Direct perturbations of aggregate excess demand

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Abstract

We establish that an exchange economy, i.e., preferences and endowments, that generates a given aggregate excess demand (AED) function is close to the economy generating the AED obtained by an arbitrary perturbation of the original one.

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1 Introduction

Genericity results — such as establishing that the set of equilibrium prices constitutes a manifold of a certain dimension or that the number of regular equilibria is finite and stable — are obtained by perturbation techniques, where the underlying primitives (e.g., preferences and endowments) are subjected to infinitesimal changes.

Debreu's [3] proof of local isolation of regular economies, for example, requires a perturbation of endowments. Extending this result to critical economies requires additional effort. Mas-Colell [10, Proposition 8.8.3], who shows that for a one-dimensional parametrization of economies, a "flat" aggregate excess demand (AED) is not generic in the first agent's utility, must resort not only to a perturbation of endowments, but also to a *quadratic* perturbation of (indirect) utility.

Part of the challenge, of course, is to show that perturbations of AED functions correspond to "legal" perturbations of preferences, i.e, perturbed utility functions must continue to satisfy the canonical properties of utility functions.

Furthermore, the relation between AED and *deeper* economic primitives such as preferences and endowments requires a clear statement. For instance, Allen [1] establishes finiteness for *multi*-dimensional parametrization of economies by employing a theorem by Tougeron [13], according to which local finiteness of the number of pre-images is a generic property of smooth functions. Notably, Allen uses AED functions as primitives rather than the agents' underlying preferences and endowments. Mas-Colell and Nachbar [11] obtain a comparable result (albeit finiteness of *critical* equilibria only), but chose to work with preferences and endowments as their economic primitives.

In this paper, we show that the two approaches to obtaining genericity results are equivalent in the following sense: we establish that small variations in deep primitives of an economy, i.e., preferences and endowments, give rise to small variations in the AED function representing the economy. Conversely, we also establish that an economy, defined by its deep primitives, that generates a given AED function can be taken to be close to the economy that generates the AED obtained by an arbitrary perturbation of the initial AED.[1] Indeed, we prove that preferences are related to AEDs by

[1] Lehmann-Waffenschmidt [9] obtains a result in the same spirit for the family of one-parametrized exchange economies with an equal number of goods and agents.

a continuous map. This map is open provided preferences are smooth. This restriction is not a serious one, however, as continuous preferences can be approximated by smooth ones.

Our results open wider the door to further genericity and determinacy research based on transversality arguments since results obtained by direct perturbation of AED are as strong as results obtained through the often much more tedious exercise of perturbing preferences and endowments. Castro and Dakhlia [2], for instance make full use of this result to establish that *generically in the space of preferences and endowments*, AED is Thom-Boardman stratified, a result that requires high-order perturbations that would have been more difficult to obtain by conventional means. The stratification result, in turn, provides an alternative proof of generic finiteness of all equilibria, including critical ones.

This paper is organized as follows: in the next section we introduce notation and some preliminary results to be used throughout the paper. In section 3 we establish results that allow us to use preferences and utilities interchangeably. Section 4 contains our main theorem concerning how perturbations of an AED function are related to perturbations of the underlying economy, and vice-versa. As a corollary, we prove that, provided the utility describing the preferences is at least C^2, perturbations of AED and of the corresponding economy are equivalent. The proof of the theorem relies on several lemmas, which finish the section. In the subsequent section we provide a simple proof, using convolution with a smooth kernel, that continuous utility functions may be approximated by smooth ones. The final section concludes.

2 Preliminary results and notation

Consider an economy with L commodities ($\ell = 1, \ldots, L$) and I agents ($i = 1, \ldots, I$). Let Ω be the non-negative orthant of $\mathbb{R}^L$ and let each agent i be defined by her endowment $\omega^i \in \Omega$ and her preferences $\succsim_i$, a complete order on Ω with the following "rationality" properties:

(P1) completeness, reflexivity, and transitivity.

If $x \succsim_i y$ and $y \succsim_i x$, then x is indifferent to y and we write $x \sim_i y$. If $x \succsim_i y$ but not $x \sim_i y$, then x is strictly preferred to y and we write $x \succ_i y$. We call the partial preference order $\succ_i$ continuous if it satisfies:

(P2) continuity ($\{x : y \succ_i x\}$ and $\{y : y \succ_i x\}$ are open).

In addition, we shall assume non-satiation and strict convexity:

(P3) non-satiation ($x \geq y$ ($x_\ell \geq y_\ell$, $\forall \ell = 1, \ldots, L$) and $x \neq y \Rightarrow x \succ_i y$);

(P4) strict convexity ($x \sim_i y$ and $x \neq y \Rightarrow \forall \alpha \in (0,1)$, $\alpha x + (1-\alpha) y \succ_i x$).

Let Ξ denote the space of all such preference orders. Following Kannai [7], every $\succ_i \in \Xi$ may be represented by a unique continuous utility function $u_i : \Omega \to \mathbb{R}$ defined as follows: for any $x \in \Omega$, there exists a unique $\hat{x}$ in the principal diagonal of Ω such that $x \sim_i \hat{x}$ (i.e., agent i is indifferent between x and $\hat{x}$). Then, let $u_i(x) \equiv \|\hat{x}\|$, where $\|.\|$ is the Euclidean norm. Denote by C^* the class of utility functions thus defined. We then have a bijective correspondence between Ξ and C^*.

Let $u_1, u_2 \in C^*$ represent the preferences $\succ_1, \succ_2 \in \Xi$ of agents 1 and 2. The metric

$$\rho(\succ_1, \succ_2) = \max_{x \in \Omega} \frac{|u_1(x) - u_2(x)|}{1 + \|x\|^2} \tag{1}$$

induces a topology on Ξ which, as shown by Kannai [7], is natural in the sense that it is minimal with the property that $A \equiv \{(x, y, \succ) : x \succ y\}$ is open in $\Omega \times \Omega \times \Xi$.

The space of utilities C^* also has a natural topology as a subspace of $C^0(\Omega, \mathbb{R})$, the space of continuous functions $u \colon \Omega \to \mathbb{R}$. Indeed, it can be endowed with the subspace topology induced from the *compact–open topology* on $C^0(\Omega, \mathbb{R})$: a basis of open sets for this topology is given by

$$V(u_0; K, \varepsilon) = \{u \ : \ \max_{x \in K} |u(x) - u_0(x)| < \varepsilon\}$$

where $u_0 \in C^0(\Omega, \mathbb{R})$, $K \subset \Omega$ is compact and $\varepsilon > 0$. Thus u_1 is close to u_2 if the Euclidean distance $|u_1(x) - u_2(x)|$ is bounded by a small number for all x in any compact set. In section 3 below we show that the natural topologies on Ξ and C^* correspond under the identification of these two spaces.

Denote by $C^k(\Omega, \mathbb{R})$ the space of k times continuously differentiable functions $u \colon \Omega \to \mathbb{R}$, endowed with the compact–open topology. In particular, $C^0(\Omega, \mathbb{R})$ is the space of continuous functions. We denote by $C^k \subset C^k(\Omega, \mathbb{R})$ the subspace of utilities, i.e., the functions $u \colon \Omega \to \mathbb{R}$ of class C^k such that the associated partial preference order $\succ$ on Ω satisfies (P1)–(P4). This happens if and only if u is strictly increasing in each coordinate and strictly quasi-concave. The latter condition is equivalent to the level hypersurfaces of u being strictly convex. Note that $C^k = C^0 \cap C^k(\Omega, \mathbb{R})$ and that $C^* \subset C^0$. Correspondingly, we let

$$\Xi^k = \{\succ \in \Xi \ : \ u \in C^k\}$$

be the set of preferences that may be represented by utilities in C^k. We shall define a C^k *economy* as one defined by preferences $\succ \in \Xi^k$.

Finally, we define a *perturbation* of a point x_0 in a topological space X as a point $x \in X$ which is contained in an arbitrarily small open neighborhood of the original point x_0. Thus, a map between topological spaces $f \colon X \to Y$ is continuous if and only if, for a perturbation x of any point $x_0 \in X$, the image $f(x)$ is a perturbation of $f(x_0)$. Equivalently, this means that the preimage $f^{-1}(V) \subset X$ is open for any open set $V \subset Y$. Conversely, if for any perturbation y of $f(x_0)$, there is a perturbation x of x_0 such that $f(x) = y$, then this is equivalent to saying that the map f is *open*, i.e., $f(V) \subset Y$ is open for any open set $V \subset X$.

3 Preferences and utilities

The purpose of this section is to show that a perturbation of preferences corresponds to a perturbation of utility, and vice-versa.

As explained in the previous section, any preference ordering in Ξ can be represented by exactly one utility function in C^*. Give Ξ the topology induced by the metric ρ defined in (1) and C^* the subspace topology induced from the compact–open topology on $C^0(\Omega, \mathbb{R})$.

Proposition 3.1. *The bijective correspondence between the spaces Ξ and C^* is a homeomorphism.*

Proof. Immediate from Lemmas 3.2 and 3.3. □

In light of this proposition, we can henceforth interchangeably work with preference and C^*-utility perturbations, depending on which is more convenient.

Lemma 3.2. *Consider preferences $\succ_0 \in \Xi$ represented by a utility function $u_0 \in C^*$. Let $u \in C^*$, representing preferences $\succ$, be a perturbation of u_0. Then $\succ \in \Xi$ is a perturbation of $\succ_0$. In other words, the correspondence $C^* \to \Xi$ is continuous.*

Proof. There exists a constant C, depending only on $L = \dim \Omega$, such that for any $u \in C^*$, one has $0 \leq u(x) < C\|x\|$ (see Kannai [7, p. 798]). We have

$$\forall\, x \in \Omega, \frac{|u(x) - u_0(x)|}{1 + \|x\|^2} \leq \frac{|u(x)| + |u_0(x)|}{1 + \|x\|^2} \leq \frac{2C\|x\|}{1 + \|x\|^2},$$

which converges to zero as $\|x\|$ approaches infinity, that is,

$$\forall\, \varepsilon > 0, \quad \exists R > 0 : \quad \|x\| > R \Rightarrow \frac{2C\|x\|}{1 + \|x\|^2} < \varepsilon. \tag{2}$$

Given $\varepsilon > 0$, take K to be the compact set $\overline{B(0,R)}$, the closure of the ball of radius R about the origin. Then, for any u such that $\max_{x \in K} |u(x) - u_0(x)| < \varepsilon$, we have by (2) that

$$\rho(\succ, \succ_0) = \max_{x \in \Omega} \frac{|u(x) - u_0(x)|}{1 + \|x\|^2} < \varepsilon.$$

□

Lemma 3.3. *Consider preferences $\succ_0 \in \Xi$ represented by a utility function $u_0 \in C^*$. Let $\succ \in \Xi$, represented by a utility $u \in C^*$, be a perturbation of $\succ_0$. Then u is a perturbation of u_0. In other words, the correspondence $\Xi \to C^*$ is continuous.*

Proof. We need to establish that for any compact subset $K \subset \Omega$,

$$\forall\, \varepsilon > 0, \quad \exists \delta > 0 : \quad \rho(\succ, \succ_0) < \delta \;\Rightarrow\; \max_{x \in K} |u(x) - u_0(x)| < \varepsilon.$$

From Kannai's [7, equation 3.2, p. 799], we have

$$\{\succ : \; \rho(\succ, \succ_0) < \delta\} = \left\{ \succ : \max_{x \leq R} \frac{|u(x) - u_0(x)|}{1 + \|x\|^2} < \delta \right\}.$$

Since K is compact, we can choose R such that $K \subset B(0,R)$ so that $\|x\|^2 \leq R^2$. Then

$$|u(x) - u_0(x)| < \delta(1 + \|x\|^2) \leq \delta(1 + R^2).$$

In order to have $|u(x) - u_0(x)| < \varepsilon$, it suffices to choose δ such that $\delta < \varepsilon/(1 + R^2)$. □

Given a utility u in C^0, there is a unique utility u^* in C^* representing the same preferences as u. This defines a map

$$\pi\colon C^0 \to C^*.$$

Let $\Delta \subset \Omega$ be the principal diagonal; we shall identify $\mathbb{R}_+$ with Δ via $t \mapsto t \cdot \frac{1}{\sqrt{L}}(1, \ldots, 1)$. Write $u_{|\Delta}(t)$ for the restriction of u to Δ, which is of course a strictly increasing function. Since, in this notation, u^* is defined by the property $u^*_{|\Delta}(t) = t = \|t \cdot \frac{1}{\sqrt{L}}(1, \ldots, 1)\|$, we have that

$$u^*(x) = u_{|\Delta}^{-1} \circ u(x),$$

where $u_{|\Delta}^{-1}$ is the inverse function of the restriction of u to Δ. Note that this shows that $u^* \in C^k$, whenever $u \in C^k$. Therefore we have the following.

Proposition 3.4. *For any $k \geq 0$, the space of preferences Ξ^k which can be represented by utilities in C^k coincides with the space of preferences which can be represented by utilities in $C^* \cap C^k$.*

4 Perturbations of aggregate excess demand

We now turn to the excess demand of an agent endowed with $\omega^i \in \Omega$. Agent i solves Utility Maximization Problem (UMP)

$$\max_{x^i \in \Omega} u_i(x^i) \text{ such that } p \cdot x^i \leq p \cdot \omega^i. \tag{3}$$

Strict convexity of preferences ensures that the solution, $x^i(p, p \cdot \omega^i)$, is unique and a continuous function of both price vector p and endowment ω^i. Non-satiation guarantees that the budget constraint is binding and can thus be written as

$$p \cdot x^i = p \cdot \omega^i \Leftrightarrow p \cdot (x^i - \omega^i) = 0. \tag{4}$$

Geometrically, the constraint is the hyperplane through ω^i orthogonal to p,

$$H(p, \omega) = \{x \in \Omega : \ p \cdot (x - \omega^i) = 0\},$$

while the solution to the UMP corresponds to the point of tangency between the level curves of u_i and the hyperplane. For smooth level curves, the point of tangency is located where ∇u_i, the gradient of u_i, is parallel to p.

The excess demand for agent i is defined as

$$z^i(p) = x^i(p, p \cdot \omega^i) - \omega^i,$$

and the AED for the economy is given by $z(p) = \sum_{i=1}^{n} z^i(p)$.

The main result of this section, Theorem 4.1, establishes that a perturbation of AED is equivalent to a perturbation of economic primitives, that is, preferences and endowments. Note that, from the definition of AED, it suffices to consider the perturbation of the excess demand of a single agent. We will thus drop the agent-specific superscript, when no confusion is possible.

In more formal terms, we are considering the following: for a fixed price vector p, demand for a given agent (say, the first) is a map

$$\begin{aligned} \xi \colon \Xi \times \Omega &\to \Omega \\ (\succ, \omega) &\mapsto \xi(\succ, \omega) = x(p). \end{aligned}$$

Analogously, excess demand is a map

$$\begin{aligned} \zeta \colon \Xi \times \Omega &\to H(p, 0) \\ (\succ, \omega) &\mapsto \zeta(\succ, \omega) = z(p), \end{aligned}$$

where $H(p, 0)$ is the hyperplane through the origin perpendicular to p. We can also consider demand and excess demand as maps defined on any of the spaces of utilities C^* or C^k and we shall denote these maps by the same letters ξ and ζ, taking care to make the domain clear in each case. Furthermore, for a fixed endowment ω, we can think of ξ and ζ as functions of just the utility u. Thus, for example, we shall use the notation

$$\begin{aligned} \xi \colon C^* &\to H(p, \omega) \\ u &\mapsto \xi(u), \end{aligned}$$

where $\xi(u)$ is the demand $x(p)$ that solves the UMP defined by the utility $u \in C^*$, and fixed endowment ω and price vector p.

We can now state our main result.

Theorem 4.1. *Let $z_0(p)$ be the AED for an economy with L goods and I agents characterized by preferences $\succ_0^i$ satisfying (P1)–(P4) and endowments ω_0^i, $i = 1, \ldots, L$.*

(I) *Any perturbation of preferences $\succ_0^i$ and endowments ω_0^i gives rise to a perturbation of the AED $z_0(p)$. In other words, the map $\zeta\colon \Xi \times \Omega \to H(p,0)$ is continuous.*

(II) *Conversely, if the preferences of the first agent can be represented by a twice continuously differentiable utility, then any perturbation of the AED $z_0(p)$ arises from a perturbation of preferences $\succ_0^1$ and endowment ω_0^1. In other words, the map $\zeta\colon \Xi^2 \times \Omega \to H(p,0)$ is open.*

Proof. It is immediate from the definition of AED that it suffices to prove the analogous statements for the demand function $\xi\colon (\succ, \omega) \mapsto x(p)$ of a single agent. The remainder of this section is devoted to this task.

As already observed, demand is continuous in ω and p. From Proposition 3.1 we know that Ξ and C^* are homeomorphic. Thus, to establish (I), it suffices to show that demand is a continuous function of utilities in C^*. Since $C^* \subset C^0$ is a subspace of the space of all continuous utilities, this follows from Lemma 4.3.

To prove that demand is an open map, i.e., that sufficiently small perturbations of demand arise from perturbations of preferences and endowments, we first consider preferences represented by twice continuously differentiable utilities (not necessarily in C^*). This allows us to prove Theorem 4.4, which establishes openness of demand as a function of utilities in C^2.

Finally, to complete the argument for Ξ^2 (identified with $C^* \cap C^2$ by Proposition 3.4), we shall use the constructions and notation of the proof of Theorem 4.4. Let $\bar{u}^* = \pi_{|C^2}(\bar{u})$ be the $C^* \cap C^2$ utility defining the same preferences as $\bar{u}$. Then $\bar{u}^*(x) = \|\hat{x}\|$, where $\hat{x}$ is the unique element of the principal diagonal of Ω with $\bar{u}(x) = \bar{u}(\hat{x})$. Since $u \in C^*$, we have $\bar{u}^*(x) = \|\hat{x}\| = u(\hat{x})$. Therefore, for all $x \in \Omega$ and all $\varepsilon > 0$,

$$\begin{aligned}
|\bar{u}^*(x) - u(x)| &\le |\bar{u}^*(x) - \bar{u}(x)| + |\bar{u}(x) - u(x)| \\
&= |u(\hat{x}) - \bar{u}(\hat{x})| + |\bar{u}(x) - u(x)| \\
&< \varepsilon + \varepsilon = 2\varepsilon,
\end{aligned}$$

where the last inequality comes from (9). □

We note that part (II) is more restrictive than part (I) since it requires twice differentiability, whereas part (I) needs only continuity in order to hold.

An immediate consequence is the following:

Corollary 4.2. *Let $z_0(p)$ be the AED for a C^2 economy with L goods and I agents characterized by C^2-preferences $\succ_0^i$ satisfying (P1)–(P4) and endowments ω_0^i, $i = 1, \ldots, L$. An AED $z(p)$ is a perturbation of $z_0(p)$ if and only if $z(p)$ is the AED for an economy with L goods and I agents such that the new preferences $\succ^1$ of the first agent are perturbations of $\succ_0^1$ and the new endowments ω^1 are perturbations of ω_0^1.*

Thus, in the context of smooth (at least C^2) pure exchange economies, it is equivalent to perturb AED or the underlying economy.

The remainder of this section is devoted to the statement and proof of all the results needed in the proof of Theorem 4.1.

Lemma 4.3. *Let x be the unique solution to the UMP defined by a utility u and endowment ω. Then x is a continuous function of u, i.e., $\xi\colon C^0 \to \Omega$ is continuous.*

Proof. We proceed by contradiction. Let u_n be a sequence of utility functions such that as $n \to +\infty$, we have $u_n \to u$. Let $x_n = x_n(p)$ be the unique maximum of u_n on $H(p, \omega)$. Assume that the sequence x_n does not converge to x. By compactness of $H(p, \omega)$, we may assume that $x_n \to x^* \neq x$, passing to a subsequence if necessary.

Define $\delta = |u(x) - u(x^*)|$. Then $\delta > 0$ because x is the unique maximum of u on $H(p, \omega)$.

Since u_n converges uniformly to u on the compact set $H(p, \omega)$, the sequence of maxima $u_n(x_n)$ converges to the maximum $u(x)$ of u. Thus

$$\exists\, N_1 \in \mathbb{N} : \quad n \geq N_1 \Rightarrow |u(x) - u_n(x_n)| < \frac{\delta}{3}. \tag{5}$$

Continuity of u in conjunction with the hypothesis that $x_n \to x^*$ implies that

$$\exists\, N_2 \in \mathbb{N} : \quad n \geq N_2 \Rightarrow |u(x_n) - u(x^*)| < \frac{\delta}{3}. \tag{6}$$

Finally, again by uniform convergence, we have that

$$\exists\, N_3 \in \mathbb{N} : \quad n \geq N_3 \Rightarrow |u(y) - u_n(y)| < \frac{\delta}{3} \ \ \forall y \in H(p, \omega)$$

and so, taking $y = x_n$,

$$\exists\, N_3 \in \mathbb{N} : \quad n \geq N_3 \Rightarrow |u_n(x_n) - u(x_n)| < \frac{\delta}{3}. \tag{7}$$

Combining (5), (6) and (7), we obtain for $n \geq \max\{N_1, N_2, N_3\}$ that

$$\begin{aligned}\delta &= |u(x) - u(x^*)| \\ &\qquad \leq |u(x) - u_n(x_n)| + |u_n(x_n) - u(x_n)| + |u(x_n) - u(x^*)| < \delta,\end{aligned}$$

which is a contradiction. Hence $x^* = x$, proving that demand is a continuous function of utility. $\square$

Theorem 4.4. *Consider an agent with endowment $\omega \in \Omega$ and preferences satisfying (P1)–(P4) that are represented by a C^2 utility function u (not necessarily in C^*). Let demand $x_0(p)$ be the unique solution to the UMP (3). Then a perturbation $\bar{x}(p)$ of $x_0(p)$ is the unique solution of the UMP defined by a perturbation $(\bar{u}, \bar{\omega})$ of (u, ω). In other words, $\xi\colon C^2 \times \Omega \to \Omega$ is open.*

Proof. We shall show that a perturbation of the solution must originate in a perturbation of utility and endowments. In other words, given $\varepsilon > 0$ and a compact subset $K \subset \Omega$, we must find a $\delta > 0$ such that

$$\begin{aligned}&\|\bar{x} - x_0\| < \delta \\ &\qquad \Longrightarrow \exists(\bar{u}, \bar{\omega}) \ : \ \max_{x \in K} |\bar{u}(x) - u(x)| + \|\bar{\omega} - \omega\| < \varepsilon \quad \text{and} \quad \xi(\bar{u}, \bar{\omega}) = \bar{x}.\end{aligned}$$

At the solution $x_0(p)$, the budget constraint (4) is tangent to the indifference curve containing x_0, which means that the gradient of u at x_0, $\nabla u(x_0)$, is orthogonal to $p \cdot (x - \omega) = 0$. As such, $\nabla u(x_0)$ is parallel to p.

Let $\bar{x}(p)$ be a perturbation of $x_0(p)$, say $\|\bar{x} - x_0\| < \delta$ for small $\delta > 0$. If $\bar{x}$ does not satisfy the budget constraint, we perturb ω, in the direction of p, to $\bar{\omega}$ so that $p \cdot (\bar{x} - \bar{\omega}) = 0$. In addition, since p remains unchanged, $\bar{\omega}$ is such that the hyperplane described by $p \cdot (x - \bar{\omega}) = 0$ is parallel to the original one. Note that $\|\bar{\omega} - \omega\| \leq \|x_0 - \bar{x}\|$. Denote the solution to the UMP defined by u and the new restriction by $x_0'(p)$. We have

$$\|x_0' - \bar{x}\| \leq \|x_0' - x_0\| + \|x_0 - \bar{x}\|,$$

where $\|x_0 - \bar{x}\| < \delta$ and $\|x_0' - x_0\|$ can be made small by uniform continuity of the demand with ω in the compact set K, since $\|\bar{\omega} - \omega\| \leq \|x_0 - \bar{x}\| < \delta$. Hence, $\bar{x}$ is also a perturbation of x_0' and we can henceforth suppose that the demand for the unperturbed problem and $\bar{x}$ belong to the same budget constraint, dropping the use of the prime.

It remains to show that we can perturb the utility so that $\bar{x}(p)$ is the solution to the UMP.

Fix a $\delta_1 > 0$ such that the closed ball $\overline{B(x_0, 2\delta_1)}$ of radius $2\delta_1$ centered at x_0 is contained in Ω. For a given $\bar{x}$ such that $\|\bar{x}-x_0\| < \delta_1$, define $\Phi : \ \Omega \to \Omega$ by

$$\Phi(x) = x - \varphi(\|x - x_0\|)(\bar{x} - x_0),$$

where $\varphi : \ \mathbb{R} \to \mathbb{R}$ is constant and equal to 1 for $0 \le t \le \delta_1$, $\varphi(t)$ is constant and equal to zero for $t \ge 2\delta_1$ and $\varphi(t)$ is smooth and decreasing for $\delta_1 \le t \le 2\delta_1$. (See Figure 1.)

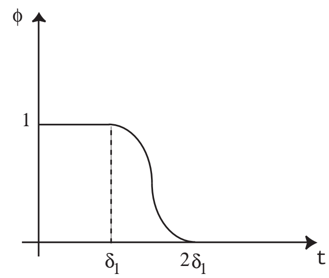

Figure 1: Graph of a function satisfying the conditions imposed on φ.

Note that there are constants A and B such that

$$|\varphi'(t)| \le \frac{A}{\delta_1} \text{ and } |\varphi''(t)| \le \frac{B}{\delta_1^2} \tag{8}$$

for some constants A and B. Clearly, Φ is the identity for $\|x - x_0\| \ge 2\delta_1$ and a translation by $\bar{x} - x_0$ for $\|x - x_0\| \le \delta_1$.

Next, define

$$\bar{u}(x) = u(\Phi(x)) = u(x - \varphi(\|x - x_0\|)(\bar{x} - x_0)).$$

We shall show that, for $\|\bar{x} - x_0\|$ sufficiently small, the function $\bar{u}$ is a utility in C^2 with associated demand $\bar{x}$.

We have

$$\begin{aligned} \bar{u}(\bar{x}) = u(\Phi(\bar{x})) &= u(\bar{x} - \varphi(\|\bar{x} - x_0\|)(\bar{x} - x_0)) \\ &= u(x_0) \end{aligned}$$

and $\nabla\bar{u}(\bar{x}) = \nabla u(x_0)$ (see Lemma 4.5 below). The level hypersurface of $\bar{u}$ at $\bar{x}$ is thus tangent to $p \cdot (x - \omega) = 0$ and hence, $\bar{x}$ is a solution to the UMP defined by $\bar{u}$.

Since u is uniformly continuous on the compact set

$$\overline{B(x_0, 2\delta_1)} \cup \Phi(\overline{B(x_0, 2\delta_1)}),$$

there is a $\delta > 0$ such that, for all x and x' in this set,

$$\|x' - x\| < \delta \implies |u(x') - u(x)| < \varepsilon.$$

On the other hand, it is clear from the definition of Φ that $\|\Phi(x) - x\| \leq \|\bar{x} - x_0\|$ for all x. Hence, for $\bar{x}$ such that $\|\bar{x} - x_0\| < \delta$, we have

$$|\bar{u}(x) - u(x)| = |u(\Phi(x)) - u(x)| < \varepsilon \; \forall x \in \Omega. \tag{9}$$

Lemmas 4.6 and 4.7 below show that, by further decreasing δ if necessary, we can guarantee that $\bar{u}$ is increasing in x and that its indifference curves are strictly convex, thus ensuring that the underlying preferences satisfy (P1)–(P4), thereby completing the proof. □

Lemma 4.5. *Let $\delta < \delta_1$ and $\|\bar{x} - x_0\| < \delta$. Let u and $\bar{u}$ be as defined in the proof of Theorem 4.4. Then we have $\nabla u(x_0) = \nabla \bar{u}(\bar{x})$.*

Proof. We use the definition of $\bar{u}$ to calculate partial derivatives and obtain

$$\frac{\partial \bar{u}}{\partial x_j}(x) = \sum_{i=1}^{L} \frac{\partial u}{\partial x_i}(\Phi(x)) \frac{\partial \Phi_i}{\partial x_j}(x)$$

and

$$\frac{\partial \Phi_i}{\partial x_j}(x) = \delta_{ij} - \varphi'(\|x - x_0\|)(\bar{x}_i - x_{0i}) \frac{\partial \|x - x_0\|}{\partial x_j},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. At $x = \bar{x}$, $\varphi'(\|\bar{x} - x_0\|) = 0$ because $\varphi(\|\bar{x} - x_0\|) = 1$ for $0 \leq \|\bar{x} - x_0\| \leq \delta_1$. Hence,

$$\frac{\partial \bar{u}}{\partial x_j}(\bar{x}) = \frac{\partial \bar{u}}{\partial x_j}(\Phi(\bar{x})) = \frac{\partial \bar{u}}{\partial x_j}(x_0).$$

□

Lemma 4.6. *There is a $\delta > 0$ such that the following holds. For $\|\bar{x} - x_0\| < \delta$, let $\bar{u}$ be as defined in the proof of Theorem 4.4. Then $\bar{u}$ is increasing in each of its arguments.*

Proof. From the proof of Lemma 4.5, we know that

$$\frac{\partial \bar{u}}{\partial x_j}(x) = \sum_{i=1}^{L} \frac{\partial u}{\partial x_i}(\Phi(x)) \frac{\partial \Phi_i}{\partial x_j}(x) \tag{10}$$

and

$$\begin{aligned} \frac{\partial \Phi_i}{\partial x_j}(x) &= \delta_{ij} - \varphi'(\|x - x_0\|)(\bar{x}_i - x_{0i}) \frac{\partial \|x - x_0\|}{\partial x_j} \\ &= \delta_{ij} - \varphi'(\|x - x_0\|)(\bar{x}_i - x_{0i}) \frac{x_j - x_{0j}}{\|x - x_0\|}. \end{aligned} \tag{11}$$

By taking $\|\bar{x} - x_0\| < \delta$, we have

$$\begin{aligned} \left| \varphi'(\|x - x_0\|)(\bar{x}_i - x_{0i}) \frac{x_j - x_{0j}}{\|x - x_0\|} \right| &\leq |\varphi'(\|x - x_0\|)(\bar{x}_i - x_{0i})| \\ &\leq \frac{A}{\delta_1} \delta. \end{aligned} \tag{12}$$

Again, if $\|x - x_0\| \leq \delta_1$ or $\|x - x_0\| \geq 2\delta_1$, we have

$$\frac{\partial \bar{u}}{\partial x_j}(x) = \frac{\partial u}{\partial x_j}(\Phi(x)).$$

Otherwise, x is in the compact annular region

$$\{x \in \Omega : \ \delta_1 \leq \|x - x_0\| \leq 2\delta_1\}$$

bounded by the discs of radius δ_1 and $2\delta_1$. In this compact set $\frac{\partial u}{\partial x_j}$ is bounded away from zero (as u is increasing in each of its arguments), and the absolute value of the partial derivatives $\frac{\partial u}{\partial x_i}$ is bounded for $i \neq j$. Hence, if the absolute value of $\frac{\partial \Phi_i}{\partial x_j}$ is sufficiently small for $i \neq j$ and close to one for $i = j$, then the jth term of the sum (10) dominates, and we obtain

$$\frac{\partial \bar{u}}{\partial x_j}(x) > 0.$$

for all x. It follows from (11) and (12) that this can be achieved by choosing $\delta > 0$ sufficiently small. $\square$

Lemma 4.7. *There is a $\delta > 0$ such that the following holds. For $\|\bar{x} - x_0\| < \delta$, the indifference curves of $\bar{u}$ as defined in the proof of Theorem 4.4 are convex.*

Proof. From Thorpe [12], we know that the normal curvature of a level hypersurface of a function u in the direction of a vector v ($\|v\| = 1$) perpendicular to $\nabla u(x)$ is given by

$$-\frac{1}{\|\nabla u(x)\|} < v, H_u(x)v >,$$

where $H_u(x)$ is the Hessian of u at x and $< v, H_u(x)v >$ represents the quadratic form defined by $H_u(x)$ (see [12, exercise 2.1] and Gladiali and Grossi [5, section 2]). Thus, the convexity of the level hypersurfaces of u is equivalent to $H_u(x)$ being positive definite for all x. We shall show that, for δ sufficiently small, $\|\bar{x} - x_0\| < \delta$ implies the entries of $H_{\bar{u}}(x)$ are close to the entries of $H_u(\Phi(x))$. Hence, $H_{\bar{u}}(x)$ is also positive definite and the level hypersurfaces of $\bar{u}$ are therefore convex.

By differentiating the first derivatives obtained in the proof of Lemma 4.5, we obtain

$$\begin{aligned}
\frac{\partial^2 \bar{u}(x)}{\partial x_k \partial x_j} &= \frac{\partial}{\partial x_k}\left[\sum_{i=1}^{L} \frac{\partial u}{\partial x_i}(\Phi(x)) \frac{\partial \Phi_i}{\partial x_j}(x)\right] = \\
&= \sum_{i,l} \frac{\partial^2 u}{\partial x_l \partial x_i}(\Phi(x)) \frac{\partial \Phi_l(x)}{\partial x_k} \frac{\partial \Phi_i(x)}{\partial x_j} + \sum_i \frac{\partial u}{\partial x_i}(\Phi(x)) \frac{\partial^2 \Phi_i(x)}{\partial x_k \partial x_j} = \\
&= \frac{\partial^2 u}{\partial x_k \partial x_j}(\Phi(x)) + \frac{\partial^2 u}{\partial x_k \partial x_j}(\Phi(x))[\varphi'(\|x - x_0\|)]^2. \\
&\quad .(\bar{x} - x_0)_j (\bar{x} - x_0)_k \frac{\partial \|x - x_0\|}{\partial x_k} \frac{\partial \|x - x_0\|}{\partial x_j} + \\
&\quad + \sum_{i \neq j, l \neq k} \frac{\partial^2 u}{\partial x_l \partial x_i}(\Phi(x)) \frac{\partial \Phi_l(x)}{\partial x_k} \frac{\partial \Phi_i(x)}{\partial x_j} + \sum_i \frac{\partial u}{\partial x_i}(\Phi(x)) \frac{\partial^2 \Phi_i(x)}{\partial x_k \partial x_j}.
\end{aligned}$$

Note that

$$\frac{\partial^2 \Phi_i(x)}{\partial x_k \partial x_j}$$

depends on terms of the form $(\bar{x} - x_0)_j$. Note also that for x such that either $0 \le \|x - x_0\| \le \delta_1$ or $\|x - x_0\| \ge 2\delta_1$, the derivatives of φ are zero (because φ is constant) and therefore, the second derivatives of u and $\bar{u}$ coincide. Since

$$\{x \in \Omega : \ \delta_1 \le \|x - x_0\| \le 2\delta_1\}$$

is compact, the derivatives of u are bounded. We may then use our choice of $\delta < \delta_1$ to bound the remaining derivatives in the same way as in the proof of Lemma 4.6, so that the products involving derivatives of φ, and hence of Φ, become small. $\square$

5 Smooth approximation of continuous utilities

It was proved by Kannai [8] that any preference in Ξ can be appoximated arbitrarily well by a preferences represented by a smooth utility function. Here, using convolution as an alternative method, we extend this result to show that any C^0 utility function can appoximated arbitrarily well by a smooth one. This shows that the restriction to C^2 economies in *(II)* of Theorem 4.1 is a minor one.

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function. Then the notation $x > y$ means, as usual, that

$$\exists\, i: \;\; x_i > y_i \;\text{ and }\;\; x_j \geq y_j, \;\text{ for }\; i \neq j.$$

In the proof of Theorem 5.2, we need the following:

Lemma 5.1. *Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous and such that $x > y \Rightarrow f(x) > f(y)$. Then f is strictly quasi-concave if and only if f is strictly concave on chords of level sets, that is, if x_1, x_2 are such that $f(x_1) = f(x_2)$ we have*

$$\forall\, \lambda \in (0,1): \; f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda) f(x_2). \tag{13}$$

Proof. The *if* part of the statement is trivial. To show the converse, suppose that (13) holds. Note that $\lambda f(x_1) + (1-\lambda)f(x_2) = f(x_1) = f(x_2)$. We want to show that

$$\forall\, \lambda \in (0,1) \;\; \forall\, x, y \;\; f(\lambda x + (1-\lambda)y) = f(y_\lambda) > \; \min\{f(x), f(y)\},$$

which holds trivially if $f(x) = f(y)$. Let $f(x) \neq f(y)$ and assume $f(x) < f(y)$. Denote by $L_z = \{x: \; f(x) = f(z)\}$ the level set of z. By monotonicity, $L_x \cap L_y = \emptyset$.

Take $w \in L_y$ such that $w > x$, that is, $f(w) > f(x)$. Consider the points on the segment connecting x to w,

$$w_t = tx + (1-t)w, \;\text{ for }\;\; t \in (0,1).$$

By monotonicity, L_{w_t} lies between L_x and L_w and $f(w_t) > f(x)$. Since $y_\lambda \in L_{w_t}$ for some t, we have $f(y_t) > f(x)$, concluding the proof. □

Theorem 5.2. *Let $u \in C^0$ be a utility representing preferences that satisfy (P1)–(P4). Endow C^0 with the uniform norm on compact sets. There exists a C^∞ perturbation $\tilde{u}$ of u representing preferences that satisfy (P1)–(P4).*

Proof. We need to show that u can be uniformly approximated on compact subsets by $\tilde{u}$ which is C^2, strictly quasi-concave and increasing in each coordinate.

We use a *convolution kernel* or *mollifier*, $\theta_\varepsilon : \mathbb{R}^L \to \mathbb{R}$, as in Hirsch [6, Chapter 2], or Ghomi [4]. The mollifier is a non-negative function which takes the value zero outside a ball of radius $\varepsilon > 0$ and such that $\int_{\mathbb{R}^L} \theta_\varepsilon = 1$. An explicit construction for such a θ_ε is given in Ghomi [4]. Define

$$\tilde{u}(x) = \int_{\mathbb{R}^L} u(x-y)\theta_\varepsilon(y)dy.$$

Hirsch [6, Theorem 2.3] asserts that $\tilde{u}$ and u are close on compact sets.

Note that we can use Tietze's theorem to extend u to a continuous function on a neighborhood of Ω and this guarantees that $\tilde{u}$ is smooth in Ω. However, since we cannot control convexity and monotonicity properties of the extension of u, we shall additionally require a modification of θ_ε for the arguments below to be correct: by a translation in the argument of θ_ε we can achieve that it has support in the non-negative orthant of $\mathbb{R}^L$, and this modification guarantees that the various integrands vanish when the argument of u is outside of Ω.

Since u is strictly quasi-concave, using Lemma 5.1, we have, for $\lambda \in (0,1)$ (cf. Ghomi [4, p. 2257]),

$$\begin{aligned}
\tilde{u}(\lambda x_1 + (1-\lambda)x_2) &= \int_{\mathbb{R}^L} u(\lambda x_1 + (1-\lambda)x_2 - y)\theta_\varepsilon(y)dy \\
&= \int_{\mathbb{R}^L} u(\lambda(x_1 - y) + (1-\lambda)(x_2 - y))\theta_\varepsilon(y)dy \\
&> \int_{\mathbb{R}^L} [\lambda u((x_1 - y) + (1-\lambda)u(x_2 - y)]\,\theta_\varepsilon(y)dy \\
&= \lambda\tilde{u}(x_1) + (1-\lambda)\bar{u}(x_2).
\end{aligned}$$

Again by Lemma 5.1, $\tilde{u}$ is strictly quasi-concave and, since monotonicity is preserved by integration, it is monotonous.

□

Corollary 5.3. *The space Ξ^∞ of smooth preferences is dense in the space Ξ of all continuous preferences.*

Proof. This follows by applying Theorem 5.2 to utilities in C^* and using the same argument as in the last paragraph of the proof of Theorem 4.1. $\square$

6 Conclusion

We clarify the relation between AED functions and the underlying economic primitives of agent preferences and endowments by showing that perturbations of one correspond to perturbations of the other. While our results are stated for smooth economies only, we show that these are dense among all continuous economies.

Furthermore, our results imply that the Sonnenschein-Mantel-Debreu results (roughly, if it "looks" like an AED, it is an AED for some economy) are stable in the following sense: the economy underlying a *perturbed* AED function can be taken to be close to the economy underlying the original AED. Last but not least, the two economies only need to differ in the preferences and endowments of the first agent.

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