A CONSTRUCTION FOR COISOTROPIC SUBALGEBRAS OF LIE BIALGEBRAS

MARCO ZAMBON

Abstract. Given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, we present an explicit procedure to construct coisotropic subalgebras, i.e. Lie subalgebras of $\mathfrak{g}$ whose annihilator is a Lie subalgebra of $\mathfrak{g}^*$. We write down families of examples for the case that $\mathfrak{g}$ is a classical complex simple Lie algebra. The construction follows naturally from considerations about pre-Poisson maps between Poisson manifolds.

Contents

1. Introduction

A *Lie bialgebra* [5] structure on a Lie algebra $(\mathfrak{g}, [\bullet, \bullet])$ is a degree 1 derivation δ of $\wedge^\bullet \mathfrak{g}$ which squares to zero and satisfies $\delta([X,Y]) = [\delta(X), Y] + [X, \delta(Y)]$. Dualizing $\delta|_{\mathfrak{g}} \colon \mathfrak{g} \to \wedge^2 \mathfrak{g}$ one obtains a Lie bracket on $\mathfrak{g}^*$, encoding δ, so that the Lie algebra structures on $\mathfrak{g}$ and $\mathfrak{g}^*$ are compatible. The aim of this paper is to construct Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ with the property that $\mathfrak{h}^\circ$, the subspace of $\mathfrak{g}^*$ consisting of elements that vanish on $\mathfrak{h}$, is a Lie subalgebra of $\mathfrak{g}^*$. Such an $\mathfrak{h}$ is called *coisotropic subalgebra*.

Our main result (Thm. 4.3) is a explicit and computationally friendly construction that works for Lie bialgebras arising from r-matrices. Recall that any r-matrix on a Lie algebra $\mathfrak{g}$, i.e. any $\pi \in \wedge^2 \mathfrak{g}$ such that $[\pi, \pi]$ is ad-invariant, gives rise to a Lie bialgebra by setting $\delta = [\pi, \bullet]$. Our result can be phrased as follows:

Theorem. *Let $\mathfrak{g}$ be a Lie bialgebra arising from an r-matrix π. Suppose $X \in \mathfrak{g}$ satisfies*

$$[X, [X, \pi]] = \lambda [X, \pi] \text{ for some } \lambda \in \mathbb{R}.$$

Then the image of the map $\mathfrak{g}^* \to \mathfrak{g}$ given by contraction with $[X, \pi] \in \wedge^2 \mathfrak{g}$ is a coisotropic subalgebra of $\mathfrak{g}$.

We remark that the coisotropic subalgebras that arise as in the theorem are all even dimensional, therefore they are by no means all coisotropic subalgebras. Using this theorem we produce in a straightforward way families of coisotropic subalgebras when $\mathfrak{g}$ is one of the four classical simple complex Lie algebras or one of their split real forms.

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We point out a few reasons for the relevance of coisotropic subalgebras. First, via $\mathfrak{k} \mapsto \mathfrak{k} \oplus \mathfrak{k}^\circ$ they correspond to lagrangian subalgebras of the Drinfeld double $\mathfrak{g} \oplus \mathfrak{g}^*$ and hence give rise to Poisson homogeneous spaces [7]. Second, coisotropic subalgebras are interesting because they have a counterpart in the Hopf algebra setting after quantization [4].

The paper is organized as follows. In Section 2 we make general considerations about maps between Poisson manifolds. Given a Lie bialgebra $\mathfrak{g}$, making a choice of element g of a Poisson-Lie group G integrating $\mathfrak{g}$, in Section 3 we construct a subspace $\mathfrak{h}^g$ of $\mathfrak{g}$. The considerations of Section 2, applied to the left translation $L_g : G \to G$, imply that *if* $\mathfrak{h}^g$ is a Lie subalgebra of $\mathfrak{g}$ then automatically it is a coisotropic subalgebra. In Section 4 we restrict our attention to Lie bialgebras arising from r-matrices and elements g of the form $exp(X)$, proving the theorem stated above. Section 5 is devoted to explicit examples in which $\mathfrak{g}$ is a semi-simple Lie algebra.

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2. Pre-poisson maps

In this section we make some considerations about maps between Poisson manifolds.

Recall that a *Poisson manifold* is a manifold P endowed with a bivector field $\Lambda \in \Gamma(\wedge^2 TP)$ satisfying $[\Lambda, \Lambda] = 0$, where $[\bullet, \bullet]$ denotes the Schouten bracket on multivector fields. We denote by $\Lambda^\sharp \colon T^*P \to TP$ the map given by contraction with Λ.

A submanifold C of a Poisson manifold P is called *coisotropic* if $\Lambda^\sharp N^*C \subset TC$, where N^*C (the conormal bundle of C) is defined as the annihilator of TC. Here we need a generalization of the notion of coisotropic submanifold:

Definition 2.1. A submanifold C of a Poisson manifold (P, Λ) is called *pre-Poisson* [2] if the rank of $TC + \Lambda^\sharp N^*C$ is constant along C, or equivalently if $pr_{NC} \circ \Lambda^\sharp \colon N^*C \to TP|_C \to NC := TP|_C / TC$ has constant rank.

A map $\phi \colon (P_1, \Lambda_1) \to (P_2, \Lambda_2)$ between Poisson manifolds is a *pre-Poisson map* if $graph(\phi)$ is a pre-Poisson submanifold of the product $P_1 \times \bar{P}_2$, where $\bar{P}_2$ denotes the Poisson manifold $(P_2, -\Lambda_2)$.

A map between Poisson manifolds is a Poisson map iff its graph is coisotropic, hence we see that pre-Poisson maps generalize the notion of Poisson map. We make more explicit what it means to be a pre-Poisson map.

Lemma 2.2. *A map $\phi \colon (P_1, \Lambda_1) \to (P_2, \Lambda_2)$ is pre-Poisson iff for all $x \in P_1$ the rank of*

$$E(x) = \{(\Lambda_2 - \phi_* \Lambda_1)^\sharp \xi : \xi \in T^*_{\phi(x)} P_2\} \subset T_{\phi(x)} P_2$$

is constant. Here $\phi_* \colon T_x P_1 \to T_{\phi(x)} P_2$.

Proof. Let $\Gamma := graph(\phi) \subset P_1 \times \bar{P}_2$ and $x \in P_1$. We have

$$\begin{aligned} T_{(x,\phi(x))}\Gamma + (\Lambda_1 - \Lambda_2)^\sharp N^*_{(x,\phi(x))}\Gamma &= \{(X, \phi_* X) : X \in T_x P_1\} + \{(\Lambda_1^\sharp \phi^* \xi, \Lambda_2^\sharp \xi) : \xi \in T^*_{\phi(x)} P_2\} \\ &= \{(X, \phi_* X) : X \in T_x P_1\} + \{(0, \Lambda_2^\sharp \xi - \phi_*(\Lambda_1^\sharp \phi^* \xi)) : \xi \in T^*_{\phi(x)} P_2\} \\ &= \{(X, \phi_* X) : X \in T_x P_1\} + \{0\} \times E(x). \end{aligned}$$

A complement of this subspace in $T_{(x,\phi(x))}(P_1 \times P_2)$ is $(0, R(x))$, where $R(x)$ is a complement to $E(x)$ in $T_{\phi(x)}P_2$. Hence Γ is a pre-Poisson submanifold iff $R(x)$, or equivalently $E(x)$, has constant rank as x varies over all points of P_1. □

Remark 2.3. 1) The composition of pre-Poisson maps is *not* pre-Poisson. Let $P_1 = (\mathbb{R}^2, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$, $P_2 = (\mathbb{R}^2, 0)$ and $P_3 = (\mathbb{R}^2, (1 + x^2 + y^2)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$. The identity maps $id\colon P_1 \to P_2$ and $id\colon P_2 \to P_3$ are pre-Poisson maps (this is seen easily using Lemma 2.2), however the composition is not.

2) Let P_1, P_2 be Poisson manifolds and $\phi\colon P_1 \to P_2$ be a submersive *Poisson* map. If $C \subset P_2$ is a pre-Poisson submanifold (for example a point), then $f^{-1}(C)$ is a pre-Poisson submanifold of P_1 [3]. When ϕ is just a submersive *pre-Poisson* map this statement is not longer true: the projection $\phi : (\mathbb{R}^3, -z^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}) \to (\mathbb{R}^2, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y})$ onto the first two components is a pre-Poisson map, but $\phi^{-1}(0) = \{(0, 0, z) : z \in \mathbb{R}\}$ is not a pre-Poisson submanifold.

From now on we consider only the case when the map ϕ of Lemma 2.2 is a *diffeomorphism*. Then $D_y := E(\phi^{-1}(y))$ defines a singular distribution on P_2 which measures how ϕ fails to be a Poisson map.

Definition 2.4. Given a diffeomorphism $\phi\colon (P_1, \Lambda_1) \to (P_2, \Lambda_2)$ between Poisson manifolds, the *deficit distribution* associated to ϕ is the singular distribution on P_2 given by

$$D = \{(\Lambda_2 - \phi_* \Lambda_1)^\sharp \xi : \xi \in T^* P_2\}.$$

The deficit distribution D singles out an interesting subalgebra of $C^\infty(P_2)$:

Lemma 2.5. *Let $\phi\colon (P_1, \Lambda_1) \to (P_2, \Lambda_2)$ be a diffeomorphism. Then the set of D-invariant functions $\{f : d_y f|_{D_y} = 0$ for all $y \in P_2\}$ coincides with*

$$\big\{f : \phi^*\{f, g\} = \{\phi^* f, \phi^* g\} \text{ for all } g \in C^\infty(P_2)\big\}, \tag{1}$$

and is a Poisson subalgebra of $C^\infty(P_2)$.

Proof. Expressing D in terms of hamiltonian vector fields we have $D = \{X_g^{P_2} - \phi_*(X^{P_1}_{\phi^* g}) : g \in C^\infty(P_2)\}$. The claimed equality follows from

$$d_y f(X_g^{P_2} - \phi_*(X^{P_1}_{\phi^* g})) = \{f, g\}_y - d_{\phi^{-1}(y)}(\phi^* f) X^{P_1}_{\phi^* g} = (\phi^*\{f, g\} - \{\phi^* f, \phi^* g\})_{\phi^{-1}(y)}$$

for all $y \in P_2$.

To show that (1) is a Poisson subalgebra we compute for D-invariant functions f and $\tilde{f}$ on P_2 and for $g \in C^\infty(P_2)$ that

$$\phi^*\{\{f, g\}, \tilde{f}\} = \{\phi^*\{f, g\}, \phi^* \tilde{f}\} = \{\{\phi^* f, \phi^* g\}, \phi^* \tilde{f}\}.$$

Hence using twice the Jacobi identity we obtain

$$\begin{aligned} \phi^*\{\{f, \tilde{f}\}, g\} &= \phi^*\{\{f, g\}, \tilde{f}\} + \phi^*\{f, \{\tilde{f}, g\}\} \\ &= \{\{\phi^* f, \phi^* g\}, \phi^* \tilde{f}\} + \{\phi^* f, \{\phi^* \tilde{f}, \phi^* g\}\} = \{\{\phi^* f, \phi^* \tilde{f}\}, \phi^* g\} = \{\phi^*\{f, \tilde{f}\}, \phi^* g\}. \end{aligned}$$

□

Summarizing the results obtained in this section we have

Proposition 2.6. *A diffeomorphism $\phi\colon (P_1,\Lambda_1)\to(P_2,\Lambda_2)$ is a pre-Poisson map iff $\Lambda_2-\phi_*\Lambda_1$ is a constant rank bivector on P_2, i.e. iff D is a smooth constant rank distribution on P_2. If D is integrable and the leaf space P_2/D is smooth, then P_2/D has a Poisson structure induced by the projection map $\pi\colon P_2\to P_2/D$. In this case the composition $\pi\circ\phi\colon P_1\to P_2/D$ is a Poisson map.*

Proof. ϕ is a pre-Poisson map by Lemma 2.2. By the second part of Lemma 2.5 the D-invariant functions on P_2 form a Poisson subalgebra of $C^\infty(P_2)$, so P_2/D has an induced Poisson structure. By the first part of Lemma 2.5 in particular $\phi^*\{f,\tilde{f}\}=\{\phi^*f,\phi^*\tilde{f}\}$ for all D-invariant functions $f,\tilde{f}$ on P_2, so $\pi\circ\phi$ is a Poisson map. □

3. Coisotropic subalgebras

We recall some notions from the theory of Poisson Lie groups; we refer to the expositions [13, 11, 12] for more details.

Definition 3.1. A *Poisson Lie group* is a Lie group G equipped with a Poisson bivector Λ such that the multiplication map $m\colon G\times G\to G$ is a Poisson map, or equivalently such that

$$\Lambda(gh)=(L_g)_*\Lambda(h)+(R_h)_*\Lambda(g) \text{ for all } g,h\in G. \tag{2}$$

To every element g of the Poisson Lie group G we associate a *subspace* of its Lie algebra $\mathfrak{g}$ as follows:

$$\mathfrak{h}^g := (\eta^g)^\sharp\,\mathfrak{g}^*, \tag{3}$$

where we use the short-hand notation

$$\eta^g := (L_g)_*\Lambda(g^{-1})\in\wedge^2\mathfrak{g}. \tag{4}$$

The subspace $\mathfrak{h}^g$ is the left-translation to the identity of $T_{g^{-1}}\mathcal{O}$, where $\mathcal{O}$ denotes the symplectic leaf of (G,Λ) through g^{-1}; in particular it is always even dimensional.

The importance of the subspace $\mathfrak{h}^g$ lies in the fact that it generates the deficit distribution of the left translation $L_g\colon G\to G$.

Lemma 3.2. *a) $L_g\colon G\to G$ is a pre-Poisson map.*
b) Its deficit distribution is $\overrightarrow{\mathfrak{h}^g}$, the right-invariant distribution obtained translating $\mathfrak{h}^g\subset T_eG$.

Proof. a) By Prop. 2.6 we have to show that $\Lambda-(L_g)_*\Lambda$ is a constant rank bivector on G. This bivector field at the point $k\in G$ is

$$\Lambda(k)-(L_g)_*[\Lambda(g^{-1}k)]=-(L_g)_*(R_k)_*\Lambda(g^{-1})=-(R_k)_*\eta^g, \tag{5}$$

where we have used (2) applied to $\Lambda(g^{-1}k)$ in the first equality. For all $k\in G$ the map $(R_k)_*$ is injective, hence the rank of the above bivector field at k is equal to the rank of η^g, which is independent of k.

b) The deficit distribution is defined as $[\Lambda-(L_g)_*\Lambda]^\sharp T^*G$. Using (5) we see that at the point k it is

$$[(R_k)_*\eta^g]^\sharp T_k^*G=(R_k)_*[(\eta^g)^\sharp\mathfrak{g}^*]=(R_k)_*\mathfrak{h}^g.$$

□

Remark 3.3. We an alternative proof of Lemma 3.2 a). Since $m\colon G\times G\to G$ is a submersive Poisson map, the result recalled in Remark 2.3 implies that $m^{-1}(i(g)) = graph(i\circ L_g)$ is pre-Poisson as a submanifold of $G\times G$, i.e. that $i\circ L_g\colon G\to \bar{G}$ is a pre-Poisson map. Here i is the inversion map on G, which viewed as a map $i\colon G\to\bar{G}$ is a Poisson diffeomorphism, hence it follows that $L_g\colon G\to G$ is a pre-Poisson map.

Definition 3.4 (Sec. 3.1 of [13])**.** Let $\mathfrak{g}$ be a Lie bialgebra. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *coisotropic*[1] if its annihilator $\mathfrak{h}^\circ$ is a Lie subalgebra of $\mathfrak{g}^*$.

Since we realized $\overrightarrow{\mathfrak{h}^g}$ as a deficit distribution we obtain

Proposition 3.5. *Let G be a Poisson Lie group and $g\in G$. If $\mathfrak{h}^g\subset\mathfrak{g}$ is a Lie subalgebra then it is automatically a coisotropic subalgebra.*

Proof. For any $f_1, f_2\in C^\infty(G)$ and $X\in\mathfrak{g}$ we have (see [13], Ch. 2.3)

$$\langle [d_e f_1, d_e f_2], X\rangle = X\{f_1, f_2\}. \tag{6}$$

Any element of $(\mathfrak{h}^{\mathfrak{g}})^\circ$ can be realized as $d_e f$ where f is a function on G which is invariant along the integrable distribution obtained right-translating $\mathfrak{h}^g$. This distribution coincides with the deficit distribution of $L_g\colon G\to G$ by Lemma 3.2 b). Hence, if f_1 and f_2 are invariant functions, by Lemma 2.5 $\{f_1,f_2\}$ is also invariant. Therefore the right hand side of (6) vanishes for all $X\in\mathfrak{h}^g$, from which we deduce that $[d_e f_1, d_e f_2]\in(\mathfrak{h}^g)^\circ$. □

The set $\{g\in G : \mathfrak{h}^g \text{ is a Lie subalgebra}\}$ is closed under inversion but is not a subgroup of G (see Remark 4.7). Further $(\mathfrak{h}^g)^\circ$ is usually not an ideal[2] in $\mathfrak{g}^*$ (see Remark 5.9).

We conclude with two remarks on Poisson actions which will not affect the rest of this note.

Remark 3.6. The considerations of Lemma 3.2 can be extended to locally free left *Poisson actions* (i.e. actions for which $\sigma : G\times P\to P$ is a Poisson map, where $G\times P$ is equipped with the product Poisson structure). In this case we obtain:
a) for all $g\in G$, $\sigma_g\colon P\to P$ is a pre-Poisson map.
b) the deficit distribution of σ_g is generated by the infinitesimal action of $\mathfrak{h}^g\subset\mathfrak{g}$.
If $\mathfrak{h}^g$ is a Lie subalgebra of $\mathfrak{g}$ and P/H^g is a smooth manifold, where H^g the connected subgroup of G integrating $\mathfrak{h}^g$, then P/H^g has a Poisson structure for which the projection map $\pi\colon P\to P/H^g$ is Poisson. This is a well-known fact (see Thm. 6 of [14] or Prop. 3.4 of [13]). Prop. 2.6 in addition tells us that $\pi\circ\sigma_g\colon P\to P/H^g$ is also a Poisson map.

Remark 3.7. Recall that a right Poisson homogeneous space for G is a Poisson manifold X with a transitive right action $X\times G\to X$ which is a Poisson map. Consider the action by left multiplication G on itself, and let $g\in G$ so that $\mathfrak{h}^g$ is a Lie subalgebra of $\mathfrak{g}$. Then $H^g\backslash G$ (if smooth), together with the action of G by right multiplication, is a right Poisson homogeneous space. Further both the projection π and $\pi\circ L_g\colon G\to H^g\backslash G$ are Poisson maps which are equivariant for the G-actions by right multiplication.

[1] A Lie subalgebra $\mathfrak{h}$ is coisotropic iff the connected subgroup H integrating it is a coisotropic subgroup of (G,Λ) (see for instance [4]).

Another equivalent characterization of the fact that $\mathfrak{h}$ is a coisotropic Lie subalgebra is the following: $\mathfrak{h}$ is a coisotropic submanifold of $\mathfrak{g}$, endowed with the linear Poisson structure induced by the Lie algebra $\mathfrak{g}^*$, and $\mathfrak{h}^\circ$ is a coisotropic submanifold of the linear Poisson manifold $\mathfrak{g}^*$.

[2] It is an ideal in $\mathfrak{g}^*$ iff the connected subgroup integrating it is a Poisson subgroup of (G,Λ) (see for instance [4]).

4. Poisson Lie groups arising from r-matrices

Let (G, Λ) be a Poisson Lie group. In this section we determine elements $g \in G$ for which the subspace $\mathfrak{h}^g \subset \mathfrak{g}$ of eq. (3) is a Lie subalgebra, for Prop. 3.5 tells us that then it is a coisotropic subalgebra.

Lemma 4.1. *If $[\eta^g, \eta^g] = 0 \in \wedge^3 \mathfrak{g}$ then $\mathfrak{h}^g$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. By equation (5) $\Lambda - (L_g)_*\Lambda$ equals $-\overrightarrow{\eta^g}$, the right-invariant bivector on G whose value at the identity is $-\eta^g$. Hence $[\eta^g, \eta^g] = 0$ iff $\Lambda - (L_g)_*\Lambda$ is a Poisson bivector. In this case the right-invariant distribution $(\Lambda - (L_g)_*\Lambda)^\sharp\, T^*G$ is integrable. Hence its value at the identity, which by Lemma 3.2 b) is $\mathfrak{h}^g$, is a Lie subalgebra of $\mathfrak{g}$. ☐

Definition 4.2. Let $\mathfrak{g}$ be a Lie algebra. An *r-matrix* is an element $\pi \in \wedge^2 \mathfrak{g}$ such that $[\pi, \pi]$ is *ad*-invariant.

It is known [6] that if π is an r-matrix for the Lie algebra $\mathfrak{g}$ then $\Lambda := \overleftarrow{\pi} - \overrightarrow{\pi}$ makes G, any Lie group integrating $\mathfrak{g}$, into a Poisson Lie group. From now on we restrict ourselves to such Poisson Lie groups. Notice that from definition (4) we get

$$\eta^g = \pi - Ad_g \pi. \tag{7}$$

Now we are able to state the main result of this paper.

Theorem 4.3. *Let G be a Poisson Lie group corresponding to an r-matrix π, $X \in \mathfrak{g}$, $g := exp(X)$. Assume that*

$$[X, [X, \pi]] = \lambda [X, \pi] \text{ for some } \lambda \in \mathbb{R}. \tag{8}$$

Then $\mathfrak{h}^g$ is a coisotropic subalgebra of $\mathfrak{g}$. Further

$$\mathfrak{h}^g = [X, \pi]^\sharp \mathfrak{g}^*. \tag{9}$$

Proof. Notice that

$$Ad_{exp(X)}\pi = e^{ad_X}\pi = \pi + [X, \pi] + \frac{1}{2}[X, [X, \pi]] + \frac{1}{3!}[X, [X, [X, \pi]]] + \cdots = \pi + \frac{e^\lambda - 1}{\lambda}[X, \pi].$$

Therefore

$$\eta^g = \pi - Ad_g\pi = \pi - (\pi + \frac{e^\lambda - 1}{\lambda}[X, \pi]) = -\frac{e^\lambda - 1}{\lambda}[X, \pi].$$

Now we use twice the fact that $[\pi, [X, \pi]] = \frac{1}{2}[X, [\pi, \pi]] = 0$ (by the graded Jacobi identity) to show that

$$[[X, \pi], [X, \pi]] = [X, [\pi, [X, \pi]]] - [\pi, [X, [X, \pi]]] = 0 - \lambda \cdot 0 = 0.$$

This means that $[\eta^g, \eta^g] = 0$, and by Lemma 4.1 and Prop. 3.5 $\mathfrak{h}^g$ is a coisotropic subalgebra. The last part of the theorem follows since the function $\frac{e^\lambda - 1}{\lambda}$ never vanishes. ☐

Remark 4.4. If $X \in \mathfrak{g}$ satisfies condition (8) then $\Lambda = \overleftarrow{\pi} - \overrightarrow{\pi}$ and $\overrightarrow{\eta^g}$ (or $\overleftarrow{\eta^g}$) are commuting Poisson structures on G. This follows at once from the computations of the proof of Thm 4.3, noticing that η^g is a multiple of $[X, \pi]$. Here at usual $g := exp(X)$.

We now display two very simple examples.

Example 4.5. Let $\mathfrak{g} = \mathfrak{su}(2, \mathbb{R})$, so that for a suitable basis we have $[e_1, e_2] = e_3, [e_2, e_3] = 1, [e_3, e_1] = e_2$, and take the r-matrix $\pi = 2e_2 \wedge e_3$ as in Ex. 2.10 of [13]. Then the only elements of $\mathfrak{su}(2, \mathbb{R})$ that satisfy eq. (8) are the multiples X of e_1, and applying (9) we see that they all give $\mathfrak{h}^{exp(X)} = \{0\}$.

Example 4.6. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, with basis

$$e_1 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Then $[e_1,e_2]=e_3, [e_2,e_3]=e_1, [e_3,e_1]=-e_2$, and $\pi = 2e_2\wedge e_3$ is an r-matrix (Ex. 2.9 of [13]). The vectors X of $\mathfrak{sl}(2,\mathbb{R})$ that satisfy eq. (8) are exactly those of the form $\alpha e_1+\beta(e_2+e_3)$ (the upper triangular matrices) and $\alpha e_1+\beta(e_2-e_3)$ (the lower triangular matrices). Applying Thm. 4.3 we obtain coisotropic subalgebras $span\{e_1, e_2-e_3\}$, $span\{e_1,e_2+e_3\}$ and $\{0\}$.

Using (4) one can compute directly all the elements $g \in G = SL(2\,\mathbb{R})$ for which $[\eta^g,\eta^g]=0$: they those of the form $\left(\begin{smallmatrix}a & b\\ 0 & a^{-1}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}a & 0\\ c & a^{-1}\end{smallmatrix}\right)$. By Lemma 4.1 and Prop. 3.5 these group elements g give rise to a coisotropic subalgebra of $\mathfrak{g}$. The first class of elements g with $b\neq 0$ all give rise to $span\{e_1,e_2-e_3\}$, the second class of elements g with $c\neq 0$ all give rise to $span\{e_1,e_2+e_3\}$, and the diagonal matrices give rise to the trivial subalgebra $\{0\}$, i.e. we obtain exactly the same coisotropic subalgebras as above.

Remark 4.7. We show that $\{g\in G : \mathfrak{h}^g \text{ is a Lie subalgebra}\}$ is closed under the inversion map but not under multiplication. Indeed notice that $\eta^{g^{-1}} = -Ad_{g^{-1}}\eta^g$ by (2), so $\mathfrak{h}^{g^{-1}} = Ad_{g^{-1}}\mathfrak{h}^g$, and since $Ad_{g^{-1}}$ is a Lie algebra isomorphism the first statement follows.

To show the second statement consider $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$ as in Example 4.6. The elements $g = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right)$ of $G=SL(2,\mathbb{R})$ have the property that $\mathfrak{h}^g$ and $\mathfrak{h}^h$ are Lie subalgebras, by Example 4.6. However $\eta^{gh} = \pi - Ad_{gh}\pi = 2(e_1\wedge e_2 + 2e_2\wedge e_3 - e_1\wedge e_3)$, implying that $\mathfrak{h}^{gh}$ is not a Lie subalgebra of $\mathfrak{g}$.

5. Examples: semi-simple complex Lie algebras

In this section we consider the standard Lie bialgebra structure on a semi-simple *complex* Lie algebra, and out of its roots, using Thm. 4.3 we construct families of coisotropic subalgebras. We write down explicitly[3] the resulting families for the classical simple Lie algebras $\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{so}(2n+1,\mathbb{C}), \mathfrak{sp}(2n,\mathbb{C}), \mathfrak{so}(2n,\mathbb{C})$ and for their split real forms $\mathfrak{sl}(n+1,\mathbb{R}), \mathfrak{so}(n+1,n), \mathfrak{sp}(2n,\mathbb{R}), \mathfrak{so}(n,n)$. We refer to Ch. 2.6 of [1], to [9] and to [10] for background material about semi-simple complex Lie algebras and their real forms.

Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$, and fix a Cartan subalgebra $\mathfrak{h}$. There is a decomposition $\mathfrak{g} = \mathfrak{h}\oplus_{\alpha\in R}\mathfrak{g}^\alpha$ where g^α denotes the one dimensional eigenspace for the adjoint action of $\mathfrak{h}$ associated to the "eigenvalue" $\alpha\in\mathfrak{h}^*$. The set $R\subset\mathfrak{h}^*$ is called root system; make a choice R_+ of positive roots. For each $\alpha\in R_+$ choose non-zero $e_\alpha\in\mathfrak{g}^\alpha$ and $f_\alpha\in\mathfrak{g}^{-\alpha}$.

Then an r-matrix is given by

$$\pi := \sum_{\alpha\in R_+}\lambda_\alpha e_\alpha\wedge f_\alpha \tag{10}$$

where $\lambda_\alpha := \frac{1}{B(e_\alpha,f_\alpha)}$ (see Ex. 2.10 of [12]). Notice that, since the subspaces $\mathfrak{g}^\alpha$ are one dimensional and the Killing form B is $\mathbb{C}$-bilinear, the above r-matrix depends only on the choice of Cartan subalgebra[4].

[3] One reason for doing this is that we were not able to find any explicit families of examples of coisotropic subalgebras in the literature.

[4] It would be interesting to study the variety of Lagrangian subalgebras of the Drinfeld double $\mathfrak{g}\oplus\mathfrak{g}^*$, since the coisotropic subalgebras we are constructing in this section are points of this variety. Evens and Lu [8] study the variety of Lagrangian subalgebras of the direct sum Lie algebra $\mathfrak{g}\oplus\mathfrak{g}$ endowed with a natural

Lemma 5.1. *Let $X \in \mathfrak{g}$ and assume that for all $\alpha \in R_+$*

1) $[X, [X, e_\alpha]] \wedge f_\alpha = 0$
2) $[X, [X, f_\alpha]] \wedge e_\alpha = 0$
3) $[X, e_\alpha] \wedge [X, f_\alpha] = 0$.

Then X satisfies condition (8) *(with $\lambda = 0$).*

Proof. We compute

$$[X, \pi] = \sum_{\alpha \in R_+} \lambda_\alpha([X, e_\alpha] \wedge f_\alpha + e_\alpha \wedge [X, f_\alpha]),$$

so

$$[X, [X, \pi]] = \sum_{\alpha \in R_+} \lambda_\alpha([X, [X, e_\alpha]] \wedge f_\alpha + 2[X, e_\alpha] \wedge [X, f_\alpha] + e_\alpha \wedge [X[X, f_\alpha]]),$$

each term of which vanishes by our assumptions. □

Proposition 5.2. *Let $\beta \in R_+$ satisfy this condition:*

(11) *For all $\alpha \in R$: $\quad (\alpha + \mathbb{Z}\beta) \cap R$ does not contain a string of 3 consecutive elements.*

Then e_β and f_β satisfy condition (8).

Proof. We check that $X = e_\beta$ satisfies the assumptions of Lemma 5.1; the proof for f_β is similar. Let $\alpha \in R$.

Suppose that $[e_\beta, [e_\beta, e_\alpha]] \neq 0$. Then $\alpha, \alpha + \beta$ and $\alpha + 2\beta$ form a string of 3 consecutive elements in $(\alpha + \mathbb{Z}\beta) \cap (R \cup \{0\})$. Since the intersection of R with any line through the origin is either empty or of the form $\{\alpha, -\alpha\}$ (Prop. 2.20 of [1]) it follows that $\beta = -\alpha$. So $[e_\beta, [e_\beta, e_\alpha]]$ is a multiple of f_α, and assumption 1) of Lemma 5.1 is satisfied.

Similarly, if $[e_\beta, [e_\beta, f_\alpha]] \neq 0$, then $-\alpha, -\alpha + \beta$ and $-\alpha + 2\beta$ form a string of 3 consecutive elements in $(\alpha + \mathbb{Z}\beta) \cap (R \cup \{0\})$, so we must have $\beta = \alpha$. So $[e_\beta, [e_\beta, f_\alpha]]$ is a multiple of e_α, and assumption 2) of Lemma 5.1 is satisfied.

At most one of $\alpha + \beta$ or $\alpha - \beta$ lie in R: if they both did then $\{\alpha - \beta, \alpha, \alpha + \beta\}$ would be a string of 3 consecutive elements in $(\alpha + \mathbb{Z}\beta) \cap R$, contradicting our assumption. If $\alpha - \beta \notin R$ then either $\alpha - \beta = 0$, in which case $[e_\alpha, e_\beta] = 0$, or $[e_\alpha, f_\beta] \in \mathfrak{g}^{\alpha - \beta} = \{0\}$. A similar reasoning holds for $\alpha + \beta$, so we conclude that assumption 3) of Lemma 5.1 holds. □

Corollary 5.3. *Assume the notation above and assume that $\beta \in R_+$ satisfy condition* (11)*. Let $\mathfrak{g}_\mathbb{R}$ denote $\mathfrak{g}$ viewed as a real Lie algebra. Then $[e_\beta, \pi]^\sharp \mathfrak{g}_\mathbb{R}^*$ and $[f_\beta, \pi]^\sharp \mathfrak{g}_\mathbb{R}^*$*

- *are coisotropic subalgebras of $\mathfrak{g}_\mathbb{R}$*
- *their complexifications are coisotropic subalgebras of the complex Lie bialgebra $\mathfrak{g}$.*

Proof. The first statement follows from Prop. 5.2 and applying Thm. 4.3 to $\mathfrak{g}_\mathbb{R}$.

Now choose $\tilde{e}_\alpha \in \mathfrak{g}^\alpha$ and $\tilde{f}_\alpha \in \mathfrak{g}^{-\alpha}$ to be part of a Chevalley basis (Ch. 2.6 of [1]) of $\mathfrak{g}$, so that

$$\mathfrak{g}_0 := \{h \in \mathfrak{h} : \alpha(h) \in \mathbb{R} \text{ for all } \alpha \in R_+\} \oplus_{\alpha \in R_+} span_\mathbb{R}\{\tilde{e}_\alpha, \tilde{f}_\alpha\}$$

is a Lie subalgebra of $\mathfrak{g}_\mathbb{R}$, namely a split real form of $\mathfrak{g}$ ([10] p. 296). Since $\pi \in \wedge^2 \mathfrak{g}_0$ and $\tilde{e}_\beta \in \mathfrak{g}_0$, applying Thm. 4.3 to $\mathfrak{g}_0$ we deduce that $[\tilde{e}_\beta, \pi]^\sharp \mathfrak{g}_0^*$ is a coisotropic subalgebra of $\mathfrak{g}_0$. The complexification of $[\tilde{e}_\beta, \pi]^\sharp \mathfrak{g}_0^* = [\tilde{e}_\beta, \pi]^\sharp \mathfrak{g}_\mathbb{R}^*$ coincides with the complexification of $[e_\beta, \pi]^\sharp \mathfrak{g}_\mathbb{R}^*$, hence the second statement follows. □

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pairing (for $\mathfrak{g}$ a semi-simple Lie algebra) and endow it with a Poisson structure. However it seems that our Drinfeld double $\mathfrak{g} \oplus \mathfrak{g}^*$ is not isomorphic to Evens and Lu's $\mathfrak{g} \oplus \mathfrak{g}$.

Our main references for the computation of the examples below are [9](part III) and [15]. Two remarks about the derivation of the examples are in order.

Remark 5.4. 1) We use the fact that the Killing form $B(A_1, A_2)$ is a non-zero real multiple of $Tr(A_1A_2)$ (see Ex. 14.36 of [9]). Since the elements e_α and f_α we choose are always *real* matrices, the bivector π is also real, and the coisotropic subalgebras of $\mathfrak{g}_\mathbb{R}$ we obtain are also coisotropic subalgebras of $\mathfrak{g} \cap Mat(n, \mathbb{R})$, which agrees with the split real form of $\mathfrak{g}$.

2) The coisotropic subspace associated to f_β will be obtained just applying the transposition map to the one associated to e_β. Indeed in all the examples below the transposition map $\bullet^T$ is an anti-homomorphism of $\mathfrak{g}$ which switches the e_α's and the f_α's, so it maps π to $-\pi$ and $[e_\beta, \pi]$ to $[f_\beta, \pi]$.

Example 5.5 (A_n). Let $\mathfrak{g} = \mathfrak{sl}(n + 1, \mathbb{C})$ with Cartan subalgebra $\mathfrak{h}$ given by the diagonal matrices, so that as roots we obtain $R = \{L_i - L_j\}_{(i \neq j)} \subset \mathbb{R}^{n+1}$, where $L_1, \cdots, L_{n+1}$ denotes the standard basis of $\mathbb{R}^{n+1}$. It is easy to check that all roots satisfy assumption (11).

For a root $\alpha = L_i - L_j$ with $i < j$ we choose $e_\alpha := E_{ij} \in \mathfrak{g}^{L_i - L_j}$ and $f_\alpha := E_{ji} \in \mathfrak{g}^{-L_i + L_j}$, where E_{ij} denotes the matrix with 1 in the (i, j)-entry and zeros elsewhere. We have $\pi \sim \sum_{i<j} E_{ij} \wedge E_{ji}$, where "$\sim$" means "is a non-zero real multiple of". Fix a root $\beta = L_i - L_j$ with $i < j$. A computation shows that

$$[E_{ij}, \pi] \sim \Big(\sum_{i<k\leq j} + \sum_{i\leq k<j} \Big) E_{ik} \wedge E_{kj} = 2 \sum_{i<k<j} E_{ik} \wedge E_{kj} - E_{ij} \wedge (H_i - H_j),$$

where $H_i := E_{ii}$, so for all $i < j$ we obtain a coisotropic subalgebra of $\mathfrak{g}$ spanned by

$$\boxed{E_{ij}, \quad H_i - H_j, \quad \{E_{kj}\}_{i<k<j} \text{ and } \{E_{ik}\}_{i<k<j}}.$$

For instance, letting $n = 2$ and taking $e_\beta = E_{13}$ leads to the coisotropic subalgebra

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & -a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

The coisotropic subalgebra we obtain from $f_\beta = E_{ji}$ $(i < j)$ is spanned by

$$\boxed{E_{ji}, \quad H_i - H_j, \quad \{E_{ki}\}_{i<k<j} \text{ and } \{E_{jk}\}_{i<k<j}}.$$

All of the above are also coisotropic subalgebras of the split real form $\mathfrak{sl}(n + 1, \mathbb{R})$.

Example 5.6 (B_n). Let $\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})$, with Cartan subalgebra given by the diagonal matrices. Then $R = \{\pm L_i \pm L_j\}_{(i<j)} \cup \{\pm L_i\} \subset \mathbb{R}^n$. The roots that satisfy assumption (11) are exactly those of the form $\pm L_i \pm L_j$ $(i < j)$.

The root space of a root $L_i - L_j$ (with $i \neq j$) is spanned by $X_{ij} = E_{i,j} - E_{n+j,n+i}$. The root space of a root $L_i + L_j$ is spanned by $Y_{ij} = E_{i,j+n} - E_{j,n+i}$, the one of $-L_i - L_j$ is spanned by $Z_{ij} = E_{n+i,j} - E_{n+j,i}$. Finally, the root space of L_i is spanned by $U_i = E_{i,2n+1} - E_{2n+1,n+i}$ and the one of $-L_i$ is spanned by $V_i = E_{n+i,2n+1} - E_{2n+1,i}$. As earlier, E_{ij} denotes the matrix with 1 in the (i, j)-entry and zeros elsewhere. The r-matrix of eq. (10) satisfies

$$\pi \sim \frac{1}{2} \Big(\sum_{i<j} X_{ij} \wedge X_{ji} - \sum_{i<j} Y_{ij} \wedge Z_{ij} - \sum_i U_i \wedge V_i \Big).$$

Given a root $\beta = L_i - L_j$ (with $i < j$), a lengthy but straightforward computation shows

$$[X_{ij}, \pi] \sim -2 \sum_{i<k<j} \big(X_{ik} \wedge X_{kj} \big) + X_{ij} \wedge (H_i - H_j).$$

So for all $i < j$ we obtain a coisotropic subalgebra spanned by

$$\boxed{\{X_{ik}, X_{kj}\}_{(i<k<j)}, \quad X_{ij}, \quad H_i - H_j}$$

where $H_i := E_{i,i} - E_{n+i,n+i} \in \mathfrak{h}$. The negative root vector $f_\beta = X_{ji}$ delivers the coisotropic subalgebra spanned by

$$\boxed{\{X_{ki}, X_{jk}\}_{(i<k<j)}, \quad X_{ji}, \quad H_i - H_j}.$$

If instead we pick a root $\beta = L_i + L_j$ (with $i < j$) we obtain

$$[Y_{ij}, \pi] = -2 \sum_{i<k\neq j} (X_{ik} \wedge Y_{kj}) + 2 \sum_{j<k} (X_{jk} \wedge Y_{ki}) + Y_{ij} \wedge (H_i - H_j) + 2U_i \wedge U_j,$$

giving rise to a coisotropic subalgebra spanned by

$$\boxed{\{X_{ik}, Y_{kj}\}_{(i<k\neq j)}, \quad \{X_{jk}, Y_{ki}\}_{(j<k)}, \quad Y_{ij}, \quad H_i - H_j, \quad U_i, \quad U_j}.$$

The root $-(L_i + L_j)$ (with $i < j$) delivers the coisotropic subalgebra spanned by

$$\boxed{\{X_{ki}, Z_{kj}\}_{(i<k\neq j)}, \quad \{X_{kj}, Z_{ki}\}_{(j<k)}, \quad Z_{ij}, \quad H_i - H_j, \quad V_i, \quad V_j}.$$

All of the above are also coisotropic subalgebras of the split real form $\mathfrak{so}(n+1, n)$.

Example 5.7 (C_n). Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. Then, choosing the diagonal matrices as Cartan subalgebra, $R = \{\pm L_i \pm L_j\} \subset \mathbb{R}^n$. The only roots that satisfy assumption (11) are those of the form $\pm 2L_i$.

For $i \neq j$ the root space of a root $L_i - L_j$ is spanned by $X_{ij} = E_{i,j} - E_{n+j,n+i}$, as in Ex. 5.6; the root space of a root $L_i + L_j$ is spanned by $Y_{ij} = E_{i,n+j} + E_{j,n+i}$, the one of $-L_i - L_j$ is spanned by $Z_{ij} = E_{n+i,j} + E_{n+j,i}$. Finally, the root space of $2L_i$ is spanned by $U_i = E_{i,n+i}$ and the one of $-2L_i$ is spanned by $V_i = E_{n+i,i}$. We obtain the r-matrix

$$\pi \sim \frac{1}{2} \sum_{i<j} X_{ij} \wedge X_{ji} + \frac{1}{2} \sum_{i<j} Y_{ij} \wedge Z_{ij} + \sum_i U_i \wedge V_i.$$

Let us consider the root $2L_i$. A computation shows

$$[U_i, \pi] \sim \sum_{i<k} (Y_{ik} \wedge X_{ik}) + U_i \wedge H_i,$$

where $H_i := E_{ii} - E_{n+i,n+i}$, so as coisotropic subspace we obtain the span of

$$\boxed{\{Y_{ik}, X_{ik}\}_{i<k}, \quad U_i, \quad H_i}.$$

For instance, when $n = 2$, taking $e_\beta = U_2 = E_{24}$ and $e_\beta = U_1 = E_{13}$ we obtain the coisotropic subalgebras of $\mathfrak{sp}(4, \mathbb{C})$

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} : a, b \in \mathbb{R} \right\} \text{ and } \left\{ \begin{pmatrix} a & c & b & d \\ 0 & 0 & d & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & -c & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

For the root $-2L_i$, whose root space is spanned by V_i, as coisotropic subspace we obtain the span of

$$\boxed{\{Z_{ik}, X_{ki}\}_{i<k}, \quad V_i, \quad H_i}.$$

All of the above are also coisotropic subalgebras of the split real form $\mathfrak{sp}(2n, \mathbb{R})$.

Example 5.8 (D_n). Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$. Then $R = \{\pm L_i \pm L_j\}_{\{i<j\}} \subset \mathbb{R}^n$, and the same computation as in Ex. 5.6 shows that all roots satisfy assumption (11). The root spaces of

$L_i - L_j, L_i + L_j$ and $-L_i - L_j$ are spanned by elements X_{ij}, Y_{ij} and Z_{ij} defined by the same formulae as in Ex. 5.6, and the r-matrix of eq. (10) satisfies

$$\pi \sim \frac{1}{2}\Big(\sum_{i<j} X_{ij} \wedge X_{ji} - \sum_{i<j} Y_{ij} \wedge Z_{ji}\Big)$$

(it consists of the first two summands of the r-matrix for the B_n case).

The same computations as in Ex. 5.6 show that (with $i < j$) from the root $L_i - L_j$ we obtain the coisotropic subalgebras spanned by

$$\boxed{\{X_{ik}, X_{kj}\}_{(i<k<j)}, \quad X_{ij}, \quad H_i - H_j}$$

and

$$\boxed{\{X_{ki}, X_{jk}\}_{(i<k<j)}, \quad X_{ji}, \quad H_i - H_j},$$

whereas from the root $L_i + L_j$ we obtain the coisotropic subalgebras spanned by

$$\boxed{\{X_{ik}, Y_{kj}\}_{(i<k\neq j)}, \quad \{X_{jk}, Y_{ki}\}_{(j<k)}, \quad Y_{ij}, \quad H_i - H_j}$$

and

$$\boxed{\{X_{ki}, Z_{kj}\}_{(i<k\neq j)}, \quad \{X_{kj}, Z_{ki}\}_{(j<k)}, \quad Z_{ij}, \quad H_i - H_j}.$$

(Here $H_i := E_{i,i} - E_{n+i,n+i}$). All of the above are also coisotropic subalgebras of the real form $\mathfrak{so}(n, n)$.

Remark 5.9. In Example 5.5, taking $n = 2$ and $g = exp(E_{13})$, we showed that $\mathfrak{h}^g = span_{\mathbb{R}}\{E_{12}, E_{13}, E_{23}, H_1 - H_3\}$ is a coisotropic subalgebra of $\mathfrak{sl}(3, \mathbb{R})$. In particular its annihilator $(\mathfrak{h}^g)^\circ$ is a Lie subalgebra, but it is *not* a Lie ideal. Indeed, taking the basis of $\mathfrak{sl}(3, \mathbb{R})$ given by $\{E_{ij}\}_{(i\neq j)}$, $H_1 - H_2$, $H_1 - H_3$ and considering the dual basis, we have $(H_1 - H_2)^* \in (\mathfrak{h}^g)^\circ$ but $\langle [(E_{12})^*, (H_1 - H_2)^*], E_{12}\rangle \neq 0$.

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Universidade do Porto, Departamentos de Matematica Pura, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

E-mail address: `mzambon@fc.up.pt`