \{\neg x : x \in X\}$. The set of colors $C = \{1, \ldots, m, t_1, \ldots, t_n\}$ corresponds to the set of clauses and the set of variables. The set of edges contains all pairs $\{x, \neg x\}$ for $x \in X$ and $t_j \in \varphi(\{x_j, \neg x_j\})$ for all $j \in \{1, \ldots, n\}$. Moreover for each color $i \in \{1, \ldots, m\}$ suppose that the clause $\mathcal{C}_i = l_1^i \vee l_2^i \vee l_3^i$. Then $\{l_1^i, l_2^i\}, \{l_2^i, l_3^i\} \in E$ and $i \in \varphi(\{l_1^i, l_2^i\}), i \in \varphi(\{l_2^i, l_3^i\})$. Clearly (G, φ) is a 2-bounded list colored graph. Let us prove that φ is list splittable if and only if there is a truth assignment such that in no clause all the three literals are all equal in truth value.
Suppose that φ is list splittable, and consider the corresponding partition of V into two disjoint sets V_1, V_2. Since for each $i = 1, \ldots, n$, t_i is contained only in the list coloring of the the edge $\{x_i, \neg x_i\}$, it is clear that if a literal $l \in V_1$ then $\neg l \in V_2$. Therefore there is a truth assignment that makes (for instance) all the literals of V_1 true and all the literals of V_2 false. Since φ is list splittable, for each clause $\mathcal{C}_i = l_1^i \vee l_2^i \vee l_3^i$ there is one edge $\{\alpha, \beta\}$ among $\{l_1^i, l_2^i\}, \{l_2^i, l_3^i\}$ such that $\alpha \in V_1, \beta \in V_2$. Hence in each clause there is a literal which is true and one which is false.
Conversely suppose that there is a truth assignment such that in no clause all the three literals are equal in truth value. Let V_1 be the set of literals in V that are true and $V_2 = V \setminus V_1$ be the one that are false. Clearly if $l \in V_1$ then $\neg l \in V_2$ thus the colors $t_i, i = 1, \ldots, n$ which are contained only in the edges $\{x_i, \neg x_i\}$ are clearly splitted. Moreover in each clause $\mathcal{C}_i = \{l_1^i, l_2^i, l_3^i\}$ there are two literals say $\alpha, \beta \in C_i$ which are respectively true and false, whence $\alpha \in V_1, \beta \in V_2$. Since $\{l_1^i, l_2^i\}, \{l_2^i, l_3^i\} \in E$ it is not difficult to see that there is an edge $\{\alpha', \beta'\} \in \{\{l_1^i, l_2^i\}, \{l_2^i, l_3^i\}\}$ with $\alpha' \in V_1, \beta' \in V_2$ and so the color i is splitted. Since this holds for all the colors $i \in \{1, \ldots, m\}$ we can conclude that V_1, V_2 splits φ.
By now we have reduced `NAESAT` to `LIST-SPLITTABLE` with a 2-bounded list coloring graph. Hence by Proposition 4 we can reduce `NAESAT` to 2-`SPLITTABLE`. □

Let (G, φ) be a list colored graph with $G = (V, E)$ and $\varphi : E \to 2^C$ for some set of colors C. We say that φ is *chromatic-transitive* if for any $i \in C$ the connected components of $G(i)$ are complete subgraphs. Equivalently iff the associated relation $\mathcal{R}(G(i))$ is an equivalence relation on V. Therefore we can define the chromatic-transitive closure of (G, φ) as the list colored graph $(\overline{G}, \overline{\varphi})$ with vertex set V and whose maximal monochromatic components are

$$\left\{ \mathcal{G}\left(\overline{\mathcal{R}(G(i))}^{tr} \right) \right\}_{i \in C}$$

where $\overline{\cdot}^{tr}$ is the transitive closure operator. The definition of chromatic-transitive closure is interesting under the following proposition.

Proposition 5. *Let (G, φ) be a list colored graph. Thus (G, φ) is list splittable if and only if $(\overline{G}, \overline{\varphi})$ is list splittable.*

Proof. Since G is a subgraph of $\overline{G}$ and $\overline{\varphi}$ is an extension of φ, then if (G, φ) is list splittable then also $(\overline{G}, \overline{\varphi})$ is list splittable. Conversely, assume that $(\overline{G}, \overline{\varphi})$ is list splittable and let V_1, V_2 be a corresponding partition of V. Given a color $i \in C$, we have an edge $\{v_0, w\}$ in $\overline{G}(i)$ with $v_0 \in V_1$ and $w \in V_2$. Thus we have a sequence of edges $\{v_0, v_1\}, \ldots, \{v_{m-1}, v_m\}$ in $G(i)$ with $v_m = w$. Since $v_0 \in V_1$ and $w \in V_2$, there is a $j \in \{1, \ldots, m\}$ such that $(v_{j-1}, v_j) \in V_1 \times V_2 \cup V_2 \times V_1$ and so (G, φ) is list splittable by considering the partition V_1, V_2. $\square$

In view of this proposition and the fact that a list coloring is determined by its maximal monochromatic components we define the problem `SEPARATING`. Given a set $\mathfrak{R}$ of equivalence relations on a set V as input, `SEPARATING` is the problem of checking whether or not there is a set $F \subseteq V$ which is not saturated by any equivalence relation of $\mathfrak{R}$, i.e. for any $\sigma \in \mathfrak{R}$, F is not union of equivalence classes of σ. The following lemma is an easy consequence of the definitions.

Lemma 2. *Let $\mathfrak{R} = \{\sigma_1, \ldots, \sigma_k\}$ be a family of equivalence relations on a set V and let (G, φ) be the associated list colored graph:*

$$G = \mathcal{G}(\cup_{\sigma \in \mathfrak{R}} \sigma),\ \varphi(\{x, y\}) = \{i \in \{1, \ldots, k\} : (x, y) \in \sigma_i\}$$

There is a set $F \subseteq V$ which is not saturated by any $\sigma \in \mathfrak{R}$ if and only if (G, φ) is splittable.

By Proposition 5 and Lemma 2 it immediately follows that `LIST-SPLITTABLE` and `SEPARATING` are equivalent problems. The problem `ORTHOGONAL-SEPARATING` is the analogous problem but with the difference that the input $\mathfrak{R}$ is a family of orthogonal equivalence relations. This problem has been introuced in [1] under the name `DISJUNCTIVE SET` as a clue of the possible **NP**-completeness of the `SYNTACTIC MONOID` problem. In the same article it is provided a proof of the **NP**-completeness of `DISJUNCTIVE SET` when are considered families of the following kind:

1. (Corollary 2.5 [1]) Orthogonal families $\mathfrak{R}$ such that each equivalence relation $\sigma \in \mathfrak{R}$ has at most three non-singleton classes and these non-singleton classes have exactly two elements.
2. (Corollary 2.4 [1]) Orthogonal families $\mathfrak{R}$ such that each equivalence relation $\sigma \in \mathfrak{R}$ has exactly one non-singleton class and this non-singleton class has at most four elements.

We remark that `ORTHOGONAL-SEPARATING` restricted to the case (2) with families $\mathfrak{R}$ such that each equivalence relation $\sigma \in \mathfrak{R}$ has exactly one non-singleton class and this non-singleton class has at most two elements elements is equivalent to 1-`SPLITTABLE`, i.e. to check whether or not the associated colored graph as in Lemma 2 is bipartite. Therefore in this case `ORTHOGONAL-SEPARATING` is in **P**. The same occurs if we restrict it to the case (1) allowing the equivalence relations to have at most one non-trivial equivalence class. The following theorem establishes the exact borderline for which `ORTHOGONAL-SEPARATING` turns out to be **NP**-complete.

Theorem 3. *`ORTHOGONAL-SEPARATING` is still **NP**-complete if we assume*

1. *Orthogonal families $\mathfrak{R}$ such that each equivalence relation $\sigma \in \mathfrak{R}$ has at most two non-singleton classes and these non-singleton classes have exactly two elements.*
2. *Orthogonal families $\mathfrak{R}$ such that each equivalence relation $\sigma \in \mathfrak{R}$ has exactly one non-singleton class and this non-singleton class has at most three elements.*

Proof. We prove the statement (1). Case (2) can be obtained analogously by the structure of the obtained graph of Theorem 2, Lemma 2 and a similar technique of Proposition 4 to pass from a list coloring to a coloring.
By Theorem 2, 2-`SPLITTABLE` is **NP**-complete. Thus given a 2-bounded coloring graph (G, φ) with $G = (V, E)$ and $\varphi : E \to C$, by Proposition 4 we can suppose without loss of generality that φ is anti-incidence. Therefore for each color $i \in C$, $\overline{\mathcal{R}(G(i))}^{tr} = \mathcal{R}(G(i))$ and so the transitive closure $(\overline{G}, \overline{\varphi})$ is equal to (G, φ). Moreover since (G, φ) is a colored graph, the associated family of equivalence relations $\mathfrak{R} = \{\mathcal{R}(G(i)) : i \in C\}$ is orthogonal. By Lemma 2, $\mathfrak{R}$ is in `ORTHOGONAL-SEPARATING` if and only if φ splits. Since φ is a 2-bounded coloring and $(\overline{G}, \overline{\varphi}) = (G, \varphi)$, it is also evident that the family $\mathfrak{R} = \{\mathcal{R}(G(i)) : i \in C\}$ is formed by equivalence classes composed by at most two elements and there are at most two equivalence classes which are not singletons. □

4 NP-completeness of co-`NEVER-MINIMAL`

In this section we show that, given an orthogonal family $\mathfrak{R}$ of equivalence relations, we can always build a strongly connected DFA $\mathscr{A}$ having $\mathfrak{R}$ as the set of minimal congruences of $\mathscr{A}$. Indeed we have the following theorem.

Theorem 4. *Let $\mathfrak{R} = \{\sigma_1, \ldots, \sigma_m\}$ be an orthogonal family of equivalence relations on a set Q and let $G = \mathcal{G}(\cup_{\sigma \in \mathfrak{R}} \sigma)$ with $G = (Q, E)$. There is a strongly connected DFA $\mathscr{A}_{\mathfrak{R}} = \langle Q, \Sigma, \delta \rangle$ with*

$$|\Sigma| \leq 3 \sum_{\sigma \in \mathfrak{R}} |Q/\sigma| + \binom{|Q|}{2} - |E|$$

such that $\mathcal{M}(\mathscr{A}_{\mathfrak{R}}) = \mathfrak{R}$.

Proof. For each $\sigma \in \mathfrak{R}$ suppose that in Q/σ there are $[q_1]_\sigma, \ldots, [q_t]_\sigma$ non singleton classes and $[q_{t+1}]_\sigma, \ldots, [q_n]_\sigma$ singleton classes, where $n = |Q/\sigma|$. Putting $[q_i]_\sigma = \{q_i^1, \ldots, q_i^{n_i}\}$ for $i = 1, \ldots, t$, we define an alphabet

$$\Sigma_\sigma = \{a(\sigma)_1, b(\sigma)_1, \ldots, a(\sigma)_t, b(\sigma)_t, c(\sigma)_1, \ldots c(\sigma)_n\}$$

The action is defined by the following rules

$$\delta(q_i^j, a(\sigma)_i) = q_i^{j+1 \bmod n_i},$$

$$\delta(q_i^1, b(\sigma)_i) = q_i^1, \delta(q_i^{n_i}, b(\sigma)_i) = q_i^1, \delta(q_i^s, b(\sigma)_i) = q_i^{s+1} \text{ for } s = 2, \ldots, n_i - 1,$$

$$\delta(q_i^1, c(\sigma)_i) = q_{i+1 \bmod t}^1,\ \delta(q_i^2, c(\sigma)_i) = q_{i+1 \bmod t}^2,$$

$$\delta(q_i^1, c(\sigma)_i) = q_{i+1}^1 \text{ for } t \le i < n, \delta(q_n^1, c(\sigma)_n) = q_t^1$$

The alphabet $\Sigma = \cup_{\sigma \in \mathfrak{R}} \Sigma_\sigma \cup \Sigma'$ is the disjoint union of the alphabets $\Sigma_\sigma, \sigma \in \mathfrak{R}$ and an alphabet Σ' used to satisfy the minimality condition $\mathcal{M}(\mathscr{A}_{\mathfrak{R}}) = \mathfrak{R}$. The action of Σ' is defined in the following way. For any pair p, q of distinct elements such that (p, q) do not belong to $\cup_{\sigma \in \mathfrak{R}} \sigma$ we have to satisfy the condition $\sigma \subseteq \sigma_{p,q}$ for some $\sigma \in \mathfrak{R}$. Therefore we define $\Sigma' = \{a(p, q) : \{p, q\} \notin E\}$ where $G = \mathcal{G}(\cup_{\sigma \in \mathfrak{R}} \sigma)$. Regarding the action we first fix two states $\overline{q}, \overline{q}'$ such that $(\overline{q}, \overline{q}') \in \overline{\sigma}$ for some $\overline{\sigma} \in \mathfrak{R}$ and we put

$$\delta(p, a(p, q)) = \overline{q},\ \delta(q, a(p, q)) = \overline{q}'.$$

It is not difficult to see that $\mathscr{A}$ is strongly transitive (in particular it is strongly transitive even if we restrict the action to Σ_σ for any $\sigma \in \mathfrak{R}$). Moreover the action δ is transitive on each $\sigma \in \mathfrak{R}$ in the sense that if $(x, y), (x', y') \in \sigma$ are two distinct pairs with $x \neq y$ and $x' \neq y'$, then there is a word $w \in \Sigma_\sigma^*$ such that $\delta(\{x, y\}, w) = \{x', y'\}$. Moreover since $\mathfrak{R}$ is orthogonal the action of any letter in $\Sigma \setminus \Sigma_\sigma$ brings any pair of distinct elements $\{x, y\}$ with $(x, y) \in \sigma$ into a singleton. Thus for each $\sigma \in \mathfrak{R}$ and for each pair $(x, y) \in \sigma$ with $x \neq y$, we have $\sigma_{x,y} = \sigma$. The minimality condition is also satisfied since for any $\{p, q\} \notin E$ we have $\mathfrak{G}_{\overline{q}, \overline{q}'} \prec \mathfrak{G}_{p,q}$, i.e. $\overline{\sigma} \subseteq \sigma_{p,q}$. Hence $\mathcal{M}(\mathscr{A}_{\mathfrak{R}}) = \mathfrak{R}$ and a simple computation gives the bound for $|\Sigma|$. $\square$

This theorem gives also a way to build never-minimal automata. Indeed to build a never-minimal DFA, by Proposition 3, it is enough to consider an orthogonal family $\mathfrak{R}$ which is not in `ORTHOGONAL-SEPARATING`, and then apply the construction of Theorem 4 to $\mathfrak{R}$. A very simple class of orthogonal families which are not `ORTHOGONAL-SEPARATING` are the families obtained from graphs which are not bipartite. Indeed consider a non-bipartite graph $G = (V, E)$, color it using the identity map $1_E : E \to E$ where E is now the set of colors. Therefore the set

$$\mathfrak{R}_G = \{\mathcal{R}(G(e)) : e \in E\}$$

is clearly an orthogonal family of equivalence relations which saturates all the subsets of V since, by Lemma 2, $(G, 1_E)$ is not splittable. We also remark that condition C_3 defined in [4] is translated to the condition that the colored graph associated to the DFA which satisfies C_3 contains a rainbow triangle and no other edge is colored using colors of this triangle, whence this graph is clearly non-splittable and so the automaton is never-minimal.
We conclude the section with the following consequences of Theorem 4.

Theorem 5. *co-`NEVER-MINIMAL` is equivalent to `ORTHOGONAL-SEPARATING`.*

Proof. By Proposition 3, $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is not never-minimal if and only if $\mathcal{M}(\mathscr{A})$ is a family of orthogonal relations belonging to `ORTHOGONAL-SEPARATING`. It is straightforward to check that computing $\mathcal{M}(\mathscr{A})$ is a polynomial time and space task since the construction of the syntactic graphs is polynomial and there are at most $\binom{|Q|}{2}$ such graphs. Thus co-`NEVER-MINIMAL` is reducible to `ORTHOGON AL-SEPARATING`. Conversely given an orthogonal family $\mathfrak{R} = \{\sigma_1, \ldots, \sigma_m\}$ of equivalence relations on a set Q by Theorem 4 we can build in polynomial space an automaton $\mathscr{A}_{\mathfrak{R}} = \langle Q, \Sigma, \delta \rangle$ with $\mathcal{M}(\mathscr{A}_{\mathfrak{R}}) = \mathfrak{R}$. By Proposition 3, $\mathfrak{R}$ is in `ORTHOGONAL-SEPARATING` if and only if $\mathscr{A}_{\mathfrak{R}}$ is not never-minimal. □

From Theorems 3, 5 we have also the following consequence.

Corollary 1. *co-`NEVER-MINIMAL` is **NP**-complete.*

5 Connections with the syntactic monoid problem

As already mentioned in [4], `NEVER-MINIMAL` is related to the `SYNTACTIC MONOID` problem (cf. [1]). A finite monoid $(M, \cdot)$ is called syntactic if there is a $P \subseteq M$ such that the congruence $\sim_P$ on M defined by

$$x \sim_P y \Leftrightarrow \forall a, b \in M (axb \in M \Leftrightarrow ayb \in M)$$

is the identity congruence on M. If a monoid M is the transition monoid of a DFA $\mathscr{A}$, and it is not syntactic, then $\mathscr{A}$ is never-minimal. However the problem of the positioning among the complexity classes of this problem is still open.
To conclude the paper we give a characterization of the monoids which are not syntactic in term of never-minimal automata. We consider complete automata here, hence an automaton $\langle Q, \Sigma, \delta, i, F \rangle$ is minimal if and only if every vertex is accessible from the initial vertex i and the Nerode equivalence defined by $p \mathcal{N} q$ if $p^{-1}F = q^{-1}F$ is the identity relation on Q.
Let $(M, \cdot)$ be a finite monoid and let $A \subseteq M$ be a set of generators for M. The *two-side Cayley automaton* of M is the automaton $\hat{\Gamma}_A(M) = \langle M, A \cup A', \delta \rangle$ where $A' = \{a' : a \in A\}$ is a disjoint copy of A and $\circ'$ is an involution in $(A \cup A')^*$ such that $(uv)' = v'u'$ for all $u, v \in (A \cup A')^*$. The action δ is defined by:

$$\forall a \in A,\ \delta(u, a) = u \cdot a,\ \ \forall a \in A',\ \delta(u, a') = a \cdot u$$

We have the following characterization.

Theorem 6. *M is not syntactic if and only if $\hat{\Gamma}_A(M)$ is never-minimal.*

Proof. Assume first that M is syntactic. Then there exists some $P \subseteq M$ such that $\sim_P = 1_M$. We claim that $(\hat{\Gamma}_A(M), 1, P)$ is a minimal automaton. Since every vertex is accessible from 1, it remains to show that the Nerode equivalence is trivial. Let $u, v \in M$ be different. Then $(x, y) \notin \sim_P$, hence we may assume without loss of generality that $xuy \in P$ and $xvy \notin P$ for some $x, y \in M$. It follows that $\delta(u, x'y) = xuy \in P$ and $\delta(v, x'y) = xvy \notin P$, hence $(u, v) \notin \mathcal{N}$ as

required.
Conversely suppose that $(\hat{\Gamma}_A(M), i, P)$ is a minimal automaton. We claim that $\sim_P = 1_M$. Indeed, take $u, v \in M$ distinct. Since $(u, v) \notin \mathcal{N}$, we may assume that $\delta(u, w) \in P$ and $\delta(v, w) \notin P$ for some $w \in (A \cup A')^*$. Now there exist $x, y \in M$ such that $\delta(u, w) = xuy$ and $\delta(v, w) = xvy$, hence $u \not\sim_P v$ and so $\sim_P = 1_M$. Therefore M is syntactic. □

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