New results on the Bochner condition about classical orthogonal polynomials

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Abstract

The classical polynomials (Hermite, Laguerre, Bessel and Jacobi) are the only orthogonal polynomial sequences (OPS) whose elements are eigenfunctions of the Bocnher second-order differential operator $\mathscr{F}$ [1]. In [16] these polynomials were described as eigenfunctions of an even order differential operator $\mathscr{F}_k$ with polynomial coefficients defined by a recursive relation. Here, an explicit expression of $\mathscr{F}_k$ for any positive integer k is given. The main purpose of this work lies in explicitly establish sums relating any power of $\mathscr{F}$ with $\mathscr{F}_k$, $k \geqslant 1$. To accomplish this goal, we introduce and develop the concept of the so called *A-modified Stirling numbers*, which could also be called as *Bessel* or *Jacobi-Stirling* numbers, depending on the context and the values of the parameter A.

Key words: classical orthogonal polynomials, Bochner differential equation, Stirling numbers, Bessel-Stirling numbers, Jacobi-Stirling numbers, inverse relations
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1. Introduction and preliminaries

The classical polynomial sequences (Hermite, Laguerre, Bessel and Jacobi) are the only orthogonal polynomial sequences (OPS), whose elements are solutions of a certain second order differential equation with polynomial coefficients given in (1.8)-(1.9), which is commonly known as *Bochner's differential equation*, in honour to the mathematician S. Bochner, for his work in 1929 [1]. Notwithstanding that, Bochner did not consider the polynomial sequence now called as Bessel polynomials (after the work of H. L. Krall and O. Frink in 1949 [12]) to be an OPS, but he realised they were also solution of such differential equation. Also W. Hahn in 1935 [8] and H.L. Krall in 1938 [11] considered Bessel polynomials to be orthogonal but in a generalised sense.

In 1938, Krall [11] showed that if the elements of a classical polynomial sequence are eigenfunctions of a differential operator, then it must be of even order. This result motivated a generalisation on Bochner's condition characterising the elements of a classical polynomial sequence [13,14]. However, in the cited works, an explicit and precise expression for the generalised equation is not given. Later, in [16], the authors build the necessary differential equation with polynomial coefficients having classical polynomials as solution (we review this result: see theorem 2.1). Such polynomial coefficients were then defined through a recursive relation. In the present work this result is improved by bringing their explicit expression (see theorem 2.2 below). Therefore, at the end of section 2.1 the elements of a classical sequence will be described as eigenfunctions of an explicitly determined even order differential operator $\mathscr{F}_k$ for any given positive integer k.

On the other hand, in [20] Miranian has shown that any even order differential operator having classical polynomials as eigenfunctions must be a polynomial with constant coefficients in the Bochner's differential operator, say $\mathscr{F}$, given in (1.9). Again, the methodology adopted is not constructive. In section 2.2, it is thoroughly explained how the $2k$-order differential operator $\mathscr{F}_k$ may be written as a polynomial with constant coefficients in $\mathscr{F}$, and, conversely how any power of $\mathscr{F}$ may be described as a sum in $\mathscr{F}_\tau$ with $0 \leqslant \tau \leqslant k$. The bridge between these two operators can be done through the Stirling numbers. Therefore, in section 3 we review this concept, which is sufficient to study the cases of Hermite and Laguerre families, whereas the cases of Bessel or Jacobi polynomials required the introduction of the concept of the so-called *A-modified Stirling numbers*, where A represents a complex parameter. On account of these sets of numbers, we fulfil our primary objective: to explicitly establish a somewhat "inverse" relation between any power of $\mathscr{F}$ and the operators $\mathscr{F}_k$. The analysis is guided separately for each classical family in section 4. Other authors have dealt with this problem but from a rather different point of view [2,6,15]. As a matter of fact, in [6,15] an explicit expression for any power of the Bochner's operator in the cases of Hermite, Laguerre and Jacobi families can be viewed, but in a self adjoint form. As far as we are concerned, the case of Bessel polynomials has not yet been considered.

First, we review preliminary results needed for the sequel. The vector space of the polynomials with coefficients in $\mathbb{C}$ (the field of complex numbers) is denoted by $\mathscr{P}$ and by $\mathscr{P}'$ its dual space, whose elements are called *forms*. The set of all the nonnegative integers will be denoted as $\mathbb{N}$ and by $\mathbb{N}^*$ we mean $\mathbb{N}-\{0\}$. The action of $u \in \mathscr{P}'$ on $f \in \mathscr{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \in \mathbb{N}$ the moments of u. Since a linear operator $T : \mathscr{P} \to \mathscr{P}$ has a transpose ${}^tT : \mathscr{P}' \to \mathscr{P}'$ defined by $\langle {}^tT(u), f\rangle = \langle u, T(f)\rangle$, $\forall u \in \mathscr{P}'$, $f \in \mathscr{P}$, then for any form u, any polynomial g, the forms $Du = u'$ and gu are as usual defined by duality according to

$$\langle u', f\rangle := -\langle u, f'\rangle \quad , \quad \langle gu, f\rangle := \langle u, gf\rangle, \quad f \in \mathscr{P},$$

where D represents the differential operator. Thus, the differential operator D on forms is minus the transpose of the differential operator D on polynomials. Throughout the text, the k-th derivative of $p \in \mathscr{P}$ is denoted either as $D^k p$ or $(p)^{(k)}$. For any $f \in \mathscr{P}$ and $u \in \mathscr{P}'$, we have:

$$D^k(fu) = \sum_{\nu=0}^{k}\left[\binom{k}{\nu}(D^\nu f)(D^{k-\nu}u)\right], k \geqslant 1. \qquad \text{(Leibniz derivation formula)}$$

We will only consider sequences of polynomials $\{P_n\}_{n\in\mathbb{N}}$ such that $\deg P_n \leqslant n$, $n \in \mathbb{N}$. If the set $\{P_n\}_{n\in\mathbb{N}}$ spans $\mathscr{P}$, which occurs when $\deg P_n = n$, $n \in \mathbb{N}$, then it will be called a *polynomial sequence* (PS). Along the text we will only deal with PS whose elements are monic, that is, *monic polynomial sequences* (MPS). It is always possible to associate to $\{P_n\}_{n\in\mathbb{N}}$ a unique sequence $\{u_n\}_{n\in\mathbb{N}}$, $u_n \in \mathscr{P}'$, $n \in \mathbb{N}$, called the *dual sequence* of $\{P_n\}_{n\in\mathbb{N}}$, and such that $\langle u_n, P_m\rangle := \delta_{n,m}$, $n, m \in \mathbb{N}$, where $\delta_{n,m}$ represents the Kronecker's symbol. We recall a result.

Lemma 1.1 *For any $u \in \mathscr{P}'$ and any integer $m \geqslant 1$, the following statements are equivalent.*
(i) $\langle u, P_{m-1}\rangle \neq 0$, $\langle u, P_n\rangle = 0$, $n \geqslant m$.
(ii) $\exists \lambda_\nu \in \mathbb{C}$, $0 \leqslant \nu \leqslant m-1$, $\lambda_{m-1} \neq 0$ *such that* $\quad u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.
Furthermore, $\lambda_\nu = \langle u, P_\nu\rangle$, $0 \leqslant \nu \leqslant m-1$.

We will denote as $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ the MPS obtained from a given MPS through a single differentiation, precisely, $P_n^{[1]}(x) := \frac{1}{n+1} P'_{n+1}(x)$, $n \in \mathbb{N}$, and we call it the normalised derivative of the original sequence. The normalised derivative sequences of higher orders, say $k \geqslant 1$, denoted as $\{P_n^{[k]}\}_{n\in\mathbb{N}}$, are recursively defined by

$$P_n^{[k+1]}(x) = \frac{\left(P_{n+1}^{[k]}(x)\right)'}{n+1}, \; n \in \mathbb{N}. \tag{1.1}$$

As a consequence of lemma 1.1, the dual sequence associated to $\{P_n^{[k]}\}_{n\in\mathbb{N}}$, say $\{u_n^{[k]}\}_{n\in\mathbb{N}}$, fulfils the recurrence relation

$$D\left(u_n^{[k]}\right) = -(n+k)\, u_{n+1}^{[k-1]}, \qquad \text{with} \qquad u_n^{[0]} = u_n, \; n \in \mathbb{N}. \tag{1.2}$$

It may be easily derived by finite induction that

$$D^k\left(u_n^{[k]}\right) = (-1)^k \prod_{\mu=1}^{k}(n+\mu)\; u_{n+k}\,, \quad n \in \mathbb{N}, \; k \geqslant 1. \tag{1.3}$$

The MPS $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to $u \in \mathscr{P}'$ when the following conditions hold: $\langle u, P_n P_m\rangle = k_n \delta_{n,m}$ with $k_n \neq 0$, for all the integers $n,m \in \mathbb{N}$ [3,17]. In this case, we say that $\{P_n\}_{n\in\mathbb{N}}$ is a *monic orthogonal polynomial sequence* (MOPS) and the form u is *regular*. Necessarily, u is proportional to u_0. Furthermore, we have

$$u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n u_0, \quad n \in \mathbb{N}, \tag{1.4}$$

and the MOPS $\{P_n\}_{n\in\mathbb{N}}$ fulfils the second order recurrence relation given by

$$P_0(x) = 1 \qquad P_1(x) = x - \beta_0 \tag{1.5}$$

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \in \mathbb{N}. \tag{1.6}$$

with $\beta_n = \dfrac{\langle u_0, xP_n^2\rangle}{\langle u_0, P_n^2\rangle}$ and $\gamma_{n+1} = \dfrac{\langle u_0, P_{n+1}^2\rangle}{\langle u_0, P_n^2\rangle} \neq 0$, $n \in \mathbb{N}$. For any regular form u and any polynomial A such that $A\,u = 0$, we necessarily have $A = 0$ [17].

The MOPS $\{P_n\}_{n\in\mathbb{N}}$ is said to be <u>*classical*</u> if $\{P_n^{[1]}\}_{n\in\mathbb{N}}$ is also orthogonal (Hahn's property, [8,9]) and u_0 is called a *classical form* (Hermite, Laguerre, Bessel and Jacobi). Among all the well known characterisations of the classical sequences we recall the following:

Theorem 1.2 *For any MPS $\{P_n\}_{n\in\mathbb{N}}$ orthogonal with respect to u_0, the following statements are equivalent:*

(i) *$\{P_n\}_{n\in\mathbb{N}}$ is a classical sequence.*

(ii) *There exists $k \geqslant 1$ such that $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is orthogonal (Hahn's theorem),* [9].

(iii) *There exist two polynomials Φ and Ψ such that the associated regular form u_0 satisfies*

$$D\big(\Phi u_0\big) + \Psi u_0 = 0\,, \tag{1.7}$$

where $\deg\Phi \leqslant 2$ *(Φ monic) and* $\deg(\Psi) = 1$ *[7,17,18].*

(iv) *There exist two polynomials, Φ monic,* $\deg\Phi \leqslant 2$*, Ψ,* $\deg\Psi = 1$ *and a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ with $\chi_0 = 0$ and $\chi_{n+1} \neq 0$, $n \in \mathbb{N}$, such that*

$$\mathscr{F}\big(P_n(x)\big) = \chi_n P_n,\ \ n \geqslant 0, \quad \textit{(Bochner's condition [1])} \tag{1.8}$$

where

$$\mathscr{F} = \Phi(x)\,D^2 - \Psi(x)\,D\,. \tag{1.9}$$

Corollary 1.3 [17,18] *If the MOPS $\{P_n\}_{n\in\mathbb{N}}$ is classical, then so is $\{P_n^{[k]}\}_{n\in\mathbb{N}}$, whenever $k \geqslant 1$, and any polynomial $P_{n+1}^{[k]}$ fulfils the following differential equation:*

$$\Phi\left(P_n^{[k]}\right)'' - (\Psi - k\Phi')\left(P_n^{[k]}\right)' = \chi_n^{[k]}\left(P_n^{[k]}\right)\,, \quad n \in \mathbb{N}, \tag{1.10}$$

where $\Phi, \Psi \in \mathscr{P}$ (with Φ monic and $\deg\Phi \leqslant 2$, $\deg\Psi = 1$*) and $\chi_0^{[k]} = 0$,*

$$\chi_{n+1}^{[k]} = (n+1)\,\left\{\frac{n+2k}{2}\,\Phi''(0) - \Psi'(0)\right\} \neq 0,\ n \in \mathbb{N}.$$

Thus, if $\{P_n\}_{n\in\mathbb{N}}$ is a classical MOPS with respect to u_0, so is $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ and the associated classical form is:

$$u_0^{[k]} = \zeta_k \, \Phi^k \, u_0 \,, \tag{1.11}$$

for some $\zeta_k \neq 0$.

In Table 1 we precise the expressions for the polynomials Φ and Ψ and the constants χ_n and ϑ_n, presented respectively in (1.8), for each one of the classical families.

Table 1
Expressions for Φ, Ψ and χ_n , $n \geqslant 0$, for each classical family.

	:	**Hermite**	**Laguerre**	**Bessel**	**Jacobi**
Regularity conditions $n\in\mathbb{N}$	:		$\alpha \neq -(n+1)$	$\alpha \neq -\frac{n}{2}$	$\alpha,\beta \neq -(n+1)$ $\alpha+\beta \neq -(n+2)$
$\Phi(x)$	:	1	x	x^2	x^2-1
$\Psi(x)$	:	$2x$	$x-\alpha-1$	$-2\,(\alpha x+1)$	$-(\alpha+\beta+2)x+(\alpha-\beta)$
χ_n	:	$-2n$	$-n$	$n(n+2\alpha-1)$	$n(n+\alpha+\beta+1)$

During the text the *Gamma* function will be represented, as usual, by $\Gamma(z)$. For $k\in\mathbb{N}$, and we will consider the shifted factorials

$$\{x\}_{(\mathbf{k})} = \begin{cases} 1 & \text{if } k=0 \\ \displaystyle\prod_{\nu=0}^{k-1}(x-\nu) = x(x-1)\ldots(x-k+1) & \text{if } k\in\mathbb{N}^* \end{cases} \tag{1.12}$$

$$(x)_k = \begin{cases} 1 & \text{if } k=0 \\ \displaystyle\prod_{\nu=0}^{k-1}(x+\nu) = x(x+1)\ldots(x+k-1) & \text{if } k\in\mathbb{N}^* \end{cases} \tag{1.13}$$

Naturally $\{x\}_{(\mathbf{k})} = \frac{\Gamma(x+1)}{\Gamma(x-k+1)}$ and $(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}$ for $k\in\mathbb{N}$. The symbol $\{x\}_{(\mathbf{k})}$ is sometimes called *falling factorial* (of order k), and $(x)_k$ is sometimes called as *rising factorial* (of order k) or also *Pochammer* symbol. The symbol $(x)_k$ is sometimes used, among combinatorialists, to denote the falling factorial instead of Pochhammer symbol. Despite in the text we will make reference to the book of Comtet [5] or Riordan [21,22], we will keep the notation that almost everybody use in what concerns the Pochhammer symbol. Moreover, one may easily observe that $\{x\}_{(\mathbf{k})} = (x-k+1)_k$ and $(x)_k = \{x+k-1\}_{(\mathbf{k})}$ for any $k\in\mathbb{N}$.

2. Generalisation on Bochner's condition about the classical orthogonal polynomials

This section concerns with generalisations on the Bochner's condition about classical orthogonal polynomials. We begin by reviewing a result of [16], where a differential equation of any even order having classical orthogonal polynomials as solutions is given (see theorem 2.1 below). In theorem 2.2, we improve this result by giving an explicit expression for the polynomial coefficients of the differential operator. Later on, in corollary 2.5, we show how the classical polynomials are eigenfunctions of any polynomial (with constant coefficients) in the operator $\mathscr{F}$ given by (1.9). At the end of this section we expound the relation between the two even order differential operators: the one given in theorem 2.1 and the one given in corollary 2.5.

2.1. *Generalised Bochner's condition*

For the sake of simplicity, whenever there is no danger of confusion, we will adopt the notation $Q_n := P_n^{[k]}$ with $k \geqslant 1$ and the elements of the dual sequence associated to $\{Q_n\}_{n\in\mathbb{N}}$ will be denoted as v_n, instead of $u_n^{[k]}$, $n \in \mathbb{N}$.

Theorem 2.1 *[16] Let $\{P_n\}_{n\in\mathbb{N}}$ be an OPS. Suppose there is an integer $k \geqslant 1$ such that $\left\{P_n^{[k]}\right\}_{n\in\mathbb{N}}$ is an OPS. Then any polynomial P_{n+k} fulfils the following differential equation of order $2k$:*

$$\sum_{\nu=0}^{k} \Lambda_\nu\,(k;x)\; D^{k+\nu} P_{n+k}\,(x) = \Xi_n\,(k)\, P_{n+k}\,(x)\,,\; n \in \mathbb{N}, \tag{2.1}$$

where

$$\Lambda_\nu\,(k;x) = \frac{1}{\nu!}\sum_{\mu=0}^{\nu} \lambda_\mu^k \Omega_{\nu-\mu}^k\,(\nu;x)\, P_{k+\mu}\,(x)\,, \quad 0 \leqslant \nu \leqslant k; \tag{2.2}$$

$$\Xi_n\,(k) = \lambda_n^k\,\{n+k\}_{(\mathbf{k})}\,, \quad n \in \mathbb{N}; \tag{2.3}$$

$$\lambda_n^k = (-1)^k\,\frac{\left\langle\, v_0, {Q_n}^2 \,\right\rangle}{\left\langle\, u_0, P_{n+k}^2 \,\right\rangle}\,\{n+k\}_{(\mathbf{k})}\,, \quad n \in \mathbb{N}; \tag{2.4}$$

and

$$\begin{cases} \Omega_0^k(0;\cdot) = 1, \\ \Omega_0^k\,(\mu+1;\cdot) = 1, \quad \mu \in \mathbb{N}, \\ \Omega_{\mu+1-\xi}^k\,(\mu+1;\cdot) = -\sum\limits_{\nu=\xi}^{\mu} \dfrac{1}{\nu!}\left(Q_{\mu+1}\right)^{(\nu)} \Omega_{\nu-\xi}^k\,(\nu;\cdot)\,, \quad 0 \leqslant \xi \leqslant \mu, \end{cases} \tag{2.5}$$

with $\{n+k\}_{(\mathbf{k})}$ is defined according to (1.12).

More information concerned with the differential equation (2.1) were obtained in [16] for each classical family: namely, the explicit expressions for λ_n^k, $n \geqslant 0$, given by (2.4). Such expressions are summarised in Table 2.

Table 2
Expressions for λ_n^k, with $n \in \mathbb{N}$, given by (2.4), for each classical family. (Note the regularity conditions already mentioned in Table 1)

	Hermite	Laguerre	Bessel	Jacobi
λ_n^k	$(-2)^k$	$\dfrac{(-1)^k}{(\alpha+1)_k}$	$C(k,\alpha)\,(2\alpha-1+k+n)_k$	$C(k,\alpha,\beta)\,(\alpha+\beta+1+k+n)_k$

with $C(k,\alpha) = 4^{-k}(2\alpha)_{2k}$ and $C(k,\alpha,\beta) = \frac{(-4)^{-k}\,(\alpha+\beta+2)_{2k}}{(\alpha+1)_k\,(\beta+1)_k}$.

In [16] were obtained the explicit expressions of (2.1) for each one of the classical families and for the first values of k with the help of symbolic computations made in *Mathematica*. The next result allows us to obtain the explicit expressions for the polynomial coefficients presented in (2.1) for any integer $k \geqslant 1$ and for each classical family.

Theorem 2.2 *Under the same assumptions of theorem 2.1, the polynomials $\Lambda_\nu(k;\cdot)$, with $0 \leqslant \nu \leqslant k$, given by (2.2) may also be expressed by:*

$$\Lambda_\nu(k;x) = \frac{\lambda_0^k\ \omega_{k,\nu}}{\nu!}\ \Phi^\nu(x)\ \Big(P_k(x)\Big)^{(\nu)}, \quad 0 \leqslant \nu \leqslant k, \tag{2.6}$$

with

$$\omega_{k,\nu} = \begin{cases} \big(-\Psi'(0)\big)^{-\nu} & \text{if } 0 \leqslant \deg\Phi \leqslant 1\,, \\[2ex] \dfrac{1}{(\,k-1-\Psi'(0)\,)_\nu} & \text{if } \deg\Phi = 2\,, \end{cases} \tag{2.7}$$

where Φ represents a monic polynomial with $\deg\Phi \leqslant 2$, Ψ a one-degree polynomial, and the elements of the two nonzero sequences $\{\lambda_n^k\}_{n\in\mathbb{N}}$ and $\{\,\Xi_n(k)\,\}_{n\in\mathbb{N}}$, are respectively given in (2.4) and (2.3) .

In section 4 will be given the precise expressions for $\Lambda_\nu(k;\cdot)$, with $0 \leqslant \nu \leqslant k$, for each classical family.

PROOF. Since $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ are two OPS, from (1.4) we have

$$u_n = \big(\langle u_0, P_n^2\rangle\big)^{-1} P_n\, u_0,\ n \in \mathbb{N}, \tag{2.8}$$
$$v_n = \big(\langle v_0, {Q_n}^2\rangle\big)^{-1} Q_n\, v_0,\ n \in \mathbb{N}. \tag{2.9}$$

By virtue of (2.8)-(2.9), the relation given by (1.3) becomes like:

$$(Q_n\, v_0)^{(k)} = \lambda_n^k\, P_{n+k} u_0, \quad n \in \mathbb{N}, \tag{2.10}$$

with

$$\lambda_n^k = (-1)^k\, \frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_{n+k}^2\rangle} \prod_{\mu=1}^{k} (n+\mu)\ , \quad n \in \mathbb{N}. \tag{2.11}$$

Using the Leibniz derivation formula, we have

$$(Q_n\, v_0)^{(k)} = \sum_{\nu=0}^{k} \binom{k}{\nu} (Q_n)^{(\nu)} (v_0)^{(k-\nu)}, \quad n \in \mathbb{N}, \tag{2.12}$$

which allows to transform (2.10) into

$$\sum_{\nu=0}^{k} \binom{k}{\nu} (Q_n)^{(\nu)} (v_0)^{(k-\nu)} = \lambda_n^k\, P_{n+k}\, u_0, \quad n \in \mathbb{N}. \tag{2.13}$$

The fact that $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ are both orthogonal provides the classical character of $\{P_n\}_{n\in\mathbb{N}}$, so there exist a monic polynomial Φ and a polynomial Ψ, with $\deg\Phi \leqslant 2$ and $\deg\Psi = 1$, such that the regular form u_0 fulfils (1.7). By virtue of corollary 1.3, $\{Q_n\}_{n\in\mathbb{N}}$ is a classical MOPS with respect to $v_0 = \zeta_k\, \Phi^k\, u_0$, where ζ_k represents a nonzero constant, and $\{P_n^{[j]}\}_{n\in\mathbb{N}}$ is also a MOPS whose elements fulfil the differential equation (1.10) with k replaced by j and n by $n+1$, which, according to the definition of $\{P_n^{[j]}\}_{n\in\mathbb{N}}$, may be written as follows:

$$\Phi(x) \left(P_n^{[j+1]}(x)\right)' - \left\{\Psi(x) - j\,\Phi'(x)\right\} P_n^{[j+1]}(x) = \widetilde{\chi}_{n,j} P_{n+1}^{[j]}(x),\ 1 \leqslant j \leqslant k,\ n \in \mathbb{N}. \tag{2.14}$$

Thus, differentiating both members of $v_0 = \zeta_k \Phi^k\, u_0$ and then taking into consideration (1.7), we obtain the identity

$$(v_0)' = \zeta_k \left\{(k-1)\Phi'\, \Phi^{k-1}\, u_0 - \Phi^{k-1}\, \Psi\, u_0\right\},$$

which, on attempt of (2.14) with $n = 0$ and $j = k-1$, may be written like

$$(v_0)' = \zeta_k\, \Phi^{k-1} \left\{(k-1)\Phi''(0) - \Psi(0)\right\} P_1^{[k-1]}\, u_0.$$

By finite induction it is not hard to prove that

$$\left(v_0\right)^{(j)} = \zeta_k \left(\prod_{\tau=1}^{j} \widetilde{\chi}_{\tau-1,k-\tau}\right) P_j^{[k-j]}\, \Phi^{k-j}\, u_0, \quad 1 \leqslant j \leqslant k, \tag{2.15}$$

where

$$\widetilde{\chi}_{\mu,\sigma} = \frac{\mu + 2\sigma}{2}\, \Phi''(0) - \Psi'(0), \quad \mu, \sigma \in \mathbb{N}.$$

Indeed, differentiating both members of (2.15) once, leads to

$$\begin{aligned}(v_0)^{(j+1)} = \zeta_k \left(\prod_{\tau=1}^{j} \widetilde{\chi}_{\tau-1,k-\tau}\right) & \left\{ \left(P_j^{[k-j]}\right)' \Phi^{k-j}\, u_0 \right. \\ & \left. + P_j^{[k-j]} \left((k-j-1)\Phi'\, \Phi^{k-j-1}\, u_0 + \Phi^{k-j-1}\left(\Phi\, u_0\right)'\right)\right\}\end{aligned}$$

which, on account (1.7), becomes like:

$$\begin{aligned}&(v_0)^{(j+1)} \\ &= \zeta_k \left\{\prod_{\tau=1}^{j} \widetilde{\chi}_{\tau-1,k-\tau}\right\} \Phi^{k-j-1} \left\{ \Phi \left(P_j^{[k-j]}\right)' + \left((k-j-1)\Phi' - \Psi\right) P_j^{[k-j]} \right\} u_0.\end{aligned}$$

By virtue of (2.14) with the pair (n, j) replaced by $(j, k-j-1)$, we conclude that the previous identity corresponds to (2.15) with $j+1$ instead of j, whence we conclude that (2.15) holds for each positive integer j. In particular, when $j = k$, (2.15) becomes like

$$(v_0)^{(k)} = \zeta_k \left(\prod_{\tau=1}^{k} \widetilde{\chi}_{\tau-1,k-\tau} \right) P_k \, u_0 \, . \tag{2.16}$$

On the other hand, if we consider $n = 0$ in (2.13) we also obtain

$$(v_0)^{(k)} = \lambda_0^k \, P_k \, u_0 \, . \tag{2.17}$$

From the comparison between (2.16) and (2.17) we achieve the conclusion:

$$\zeta_k = \left(\prod_{\tau=1}^{k} \widetilde{\chi}_{\tau-1,k-\tau} \right)^{-1} \lambda_0^k \, .$$

Bringing this information into (2.15) with j replaced by $k-\nu$, we obtain:

$$(v_0)^{(k-\nu)} = \omega_{k,\nu} \, \lambda_0^k \, \Phi^\nu \, P_{k-\nu}^{[\nu]} \, u_0 \tag{2.18}$$

where

$$\omega_{k,\nu} = \begin{cases} \left(\displaystyle\prod_{\tau=k-\nu+1}^{k} \widetilde{\chi}_{\tau-1,k-\tau} \right)^{-1} & , \quad 1 \leqslant \nu \leqslant k \\ \\ 1 & , \quad \nu = 0 \, . \end{cases}$$

Based on the definition of $\widetilde{\chi}_{k-\tau-1,\tau}$, the coefficients $\omega_{k,\nu}$ may be expressed like:

$$\omega_{k,\nu} = \begin{cases} \left[\displaystyle\prod_{\tau=0}^{\nu-1} \left(\frac{k+\tau-1}{2} \, \Phi''(0) - \Psi'(0) \right) \right]^{-1} & , \ 1 \leqslant \nu \leqslant k \\ \\ 1 & , \ \nu = 0 \, , \end{cases}$$

Since Φ is a monic polynomial with $\deg \Phi \leqslant 2$, then, recalling (1.13), it is also possible to express $\omega_{k,\nu}$ as in (2.7). Hence, on account (2.18), the relation (2.13) may be transformed into

$$\sum_{\nu=0}^{k} \binom{k}{\nu} (Q_n)^{(\nu)} \ \lambda_0^k \, \omega_{k,\nu} \, P_{k-\nu}^{[\nu]} \, \Phi^\nu \, u_0 = \lambda_n^k \, P_{n+k} \, u_0 \, , \, n \in \mathbb{N}. \tag{2.19}$$

which, owing to the regularity of u_0, provides

$$\sum_{\nu=0}^{k} \binom{k}{\nu} \left\{ \lambda_0^k \, \omega_{k,\nu} \, P_{k-\nu}^{[\nu]}(x) \, \Phi^\nu(x) \right\} \big(Q_n(x) \big)^{(\nu)} = \lambda_n^k \, P_{n+k}(x) \, , \, n \in \mathbb{N}. \tag{2.20}$$

Following the definition of the polynomials Q_n, one has

$$\big(Q_n(x) \big)^{(\nu)} = \frac{n!}{(n+k)!} \, \big(P_{n+k}(x) \big)^{(k+\nu)}, \, n \in \mathbb{N}, \tag{2.21}$$

so, (2.20) may actually be written as

$$\sum_{\nu=0}^{k} \widehat{\Lambda}_\nu(k;x) \, D^{k+\nu} \big(P_{n+k}(x) \big) = \Xi_n(k) \, P_{n+k}(x) \, , \, n \in \mathbb{N}. \tag{2.22}$$

where

$$\widehat{\Lambda}_\nu(k;x) = \binom{k}{\nu} \lambda_0^k \, \omega_{k,\nu} \, P_{k-\nu}^{[\nu]}(x) \, \Phi^\nu(x), \, 0 \leqslant \nu \leqslant k,$$

and $\Xi_n(k)$ is given by (2.3). Clearly, under the definition of $P_{k-\nu}^{[\nu]}(\cdot)$, we easily observe that

$$\widehat{\Lambda}_\nu(k;x) = \frac{\lambda_0^k \, \omega_{k,\nu}}{\nu!} \, \Phi^\nu(x) \, \Big(P_k(x)\Big)^{(\nu)}, \quad 0 \leqslant \nu \leqslant k.$$

Now, comparing (2.1) with (2.22) and representing by

$$A_\nu(k;x) = \Lambda_\nu(k;x) - \widehat{\Lambda}_\nu(k;x) \, , \quad 0 \leqslant \nu \leqslant k \, ,$$

we deduce that

$$\sum_{\nu=0}^{k} A_\nu(k;x) \, D^{k+\nu}(P_{n+k}) = 0 \, , \quad n \in \mathbb{N} \, .$$

Since $D^{k+\nu}\big(P_j(x)\big) = 0, \, 0 \leqslant j \leqslant k-1$, it is obvious that

$$\sum_{\nu=0}^{k} A_\nu(k;x) \, D^{k+\nu}(P_n) = 0 \, , \quad n \in \mathbb{N} \, .$$

Based on the fact that $\{P_n\}_{n\in\mathbb{N}}$ forms a basis of $\mathscr{P}$, we conclude from the previous equalities that

$$\sum_{\nu=0}^{k} A_\nu(k;x) \, D^{k+\nu} f = 0 \, , \quad f \in \mathscr{P}. \tag{2.23}$$

The particular choice $f(x) = x^k$ in (2.23) provides $A_0(k;\cdot) = 0$. Let us suppose that

$$A_\nu(k;\cdot) = 0, \, 0 \leqslant \nu \leqslant \mu \leqslant k-1.$$

If we consider $f(x) = x^{k+\mu+1}$ in (2.23), then, under the assumption, we easily derive that

$$A_{\mu+1}(k;x) \, (k+\mu+1)! = 0$$

which implies $A_{\mu+1}(k;x) = 0, \; 0 \leqslant \mu \leqslant k-1$. Therefore $A_\nu(k;x) = 0, \, 0 \leqslant \nu \leqslant k$, whence the result.

Remark 2.3 *Consider $\{P_n\}_{n\in\mathbb{N}}$ to be a classical MOPS. By virtue of Hahn's theorem (statement ii of theorem 1.2), there exists $k \geqslant 1$ such that $\{P_n^{[k]}\}_{n\in\mathbb{N}}$ is a MOPS, whence, if τ is an integer between 1 and k, $\{P_n^{[\tau]}\}_{n\in\mathbb{N}}$ is also orthogonal. Therefore from theorem 2.1 and theorem 2.2, we deduce that P_n stills fulfilling the differential equation (2.1) with the pair (n,k) replaced by $(n-\tau,\tau)$ and $n \geqslant \tau$.*
Besides, it can be easily seen that when $0 \leqslant n \leqslant \tau-1$, necessarily $D^{\tau+\nu}(P_n) = 0$ (with $0 \leqslant \nu \leqslant \tau$) and $\Xi_{n-\tau}(\tau) = 0$. This last equality is due to the fact that $\{n\}_{(\tau)} = 0$ when $0 \leqslant n \leqslant \tau-1$ (it is a simple consequence of the definition of the falling factorial of a number (1.12)). This allows us to conclude that each element of $\{P_n\}_{n\in\mathbb{N}}$ is also a solution of the differential equation

$$\sum_{\nu=0}^{\tau} \Lambda_\nu(k;x) \, D^{\tau+\nu} P_n(x) = \Xi_{n-\tau}(\tau) P_n(x), \quad n \geqslant 0. \tag{2.24}$$

Moreover, with the convention $P_n^{[0]} := P_n$, there is no danger to consider in (2.24) the case where $\tau = 0$ since it is identically satisfied.

Remark 2.4 *The differential equation (2.1) characterises the classical polynomials [16, theorem 3.3].*

2.2. *An extension of Bochner's differential equation*

From now on, the k-th power of the second order differential operator $\mathscr{F}$ given in (1.9) will be denoted by $\mathscr{F}^k$ and is inductively defined through $\mathscr{F}^k[y](x) = \mathscr{F}\left(\mathscr{F}^{k-1}[y](x)\right)$, for $k \in \mathbb{N}^*$ and $\mathscr{F}^0$ denote the identity operator.

As a direct consequence of the Bochner's property for the classical polynomial sequences (statement (iv) of theorem 1.2), we present the following result.

Corollary 2.5 *Let $\{P_n\}_{n\in\mathbb{N}}$ be a classical OPS and k a positive integer. Consider the differential operator $\mathscr{F}$ given by (1.9) where Φ represents a monic polynomial with $\deg\Phi \leqslant 2$, and Ψ a polynomial such that $\deg\Psi = 1$. Then, for any set $\{c_{k,\mu}\ :\ 0 \leqslant \mu \leqslant k\}$ of complex numbers not depending on n, each element of $\{P_n\}_{n\in\mathbb{N}}$ fulfils the differential equation given by*

$$\sum_{\mu=0}^{k} c_{k,\mu}\, \mathscr{F}^{\mu} P_n(x) = \sum_{\mu=0}^{k} c_{k,\mu}\, (\chi_n)^{\mu}\, P_n(x)\ , \quad n \in \mathbb{N}, \tag{2.25}$$

where $\{\chi_n\}_{n\geqslant 1}$ represents a sequence of nonzero complex numbers.

PROOF. Since $\{P_n\}_{n\in\mathbb{N}}$ is a classical OPS, then, according to theorem statement iv of 1.2, there is a monic polynomial Φ with $\deg\Phi \leqslant 2$, a polynomial Ψ with $\deg\Psi = 1$ and a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ with $\chi_0 = 0$ and $\chi_{n+1} \neq 0$, $n \in \mathbb{N}$, such that (1.8) holds. Let us suppose that, for $\nu - 1 \geqslant 1$, P_n is a solution of the differential equation given by $\mathscr{F}^{\nu-1} P_n(x) = (\chi_n)^{\nu-1} P_n(x)\, ,\ n \in \mathbb{N}$.
Under the assumption we have $\mathscr{F}^{\nu} P_n(x) = \mathscr{F}\left(\mathscr{F}^{\nu-1} P_n(x)\right) = \mathscr{F}\left((\chi_n)^{\nu-1} P_n(x)\right)$.
On account of (1.8) we easily deduce that

$$\mathscr{F}^{\nu} P_n(x) = (\chi_n)^{\nu}\, P_n(x), \quad n \in \mathbb{N},$$

holds for any integer $\nu \geqslant 1$. If $\{c_{k,\mu}\}_{0\leqslant\mu\leqslant k}$ represents any set of complex numbers not depending on n, (2.25) is trivially verified.

2.3. *Relation between the generalisations of the Bochner's differential equation*

As a consequence of theorem 2.1 and corollary 2.5 we present the following result.

Corollary 2.6 *Let $\{P_n\}_{n\in\mathbb{N}}$ be a classical sequence and k a positive integer. If there exist coefficients $d_{k,\mu}$ and $\widetilde{d}_{k,\mu}$ $0 \leqslant \mu \leqslant k$, not depending on n, such that*

$$\Xi_{n-k}(k) = \sum_{\tau=0}^{k} d_{k,\tau}\,(\chi_n)^{\tau}\,, \quad n \geqslant 0, \tag{2.26}$$

$$(\chi_n)^k = \sum_{\tau=0}^{k} \widetilde{d}_{k,\tau}\,\Xi_{n-\tau}(\tau)\,, \quad n \geqslant 0, \tag{2.27}$$

where χ_n and $\Xi_{n-\tau}(\tau)$, $1 \leqslant \tau \leqslant k$, $n \geqslant 0$, are respectively the ones presented in (1.8) and (2.3), then the two following equalities hold:

$$\sum_{\nu=0}^{k} \Lambda_k(k;x) D^{k+\nu} = \sum_{\tau=0}^{k} d_{k,\tau}\,\mathscr{F}^{\tau}\,, \tag{2.28}$$

$$\mathscr{F}^k = \sum_{\tau=0}^{k} \widetilde{d}_{k,\tau} \left\{ \sum_{\nu=0}^{\tau} \Lambda_\nu(\tau;x) D^{\tau+\nu} \right\} \tag{2.29}$$

where $\mathscr{F}$ is given by (1.9) and $\left\{ \sum_{\nu=0}^{\tau} \Lambda_\nu(\tau;x) D^{\nu+\tau} \right\}$ *the one presented in (2.24).*

PROOF. Let $\{P_n\}_{n\in\mathbb{N}}$ be a classical MOPS and $k \geqslant 1$. First we are going to show how (2.26) implies (2.28) and afterwards how (2.27) implies (2.29). According to theorem 2.1, P_n fulfils the equation

$$\sum_{\nu=0}^{k} \Lambda_\nu(k;x) D^{\nu+k} P_n(x) = \Xi_{n-k}(k)\,P_n(x)\,, \quad n \geqslant k.$$

It is clear, from (2.3), that whenever n is a integer such that $0 \leqslant n \leqslant k-1$, $\Xi_{n-k}(k) = 0$. So, we actually deduce from theorem 2.1, that

$$\sum_{\nu=0}^{k} \Lambda_\nu(k;x) D^{\nu+k} P_n(x) = \Xi_{n-k}(k)\,P_n(x)\,, \quad n \geqslant 0.$$

If $\{d_{k,\tau} : 0 \leqslant \tau \leqslant k\}$ represents a set of coefficients such that (2.26) holds, then we have

$$\sum_{\nu=0}^{k} \Lambda_\nu(k;x) D^{\nu+k} P_n(x) = \sum_{\tau=0}^{k} d_{k,\tau}\,(\chi_n)^{\tau}\,P_n(x)\,, \quad n \geqslant 0,$$

where χ_n corresponds to the eigenvalues of (1.8). On the other hand, corollary 2.5 allows us to write

$$\sum_{\mu=0}^{k} d_{k,\mu}\,(\chi_n)^{\mu}\,P_n(x) = \sum_{\mu=0}^{k} d_{k,\mu}\,\mathscr{F}^{\mu} P_n(x)\,, \quad n \geqslant 0.$$

Hence we get

$$\mathscr{L}_{2k}\,P_n(x) = 0\,, \quad n \geqslant 0. \tag{2.30}$$

where $\mathscr{L}_{2k} = \sum_{\mu=0}^{k} d_{k,\mu}\,\mathscr{F}^{\mu} - \sum_{\nu=k}^{2k} \Lambda_{\nu-k}(k;x) D^{\nu}$. Since $\{P_n\}_{n\in\mathbb{N}}$ forms a basis of $\mathscr{P}$, then (2.30) provides that $\mathscr{L}_{2k} f = 0$, for any $f \in \mathscr{P}$, whence we get (2.28).

Likewise, by virtue of corollary 2.5 and by taking into account (2.27), from (2.24) we derive

$$\mathscr{F}^k P_n(x) = \sum_{\tau=0}^{k} \widetilde{d}_{k,\tau} \left\{ \sum_{\nu=0}^{\tau} \Lambda_\nu\,(\tau;x)\; D^{\tau+\nu} \right\} P_n(x)\,,\; n \in \mathbb{N},$$

which implies the relation (2.29), regarding the fact that $\{P_n\}_{n\in\mathbb{N}}$ forms a basis of $\mathscr{P}$.

We intend to know whether it is possible to express the eigenvalues of the differential equation (2.1) as a sum of powers of the eigenvalues of the differential equation (1.8). In other words, we face the problem of finding two sets of coefficients $\{d_{k,\tau} \,:\, 1 \leqslant \tau \leqslant k\,,\, k \geqslant 1\}$ and $\{\widetilde{d}_{k,\tau} \,:\, 1 \leqslant \tau \leqslant k\,,\, k \geqslant 1\}$ realising the equalities (2.26)-(2.27). Considering the information contained in Table 1 and Table 2, one realises that the determination of those two sets of coefficients shall be done separately for each one of the classical families. Indeed, observing the nature of the eigenvalues χ_n and $\Xi_{n-\tau}(\tau)$, the problem under analysis resembles the relation between the powers of a variable and its factorials. The bridge between those two sequences can be done in a natural way through the Stirling numbers. In order to have a more clear understanding in the next section we review some basic concepts concerned with this subject. That revision is basically enough to derive the expression for $d_{k,\tau}$ and $\widetilde{d}_{k,\tau}$ (presented in the relations (2.26)-(2.27)) for the cases of Hermite and Laguerre families, while for the analysis of the cases of Bessel or Jacobi families we introduce a slight modification in the concepts of the factorial of a complex number and Stirling numbers.

3. Sums relating powers of a variable and its factorials

In this section, we begin by reviewing the definition of the Stirling numbers and its properties. Later, we will introduced the concept of *A-modified falling factorial* and also the *A-modified Stirling numbers*, motivating the reason for such names.

Representing by $s(k,\nu)$ and $S(k,\nu)$, with $k,\nu \in \mathbb{N}$, the **Stirling numbers of first and second kind**, respectively, the following equalities hold [5,21,22]:

$$\{x\}_{(\mathbf{k})} = \sum_{\nu=0}^{k} s(k,\nu) x^\nu\,. \tag{3.1}$$

and

$$x^k = \sum_{\nu=0}^{k} S(k,\nu)\{x\}_{(\nu)}\;, \tag{3.2}$$

where $\{x\}_{(\mathbf{k})}$ represent the **falling factorial** of x and is defined in (1.12). Indeed, such numbers fulfil a "*triangular*" recurrence relation, namely we have

$$\begin{cases} s(k+1,\nu+1) = s(k,\nu) - k\,s(k,\nu+1) \\ s(k,0) = s(0,k) = \delta_{k,0} \\ s(k,\nu) = 0\,, \quad \nu \geqslant k+1 \end{cases}$$

and

$$\begin{cases} S(k+1,\nu+1) = S(k,\nu) + (\nu+1)\,S(k,\nu+1) \\ S(k,0) = S(0,k) = \delta_{k,0} \\ S(k,\nu) = 0\,, \quad \nu \geqslant k+1 \end{cases}$$

with $k,\nu \in \mathbb{N}$ (see, for instance [5, Chapter V]). Moreover the Stirling numbers of first and second kind fulfil the biorthogonality conditions

$$\sum_{\tau=0}^{\max\{k,\nu\}} s(k,\tau)\,S(\tau,\nu) = \sum_{\tau=0}^{\max\{k,\nu\}} S(k,\tau)\,s(\tau,\nu) = \delta_{k,\nu}\ .$$

The matrix $\mathbf{s} := [s(k,\nu)]_{k,\nu\in\mathbb{N}}$ consisting of the Stirling numbers of the first kind is the inverse of the matrix $\mathbf{S} := [S(k,\nu)]_{k,\nu\in\mathbb{N}}$ of the Stirling numbers of the second kind ($\mathbf{s}^{-1} = \mathbf{S}$). Furthermore, the Stirling number of the second kind $S(k,\nu)$ equals:

$$S(k,\nu) = \frac{1}{\nu!}\sum_{\tau=0}^{\nu}(-1)^{\nu-\tau}\binom{\nu}{\tau}\tau^k\,, \quad 1 \leqslant \nu \leqslant k\,.$$

We now introduce a slight modification on the concept of the falling factorial (1.12).

Definition 3.1 *Let A be a number (possibly complex) and $k \in \mathbb{N}$. For any number x we define*

$$\{x\}_{(\mathbf{k};\mathbf{A})} := \begin{cases} 1 & if\ \ k=0, \\ \displaystyle\prod_{\nu=0}^{k-1}(x-\nu(\nu+A)) & if\ \ k\in\mathbb{N}^*, \end{cases} \tag{3.3}$$

to be the **A-modified falling factorial (of order** *k***)**.

It is clear that $\left\{\{x\}_{(\mathbf{n};\mathbf{A})}\right\}_{n\in\mathbb{N}}$ is a MPS, as well as $\{x^n\}_{n\in\mathbb{N}}$ is. As a result, there exist two unique sequences of numbers $\{\widehat{s}_A(k,\nu)\}_{k,\nu\in\mathbb{N}}$ and $\{\widehat{S}_A(k,\nu)\}_{k,\nu\in\mathbb{N}}$ such that

$$\{x\}_{(\mathbf{k};\mathbf{A})} = \sum_{\nu=0}^{k}\widehat{s}_A(k,\nu)\,x^\nu\ ,\ k\in\mathbb{N} \tag{3.4}$$

$$x^k = \sum_{\nu=0}^{k}\widehat{S}_A(k,\nu)\,\{x\}_{(\nu;\mathbf{A})}\ ,\ k\in\mathbb{N}, \tag{3.5}$$

The next result provides more information about these two sequences.

Proposition 3.2 *The numbers $\widehat{s}_A(k,\nu)$ defined by (3.4) satisfy the following "triangular" recurrence relation*

$$\widehat{s}_A(k+1,\nu+1)=\widehat{s}_A(k,\nu)-k(k+A)\,\widehat{s}_A(k,\nu+1)\ , \tag{3.6}$$
$$\widehat{s}_A(k,0)=\widehat{s}_A(0,k)=\delta_{k,0}\ , \tag{3.7}$$
$$\widehat{s}_A(k,\nu)=0\,,\nu\geqslant k+1\ , \tag{3.8}$$

whereas $\widehat{S}_A(k,\nu)$ defined by (3.5) satisfy the "triangular" recurrence relation given by

$$\widehat{S}_A(k+1,\nu+1)=\widehat{S}_A(k,\nu)+(\nu+1)(\nu+1+A)\widehat{S}_A(k,\nu+1)\ , \tag{3.9}$$
$$\widehat{S}_A(k,0)=\widehat{S}_A(0,k)=\delta_{k,0}\ , \tag{3.10}$$
$$\widehat{S}_A(k,\nu)=0\,,\nu\geqslant k+1\ , \tag{3.11}$$

for $k,\nu\in\mathbb{N}$.

PROOF. Suppose that the relations (3.4)-(3.5) hold. The fact that $x^0=1=\{x\}_{(\mathbf{0;A})}$ provides that $\widehat{s}_A(0,0)=\widehat{S}_A(0,0)=1$. It is clear that $\{x\}_{(\mathbf{k;A})}$, with $k\in\mathbb{N}$, is a polynomial in x and $\deg\left(\{x\}_{(\mathbf{k;A})}\right)=k$. Therefore, the relations (3.8) and (3.11) are just a consequence of (3.4) and (3.5), respectively. Meanwhile, due to (3.3), the following identity

$$\{x\}_{(\mathbf{k+1;A})}=\big(x-k(k+A)\big)\,\{x\}_{(\mathbf{k;A})}\ ,\ k\in\mathbb{N}, \tag{3.12}$$

holds. Therefore, we successively have:

$$\begin{aligned}\sum_{\nu=0}^{k+1}\widehat{s}_A(k+1,\nu)x^\nu &= \{x\}_{(\mathbf{k+1;A})}=\big(x-k(k+A)\big)\,\{x\}_{(\mathbf{k;A})}\\ &= \big(x-k(k+A)\big)\sum_{\nu=0}^{k}\widehat{s}_A(k,\nu)\,x^\nu\\ &= \sum_{\nu=1}^{k}\big\{\widehat{s}_A(k,\nu-1)-k(k+A)\widehat{s}_A(k,\nu)\big\}x^\nu\\ &\quad+\widehat{s}_A(k,k)\,x^{k+1}-k(k+A)\widehat{s}_A(k,0),\quad k\in\mathbb{N}.\end{aligned}$$

Now, the comparison of the coefficients of x^ν, $0\leqslant\nu\leqslant k$, in the first and last members of the previous equalities provides

$$\widehat{s}_A(k+1,0)=-k(k+A)\widehat{s}_A(k,0)\quad,\quad \widehat{s}_A(k+1,k+1)=\widehat{s}_A(k,k) \tag{3.13}$$

and also (3.6) with ν replaced by $\nu+1$. Clearly, (3.13) implies (3.7).

Likewise, from (3.5), we deduce

$$\sum_{\nu=0}^{k+1}\widehat{S}_A(k+1,\nu)\{x\}_{(\nu;\mathbf{A})}=x^{k+1}=x\cdot x^k=\sum_{\nu=0}^{k+1}\widehat{S}_A(k,\nu)\,x\,\{x\}_{(\nu;\mathbf{A})}$$

which, on attempt of (3.12) with k replaced by ν, becomes like

$$\sum_{\nu=0}^{k+1}\widehat{S}_A(k+1,\nu)\{x\}_{(\nu;\mathbf{A})}=\sum_{\nu=0}^{k}\widehat{S}_A(k,\nu)\big\{\{x\}_{(\nu+\mathbf{1;A})}\ +\nu(\nu+A)\{x\}_{(\nu;\mathbf{A})}\big\}$$

The fact that $\{\{x\}_{(\nu;\mathbf{A})}\}_{\nu\in\mathbb{N}}$ forms an independent system on $\mathscr{P}$ allows us to conclude that $\widehat{S}_A(k+1,0)=0$, $\widehat{S}_A(k+1,k+1)=\widehat{S}_A(k,k)$ and also (3.9) after replacing ν by $\nu+1$. Thus, we have (3.10).

Inserting (3.4) into (3.5), that is

$$x^k=\sum_{\nu=0}^{k}\sum_{\tau=0}^{\nu}\widehat{S}_A(k,\nu)\,\widehat{s}_A(\nu,\tau)x^\tau$$

shows that

$$\sum_{\nu\in\mathbb{N}}\widehat{S}_A(k,\nu)\,\widehat{s}_A(\nu,\tau)=\delta_{k,\tau}\,.$$

Conversely, if we insert (3.5) into (3.4), we derive that

$$\sum_{\nu\in\mathbb{N}}\widehat{s}_A(k,\nu)\,\widehat{S}_A(\nu,\tau)=\delta_{k,\tau}\,.$$

The similar-looking of $\widehat{s}_A(k,\nu)$ and $\widehat{S}_A(k,\nu)$ with the Stirling numbers of first and second kind, respectively, compels us to call the numbers $\widehat{s}_A(k,\nu)$ and $\widehat{S}_A(k,\nu)$ as the **A-modified Stirling numbers of first and second kind**, respectively. Several authors have studied the Stirling numbers, its generalisations or some of their analogies (among them we cite [4,10,19]), however, as far as we are concerned, the study of $\widehat{s}_A(k,\nu)$ and $\widehat{S}_A(k,\nu)$ still remains somewhat unexplored. It might be worth to explore more properties about the so called A-modified Stirling numbers. Either way, this is not the main purpose of the present paper, so we will leave the study of other interesting properties for a future work. Nevertheless, we present some few considerations specially those about the A-modified Stirling numbers of the second kind.

Corollary 3.3 *The numbers $\widehat{S}_A(k,\nu)$ presented in (3.5) equal*

$$\widehat{S}_A(k,\nu)=\frac{1}{\nu!}\sum_{\sigma=1}^{\nu}\binom{\nu}{\sigma}(-1)^{\nu+\sigma}\,\frac{(A+2\sigma)\,\Gamma(A+\sigma)}{\Gamma(A+\sigma+\nu+1)}\,\Big(\sigma(\sigma+A)\Big)^k, \tag{3.14}$$

for $1\leqslant\nu\leqslant k$.

PROOF. From proposition 3.2, it follows that (3.5) holds for all the integers $k\in\mathbb{N}$ where the numbers $\widehat{S}_A(k,\nu)$ satisfy the relations (3.9)-(3.11). Now, let

$$c_{k,\nu}(A)=\frac{1}{\nu!}\sum_{\sigma=1}^{\nu}\binom{\nu}{\sigma}(-1)^{\nu+\sigma}\,\frac{(A+2\sigma)\,\Gamma(A+\sigma)}{\Gamma(A+\sigma+\nu+1)}\,\Big(\sigma(\sigma+A)\Big)^k,\ 1\leqslant\nu\leqslant k.$$

When we take $\nu=0$ in (3.9), we get

$$\widehat{S}_A(k+1,1)=\begin{cases}1 & ,\ k=0\\ (A+1)\,\widehat{S}_A(k,1) & ,\ k\geqslant 1\end{cases}\ ,$$

therefore

$$\widehat{S}_A(k,1)=(A+1)^{k-1}\,,\ k\geqslant 1. \tag{3.15}$$

Now, the relation (3.9) with $\nu=1$ and on account of (3.15) becomes like

$$\widehat{S}_A(k+1,2)=(A+1)^{n-1}+2(A+2)\widehat{S}_A(k,2),\ k\geqslant 2,$$

from which we derive

$$\begin{aligned}\widehat{S}_A(k,2)&=\frac{\big(2(2+A)\big)^{n-1}-2(1+A)^{n-1}}{2(3+A)}\\&=\frac{1}{2}\left\{\frac{\big(2(2+A)\big)^{k}(A+4)\,\Gamma(A+2)}{\Gamma(A+5)}-2\frac{(1+A)^{k}(A+2)\,\Gamma(A+1)}{\Gamma(A+4)}\right\}\end{aligned}\tag{3.16}$$

for all the integers $k\geqslant 2$. Hence (3.15)-(3.16) show that $\widehat{S}_A(k,\nu)=c_{k,\nu}(A)$ for $\nu=1,2$ and $k\geqslant 1$. Now suppose that $\widehat{S}_A(k,\nu)=c_{k,\nu}(A)$ for $1\leqslant\nu\leqslant k$. From (3.9), we have

$$\begin{aligned}\widehat{S}_A(k+1,\nu)&=\widehat{S}_A(k,\nu-1)+\big(\nu(\nu+A)\big)\widehat{S}_A(k,\nu)\\
&=c_{k,\nu-1}(A)+\big(\nu(\nu+A)\big)\,c_{k,\nu}(A)\\
&=\frac{\nu(\nu+A)\,(A+2\nu)\,\Gamma(A+\nu)}{\nu!\,\Gamma(A+2\nu+1)}\big(\nu(\nu+A)\big)^{k}\\
&\quad+\sum_{\sigma=1}^{\nu-1}\left\{-1+\frac{\nu(\nu+A)}{(\nu-\sigma)(A+\sigma+\nu)}\right\}\frac{(-1)^{\nu+\sigma}\,(A+2\sigma)\,\Gamma(A+\sigma)\,\big(\sigma(\sigma+A)\big)^{k}}{(\nu-\sigma-1)!\,\sigma!\,\Gamma(A+\sigma+\nu)}\\
&=\frac{\nu(\nu+A)\,(A+2\nu)\,\Gamma(A+\nu)}{\nu!\,\Gamma(A+2\nu+1)}\big(\nu(\nu+A)\big)^{k}\\
&\quad+\sum_{\sigma=1}^{\nu-1}\frac{\sigma(\sigma+A)\,(-1)^{\nu+\sigma}\,(A+2\sigma)\,\Gamma(A+\sigma)\,\big(\sigma(\sigma+A)\big)^{k}}{(\nu-\sigma)!\,\sigma!\,\Gamma(A+\sigma+\nu+1)}\\
&=c_{k+1,\nu}(A)\,,\quad 1\leqslant\nu\leqslant k+1,\end{aligned}$$

whence we conclude that $\widehat{S}_A(k,\nu)=c_{k,\nu}(A)$ for all $k,\nu\in\mathbb{N}$ with $\nu\leqslant k$.

Remark 3.4 *When $x=n(n+A)$ for $n\in\mathbb{N}$ and $A\in\mathbb{C}$, its A-modified factorial (of order k) is given by:*

$$\{n(n+A)\}_{(\mathbf{k;A})}=\prod_{\nu=0}^{k-1}\Big(n(n+A)-\nu(\nu+A)\Big)=\prod_{\nu=0}^{k-1}\Big((n-\nu)(n+A+\nu)\Big)$$

which, in accordance with (1.12)-(1.13), may be expressed like

$$\{n(n+A)\}_{(\mathbf{k;A})}\ =\ \{n\}_{(\mathbf{k})}\ (n+A)_k.\tag{3.17}$$

The previous equalities bring a relation between the A-modified Stirling numbers and the Stirling numbers itself. Namely, on attempt of (3.4) and (3.1), the comparison of the first and last members of the previous equality, leads to

$$\sum_{\nu=0}^{k} \widehat{s}_A(k,\nu)\ \big(n(n+A)\big)^{\nu} = \sum_{\nu=0}^{k}\sum_{\tau=0}^{\nu} s(k,\nu)\, s(\nu,\tau)\, n^{\nu}\ (n+A+k-1)^{\tau}$$

or, equivalently,

$$\sum_{\nu=0}^{k} \widehat{s}_A(k,\nu)\ \big(n(n+A)\big)^{\nu} = \sum_{\nu=0}^{k}\sum_{\tau=0}^{\nu} (-1)^{\nu+\tau}\, s(k,\nu)\, s(\nu,\tau)\, n^{\nu}\ (n+A)^{\tau}$$

Such expression may be simplified, nevertheless we will leave the study of the properties of such numbers to a future work. Analogously, due to (3.5) and (3.2), from the relation $\big(n(n+A)\big)^k = n^k\ (n+A)^k$ *we derive*

$$\sum_{\nu=0}^{k} \widehat{S}_A(k,\nu)\ \{n(n+A)\}_{(\nu;\mathbf{A})} = \sum_{\nu=0}^{k}\sum_{\tau=0}^{\nu} S(k,\nu)\, S(\nu,\tau)\ \{n\}_{(\nu)}\ \{n+A\}_{(\tau)}\ .$$

We finish this section with two tables about the A-modified Stirling numbers.

Table 3
A list of the first **A-modified Stirling numbers of 1^{st} kind**: $\widehat{s}_A(k,\nu)$, with $1 \leqslant \nu, k \leqslant 5$.

$k \setminus \nu$	1	2	3	4	5
1	1	0	0	0	0
2	$-(1+A)$	1	0	0	0
3	$2\,(1+A)_2$	$-5-3A$	1	0	0
4	$-6\,(1+A)_3$	$49+A(48+11A)$	$-2(7+3A)$	1	0
5	$24\,(1+A)_4$	$-2(410+515A)$ $-2A^2(202+25A)$	$273+5A(40+7A)$	$-10(3+A)$	1

Table 4
A list of the first **A-modified Stirling numbers of 2^{nd} kind**: $\widehat{S}_A(k,\nu)$, with $1 \leqslant \nu, k \leqslant 5$.

$k \setminus \nu$	1	2	3	4	5
1	1	0	0	0	0
2	$1+A$	1	0	0	0
3	$(1+A)^2$	$5+3A$	1	0	0
4	$(1+A)^3$	$21+A(24+7A)$	$14+6A$	1	0
5	$(1+A)^4$	$(5+3A)(17+A(18+5A))$	$147+5A(24+5A)$	$10(3+A)$	1

4. Sums relating powers of the Bochner differential operator and the obtained differential operators of even order

In this section it will be explicitly presented the $2k$-order differential equation (2.1) given in theorem 2.1 for each classical family (Hermite, Laguerre, Bessel and Jacobi) and any integer $k \geqslant 1$. The expression for the polynomials $\Lambda_\nu(k;\cdot)$ (with $0 \leqslant \nu \leqslant k$) that will be in use is the one given in theorem 2.2, in spite of the one given by (2.2).

Following corollary 2.6 it is possible to express the even order differential operator associated to the equation (2.1) as a polynomial in $\mathscr{F}$, the Bochner differential operator, providing there is a set of numbers $\{d_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ such that the condition (2.26) holds. Conversely, if there is a set of numbers $\{\widetilde{d}_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ such that (2.27) holds, then we obtain an explicit expression for any power of the Bochner's operator according to (2.29) and considering (2.6).

The determination of the sets $\{d_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ and $\{\widetilde{d}_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ will be thoroughly revealed for each classical family, by taking into account the considerations made in section 3. To accomplish this issue, we will work separately with each one of the classical families. Naturally, it won't be necessary to compute the successive powers of the Bochner's operator $\mathscr{F}$. For the sequel we will strongly use the information contained in Table 1 and Table 2.

4.1. *Hermite case*

Let $\{P_n(\cdot)\}_{n\in\mathbb{N}}$ be an Hermite monic polynomial sequence. Based on the information given in Table 2 and according to (2.3)-(2.4), we have $\Xi_n(k) = (-2)^k\ \{n+k\}_{(\mathbf{k})}$, $n \in \mathbb{N}$. On the other hand, considering the information provided by Table 1, the relation (2.7) becomes like $\omega_{k,\nu} = (-2)^{-\nu}$, $0 \leqslant \nu \leqslant k$, and the polynomial $\Lambda_\nu(k;x)$ defined in (2.6) may be expressed as follows:

$$\Lambda_\nu(k;x) = \frac{1}{\nu!}\ (-2)^{k-\nu}\ (P_k)^{(\nu)}(x) = \binom{k}{\nu}\ (-2)^{k-\nu}\ P^{[\nu]}_{k-\nu}(x)\ , \quad 0 \leqslant \nu \leqslant k.$$

For each integer $\nu \geqslant 1$, $P^{[\nu]}_n(\cdot) = P_n(\cdot)$, $n \in \mathbb{N}$, therefore

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\ (-2)^{k-\nu}\ P_{k-\nu}(x)\, ,\ 0 \leqslant \nu \leqslant k, \tag{4.1}$$

where

$$P_{2\tau}(x) = (2\tau)!\ \sum_{\mu=0}^{\tau} \frac{(-1)^{\tau-\mu}}{2^{2(\tau-\mu)}\,(\tau-\mu)!}\ \frac{x^{2\mu}}{(2\mu)!}\, , \quad \tau \in \mathbb{N}\, ,$$
$$P_{2\tau+1}(x) = (2\tau+1)!\ \sum_{\mu=0}^{\tau} \frac{(-1)^{\tau-\mu}}{2^{2(\tau-\mu)}\,(\tau-\mu)!}\ \frac{x^{2\mu+1}}{(2\mu+1)!}\, , \quad \tau \in \mathbb{N}\, .$$

Thus, $Y(x) = P_n(x)$ is a solution of the following differential equation:

$$\sum_{\nu=0}^{k} \binom{k}{\nu} (-2)^{-\nu}\, P_{k-\nu}(x)\ D^{k+\nu}Y(x) = \{n\}_{(\mathbf{k})}\, Y(x)\,, \quad n\in\mathbb{N}\,.$$

The relation (3.1) with x replaced by n permits to successively deduce the equalities

$$\Xi_{n-k}(k) = (-2)^k \{n\}_{(\mathbf{k})} = (-2)^k \sum_{\tau=0}^{k} s(k,\tau)\, n^{\tau} = \sum_{\tau=0}^{k} (-2)^{k-\tau}\, s(k,\tau)\, \big(\chi_n\big)^{\tau},\, n\in\mathbb{N},$$

since $\chi_n = -2n$, $n\in\mathbb{N}$. Equating the first and last members of the previous equalities we meet the relation (2.26) and therefore

$$d_{k,\tau} = (-2)^{k-\tau}\, s(k,\tau)\,, \quad 0\leqslant \tau \leqslant k\,.$$

Conversely, on account of (3.2) with x replaced by n, we derive

$$(\chi_n)^k = (-2)^k \sum_{\tau=0}^{k} S(k,\tau)\{n\}_{(\tau)} = \sum_{\tau=0}^{k} (-2)^{k-\tau}\, S(k,\tau)\ \Xi_{n-\tau}(\tau), \quad n\in\mathbb{N},$$

Thus, we have just obtained (2.27) if we consider

$$\widetilde{d}_{k,\tau} = (-2)^{k-\tau}\, S(k,\tau)\,, \quad 0\leqslant \tau \leqslant k\,.$$

As a result, by virtue of corollary 2.6, we conclude

$$\begin{cases} \displaystyle\sum_{\nu=0}^{k} \Lambda_{\nu}\,(k;x)\,D^{k+\nu} = \sum_{\tau=0}^{k} (-2)^{k-\tau}\, s(k,\tau)\,\mathscr{F}^{\tau} \\[2ex] \displaystyle\mathscr{F}^{k} = \sum_{\tau=0}^{k} (-2)^{k-\tau}\, S(k,\tau) \sum_{\nu=0}^{\tau} \Lambda_{\nu}\,(\tau;x)\,D^{\tau+\nu} \end{cases}, \tag{4.2}$$

where $\Lambda_\nu(k;x)$ is given in (4.1) and, considering Table 1, $\mathscr{F} = D^2 - 2xD$.

4.2. *Laguerre case*

Consider $\{P_n(\cdot;\alpha)\}_{n\in\mathbb{N}}$ with $\alpha\neq -(n+1)$, $n\in\mathbb{N}$, to be a Laguerre monic polynomial sequence. From Table 1 and in accordance with (2.3)-(2.4) we get $\Xi_n(k) = \dfrac{(-1)^k}{(\alpha+1)_k}\,\{n+k\}_{(\mathbf{k})}$, with $n\in\mathbb{N}$, while the information in Table 2 permits to obtain from (2.7) $\omega_{k,\nu} = (-1)^{-\nu}$, for $0\leqslant \nu\leqslant k$. Therefore the polynomial coefficients given by (2.6) may be expressed as follows:

$$\Lambda_\nu(k;x) = \frac{1}{\nu!}\,\frac{(-1)^{k-\nu}}{(\alpha+1)_k}\,x^{\nu}\,(P_k)^{(\nu)}(x) = \binom{k}{\nu}\frac{(-1)^{k-\nu}}{(\alpha+1)_k}\,x^{\nu}\,P^{[\nu]}_{k-\nu}(x;\alpha)$$

Since, for each integer $\nu\geqslant 1$, $P_n^{[\nu]}(\cdot;\alpha) = P_n(\cdot,\alpha+\nu)$, $n\in\mathbb{N}$, then we have

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\frac{(-1)^{k-\nu}}{(\alpha+1)_k}\,x^{\nu}\,P_{k-\nu}(x;\alpha+\nu) \tag{4.3}$$

with

$$P_{k-\nu}(x;\alpha+\nu)=\sum_{\mu=0}^{k-\nu}\binom{k-\nu}{\mu}(-1)^{k-\nu-\mu}\ \frac{\Gamma(k+\alpha+1)}{\Gamma(\mu+\alpha+\nu+1)}\ x^{\mu}\ ,\quad 0\leqslant\nu\leqslant k\,.$$

Following (2.1), $Y(x)=P_n(x;\alpha)$ is a solution of the differential equation

$$\sum_{\nu=0}^{k}\binom{k}{\nu}\left\{(-1)^{\nu}\,x^{\nu}\,P_{k-\nu}(x;\alpha+\nu)\right\}D^{k+\nu}\big(Y(x)\big)=\{n\}_{(\mathbf{k})}\,Y(x)\,,\quad n\in\mathbb{N}\,.$$

The problem of determining the two sets of coefficients $\{d_{k,\mu}:0\leqslant\mu\leqslant k\}$ and $\{\widetilde{d}_{k,\mu}:0\leqslant\mu\leqslant k\}$ realising the conditions (2.26)-(2.27) in this case, is analogous to the corresponding problem in the Hermite case. Indeed, if we replace x by n in (3.1), then the eigenvalues $\Xi_{n-k}(k)$ may be expressed in terms of χ_n given in Table 1:

$$\Xi_{n-k}(k)=\frac{(-1)^k}{(\alpha+1)_k}\sum_{\nu=0}^{k}s(k,\nu)\,n^{\nu}=\sum_{\nu=0}^{k}\frac{(-1)^k}{(\alpha+1)_k}\,s(k,\nu)\,\big(\chi_n\big)^{\nu}\ ,\quad n\in\mathbb{N},$$

providing (2.26) with

$$d_{k,\tau}=\frac{(-1)^{k-\tau}}{(\alpha+1)_k}\,s(k,\tau)\,,\quad 0\leqslant\tau\leqslant k\,.$$

Conversely, we have

$$\begin{aligned}(\chi_n)^k\ &=\ (-1)^k n^k\ =\ (-1)^k\sum_{\tau=0}^{k}S(k,\tau)\,\{n\}_{(\tau)}\\ &=\sum_{\tau=0}^{k}(-1)^k\,S(k,\tau)\left(\frac{(-1)^{\tau}}{(\alpha+1)_{\tau}}\right)^{-1}\Xi_{n-\tau}(\tau)\,,\quad n\in\mathbb{N},\end{aligned}$$

whence we attain (2.27) with

$$\widetilde{d}_{k,\tau}=(-1)^{k-\tau}\,(\alpha+1)_{\tau}\,S(k,\tau)\,,\quad 0\leqslant\tau\leqslant k\,.$$

From corollary 2.6 it follows

$$\left\{\begin{array}{l}\displaystyle\sum_{\nu=0}^{k}\Lambda_{\nu}\,(k;x)\,D^{k+\nu}=\sum_{\tau=0}^{k}\frac{(-1)^{k-\tau}}{(\alpha+1)_k}\,s(k,\tau)\,\mathscr{F}^{\tau}\\[2ex] \displaystyle\mathscr{F}^{k}=\sum_{\tau=0}^{k}(-1)^{k-\tau}\,(\alpha+1)_{\tau}\,S(k,\tau)\,\sum_{\nu=0}^{\tau}\Lambda_{\nu}\,(\tau;x)\,D^{\tau+\nu}\end{array}\right.\ ,\tag{4.4}$$

where $\Lambda_{\nu}(k;x)$ is given by (4.3) and, according to Table 1, $\mathscr{F}=xD^2-(x-\alpha-1)D$.

4.3. *Bessel case*

Let $\{P_n(\cdot;\alpha)\}_{n\in\mathbb{N}}$ with $\alpha\neq-\frac{n}{2}$, $n\in\mathbb{N}$, represent a Bessel monic polynomial sequence. Recalling the information provided in Table 2, it follows from (2.3)-(2.4) $\Xi_n(k)=\lambda_n^k\,\{n+k\}_{(\mathbf{k})}$ where

$\lambda_n^k = C(k;\alpha)\,(2\alpha-1+k+n)_k$ and $C(k,\alpha) = 4^{-k}\,(2\alpha)_{2k}$, for $n \in \mathbb{N}$. On the other hand from Table 1 and following (2.7) we have $\omega_{k,\nu} = \dfrac{1}{(2\alpha-1+k)_\nu}$, (with $0 \leqslant \nu \leqslant k$). Therefore the polynomials defined in (2.6), become like

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\ C(k;\alpha)\ (2\alpha-1+k+\nu)_{k-\nu}\ x^{2\nu}\ P_{k-\nu}^{[\nu]}(x;\alpha)\,,\quad 0 \leqslant \nu \leqslant k.$$

Since, for each integer $\nu \geqslant 1$, $P_n^{[\nu]}(\cdot;\alpha) = P_n(\cdot,\alpha+\nu)$, $n \in \mathbb{N}$, then we have

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\ C(k;\alpha)\ (2\alpha-1+k+\nu)_{k-\nu}\ x^{2\nu}\ P_{k-\nu}(x;\alpha+\nu),\ 0 \leqslant \nu \leqslant k, \tag{4.5}$$

where

$$P_{k-\nu}(x;\alpha+\nu) = \sum_{\mu=0}^{k-\nu} \binom{k-\nu}{\mu}\ \frac{2^{k-\nu-\mu}\ x^\mu}{(2\alpha-1+k+\nu+\mu)_{k-\nu-\mu}}\ ,\quad 0 \leqslant \nu \leqslant k\,.$$

Following (2.1), $Y(x) = P_n(x;\alpha)$ is a solution of the differential equation

$$\begin{aligned}&\sum_{\nu=0}^{k} \binom{k}{\nu}\ \left\{(2\alpha-1+k+\nu)_{k-\nu}\, x^{2\nu}\ P_{k-\nu}(x;\alpha+\nu)\ \right\}\ D^{k+\nu}\big(Y(x)\big)\\ &= \{n\}_{(\mathbf{k})}\ (2\alpha-1+n)_k\, Y(x)\,,\quad n \in \mathbb{N}\,.\end{aligned}$$

Now we face the problem of determining the two sets of coefficients $\{d_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ and $\{\widetilde{d}_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ realising the conditions (2.26)-(2.27) for this case. Indeed, the relation (3.17) presented in remark 3.4 under the particular choice of $A = 2\alpha-1$, yields

$$\Xi_{n-k}(k) = C(k,\alpha)\ \{n(n+2\alpha-1)\}_{(\mathbf{k};\mathbf{2\alpha-1})}$$

and, on account of (3.4), we deduce

$$\begin{aligned}\Xi_{n-k}(k) &= C(k,\alpha) \sum_{\nu=0}^{k} \widehat{s}_{2\alpha-1}(k,\nu)\ \big(n(n+2\alpha-1)\big)^\nu\\ &= C(k,\alpha) \sum_{\nu=0}^{k} \widehat{s}_{2\alpha-1}(k,\nu)\ \big(\chi_n\big)^\nu,\ n \in \mathbb{N},\end{aligned}$$

according to the expression of χ_n, $n \in \mathbb{N}$, given in Table 1. Equating the first and last members of the previous equalities, we obtain (2.26) with

$$d_{k,\tau} = C(k,\alpha)\ \widehat{s}_{2\alpha-1}(k,\tau)\,,\quad 0 \leqslant \tau \leqslant k.$$

Conversely, by virtue of (3.5) we have

$$\begin{aligned}(\chi_n)^k\ &=\ \big(n\,(n+2\alpha-1)\big)^k = \sum_{\tau=0}^{k} \widehat{S}_{2\alpha-1}(k,\tau)\ \{n\,(n+2\alpha-1)\}_{(\tau;\mathbf{2\alpha-1})}\\ &=\ \sum_{\tau=0}^{k} \big(C(\tau;\alpha)\big)^{-1}\ \widehat{S}_{2\alpha-1}(k,\tau)\ \Xi_{n-\tau}(\tau)\,,\quad n \in \mathbb{N},\end{aligned}$$

whence we achieve (2.27) with

$$\widetilde{d}_{k,\tau} = C(\tau;\alpha)^{-1}\,\widehat{S}_{2\alpha-1}(k,\tau)\,, \quad 0 \leqslant \tau \leqslant k.$$

From corollary 2.6 it follows

$$\begin{cases} \sum_{\nu=0}^{k} \Lambda_\nu\,(k;x)\,D^{k+\nu} = \sum_{\tau=0}^{k} C(k,\alpha)\,\widehat{s}_{2\alpha-1}(k,\nu)\,\mathscr{F}^{\tau} \\ \\ \mathscr{F}^{k} = \sum_{\tau=0}^{k} \big(C(\tau;\alpha)\big)^{-1}\,\widehat{S}_{2\alpha-1}(k,\tau)\,\sum_{\nu=0}^{\tau} \Lambda_\nu\,(\tau;x)\,D^{\tau+\nu} \end{cases} \tag{4.6}$$

where $\Lambda_\nu(k;x)$ is given in (4.5) and $\mathscr{F} = x^2D^2 + 2(\alpha x+1)D$.

4.4. *Jacobi case*

Let $\{P_n(\cdot;\alpha,\beta)\}_{n\in\mathbb{N}}$ with $\alpha,\beta \neq -(n+1)$, $\alpha+\beta \neq -(n+2)$, $n\in\mathbb{N}$, represent a Jacobi monic polynomial sequence. From (2.3)-(2.4) and based on Table 2, it follows $\Xi_n(k) = \lambda_n^k\,\{n+k\}_{(\mathbf{k})}$, $n\in\mathbb{N}$. Considering the information presented in Table 1 for the Jacobi case, (2.7) becomes like $\omega_{k,\nu} = \dfrac{1}{(\alpha+\beta+1+k)_\nu}$, (with $0\leqslant\nu\leqslant k$). Consequently, the polynomial $\Lambda_\nu(k;x)$, defined in (2.6), may be expressed like:

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\,C(k;\alpha,\beta)\,(\alpha+\beta+1+k+\nu)_{k-\nu}\,(x^2-1)^\nu\,P_{k-\nu}^{[\nu]}(x;\alpha,\beta)$$

Since, for each integer $\nu\geqslant 1$, $P_n^{[\nu]}(\cdot;\alpha,\beta) = P_n(\cdot,\alpha+\nu,\beta+\nu)$, $n\in\mathbb{N}$, then we have

$$\Lambda_\nu(k;x) = \binom{k}{\nu}\,C(k;\alpha,\beta)\,(\alpha+\beta+1+k+\nu)_{k-\nu}\,(x^2-1)^\nu\,P_{k-\nu}(x;\alpha+\nu,\beta+\nu) \tag{4.7}$$

where

$$P_{k-\nu}(x;\alpha+\nu,\beta+\nu) = \frac{(-2)^{k-\nu}\Gamma(k+\alpha+1)}{\Gamma(2k+\alpha+\beta+1)}\sum_{\mu=0}^{k-\nu}\Bigg\{\sum_{\tau=\mu}^{k-\nu}(-2)^{-\tau}\binom{k-\nu}{\tau}\binom{\tau}{\mu} \times\frac{\Gamma(\tau+k+\nu+\alpha+\beta+1)}{\Gamma(\tau+\alpha+\nu+1)}\Bigg\}\,x^\mu\,, \quad 0\leqslant\nu\leqslant k.$$

Following (2.1), $Y(x) = P_n(x;\alpha,\beta)$ is a solution of the following differential equation

$$\sum_{\nu=0}^{k}\binom{k}{\nu}\,\big\{(\alpha+\beta+1+k+\nu)_{k-\nu}\,(x^2-1)^\nu\,P_{k-\nu}(x;\alpha+\nu,\beta+\nu)\,\big\}\,D^{k+\nu}\big(Y(x)\big) = \{n\}_{(\mathbf{k})}\,(\alpha+\beta+1+n)_k\,Y(x)\,, \quad n\in\mathbb{N}.$$

The determination of the two sets of coefficients $\{d_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ and $\{\widetilde{d}_{k,\mu} : 0 \leqslant \mu \leqslant k\}$ realising the conditions (2.26)-(2.27) for this case is analogous to the corresponding problem in the Bessel case. In turn, the relation (3.17), with $A = \alpha+\beta+1$, yields

$$\Xi_{n-k}(k) = C(k,\alpha,\beta)\ \{n(n+\alpha+\beta+1\}_{(\mathbf{k};\alpha+\beta+\mathbf{1})}\,, \quad n \in \mathbb{N},$$

and (3.4) permits to write

$$\begin{aligned}\Xi_{n-k}(k) \;&=\; C(k,\alpha,\beta)\sum_{\nu=0}^{k} \widehat{s}_{\alpha+\beta+1}(k,\nu)\,\big(n(n+\alpha+\beta+1)\big)^{\nu}\\ &=\; C(k,\alpha,\beta)\sum_{\nu=0}^{k} \widehat{s}_{\alpha+\beta+1}(k,\nu)\,\big(\chi_n\big)^{\nu}, \quad n \in \mathbb{N}.\end{aligned}$$

whence we obtain (2.26) with

$$d_{k,\tau} = C(k,\alpha,\beta)\,\widehat{s}_{\alpha+\beta+1}(k,\tau)\,, \quad 0 \leqslant \tau \leqslant k\,.$$

Conversely, due to (3.5) we have

$$\begin{aligned}(\chi_n)^k \;&=\; \big(n\,(n+\alpha+\beta+1)\big)^k = \sum_{\tau=0}^{k} \widehat{S}_{\alpha+\beta+1}(k,\tau)\ \{n(n+\alpha+\beta+1\}_{(\tau;\alpha+\beta+\mathbf{1})}\\ &=\; \sum_{\tau=0}^{k} \big(C(\tau;\alpha,\beta)\big)^{-1}\,\widehat{S}_{\alpha+\beta+1}(k,\tau)\,\Xi_{n-\tau}(\tau)\,, \quad n \in \mathbb{N}.\end{aligned}$$

The first and last members of the previous equality correspond to (2.27) if we consider

$$\widetilde{d}_{k,\tau} = \big(C(\tau;\alpha,\beta)\big)^{-1}\,\widehat{S}_{\alpha+\beta+1}(k,\tau)\,, \quad 0 \leqslant \tau \leqslant k.$$

From corollary 2.6 it follows

$$\begin{cases}\displaystyle\sum_{\nu=0}^{k} \Lambda_\nu\,(k;x)\,D^{k+\nu} = \sum_{\tau=0}^{k} C(k,\alpha,\beta)\,\widehat{s}_{\alpha+\beta+1}(k,\tau)\,\mathscr{F}^{\tau}\\[2ex] \displaystyle\mathscr{F}^{k} = \sum_{\tau=0}^{k} \big(C(\tau;\alpha,\beta)\big)^{-1}\,\widehat{S}_{\alpha+\beta+1}(k,\tau)\,\sum_{\nu=0}^{\tau} \Lambda_\nu\,(\tau;x)\,D^{\tau+\nu}\end{cases}, \tag{4.8}$$

where $\Lambda_\nu(k;x)$ is given by (4.7), and $\mathscr{F} = (x^2-1)D^2 + \big\{(\alpha+\beta+2)x - (\alpha-\beta)\big\}D$.

5. Some final remarks

It might be worthy to bring the attention of a well known result that is, in particular, presented in [5,21,22] but mostly developed in [22, chapter VI]. Considering the differential operator $\theta = xD$. It is possible to relate the powers of θ and its “factorials”, say $\theta_j = x^j D^j$, through the following equalities:

$$\theta^k = \big(xD\big)^k = \sum_{j=0}^{k} S(k,j)\,x^j\,D^j = \sum_{j=0}^{k} S(k,j)\,\theta_j$$

$$\theta_k = x^k\,D^k = \sum_{j=0}^{k} s(k,j)\,\big(xD\big)^j = \sum_{j=0}^{k} s(k,j)\ \theta^j$$

In the previous section (more precisely in (4.2), (4.4) ,(4.6) and (4.8)) we have shown that for each of the classical families, we can establish a similar-looking "*inversion*" formula. For instance, representing by $\mathscr{F}_\tau := \sum_{\nu=0}^{\tau} \Lambda_\nu(\tau;x) D^{\tau+\nu}$, we have determined two sets of coefficients $\{d_{k,\tau}\}_{0\leqslant\tau\leqslant k}$ and $\{\widetilde{d}_{k,\tau}\}_{0\leqslant\tau\leqslant k}$ for each classical family such that

$$\mathscr{F}_k = \sum_{\tau=0}^{k} d_{k,\tau}\, \mathscr{F}^\tau \qquad \text{and} \qquad \mathscr{F}^k = \sum_{\tau=0}^{k} \widetilde{d}_{k,\tau}\, \mathscr{F}_\tau\, .$$

The A-modified Stirling numbers introduced in section 3, could also be called *Bessel-Stirling* numbers or *Jacobi-Stirling* numbers depending on the context and the values of the complex parameter A. Indeed, in [6] the authors have dealt with the Jacobi polynomials and they have already used the name *Jacobi-Stirling* numbers in the same sense as the one presented here.

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References

[1] S. Bochner, *Über Sturm-Liouvillesche Polynomsysteme*, Math. Z. **29** (1929) 730-736.

[2] D. I. Caruntu, *Self-adjoint differential equations for classical orthogonal polynomials*, *J. Comput. Appl. Math.* **180** (2005) 107-118.

[3] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[4] W.-S. Chou, L.C. Hsu, P. J.-S. Shiue, *On a class of combinatorial sums involving generalized factorials*, Intern. J. Mathematics and Mathematical Sciences Article ID 12604 (2007) 9 pages. doi:10.1155/2007/12604.

[5] L. Comtet, *Advanced Combinatorics - The art of finite and infinite expansions*, Kluwer, Dordrecht, 1974.

[6] W.N. Everitt, L.L. Littlejohn, R. Wellman, G.J.Yoon, *Jacobi–Stirling numbers, Jacobi polynomials, and the left-definite analysis of the classical Jacobi differential expression*, J. Comput. Appl. Math. **208** (2007) 29-56.

[7] J.L. Geronimus, *On polynomials orthogonal with respect to numerical sequences and on Hahn's theorem*, Izv. Akad. Nauk. **250** (1940) 215-228. (*in* Russian)

[8] W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Math. Z. **39** (1935) 634-638.

[9] W. Hahn, *Über höhere Ableitungen von Orthogonalpolynomen*, Math. Z. **43** (1937) 101.

[10] Leetsch C. Hsu, *A unified approach to generalized Stirling numbers*, Adv. Appl. Math. **20** (1998) 366-384.

[11] H.L. Krall, *Certain differential equations for Tchebyshev polynomials*, Duke Math. J. **4** (1938) 705-718.

[12] H.L. Krall, O. Frink, *A new class of orthogonal polynomials: the Bessel polynomials*, Trans. Amer. Math. Soc. **65** (1949) 100-115.

[13] K.H. Kwon, L.L. Littlejohn, B.H. Yoon, *Characterizations of orthogonal polynomial sequences satisfying differential equations*, *SIAM J. Math. Anal.* **25** (1994) 976-990.

[14] K.H. Kwon, L.L. Littlejohn, B.H. Yoon, *New characterizations of classical orthogonal polynomials*, *Indag. Math. (N.S.)* **7** (1996) 199-213.

[15] L.L. Littlejohn, R. Wellman, *A general left-definite theory for certain self-adjoint operators with applications to differential equations*, J. Differ. Equations **181** (2002) 280-339.

[16] A.F. Loureiro, P. Maroni, P., Z. Rocha, *The generalized Bochner condition about classical orthogonal polynomials revisited*, J. Math. Anal. Appl. **322** (2006) 645-667.

[17] P. Maroni, *Variations around classical orthogonal polynomials. Connected problems*, J. Comput. Appl. Math. **48** (1993) 133-155.

[18] P. Maroni, *Fonctions Eulériennes. Polynômes orthogonaux classiques.* Techniques de l'Ingénieur, traité Généralités (Sciences Fondamentales), A **154** (1994), 1-30.

[19] S. C. Milne, G. Bhatnagar, *A characterization of inverse relations*, Discrete Math. **193** (1998) 235-245.

[20] L. Miranian, *On classical orthogonal polynomials and differential operators*, J. Phys. A: Math Gen. **38** (2005) 6379-6382.

[21] J. Riordan, *An introduction to combinatorial analysis*, Dover ed., New Yok, 2002.

[22] J. Riordan, *Combinatorial identities*, Wiley, New Yok, 1968.