Random Dynamical Systems

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1 Introduction

The concept of random dynamical system is a comparatively recent development combining ideas and methods from the well developed areas of probability theory and dynamical systems.

Let us consider a mathematical model of some physical process given by the iterates $T_0^k = T_0 \circ \overset{k}{\cdots} \circ T_0$, $k \geq 1$, of a smooth transformation $T_0 : M \circlearrowleft$ of a manifold into itself. A realization of the process with initial condition x_0 is modelled by the sequence $(T_0^k(x_0))_{k\geq 1}$, the *orbit* of x_0.

Due to our inaccurate knowledge of the particular physical system or due to computational or theoretical limitations (lack of sufficient computational power, inefficient algorithms or insufficiently developed mathematical or physical theory, for example), the mathematical models never correspond exactly to the phenomenon they are meant to model. Moreover when considering practical systems we cannot avoid either external noise or measurement or *inaccuracy* errors, so every realistic mathematical model should allow for small errors along orbits not to disturb too much the long term behavior. To be able to cope with unavoidable uncertainty about the "correct" parameter values, observed initial states and even the specific mathematical formulation involved, we let randomness be embedded within the model to begin with.

We present the most basic classes of models in what follows, then define the general concept and present some developments and examples of applications.

2 Dynamics with noise

To model random perturbations of a transformation T_0 we may consider a transition from the image $T_0(x)$ to some point according to a given probability law, obtaining a Markov Chain, or, if T_0 depends on a parameter p, we may choose p at random at each iteration, which also can be seen as a Markov Chain but whose transitions are strongly correlated.

2.1 Random noise

Given $T_0 : M \circlearrowleft$ and a family $\{p(\cdot \mid x) : x \in M\}$ of probability measures on M such that the support of $p(\cdot \mid x)$ is close to $T_0(x)$, the *random orbits* are sequences $(x_k)_{k\geq 1}$ where each x_{k+1} is a random variable with law $p(\cdot \mid x_k)$. This is a Markov Chain with state space M and transition probabilities $\{p(\cdot \mid x)\}_{x\in M}$. To extend the concept of invariant measure of a transformation to this setting, we say that a probability measure μ is *stationary* if $\mu(A) = \int p(A \mid x)\, d\mu(x)$ for every measurable (Borel) subset A. This can be conveniently translated by saying that the skew-product measure $\mu \times p^{\mathbb{N}}$ on $M \times M^{\mathbb{N}}$ given by

$$
\begin{aligned}
& d(\mu \times p^{\mathbb{N}})(x_0, x_1, \dots, x_n, \dots) \\
& = d\mu(x_0) p(dx_1 \mid x_0) \cdots p(dx_{n+1} \mid x_n) \cdots
\end{aligned}
$$

is invariant by the shift map $\mathcal{S} : M \times M^{\mathbb{N}} \circlearrowleft$ on the space of orbits. Hence we may use the Ergodic Theorem and get that time averages of every continuous observable $\varphi : M \to \mathbb{R}$, i.e. writing $\underline{x} = (x_k)_{k \geq 0}$ and

$$
\begin{aligned}
\tilde{\varphi}(\underline{x}) &= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(x_k) \\
&= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\pi_0(\mathcal{S}^k(\underline{x})))
\end{aligned}
$$

exist for $\mu \times p^{\mathbb{N}}$-almost all sequences $\underline{x}$, where $\pi_0 : M \times M^{\mathbb{N}} \to M$ is the natural projection on the first coordinate. It is well known that stationary measures always exist if the transition probabilities $p(\cdot \mid x)$ depend continuously on x.

A function $\varphi : M \to \mathbb{R}$ is *invariant* if $\varphi(x) = \int \varphi(z) p(dz \mid x)$ for μ-almost every x. We then say that μ is *ergodic* if every invariant function is constant μ-almost everywhere. Using the Ergodic Theorem again, if μ is ergodic, then $\tilde{\varphi} = \int \varphi\, d\mu$, μ-almost everywhere.

Stationary measures are the building blocks for more sophisticated analysis involving e.g. asymptotic sojourn times, Lyapunov exponents, decay of correlations, entropy and/or dimensions, exit/entrance times from/to subsets of M, to name just a few frequent notions of dynamical and probabilistic/statistical nature.

Example 1 *(Random jumps).* Given $\varepsilon > 0$ and $T_0 : M \to M$, let us define

$$
p^{\varepsilon}(A \mid x) = \frac{m(A \cap B(T_0(x), \varepsilon))}{m(B(T_0(x), \varepsilon))}
$$

where m denotes some choice of Riemannian volume form on M. Then $p^{\varepsilon}(\cdot \mid x)$ is the normalized volume restricted to the ε-neighborhood of $T_0(x)$. This defines a family of transition probabilities allowing the points to "jump" from $T_0(x)$ to any point in the ε-neighborhood of $T_0(x)$ following a uniform distribution law.

2.2 Random maps

Alternatively we may choose maps $T_1, T_2, \dots, T_k$ independently at random near T_0 according to a probability law ν on the space $T(M)$ of maps, whose support is close to T_0 in some topology, and consider sequences $x_k = T_k \circ \cdots \circ T_1(x_0)$ obtained through random iteration, $k \geq 1$, $x_0 \in M$.

This is again a Markov Chain whose transition probabilities are given for any $x \in M$ by

$$
p(A \mid x) = \nu\big(\{T \in T(M) : T(x) \in A\}\big),
$$

so this model may be reduced to the first one. However in the random maps setting we may associate to each random orbit a sequence of maps which are iterated, enabling us to use *robust properties* of the transformation T_0 (i.e. properties which are known to hold for T_0 and for every nearby map T) to derive properties of the random orbits.

Under some regularity conditions on the map $x \mapsto p(A \mid x)$ for every Borel subset A, it is possible to represent random noise by random maps on suitably chosen spaces of transformations. In fact the transition probability measures obtained in the random maps setting exhibit strong spatial correlation: $p(\cdot \mid x)$ is close to $p(\cdot \mid y)$ is x is near y.

If we have a parameterized family $T : \mathcal{U} \times M \to M$ of maps we can specify the law ν by giving a probability θ on $\mathcal{U}$. Then to every sequence $T_1, \ldots, T_k, \ldots$ of maps of the given family we associate a sequence $\omega_1, \ldots, \omega_k, \ldots$ of parameters in $\mathcal{U}$ since

$$T_k \circ \cdots \circ T_1 = T_{\omega_k} \circ \cdots \circ T_{\omega_1} = T^k_{\omega_1,\ldots,\omega_k}$$

for all $k \geq 1$, where we write $T_\omega(x) = T(\omega, x)$. In this setting the shift map $\mathcal{S}$ becomes a skew-product transformation

$$\mathcal{S} : M \times \mathcal{U}^{\mathbb{N}} \circlearrowleft \quad (x, \underline{\omega}) \mapsto \big(T_{\omega_1}(x), \sigma(\underline{\omega})\big),$$

to which many of the standard methods of dynamical systems and ergodic theory can be applied, yielding stronger results that can be interpreted in random terms.

Example 2 *(Parametrical noise).* Let $T : P \times M \to M$ be a smooth map where P, M are finite dimensional Riemannian manifolds. We fix $p_0 \in P$, denote by m some choice of Riemannian volume form on P, set $T_w(x) = T(w, x)$ and for every $\varepsilon > 0$ write $\theta_\varepsilon = (m(B(p_0, \varepsilon))^{-1} \cdot (m \mid B(p_0, \varepsilon))$, the normalized restriction of m to the ε-neighborhood of p_0. Then $(T_w)_{w \in P}$ together with θ_ε defines a random perturbation of T_{p_0}, for every small enough $\varepsilon > 0$.

Example 3 *(Global additive perturbations).* Let M be a homogeneous space, i.e., a compact connected Lie Group admitting an invariant Riemannian metric. Fixing a neighborhood $\mathcal{U}$ of the identity $e \in M$ we can define a map $T : \mathcal{U} \times M \to M, (u, x) \mapsto L_u(T_0(x))$, where $L_u(x) = u \cdot x$ is the left translation associated to $u \in M$. The invariance of the metric means that left (an also right) translations are isometries, hence fixing $u \in \mathcal{U}$ and taking any $(x, v) \in TM$ we get

$$\begin{aligned} \|DT_u(x) \cdot v\| &= \|DL_u(T_0(x))(DT_0(x) \cdot v)\| \\ &= \|DT_0(x) \cdot v\|. \end{aligned}$$

In the particular case of $M = \mathbb{T}^d$, the d-dimensional torus, we have $T_u(x) = T_0(x) + u$ and this simplest case suggests the name *additive random perturbations* for random perturbations defined using families of maps of this type.

For the probability measure on $\mathcal{U}$ we may take θ_ε any probability measure supported in the ε-neighborhood of e and absolutely continuous with respect to the Riemannian metric on M, for any $\varepsilon > 0$ small enough.

Example 4 *(Local additive perturbations).* If $M = \mathbb{R}^d$ and U_0 is a bounded open subset of M strictly invariant under a diffeomorphism T_0, i.e., closure $(T_0(U_0)) \subset U_0$, then we can define an isometric random perturbation setting

- $V = T_0(U_0)$ (so that closure (V) = closure $(T_0(U_0)) \subset U_0$);
- $G \simeq \mathbb{R}^d$ the group of translations of $\mathbb{R}^d$;
- $\mathcal{V}$ a small enough neighborhood of 0 in G.

Then for $v \in \mathcal{V}$ and $x \in V$ we set $T_v(x) = x + v$, with the standard notation for vector addition, and clearly T_v is an isometry. For θ_ε we may take any probability measure on the ε-neighborhood of 0, supported in $\mathcal{V}$ and absolutely continuous with respect to the volume in $\mathbb{R}^d$, for every small enough $\varepsilon > 0$.

2.3 Random perturbations of flows

In the continuous time case the basic model to start with is an ordinary differential equation $dX_t = f(t, X_t)dt$, where $f : [0, +\infty) \to \mathcal{X}(M)$ and $\mathcal{X}(M)$ is the family of vector fields in M. We embed randomness in the differential equation basically through *diffusion*, the perturbation is given by *white noise* or *Brownian motion* “added” to the ordinary solution.

In this setting, assuming for simplicity that $M = \mathbb{R}^n$, the random orbits are solutions of stochastic differential equations

$$dX_t = f(t,X_t)dt + \varepsilon \cdot \sigma(t,X_t)dW_t,\ 0 \le t \le T, X_0 = Z,$$

where Z is a random variable, $\varepsilon, T > 0$ and both $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^k, \mathbb{R}^n)$ are measurable functions. We have written $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^n)$ for the space of linear maps $\mathbb{R}^k \to \mathbb{R}^n$ and W_t for the *white noise process* on $\mathbb{R}^k$. The solution of this equation is a stochastic process

$$X : \mathbb{R} \times \Omega \to M, \quad (t,\omega) \mapsto X_t(\omega),$$

for some (abstract) probability space Ω, given by

$$X_t = Z + \int_0^T f(s,X_s)\,ds + \int_0^T \varepsilon \cdot \sigma(s,X_s)\,dW_s,$$

where the last term is a stochastic integral in the sense of Itô. Under reasonable conditions on f and σ, there exists a unique solution *with continuous paths*, i.e.

$$[0,+\infty) \ni t \mapsto X_t(\omega)$$

is continuous for almost all $\omega \in \Omega$ (in general these paths are *nowhere differentiable*).

Setting $Z = \delta_{x_0}$ the probability measure concentrated on the point x_0, the initial point of the path is x_0 with probability 1. We write $X_t(\omega)x_0$ for paths of this type. Hence $x \mapsto X_t(\omega)x$ defines a map $X_t(\omega) : M \circlearrowleft$ which can be shown to be a homeomorphism and even a diffeomorphisms under suitable conditions on f and σ. These maps satisfy a cocycle property

$$\begin{aligned} X_0(\omega) &= Id_M \text{ (identity map of } M), \\ X_{t+s}(\omega) &= X_t(\theta(s)(\omega)) \circ X_s(\omega), \end{aligned}$$

for $s,t \ge 0$ and $\omega \in \Omega$, for a family of measure preserving transformations $\theta(s) : (\Omega,\mathbb{P}) \circlearrowleft$ on a suitably chosen probability space $(\Omega,\mathbb{P})$. This enables us to write the solution of this kind of equations also as a skew-product.

2.4 The abstract framework

The illustrative particular cases presented can all be written in skew-product form as follows.

Let $(\Omega,\mathbb{P})$ be a given probability space, which will be the model for the noise, and $\mathbb{T}$ be time, which usually means $\mathbb{Z}_+, \mathbb{Z}$ (discrete, resp. invertible system) or $\mathbb{R}_+, \mathbb{R}$ (continuous, resp. invertible system). A random dynamical system is a skew-product

$$\mathcal{S}_t : \Omega \times M \circlearrowleft,\ (\omega,x) \mapsto (\theta(t)(\omega), \varphi(t,\omega)(x)),$$

for all $t \in \mathbb{T}$, where $\theta : \mathbb{T} \times \Omega \to \Omega$ is a family of measure preserving maps $\theta(t) : (\Omega,\mathbb{P}) \circlearrowleft$ and $\varphi : \mathbb{T} \times \Omega \times M \to M$ is a family of maps $\varphi(t,\omega) : M \circlearrowleft$ satisfying the cocycle property: for $s,t \in \mathbb{T}$, $\omega \in \Omega$

$$\begin{aligned} \varphi(0,\omega) &= Id_M, \\ \varphi(t+s,\omega) &= \varphi(t,\theta(s)(\omega)) \circ \varphi(s,\omega). \end{aligned}$$

In this general setting an invariant measure for the random dynamical system is any probability measure μ on $\Omega \times M$ which is $\mathcal{S}_t$-invariant for all $t \in \mathbb{T}$ and whose *marginal* is $\mathbb{P}$, i.e. $\mu(\mathcal{S}_t^{-1}(U)) = \mu(U)$ and $\mu(\pi_\Omega^{-1}(U)) = \mathbb{P}(U)$ for every measurable $U \subset \Omega \times M$, respectively, with $\pi_\Omega : \Omega \times M \to \Omega$ the natural projection.

Example 5. In the setting of the previous examples of random perturbations of maps, the product measure $\eta = \mathbb{P} \times \mu$ on $\Omega \times M$, with $\Omega = \mathcal{U}^{\mathbb{N}}$, $\mathbb{P} = \theta_\varepsilon^{\mathbb{N}}$ and μ any stationary measure, is clearly invariant. However not all invariant measures are product measures of this type.

Naturally an invariant measure is ergodic if every $\mathcal{S}_t$-invariant function is μ-almost everywhere constant. i.e. if $\psi : \Omega \times M \to \mathbb{R}$ satisfies $\psi \circ \mathcal{S}_t = \psi$ μ-almost everywhere for every $t \in \mathbb{T}$, then ψ is μ-almost everywhere constant.

3 Applications

We avoid presenting well established applications of both probability or stochastic differential equations (solution of boundary value problems, optimal stopping, stochastic control etc) and dynamical systems (all sort of models of physical, economic or biological phenomena, solutions of differential equations, control systems etc), focusing instead on topics where the subject sheds new light on these areas.

3.1 Products of random matrices and the Multiplicative Ergodic Theorem

The following celebrated result on products of random matrices has far-reaching applications on dynamical systems theory.

Let $(X_n)_{n\geq 0}$ be a sequence of independent and identically distributed random variables on the probability space (Ω, P) with values in $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$ such that $E(\log^+ \|X_1\|) < +\infty$, where $\log^+ x = \max\{0, \log x\}$ and $\|\cdot\|$ is a given norm on $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$. Writing $\varphi_n(\omega) = X_n(\omega) \circ \cdots \circ X_1(\omega)$ for all $n \geq 1$ and $\omega \in \Omega$ we obtain a cocycle. If we set

$$\begin{aligned} B &= \{(\omega, y) \in \Omega \times \mathbb{R}^k : \lim_{n\to+\infty} \frac{1}{n} \log \|\varphi_n(\omega) y\| \\ &\quad \text{exists and is finite or is } -\infty\}, \text{ and} \\ \Omega' &= \{\omega \in \Omega : (\omega, y) \in B \text{ for all } y \in \mathbb{R}^k\}, \end{aligned}$$

then Ω' contains a subset Ω'' of full probability and there exist random variables (which might take the value $-\infty$) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ with the following properties.

(1) Let $I = \{k+1 = i_1 > i_2 > \cdots > i_{l+1} = 1\}$ be any $(l+1)$-tuple of integers and then we define

$$\begin{aligned} \Omega_I = \{\omega \in \Omega'' : \lambda_i(\omega) = \lambda_j(\omega), i_h > i, j \geq i_{h+1}, \text{ and} \\ \lambda_{i_h}(\omega) > \lambda_{i_{h+1}}(\omega) \text{ for all } 1 < h < l\} \end{aligned}$$

the set of elements where the sequence λ_i jumps exactly at the indexes in I. Then for $\omega \in \Omega_I$, $1 < h \leq l$

$$\Sigma_{I,h}(\omega) = \{y \in \mathbb{R}^k : \lim_{n\to+\infty} \frac{1}{n} \log \|\varphi_n(\omega)\| \leq \lambda_{i_h}(\omega)\}$$

is a vector subspace with dimension $i_{h-1} - 1$.

(2) Setting $\Sigma_{I,k+1}(\omega) = \{0\}$, then

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\varphi_n(\omega)\| = \lambda_{i_h}(\omega),$$

for every $y \in \Sigma_{I,h}(\omega) \setminus \Sigma_{I,h+1}(\omega)$.

(3) For all $\omega \in \Omega''$ there exists the matrix

$$A(\omega) = \lim_{n\to+\infty} \left[\left(\varphi_n(\omega)\right)^* \varphi_n(\omega) \right]^{1/2n}$$

whose eigenvalues form the set $\{e^{\lambda_i} : i = 1, \ldots, k\}$.

The values of λ_i are the random Lyapunov characteristics and the corresponding subspaces are analogous to random eigenspaces. If the sequence $(X_n)_{n\geq 0}$ is ergodic, then the Lyapunov characteristics become non-random constants, but the Lyapunov subspaces are still random.

We can easily deduce the Multiplicative Ergodic Theorem for measure preserving differentiable maps (T_0, μ) on manifolds M from this result. We assume for simplicity that $M \subset \mathbb{R}^k$ and set $p(A \mid x) = \delta_{T_0(x)}(A) = 1$ if $T_0(x) \in A$ and 0 otherwise. Then the measure $\mu \times p^{\mathbb{N}}$ on $M \times M^{\mathbb{N}}$ is σ-invariant (as defined in Section 2) and we have that $\pi_0 \circ \sigma = T_0 \circ \pi_0$, where $\pi_0 : M^{\mathbb{N}} \to M$ is the projection on the first coordinate, and also $(\pi_0)_*(\mu \times p^{\mathbb{N}}) = \mu$. Then setting for $n \geq 1$

$$\begin{array}{rccc} X: & M & \to & \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k) \\ & x & \mapsto & DT_0(x) \end{array} \quad \text{and } X_n = X \circ \pi_0 \circ \sigma^n$$

we obtain a stationary sequence to which we can apply the previous result, obtaining the existence of Lyapunov exponents and of Lyapunov subspaces on a *full measure subset for any C^1 measure preserving dynamical system*.

By a standard extension of the previous setup we obtain a random version of the multiplicative ergodic theorem. We take a family of skew-product maps $\mathcal{S}_t : \Omega \times M \circlearrowleft$ as in Subsection 2.4 with an invariant probability measure μ and such that $\varphi(t,\omega) : M \circlearrowleft$ is (for simplicity) a local diffeomorphism. We then consider the stationary family

$$X_t : \begin{array}{ccc} \Omega & \to & \mathcal{L}(TM) \\ \omega & \mapsto & D\varphi(t,\omega) : TM \circlearrowleft \end{array}, \quad t \in \mathbb{T},$$

where $D\varphi(t,\omega)$ is the tangent map to $\varphi(t,\omega)$. This is a cocycle since for all $t,s \in \mathbb{T}$, $\omega \in \Omega$ we have

$$X(s+t,\omega) = X(s,\theta(t)\omega) \circ X(t,\omega).$$

If we assume that

$$\sup_{0\le t\le 1} \sup_{x\in M} \left(\log^+ \|D\varphi(t,\omega)(x)\|\right) \in L^1(\Omega,\mathbb{P}),$$

where $\|\cdot\|$ denotes the norm on the corresponding space of linear maps given by the induced norm (from the Riemannian metric) on the appropriate tangent spaces, then we obtain a sequence of random variables (which might take the value $-\infty$) $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$, with k being the dimension of M, such that

$$\lim_{t\to+\infty} \frac{1}{t} \log \|X_t(\omega,x)y\| = \lambda_i(\omega,x)$$

for every $y \in E_i\omega,x) = \Sigma_i(\omega,x) \setminus \Sigma_{i+1}(\omega,x)$ and $i = 1,\dots,k+1$ where $(\Sigma_i(\omega,x))_i$ is a sequence of vector subspaces in T_xM as before, measurable with respect to (ω,x). In this setting the subspaces $E_i(\omega,x)$ and the Lyapunov exponents are invariant, i.e. for all $t \in \mathbb{T}$ and μ almost every $(\omega,x) \in \Omega \times M$ we have

$$\lambda_i(\mathcal{S}_t(\omega,x)) = \lambda_i(\omega,x) \text{ and } E_i(\mathcal{S}_t(\omega,x)) = E_i(\omega,x).$$

The *dependence of Lyapunov exponents on the map* T_0 has been a fruitful and central research program in dynamical systems for decades extending to the present day. The random multiplicative ergodic theorem sets the stage for the study of the stability of Lyapunov exponents under random perturbations.

3.2 Stochastic stability of physical measures

The development of the theory of dynamical systems has shown that models involving expressions as simple as quadratic polynomials (as the *logistic family* or *Hénon attractor*), or autonomous ordinary differential equations with a hyperbolic singularity of saddle-type, as the *Lorenz flow*, exhibit *sensitive dependence on initial conditions*, a common feature of *chaotic dynamics*: small initial differences are rapidly augmented as time passes, causing two trajectories originally coming from practically indistinguishable points to behave in a completely different manner after a short while. Long term predictions based on such models are unfeasible since it is not possible to both specify initial conditions with arbitrary accuracy and numerically calculate with arbitrary precision.

Physical measures

Inspired by an analogous situation of unpredictability faced in the field of Statistical Mechanics/Thermodynamics, researchers focused on the statistics of the data provided by the time averages of some observable (a continuous function on the manifold) of the system. Time averages are guaranteed to exist for a positive volume subset of initial states (also called an *observable subset*) on the mathematical model if the transformation, or the flow associated to the ordinary differential equation, admits a smooth invariant measure (a density) or a *physical* measure.

Indeed, if μ_0 is an ergodic invariant measure for the transformation T_0, then the Ergodic Theorem ensures that for every μ-integrable function $\varphi : M \to \mathbb{R}$ and for μ-almost every point x in the manifold M the time average $\tilde{\varphi}(x) = \lim_{n\to+\infty} n^{-1} \sum_{j=0}^{n-1} \varphi(T_0^j(x))$ exists and equals the space average $\int \varphi \, d\mu_0$. A *physical measure* μ is an invariant probability measure for which it is *required* that *time averages of every continuous function* φ *exist for a positive Lebesgue measure (volume) subset of the space and be equal to the space average* $\mu(\varphi)$.

We note that if μ is a density, that is, is absolutely continuous with respect to the volume measure, then the Ergodic Theorem ensures that μ is physical. However not every physical measure is absolutely continuous. To see why in a simple example we just have to consider a singularity p of a vector field which is an attracting fixed point (a sink), then the Dirac mass δ_p concentrated on p is a physical probability measure, since every orbit in the basin of attraction of p will have asymptotic time averages for any continuous observable φ given by $\varphi(p) = \delta_p(\varphi)$.

Physical measures need not be unique or even exist in general, but when they do exist it is desirable that *the set of points whose asymptotic time averages are described by physical measures* (such set is called the *basin* of the physical measures) *be of full Lebesgue measure* — only an exceptional set of points with zero volume would not have a well defined asymptotic behavior. This is yet far from being proved for most dynamical systems, in spite of much recent progress in this direction.

There are robust examples of systems admitting several physical measures whose basins together are of full Lebesgue measure, where *robust* means that there are whole open sets of maps of a manifold in the C^2 topology exhibiting these features. For typical parameterized families of one-dimensional unimodal maps (maps of the circle or of the interval with a unique critical point) it is known that the above scenario holds true for Lebesgue almost every parameter. It is known that there are systems admitting no physical measure, but the only known cases are not robust, i.e. there are systems arbitrarily close which admit physical measures.

It is hoped that conclusions drawn from models admitting physical measures to be effectively observable in the physical processes being modelled. In order to lend more weight to this expectation researchers demand stability properties from such invariant measures.

Stochastic stability

There are two main issues when we are given a mathematical model, both theoretical but with practical consequences. The first one is to describe the asymptotic behavior of most orbits, that is, to understand where do orbits go when time tends to infinity. The second and equally important one is to ascertain whether the asymptotic behavior is stable under small changes of the system, i.e. whether the limiting behavior is still essentially the same after small changes to the evolution law. In fact since models are always simplifications of the real system (we cannot ever take into account the whole state of the universe in any model), the lack of stability considerably weakens the conclusions drawn from such models, because some properties might be specific to it and not in any way resemblant of the real system.

Random dynamical systems come into play in this setting when we need to check whether a given model is stable under small random changes to the evolution law.

In more precise terms, we suppose that we are given a dynamical system (a transformation or a flow) admitting a physical measure μ_0, and we take any random dynamical system obtained from this one through the introduction of small random perturbations on the dynamics, as in Examples 1- 4 or in Subsection 2.3, with the noise level $\varepsilon > 0$ close to zero.

In this setting if, for any choice μ_ε of invariant measure for the random dynamical system for all $\varepsilon > 0$ small enough, the set of accumulation points of the family $(\mu_\varepsilon)_{\varepsilon>0}$, when ε tends to 0 — also known as *zero noise limits* — is formed by physical measures or, more generally, by convex linear combinations of physical measures, then the original unperturbed dynamical system is *stochastically stable*.

This intuitively means that the asymptotic behavior measured through time averages of continuous observables for the random system is close to the behavior of the unperturbed system.

Recent progress in one-dimensional dynamics has shown that, for typical families $(f_t)_{t\in(0,1)}$ of maps of the circle or of the interval having a unique critical point, a full Lebesgue measure subset T of the set of parameters is such that, for $t \in T$, the dynamics of f_t admits a unique stochastically stable (under additive noise type random perturbations) physical measure μ_t whose basin has full measure in the ambient space (either the circle or the interval). Therefore models involving one-dimensional unimodal maps typically are stochastically stable.

In many settings (e.g. low dimensional dynamical systems) Lyapunov exponents can be given by time averages of continuous functions — for example the time average of $\log\|DT_0\|$ gives the biggest exponent. In this case stochastic stability directly implies stability of the Lyapunov exponents under small random perturbations of the dynamics.

Example 6 *(Stochastically stable examples).* Let $T_0 : \mathbb{S}^1 \circlearrowleft$ be a map such that λ, the Lebesgue (length) measure on the circle, is T_0-invariant and ergodic. Then λ is physical.

We consider the parameterized family $T_t : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1, (t,x) \mapsto x+t$ and a family of probability measures $\theta_\varepsilon = (\lambda(-\varepsilon,\varepsilon))^{-1} \cdot (\lambda \mid (-\varepsilon,\varepsilon))$ given by the normalized restriction of λ to the ε-neighborhood of 0, where we regard $\mathbb{S}^1$ as the Lie group $\mathbb{R}/\mathbb{Z}$ and use additive notation for the group operation. Since λ is T_t-invariant for every $t \in \mathbb{S}^1$, λ is also an invariant measure for the measure preserving random system

$$\mathcal{S} : (\mathbb{S}^1 \times \Omega^{\mathbb{N}}, \lambda \times \theta_\varepsilon^{\mathbb{N}}) \circlearrowleft,$$

for every $\varepsilon > 0$, where $\Omega = (\mathbb{S}^1)^{\mathbb{N}}$. Hence (T_0,λ) is stochastically stable under additive noise perturbations.

Concrete examples can be irrational rotations, $T_0(x) = x + \alpha$ with $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, or expanding maps of the circle, $T_0(x) = b \cdot x$ for some $b \in \mathbb{N}$, $n \geq 2$. Analogous examples exist in higher dimensional tori.

Example 7 *(Stochastic stability depends on the type of noise).* In spite of the straightforward way to obtain stochastic stability in Example 6, for e.g. an expanding circle map $T_0(x) = 2 \cdot x$, we can choose a continuous family of probability measures θ_ε such that the same map T_0 is not stochastically stable.

It is well known that λ is the unique absolutely continuous invariant measure for T_0 and also the unique physical measure. Given $\varepsilon > 0$ small let us define transition probability measures as follows

$$p_\varepsilon(\cdot \mid z) = \frac{\lambda \mid [\phi_\varepsilon(z) - \varepsilon, \phi_\varepsilon(z) + \varepsilon]}{\lambda([\phi_\varepsilon(z) - \varepsilon, \phi_\varepsilon(z) + \varepsilon])},$$

where $\phi_\varepsilon \mid (-\varepsilon,\varepsilon) \equiv 0$, $\phi_\varepsilon \mid [\mathbb{S}^1 \setminus (-2\varepsilon, 2\varepsilon)] \equiv T_0$ and over $(-2\varepsilon,-\varepsilon] \cup [\varepsilon, 2\varepsilon)$ we define ϕ_ε by interpolation in order that it be smooth.

In this setting every random orbit starting at $(-\varepsilon,\varepsilon)$ never leaves this neighborhood in the future. Moreover it is easy to see that every random orbit eventually enters $(-\varepsilon, \varepsilon)$. Hence every invariant probability measure μ_ε for this Markov Chain model is supported in $[-\varepsilon, \varepsilon]$. Thus letting $\varepsilon \to 0$ we see that the only zero-noise limit is δ_0 the Dirac mass concentrated at 0, which is not a physical measure for T_0.

This construction can be done in a random maps setting, but only in the C^0 topology — it is not possible to realize this Markov Chain by random maps that are C^1 close to T_0 for ε near 0.

3.3 Characterization of measures satisfying the Entropy Formula

A lot of work has been put in recent years in extending important results from dynamical systems to the random setting. Among many examples we mention the local conjugacy between the dynamics near a hyperbolic fixed point and the action of the derivative of the map on the tangent space, the stable/unstable manifold theorems for hyperbolic invariant sets and the notions and properties of metric and topological entropy, dimensions and equilibrium states for potentials on *random (or fuzzy) sets*.

The characterization of measures satisfying the Entropy Formula is one important result whose extension to the setting of iteration of independent and identically distributed random maps has recently had interesting new consequences back into non-random dynamical systems.

Metric entropy for random perturbations

Given a probability measure μ and a partition ξ of M, except perhaps for a subset of μ-null measure, the *entropy of μ with respect to ξ* is defined to be

$$H_\mu(\xi) = -\sum_{R \in \xi} \mu(R) \log \mu(R)$$

where we convention $0 \log 0 = 0$. Given another finite partition ζ we write $\xi \vee \zeta$ to indicate the partition obtained through intersection of every element of ξ with every element of ζ, and analogously for any finite number of partitions. If μ is also a stationary measure for a random maps model (as in Subsection 2.2), then for any finite measurable partition ξ of M,

$$h_\mu(\xi) = \inf_{n \geq 1} \frac{1}{n} \int H_\mu\Big(\bigvee_{i=0}^{n-1} (T^i_{\underline{\omega}})^{-1}(\xi) \Big) \, dp^{\mathbb{N}}(\underline{\omega})$$

is finite and is called *the entropy of the random dynamical system* with respect to ξ and to μ.

We define $h_\mu = \sup_\xi h_\mu(\xi)$ as the *metric entropy* of the random dynamical system, where the supremo is taken over all μ-measurable partitions. An important point here is the following notion: setting $\mathcal{A}$ the Borel σ-algebra of M, we say that a finite partition ξ of M is a *random generating partition* for $\mathcal{A}$ if

$$\bigvee_{i=0}^{+\infty} (T^i_{\underline{\omega}})^{-1}(\xi) = \mathcal{A}$$

(except μ-null sets) for $p^{\mathbb{N}}$-almost all $\omega \in \Omega = \mathcal{U}^{\mathbb{N}}$. Then a classical result from Ergodic Theory ensures that we can calculate the entropy using only a random generating partition ξ: we have $h_\mu = h_\mu(\xi)$.

The Entropy Formula

There exists a general relation ensuring that the entropy of a measure preserving differentiable transformation (T_0, μ) on a compact Riemannian manifold is bounded from above by the sum of the positive Lyapunov exponents of T_0

$$h_\mu(T_0) \le \int \sum_{\lambda_i(x)>0} \lambda_i(x)\, d\mu(x).$$

The equality (*Entropy Formula*) was first shown to hold for diffeomorphisms preserving a measure equivalent to the Riemannian volume, and then the measures satisfying the Entropy Formula were characterized: for C^2 *diffeomorphisms the equality holds if, and only if, the disintegration of* μ *along the unstable manifolds is formed by measures absolutely continuous with respect to the Riemannian volume restricted to those submanifolds*. The *unstable manifolds* are the submanifolds of M everywhere tangent to the Lyapunov subspaces corresponding to all positive Lyapunov exponents, the analogous to "integrating the distribution of Lyapunov subspaces corresponding to positive exponents" — this particular point is a main subject of smooth ergodic theory for non-uniformly hyperbolic dynamics.

Both the inequality and the characterization of stationary measures satisfying the Entropy Formula were extended to random iterations of independent and identically distributed C^2 maps (non-injective and admitting critical points), and the inequality reads

$$h_\mu \le \int\int \sum_{\lambda_i(x,\omega)>0} \lambda_i(x,\omega)\, d\mu(x)\, dp^{\mathbb{N}}(\omega).$$

where the functions λ_i are the random variables provided by the Random Multiplicative Ergodic Theorem.

3.4 Construction of physical measures as zero-noise limits

The characterization of measures which satisfy the Entropy Formula enables us to construct physical measures as zero-noise limits of random invariant measures in some settings, outlined in what follows, obtaining in the process that the physical measures so constructed are also stochastically stable.

The physical measures obtained in this manner arguably are *natural measures* for the system, since they are both stable under (certain types of) random perturbations and describe the asymptotic behavior of the system for a positive volume subset of initial conditions. This is a significant contribution to the state-of-the-art of present knowledge on Dynamics from the perspective of Random Dynamical Systems.

Hyperbolic measures and the Entropy Formula

The main idea is that an ergodic invariant measure μ for a diffeomorphism T_0 which satisfies the Entropy Formula and whose Lyapunov exponents are everywhere non-zero (known as *hyperbolic measure*) necessarily is a *physical measure* for T_0. This follows from standard arguments of smooth non-uniformly hyperbolic ergodic theory.

Indeed μ satisfies the Entropy Formula if, and only if, μ disintegrates into densities along the unstable submanifolds of T_0. The unstable manifolds $W^u(x)$ are tangent to the subspace corresponding to every positive Lyapunov exponent at μ-almost every point x, they are an invariant family, i.e. $T_0(W^u(x)) = W^u(x)$ for μ-almost every x, and distances on them are uniformly contracted under iteration by T_0^{-1}.

If we know that the exponents along the complementary directions are non-zero, then they must be negative and smooth ergodic theory ensures that there exist *stable manifolds*, which are submanifolds $W^s(x)$ of M everywhere tangent to the subspace of negative Lyapunov exponents at μ-almost every point x, form a T_0-invariant family ($T_0(W^s(x)) = W^s(x)$, μ-almost everywhere), and distances on them are uniformly contracted under iteration by T_0.

We still need to understand that time averages are constant along both stable and unstable manifolds, and that the families of stable and unstable manifolds are absolutely continuous, in order to realize how an hyperbolic measure is a physical measure.

Given $y \in W^s(x)$ the time averages of x and y coincide for continuous observables simply because $\operatorname{dist}(T_0^n(x), T_0^n(y)) \to 0$ when $n \to +\infty$. For unstable manifolds the same holds when considering time averages for T_0^{-1}. Since forward and backward time averages are equal μ-almost everywhere, we see that the set of points having asymptotic time averages given by μ has positive Lebesgue measure if the following set

$$B = \bigcup \{W^s(y) : y \in W^u(x) \cap \operatorname{supp}(\mu)\}$$

has positive volume in M, for some x whose time averages are well defined.

Now, stable and unstable manifolds are transverse everywhere where they are defined, but they are only defined μ-almost everywhere and depend measurably on the base point, so we cannot use transversality arguments from differential topology, in spite of $W^u(x) \cap \operatorname{supp}(\mu)$ *having positive volume in* $W^u(x)$ by the existence of a smooth disintegration of μ along the unstable manifolds. However it is known for smooth (C^2) transformations that the families of stable and unstable manifolds are *absolutely continuous*, meaning that projections along leaves preserve sets of zero volume. This is precisely what is needed for measure-theoretic arguments to show that B has positive volume.

Zero-noise limits satisfying the Entropy Formula

Using the extension of the characterization of measures satisfying the Entropy Formula for the random maps setting, we can build random dynamical systems, which are small random perturbations of a map T_0, having invariant measures μ_ε satisfying the Entropy Formula for all sufficiently small $\varepsilon > 0$. Indeed it is enough to construct small random perturbations of T_0 having absolutely continuous invariant probability measures μ_ε for all small enough $\varepsilon > 0$.

In order to obtain such random dynamical systems we choose families of maps $T : \mathcal{U} \times M \to M$ and of probability measures $(\theta_\varepsilon)_{\varepsilon>0}$ as in Examples 3 and 4, where we assume that $o \in \mathcal{U}$ so that T_0 belongs to the family. Letting $T_x(u) = T(u,x)$ for all $(u,x) \in \mathcal{U} \times M$, we then have that $T_x(\theta_\varepsilon)$ is absolutely continuous. This means that sets of perturbations of positive θ_ε-measure send points of M onto positive volume subsets of M. This kind of perturbation can be constructed for every continuous map of any manifold.

In this setting we have that any invariant probability measure for the associated skew-product map $\mathcal{S} : \Omega \times M \circlearrowleft$ of the form $\theta_\varepsilon^{\mathbb{N}} \times \mu_\varepsilon$ is such that μ_ε is absolutely continuous with respect to volume on M. Then the Entropy Formula holds

$$h_{\mu_\varepsilon} = \int\int \sum_{\lambda_i(x,\omega)>0} \lambda_i(x,\omega)\, d\mu_\varepsilon(x)\, d\theta_\varepsilon^{\mathbb{N}}(\omega).$$

Having this and knowing the characterization of measures satisfying the Entropy Formula, it is natural to look for conditions under which we can guarantee that the above inequality extends to any zero-noise limit μ_0 of μ_ε when $\varepsilon \to 0$. In that case μ_0 satisfies the Entropy Formula for T_0.

If in addition we are able to show that μ_0 is a hyperbolic measure, then we obtain a physical measure for

T_0 which is stochastically stable by construction.

These ideas can be carried out completely for hyperbolic diffeomorphisms, i.e. maps admitting an continuous invariant splitting of the tangent space into two sub-bundles $E \oplus F$ defined everywhere with bounded angles, whose Lyapunov exponents are negative along E and positive along F. Recently maps satisfying weaker conditions where shown to admit stochastically stable physical measures following the same ideas.

These ideas also have applications to the construction and stochastic stability of physical measure for *strange attractors* and for all mathematical models involving ordinary differential equations or iterations of maps.

See also

Equilibrium statistical mechanics
Dynamical systems
Global analysis
Non-equilibrium statistical mechanics
Ordinary and partial differential equations
Stochastic methods
Strange attractors

Keywords

Dynamical system
Flows
Orbits
Ordinary differential equations
Markov chains
Multiplicative Ergodic Theorem
Physical measures
Products of random matrices
Random maps
Random orbits
Random perturbations
Stochastic processes
Stochastic differential equations
Stochastic flows of diffeomorphisms
Stochastic stability

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