REGULAR AND BIREGULAR MODULE ALGEBRAS

CHRISTIAN LOMP

Abstract. Motivated by the study of von Neumann regular skew groups as carried out by Alfaro, Ara and del Rio in [1] we investigate regular and biregular Hopf module algebras. If A is an algebra with an action by an affine Hopf algebra H, then any H-stable left ideal of A is a direct summand if and only if A^H is regular and the invariance functor $(-)^H$ induces an equivalence of A^H-Mod to the Wisbauer category of A as $A\#H$-module. Analogously we show a similar statement for the biregularity of A relative to H where A^H is replaced by $R = Z(A)\cap A^H$ using the module theory of A as a module over $A^e \bowtie H$ the envelopping Hopf algebroid of A and H. We show that every two-sided H-stable ideal of A is generated by a central H-invariant idempotent if and only if R is regular and A_m is H-simple for all maximal ideals m of R. Further sufficient conditions are given for $A\#H$ and A^H to be regular.

1. Introduction

Motivated by the study of von Neumann regular of skew group rings by Alfaro et al. in [1] and by the studies of the regularity of fix rings by Goursad et all in [9] we look at the regularity of Hopf module algebras, their smash products and their subrings of invariants. To achieve our goal we will work in the following more general setting:

Let k be a commutative ring. An extension $A \subseteq B$ of k-algebras is said to have an additional module structure if there exists a ring homomorphism $\Psi : B \to \mathrm{End}_k(A)$ such that $\Psi(a) = L_a$ for all $a \in A$, where L_a denotes the left multiplication of a on A. Then A is a cyclic left B-module with B-action $b \cdot a := \Psi(b)(a)$ for all $b \in B, a \in A$. Moreover $\alpha : B \longrightarrow A$ with $(b)\alpha = b \cdot 1$ is an epimorphism of left B-modules. Note that we will write homomorphisms oposite of scalars. Furthermore $\phi : \mathrm{End}_B(A) \longrightarrow A$ with $\phi(f) = (1)f$ defines a ring homomorphism whose image is denote by A^B. In particular

$$A^B = \{a \in A \mid \forall b \in B : b \cdot a = (b)\alpha a\} = \{a \in A \mid \forall b \in B \ \forall a' \in A : b \cdot (a'a) = (b \cdot a')a\}.$$

Defining for any B-module M:

$$M^B = \{m \in M \mid \forall b \in B \ \forall a \in A : b \cdot (am) = (b \cdot a)m\}$$

one also has functorial isomorphisms

$$\mathrm{Hom}_B(A, M) \longrightarrow M^B \quad f \mapsto (1)f$$

such that $\mathrm{Hom}_B(A, -)$ and $(-)^B$ are isomorphic functors (see [14] for details). In the terminology of [3], B is an A-ring with a right grouplike character.

Examples of the described situation are abundant in the theory of Hopf algebra actions where a Hopf algebra H (or more general a weak Hopf algebra) acts on an algebra A and $A \subseteq B =$

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$A\#H$ is an extension with additional module structure. This also includes group action and Lie actions. Further examples are given by the envelopping algebra $A \subseteq A^e$ or more generally by the envelopping Hopf algebroid $A^e \bowtie H$ as defined in [14] (see also [13] or [7]), k-algebras A with an involution $*$ with $B = A^e * G$ where $G = \langle \sigma \rangle$ is group generated by the automorphism σ of A^e defined by $\sigma(a \otimes b) = b^* \otimes a^*$ or certain extensions $A \subseteq B$ arrising in the study of Banach algebras (see Cabrera et al. [2]).

In this paper we will characterize regular and biregular H-module algebras, generalising some known results on the regularity skew group rings.

All rings will be associative and unital. Ring homomorphisms are supposed to respect the unit. Throughout the text k will denote a commutative ring and A a k-algebra. We denote by $A^e := A \otimes A^{op}$ the enveloping algebra of A whose multiplication is defined as $(a \otimes b)(a' \otimes b') = aa' \otimes b'b$.

2. Regular Modules

John von Neumann defined a ring R to be regular if for any element $a \in R$ there exists an element $b \in R$ such that $a = aba$. He showed in [17] that R is regular if and only if every cyclic (finitely generated) left (right) ideal of R is a direct summand. Later Auslander proved that the regularity of a ring can also be characterised by the property that any module is fla or equivalently that any submodule of a module is pure. Several author's have transfered the regularity condition to modules. A.Tuganbaev in [20] calls a left R-module M regular if any cyclic (finitely generated) submodule is a direct summand using the lattice theoretical approach, while J.Zelmanowitz in [24] followed the original elementwise definition of von Neumann and called a left R-module M regular if for any $m \in M$ there exists $f \in \mathrm{Hom}_R(M, R)$ such that $(m)fm = m$.

The module theoretic version of Auslander's charcterisation had been carried out by Fieldhouse [8] where he called a left R-module regular if any of its submodule is pure in the sense of P.M.Cohen. R.Wisbauer [21] used his ideas to define regularity for nonassociative rings (see also [22, Chapter 34]): Let R be an arbitrary ring and M a left R-module. The Wisbauer category $\sigma[M]$ is the subcategory of R-Mod whose objects are the submodules of M-generated modules, i.e. submodules of factor modules of direct sums of copies of M. A module $P \in \sigma[M]$ is called finitely presented in $\sigma[M]$ if P is finitely generated and every exact sequence in $\sigma[M]$:

$$0 \longrightarrow K \longrightarrow L \longrightarrow P \longrightarrow 0$$

with L finitely generated implies K to be finitely generated. Note that P might be finitely presented in $\sigma[M]$ but not in R-Mod, for example take any simple module $P = M$. A short exact sequence in $\sigma[M]$ is called *pure* if any finitely presented module in $\sigma[M]$ is projective with respect to this sequence and a module $N \in \sigma[M]$ is called *flat* in $\sigma[M]$ if any short exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow N \longrightarrow 0$$

in $\sigma[M]$ is pure. Finally M is called *regular* if any module in $\sigma[M]$ is flat or equivalently if any short exact sequence in $\sigma[M]$ is pure.

2.1. **Relative regularity.** Let $A \subseteq B$ be an extension with additional module structure. Our first aim will be to characterise A as a regular B-module.

Proposition. *Let $A \subseteq B$ be an extension with additional module structure. The following statments are equivalent:*

(a) *A is regular and finitely presented in $\sigma[{}_BA]$;*

(b) *Every left B-stable ideal that is finitely generated as left B-module is a direct summand and A is finitely presented in $\sigma[_BA]$.*
(c) *A^B is von Neumann regular and A is a generator in $\sigma[_BA]$.*
(d) *A^B is von Neumann regular and $()^B = \mathrm{Hom}_B(A, -) : \sigma[_BA] \to A^B{-}Mod$ is an equivalence of categories.*

In this case A is a projective generator in $\sigma[_BA]$.

Proof. $(a) \Leftrightarrow (b)$ follows from [22, 37.4]

$(a) \Rightarrow (c)$ Since A is finitely presented and regular in $\sigma[_BA]$, it is projective in $\sigma[_BA]$. By As A is a cyclic B-module, by [22, 37.8], A is a (finitely generated) projective generator in $\sigma[_BA]$ and A^B is regular.

$(c) \Leftrightarrow (d)$ is clear.

$(c) \Rightarrow (b)$ Since $_BA$ is cyclic and a generator in $\sigma[_BA]$ and since $A^B \simeq \mathrm{End}_B\,(A)$ is regular and thus A is a faithfully flat A^B-module, we have by [22, 18.5(2)] that A is self-projective (and hence projective in $\sigma[_BA]$). This implies that A is also finitely presented in $\sigma[_BA]$. If U is a finitely generated B-stable left ideal of A, then $U = AI$ for $I = U^B$. U being finitely generated as B-module, implies I being finitely generated as right ideal of A^B. Thus $I = A^Be$ for some idempotent e and $U = AI = Ae$ is a direct summand of A, i.e. A is regular by [22, 37.4].

□

2.2. **Relative biregularity.** If $A \subseteq B$ is an extension with additional modules structure $\Psi : B \to \mathrm{End}_k\,(A)$, we might identify B with its image in $\mathrm{End}_k\,(A)$ seeing it as an extension of the subalgebra generated by left multiplications of A. In order to study the two-sided B-stable ideals we might enlarge B by considering $B' = \langle B \cup M(A)\rangle \subseteq \mathrm{End}_k\,(A)$. Note that all B-submodules of A are two-sided and $A^B \subseteq Z(A)$ if $M(A) \subseteq B \subseteq \mathrm{End}_k\,(A)$. A B-stable ideal I is called prime if $JK \subseteq I$ implies $J \subseteq I$ or $K \subseteq I$ for any B-stable ideals J and K. I is semiprime if it is the intersection of prime B-stable ideals. A is B-semiprime if 0 is a prime as B-stable ideal or equivalently A does not contain any non-zero nilpotent B-stable ideal (see [13, 2.3]). If a cyclic B-stable ideal $B \cdot a$ is a direct summand of A, then there exists an idempotent $e \in A^B$ with $B \cdot a = B \cdot e = Ae$ and $A = Ae \oplus A(1-e)$. A k-algebra A is called *B-biregular* if every cyclic B-stable ideal is a direct summand of A. In particular Proposition 2.1 applies to get a characterisation of B-biregular algebras A in case A is finitely presented in $\sigma[_BA]$, namely that A is B-biregular if and only if A^B is a von Neumann regular ring with $(-)^B : \sigma[_BA] \to A^B$-Mod being an equivalence.

2.3. **Properties of relative biregular algebras.** In the next two subsections, we intend to characterise B-biregular algebras A without assuming that A is finitely presented in $\sigma[_BA]$.

Proposition. *Let $M(A) \subseteq B \subseteq \mathrm{End}_k\,(A)$. Suppose that A is B-biregular. Then*

(1) *A^B is von Neumann regular and A is B-semiprime.*
(2) *A is a A^B-Ideal Algebra, i.e. the map $I \mapsto IA$ is a bijection between the ideals I of A^B and the B-stable ideals of A, whose inverse is given by $N \mapsto \mathrm{Ann}_{A^B}(A/N) \simeq \mathrm{Hom}_B\,(A/N, A)$.*
(3) *Every finitely generated B-stable ideal of A is cyclic and is generated by some central idempotent in A^B.*
(4) *For any B-stable ideal I of A, also A/I is B/I-biregular.*
(5) *Every B-stable ideal of A is idempotent and equals the intersection of maximal B-stable ideals.*

(6) *Every prime B-stable ideal is maximal.*

Proof. (1) Let $f \in \mathrm{End}_B\,(A)$, then $(A)f = B(1)f$ is a direct sumand in A by hypothesis, i.e. $(A)f = Ae$ with $e^2 = e \in A^B \subseteq Z(A)$. Since $A(1-e) \subseteq \mathrm{Ker}\,(f) \subseteq l.ann((A)f) = A(1-e)$ also the kernel of f is a direct summand,. Hence by [23, 7.6], $\mathrm{End}_B\,(A)$ and thus A^B is regular. Since no cyclic B-stable ideal is nilpotent, A is B-semiprime.

(2) A generates all cyclic B-stable ideals, i.e. ${}_BA$ is a self-generator and since A^B is regular by (1), ${}_BA$ is intern-projective by [23, 5.6]. Since A is a cyclic B-module, the claim then follows by [23, 5.9].

(3) Let Ae and Af be cyclic B-stable ideals with idempotents $e, f \in A^B$. Then $Ae + Af = A(e + f - ef) = A(e \uplus f)$, where $\uplus$ is the addition in the boolean ring of idempotents $B(A^B)$.

(4) By (2), every B-stable ideal I can be written as $I = JA$ with J ideal in $Z := A^B$. Hence the canonical projection $A = A \otimes_Z Z \to A/I \simeq A \otimes Z/J$ can be understood as the tensoring of the canonical projection of $Z \to Z/J$ by $A \otimes_Z -$, which respects direct sums.

(5) For every cyclic B-stable ideal $B \cdot x = Ae$ we have $(Ae)^2 = A^2e^2 = Ae$. Hence $B \cdot x$ and thus any B-stable ideal is idempotent. Since there are no small B-submodules in A, we have $\mathrm{Rad}\,({}_BA) = 0$ and 0 is the intersection of maximal B-stable ideals. By (4) we can use this argument to each A/I.

(6) Suppose A is B-prime and B-biregular. Let $0 \neq I = Ae$ be a cyclic B-stable ideal with idempotent e. As $A(1-e)$ is a B-stable ideal with $A(1-e)I = 0$, we have $A(1-e) = 0$, i.e. $I = A$ and A is B-simple. □

2.4. **Characterisation of relative biregularity.** The next Proposition characterises biregular extensions $A \subseteq M(A) \subseteq B \subseteq \mathrm{End}_k\,(A)$. Denote by $\mathrm{Max}(A^B)$ the spectrum of maximal ideals of A^B and by A_m the localisation of A by a maximal ideal m of A^B. Note that if A^B is regular, then $A_m = A/mA$ by [23, 17.7] and in particular since mA is B-stable, we might consider $B \subseteq \mathrm{End}_k\,(A/mA) = \mathrm{End}_k\,(A_m)$. We say that A is B-simple if 0 and A are the only B-stable ideals of A.

Theorem. *The following statements are equivalent for an extension $M(A) \subseteq B \subseteq \mathrm{End}_k\,(A)$.*

(a) *A is B-biregular;*
(b) *A^B is regular and every maximal B-stable ideal M of A is of the form $M = AM^B$.*
(c) *A^B is regular and A_m is B-simple for all $m \in \mathrm{Max}(A^B)$.*

Proof. $(a) \Rightarrow (b)$ the properties $(i - iii)$ follow from Proposition 2.3 and (iv) follows from the fact if A is B-biregular then for any $x \in A : l.ann_A(Bx) = A(1-e)$ with $e^2 = e \in A^B$ is already a B-ideal.

$(b) \Rightarrow (c)$: Let m be a maximal ideal of A^B and let M be a maximal B-stable ideal containing $mA \subseteq M$. Since $M = M^BA$ we have

$$m \subseteq (mA)^B \subseteq M^B$$

which implies $M^B = m$ since $M \neq A$. Thus $mA = M$ and $A/M = A/mA = A_m$ is B-simple.

$(c) \Rightarrow (a)$ Let I be any B-stable ideal I of A. Then $I^BA \subseteq I$ and

$$(I^BA)_m = (I \cap A^B)_m A_m = (I_m \cap A_m^B)A_m.$$

If $I_m = A_m$ then $I_m^B = I_m \cap A_m^B = A_m^B$ and hence $(I^BA)_m = I_m$. If $I_m \neq A_m$, then $I_m = 0_m$ and therefore $I_m^B = 0_m$, i.e. $(IA^B)_m = I_m$. Since this holds for any maximal ideal m of A^B, we get $I = I^BA$ which shows that A is a self-generator as B-module.

□

2.5. **Regular subring of invariants.** Assume again that $A \subseteq B$ is any extension with additional module structure. In order to determine when the subring of invariants A^B is regular, we need first to borrow another notion from module theory.

Definition. *A left R-module M is called semi-projective, if every diagram:*

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow g & & \\ M & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

with $N \subseteq M$ can be completed by an endomorphism $h \in S := \mathrm{End}_R(M)$ such that $hf = g$. As it is easily seen: M is semi-projective if and only if $\mathrm{Hom}_R(M, Mf) = Sf$ for all $f \in S$.

Hence A is semi-projective as left B-module if $\forall x \in A^B : (Ax)^B = A^B x$.

Proposition. *Let $A \subseteq B$ be an extension with additional module structure. Then A^B is von Neumann regular if and only if*

1. *A is semi-projective as left B-module and*
2. *every cyclic left ideal generated by an B-invariant element $x \in A^B$ is a direct summand of A as B-module.*

Proof. If $A^B \simeq \mathrm{End}_B(A)$ is regular, then ${}_BA$ is semi-projective by [23, 5.9]. Furthermore since the images of B-linear maps are direct summands and are precisely the cyclic B-stable left ideals generated by a B-invariant element we are done.

On the other hand assume that ${}_BA$ is semi-projective. Let $0 \neq x \in A^B$ then $B \cdot x = Ax$ is a direct summand of A as left B-module by hypothesis. Thus $A = Ax \oplus I$ as left B-modules. But then

$$A^B = (1)(\mathrm{Hom}_B(A, Ax) \oplus \mathrm{Hom}_B(A, I)) = (Ax)^B \oplus I^B = A^B x \oplus I^B.$$

Hence every cyclic left ideal of A^B is a direct summand, i.e. A^B is von Neumann regular. □

2.6. **Large subring of invariants.** If A is finitely presented and regular in $\sigma[{}_BA]$, then by Propositon 2.1 it is a projective generator.in $\sigma[{}_BA]$. Weakening the generator conditon J.Zelmanowitz called a left R-module M *retractable* if $\mathrm{Hom}_R(M, N) \neq 0$ for all non-zero submodules $N \subseteq M$. For a module algebra extension $A \subseteq B$ we say that A^B is *large* in A if $I \cap A^B \neq 0$ for all B-stable left ideals of A or eqivalently if A is a retractable B-module. A classical theorem of Bergmann and Isaacs says that if finite group G acts on an algebra A such that A is G-semiprime and has no $|G|$-torsion, then R^G is large in R.

A purely module theoretical result by J.Zelmanowitz from [25] says now in our language:

Lemma. *Let A be projective in $\sigma[{}_BA]$ and A^B large in A, then*

1. *If A^B is left self-injective, then A is a self-injective left B-module.*
2. *If A^B is von Neumann regular, then A is a non-singular in $\sigma[{}_BA]$, i.e. if $K \subseteq L$ is an essential extension in $\sigma[{}_BA]$, then $\mathrm{Hom}_B(L/K, A) = 0$.*

Proof. Zelmanowitz calls a left R-module M *fully retractable* if $\mathrm{Hom}_R(M, N)g \neq 0$ for any $0 \neq g \in \mathrm{Hom}_R(N, M)$ and submodule $N \subseteq M$. It is easy to see that self-projective retractable modules are fully retractable. Zelmanowitz proves in [25, Proposition on page 567] that M is self-injective if M is fully retractable and left $\operatorname{End}_R(M)$ self-injective. Property (2) follows from [25, Corollary on page 568]. □

Note that a module M is non-singular in $\sigma[M]$ if and only if it is "polyform" in the sense of J.Zelmanowitz (see [23]).

2.7. As a consequence we have that if A is projective in $\sigma[{}_BA]$ and A^B large in A, then A^B is regular and left self-injective if and only if A is injective and non-singular in $\sigma[{}_BA]$, because the endomorphism ring of any self-injective polyform module is self-injective and regular by [23, 11.1].

3. Relative semisimple extensions

Let $A \subseteq B$ be an extension of k-algebras. An element $c = \sum_i c_i \otimes c^i \in B \otimes_A B$ which is B-centralising, i.e. $bc = cb$ for all $b \in B$ is called a *Casimir element* for B over A (see [19] for the terminology). We say that a Casimir element acts unitarily on an element m of a left B-module M if $\left(\sum_i c_i c^i\right) \cdot m = m$.

Proposition. *Let $A \subseteq B$ be an extension with additional module structure and suppose that B has a Casimir element over A that acts unitarily on A, then the following hold:*

(1) *c acts unitarily on any module in $\sigma[{}_BA]$.*
(2) *The k-linear map $\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_B(M,N)$ with $f \mapsto \tilde{f} : [m \mapsto \sum c_i \cdot f(c^i \cdot m)]$ splits the embedding $\operatorname{Hom}_B(M,N) \subseteq \operatorname{Hom}_A(M,N)$ for any $N, M \in \sigma[{}_BA]$.*

Proof. Let $\gamma := \sum c_i c^i$ and $\alpha : B \longrightarrow A$ with $(b)\alpha = b \cdot 1$. Then α is left B-linear and $(a)\alpha = a$ for any $a \in A$. For all $a \in A$ we have $ac = \sum ac_i \otimes c^i = \sum c_i \otimes c^i a = ca$. Then also $\left(\sum ac_ic^i\right)\alpha = \left(\sum c_ic^ia\right)\alpha$ holds. Thus

$$(*) \quad a = a(\gamma)\alpha = \left(\sum ac_ic^i\right)\alpha = \left(\sum c_ic^ia\right)\alpha = \left(\sum c_ic^i\right) \cdot (a)\alpha = \gamma \cdot a.$$

(1) Let $M \in \sigma[{}_BA]$. Then there exists a set Λ and a B-submodule $I \subseteq A^{(\Lambda)}$, such that M is isomorphic to a B-submodule of $A^{(\Lambda)}/I$. We identify M with a submodule of $A^{(\Lambda)}/I$. Let $m \in M$, then there are elements $a_\lambda \in A$ for $\lambda \in \Lambda$ such that $m = (a_\lambda)_\Lambda + I$. Now it follows with $(*)$:

$$\gamma \cdot m = \gamma \cdot [(a_\lambda)_\Lambda + I] = (\gamma \cdot a_\lambda)_\Lambda + I = (a_\lambda)_\Lambda + I = m.$$

(2) Obviously $\tilde{f}$ is B-linear for all $f : M \longrightarrow N$ since c is a Casimir element. If f was already B-linear, then using (1) we get for all $m \in M$:

$$\tilde{f}(m) = \sum c_i \cdot f(c^i \cdot m) = \left(\sum c_ic^i\right) \cdot f(m) = f(m),$$

i.e. $\tilde{f} = f$ showing that the embedding splits. □

3.1. M is a (B,A)-*semisimple* B-module if any short exact sequence in $\sigma[{}_BM]$ that splits as left A-module, also splits as left B-module (see [23, page 170]). Recall that Hirata and Sugano called a ring extension $A \subseteq B$ a semisimple extension if B is (B,A)-semisimple (see [10]).

Corollary. *If B has a Casimir element c which acts unitarily on A, then A is a (B,A)-semisimple B-module and for any $M \in \sigma[{}_BA]$*

- *If M is N-projective as A-module for $N \in \sigma[{}_BA]$, then M is also N-projective as B-module.*
- *If M is N-injective as A-module for $N \in B$-Mod, then M is also N-injective as B-module.*

In particular A is projective in $\sigma[{}_BA]$.

Proof. Let $\pi : M \longrightarrow N$ be a projection in $\sigma[{}_BA]$ with $\pi(n) = n$. Then for any $n \in N$:

$$\tilde{\pi}(n) = \sum c_i \cdot \pi(c^i \cdot n) = \left(\sum c_i c^i\right) \cdot \pi(n) = \pi(n) = n.$$

Thus $\tilde{\pi}$ splits the embedding of N into M as B-module. In the same way one proves the statements (1). For (2) note that if $f : U \to M$ is B-linear, where U is a B-submodule of N, then there exists an A-linear map $g : N \to M$ such that $g_{|U} = f$. Set as before $\tilde{g} : N \to M$ which is B-linear. Then $\tilde{g}(u) = \left(\sum c_i c^i\right) \cdot f(u) = f(u)$. □

3.2. In [10] Hirata and Sugano called a ring extension $A \subseteq B$ *separable* if there exists a Casimir element $c = \sum_i c_i \otimes c^i$ such that $\sum_i c_i c^i = 1$.

Corollary. *Let $A \subseteq B$ be an extension with additional module structure such that there exists a Casimir element in B which acts unitarily on A, then*

1. *If A is a semisimple artinian ring, then A is semisimple B-module.*
2. *If A is von Neumann regular and ${}_AB$ is finitely generated, then*
   - *A is a regular module in $\sigma[{}_BA]$;*
   - *A^B is a regular ring and*
   - *$(-)^B$ defines a Morita equivalence between A^B-Mod and $\sigma[{}_BA]$.*
3. *If $\sigma[{}_BA] = B$-Mod, then $A \subseteq B$ is a semisimple extension.*

Proof. (1) Is clear since A is (B, A)-semisimple.

(2) Since ${}_AB$ is finitely generated, A is finitely presented in $\sigma[{}_BA]$. If $B \cdot a$ is a cyclic B-submodule of A, then by hypothesis $B \cdot a$ is also finitely generated as left A-module and hence a direct summand of A as left A-module. Thus $B \cdot a$ is also a direct summand of A as left B-module since A is (B, A)-semisimple. By 2.1 A is a regular module in $\sigma[{}_BA]$. Also by 2.1 we have that $\mathrm{End}_B\,(A) \simeq A^B$ is regular and A is a progenerator in $\sigma[{}_BA]$ with equivalence $\mathrm{Hom}_B\,(A, -) \simeq (-)^B : \sigma[{}_BA] \longrightarrow \mathrm{End}_B\,(A) \simeq A^B$.

(3) If A is a subgenerator in B-Mod, then $B \in \sigma[{}_BA]$ is itself (B, A)-semisimple. □

4. Applications to Hopf algebra actions

Let H be a Hopf algebra over k acting on an algebra A, i.e. A is a left H-module algebra. The smash product of A and H is denoted by $A\#H$ whose underlying k-module is $A \otimes_k H$ and whose multiplication is defined by

$$(a\#h)(b\#g) = \sum_{(h)} a(h_1 \cdot b)\#h_2 g,$$

where $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$ is the comultiplication of h. Then $A \subseteq A\#H =: B$ is an extension with additional module structure whose module action is given by $a\#h \cdot b = a(h \cdot b)$. The subring of invariants is $A^B = A^H = \{a \in A \mid h \cdot a = \epsilon(h)a \ \forall h \in H\}$. For more details on Hopf algebra action we refer to [16].

4.1. Regularity of the subring of invariants. From 2.5 we get a characterisation of the regularity of the subring of invariants of A.

Proposition. *Let A be a k-algebra with Hopf action H. Then A^H is regular if and only if A is a semi-projective left $A\#H$-module such that any cyclic left ideal generated by an H-invariant element is generated by an H-invariant idempotent.*

4.2. In order to ensure that A is a finitely presented object in $\sigma[{}_{A\#H}A]$ we will assume some finiteness conditions on H or on its action. We say that a Hopf algebra H *acts finitely* on a k-algebra A if the image of the defining action $H \rightarrow \mathrm{End}_k(A)$ is a finitely generated k-module or equivalently if $A\#H/\mathrm{Ann}_{A\#H}(A)$ is finitely generated as left A-module. Recall that a k-algebra A is called *affine* if it is finitely generated as k-algebra.

Denote by $\epsilon : H \rightarrow k$ the counit of H. We need the following Lemma:

Lemma. *Let H be a Hopf algebra over k that is affine as k-algebra, then* $\mathrm{Ker}\,(\epsilon)$ *is a finitely generated left ideal.*

Proof. Suppose that H is affine and let $\mathcal{B} \subseteq H$ be a finite set of elements which generate H as a k-algebra. We will show that $\mathrm{Ker}\,(\epsilon) = \sum_{b\in\mathcal{B}} H(b-\epsilon(b))$. Obviously the right hand side is included in the left hand side. Note that for any word(=product) $\omega = b_1\cdots b_m$ with $b_i \in \mathcal{B}$ we might set $a_0 = \epsilon(\omega)$, $a_m = \omega$ and $a_i = b_1\cdots b_i\epsilon(b_{i+1}\cdots b_m)$ for $0 < i < m$ and conclude that $\omega - \epsilon(\omega) \in \sum_{i=1}^m H(b_i - \epsilon(b_i))$, since as a telescopic sum we have

$$\omega - \epsilon(\omega) = \sum_{i=1}^{m} a_i - a_{i-1} = \sum_{i=1} b_1\cdots b_{i-1}\epsilon(b_{i+1}\cdots b_n)(b_i - \epsilon(b_i)).$$

Take any element $h \in \mathrm{Ker}\,(\epsilon)$. Then there exist $\lambda_i \in k$ and words ω_i in $\mathcal{B}$ such that

$$h = h - \epsilon(h) = \sum_i \lambda_i[\omega_i - \epsilon(\omega_i)] \in \sum_{b\in\mathcal{B}} H(b-\epsilon(b)).$$

Thus $\mathrm{Ker}\,(\epsilon)$ is finitely generated. □

In the telescopic sum argument in the proof of the last Lemma we made use of the fact that the counit ϵ of a Hopf algebra is an algebra homomorphism. We do not know whether this Lemma holds true for affine weak Hopf algebras.

4.3. From Proposition 2.1 we deduce the next result:

Theorem. *If H is an affine k-algebra or acts finitely on A, then A is a finitely presented in $\sigma[{}_{A\#H}A]$ and the following statements are equivalent:*

1. *A is H-regular, i.e. any finitely generated H-stable left ideal is generated by an H-invariant element;*
2. *A is a projective generator in $\sigma[{}_{A\#H}A]$ and any cyclic left ideal generated by an H-invariant element is generated by an H-invariant idempotent.*
3. *A^H is von Neumann regular and $(-)^H : \sigma[{}_{A\#H}A] \rightarrow A^H$-Mod is an equivalence.*
4. *A^H is von Neumann regular and A is a projective generator in $\sigma[{}_{A\#H}A]$.*
5. *A is a regular module in $\sigma[{}_{A\#H}A]$.*

Proof. Once we showed that A is finitely presented in $\sigma[{}_{A\#H}A]$, the result follows from 2.1. If H acts finitely on A, then we might substitute $A\#H$ by $B = A\#H/\mathrm{Ann}_{A\#H}(A)$ which is finitely generated as left B-module. Also α lifts to a map $\overline{\alpha} : B \rightarrow A$ and splits as left A-module map. Thus $\mathrm{Ker}\,(\overline{\alpha})$ is finitely generated as left A-module and thus as left ideal of B, i.e. A is finitely presented in B-Mod and hence in $\sigma[{}_BA] = \sigma[{}_{A\#H}A]$.

On the other hand suppose that H is affine, then $\mathrm{Ker}\,(\epsilon)$ is a finitely generated left ideal by Lemma 4.2. For the module algebra A and $\mathrm{Ker}(\alpha : A\#H \rightarrow A)$, we have that if $x = \sum_{i=1}^n a_i\#h_i \in$

Ker (α), then

$$x=\sum_{i=1}^{n}a_i\#h_i-\left(\sum_{i=1}^{n}a_i\epsilon(h_i)\right)\#1=\sum_{i=1}^{n}a_i\#(h_i-\epsilon(h_i))\in A\#\mathrm{Ker}\,(\epsilon).$$

Thus Ker $(\alpha)=A\#\mathrm{Ker}\,(\epsilon)=\sum_{b\in\mathcal{B}}A\#H(1\#(b-\epsilon(b)))$ is a finitely generated left ideal of $A\#H$ and therefore A is finitely presented. □

4.4. Note that the notion of regularity used here is different from the concept of an H-regular module algebra as defined by [26]. There the author define an element a of an H-module algebra A to be H-regular if $a\in(H\cdot a)A(H\cdot a)$ and calls A H-regular if every element is H-regular.

4.5. The envelopping Hopf algebroid. In general a Hopf action does not extend to the envelopping algebra A^e unless H is cocommutative. In order to study the two-sided H-stable ideals of a Hopf module algebra A with Hopf action H, one defines a new product on the tensor product $A^e\otimes H$ as follows:

$$[(a\otimes b)\bowtie h][(a'\otimes b')\bowtie h']=\sum_{(h)}a(h_1\cdot a')\otimes(h_3\cdot b')b\bowtie h_2h'$$

for all $a\otimes b, a'\otimes b'\in A^e$ and $h,h'\in H$. This construction had been used by the author in [14] (see also [13]) in order to define the central closure of a module algebra A as the self-injective hull of A as $A^e\bowtie H$-module and had also been used by Connes and Moscovici in [7]). A similar construction had been used by L.Kadison in [11] which in [18] was shown to be isomorphic to the construction by Connes-Moscovici. Following Kadison, we denote this algebra on $A^e\otimes H$ by $A^e\bowtie H$ and call it the *envelopping Hopf algebroid* of A and H. For any left $A^e\bowtie H$-module M denote by

$$Z(M)^H:=\{m\in M\mid am=ma\wedge hm=\epsilon(h)m\forall a\in A, h\in H\}.$$

Then since $A\subseteq A^e\bowtie H$ is again an extension with additional module structure we have that $Z(-)^H$ is a functor from $A^e\bowtie H\to Z(A)^H$-Mod and that

$$\mathrm{Hom}_{A^e\bowtie H}\,(A,M)\to Z(M)^H\quad f\mapsto(1)f$$

is a functorial isomorphism. Note that $Z(A)^H:=Z(A)\cap A^H\simeq\mathrm{End}_{A^e\bowtie H}\,(A)$.

From 2.5 we get a characterisation of the regularity of the subring of central invariants of A.

Corollary. *$Z(A)^H$ is regular if and only if A is a semi-projective left $A^e\bowtie H$-module such that any cyclic ideal generated by a central H-invariant element is generated by a central H-invariant idempotent.*

4.6. As before we need to ensure that A is a finitely presented object in $\sigma[_{A^e\bowtie H}A]$ in order to apply 2.1.

Lemma. *If A and H are affine k-algebras, then A is a finitely presented module in $\sigma[_{A^e\bowtie H}A]$.*

Proof. Consider $\alpha:A^e\bowtie H\to A$ by $a\otimes b\bowtie h\mapsto a\epsilon(h)b$. For any $x=\sum_{i=1}^n a_i\otimes b_i\bowtie h_i\in\mathrm{Ker}\,(\alpha)$ we have $\sum_{i=1}^n a_i\epsilon(h_i)b_i=0$. Hence

$$\begin{aligned}x&=\sum_{i=1}^{n}a_i\otimes b_i\bowtie h_i-\left(\sum_{i=1}^{n}a_ib_i\epsilon(h_i)\right)\otimes1\bowtie1+\left[\sum_{i=1}^{n}a_i\otimes b_i\bowtie\epsilon(h_i)-\sum_{i=1}^{n}a_i\otimes b_i\bowtie\epsilon(h_i)\right]\\&=\sum_{i=1}^{n}a_i\otimes b_i\bowtie[h_i-\epsilon(h_i)]+\sum_{i=1}^{n}a_i\epsilon(h_i)[1\otimes b_i-b_i\otimes1]\bowtie1\\&\in A^e\bowtie\mathrm{Ker}\,(\epsilon)+A\mathrm{Ker}\,(\mu)\bowtie1\end{aligned}$$

where Ker (μ) is the augmentation ideal of the envelopping algebra, i.e. the kernel of the multiplication map $\mu : A^e \to A$. Hence we see that Ker (α) is generated as left ideal of $A^e \bowtie H$ by elements of $1 \otimes$ Ker (ϵ) and Ker $(\mu) \bowtie 1$. It is well-known that Ker (μ) is finitely generated left ideal of A^e if A is affine and by 4.2 it follows that Ker (ϵ) is finitely generated if H is affine. □

4.7. H-biregular module algebras. The last statement, 2.1 and 2.4 yield the main result in this section which generalises [4, 1.2] from group actions to Hopf actions.

Corollary. *Let A and H be affine k-algebras, then the following statements are equivalent:*

(a) *A is H-biregular, i.e. every finitely generated H-stable two-sided ideal of A is generated by a central H-invariant idempotent.*
(b) *A is a projective generator in $\sigma[{}_{A^e\bowtie H}A]$ and any ideal generated by a central H-invariant element is generated by an idempotent central H-invariant element.*
(c) *A is a regular module in $\sigma[{}_{A^e\bowtie H}A]$.*
(d) *$Z(A)^H$ is von Neumann regular and one of the following statements hold:*
   (i) *the functor $Z(-)^H : \sigma[{}_{A^e\bowtie H}A] \longrightarrow Z(A)^H$-Mod is an equivalence.*
   (ii) *A is a projective generator in $\sigma[{}_{A^e\bowtie H}A]$.*
   (iii) *every maximal H-stable ideal M of A can be written as $M = [Z(A)^H \cap M]A$.*
   (iv) *A_m is H-simple for any maximal ideal m of $Z(A)^H$.*

4.8. Relative semisimple extension. Let G be a finite group acting on an algebra A. The condition that $|G|$ is invertible is frequently used in the study of group actions because it implies that $A \subseteq A * G$ is a separable extension. The weaker condition on A of having an *element of trace* 1, i.e. an element $z \in A$ such that $t \cdot a = \sum_{g\in G}(g \cdot a) = 1$, where $t = \sum_{g\in G} g$, implies at least the projectivity of A as $A * G$-module. Here we will analyse those concepts and carry them over to Hopf algebra actions.

The antipode of a Hopf algebra H is denoted by S. An element $t \in H$ is called a right(resp. left) integral in H if $th = t\epsilon(h)$ (resp. $ht = \epsilon(h)t$) for all $h \in H$. it is well-known that $\sum_{(t)} S(t_1) \otimes t_2 h = \sum_{(t)} hS(t_1) \otimes t_2$ for all $h \in H$.

Proposition. *Let H be a Hopf algebra and A a left H-module algebra. Suppose H has a non-zero right integral t and A admits a central element z such that $S(t) \cdot z = 1$, then*

$$c := \sum_{(t)} (1\#S(t_1)) \otimes (z\#t_2)$$

is a Casimir element of $A\#H$ that acts unitarily on A. Hence A is a semisimple $(A\#H, A)$-module.

(1) *$A \subseteq A\#H$ is a semisimple extension if $A^H \subseteq A$ is a H^*-Galois extension, i.e. A is a generator in $A\#H$-Mod.*
(2) *$A \subseteq A\#H$ is separable if $z \in A^H$ or H is cocommutative*

Proof. The element $\sum_{(t)} 1\#S(t_1) \otimes z\#t_2$ is a Casimir element in $(A\#H) \otimes_A (A\#H)$ because for all $a\#h \in A\#H$:

$$\begin{aligned}
c(a\#h) &= \sum_{(t)}(1\#S(t_1))\otimes(z\#t_2)(a\#h) \\
&= \sum_{(t)}(1\#S(t_1))\otimes z(t_2\cdot a)\#t_3h \\
&= \sum_{(t)}(S(t_2)\cdot(t_3\cdot a))\#S(t_1)\otimes z\#t_4h \\
&= (a\#1)\left(\sum_{(t)}1\#S(t_1)\otimes z\#t_2h\right) \\
&= (a\#1)\left(\sum_{(t)}1\#hS(t_1)\otimes z\#t_2\right) \\
&= (a\#h\left(\sum_{(t)}1\#hS(t_1)\otimes z\#t_2\right) = (a\#h)c
\end{aligned}$$

Moreover

$$\sum_{(t)}(1\#S(t_1))(z\#t_2)\cdot 1 = \sum_{(t)}S(t_2)\cdot(z(t_2\cdot 1)) = S(t)\cdot z = 1.$$

Thus c acts unitarily on A. By 3, A is a semisimple $(A\#H, A)$-module.

If A/A^H is a H^*-Galois extension, then $\sigma[_{A\#H}A] = A\#H$ and the claim follows from 3.2.

Note that $\mu(c) = \sum_{(t)} S(t_2)z\#S(t_1)t_3$. If $z \in Z(A)^H$, then $\mu(c) = \sum_{(t)} z\#S(t_1)\epsilon(t_2)t_3 = \epsilon(S(t))z\#1 = S(t)z\#1 = 1\#1$. If H is cocommutative, then $\mu(c) = \sum_{(t)} S(t_1)z\#S(t_2)t_3 = S(t)x\#1 = 1\#1$. Hence in both cases $A\#H$ is separable over A. □

4.9. If the antipode is bijective and A has a central element of trace 1, i.e. $z \in Z(A)$ with $t\cdot z = 1$ for a left integral t of H, then $t' = S^{-1}(t)$ is a right integral, and $S(t')\cdot z = 1$ holds, i.e. the condition of 4.8 is fulfilled.

4.10. The observation that $A \subseteq A\#H$ is separable if $z \in Z(A)^H$ or H cocommutative, can also be found in [5, Theorem 1.11] or [6], but under the hypothesis of H being a Frobenius k-algebra and thus finitely generated and projective as k-module. Note that the existence of a left (or right) integral forces a Hopf algebra in many cases to be finitely generated although there are examples of non-finitely generated ones (see [12]).

4.11. Regularity of smash products. In [1] the authors studied the regularity of skew group rings. They showed in particular that a skew group ring $A * G$ is regular if A is regular, G is locally finite and for every finite subgroup H of G there exists a central element of H-trace 1. In this section we will show how much of their arguments go over to smash products.

4.12. First note the following Corollary that we get from 4.8:

Corollary. *Let H be a Hopf algebra acting on a regular module algebra A. Assume that there exists a right integral t of H and a central element z such that $S(t)\cdot z = 1$.*

1. *If H acts finitly on A, then A is regular in $\sigma[_{A\#H}A]$, A^H is regular and A^H-Mod is Morita-equivalent to $\sigma[_{A\#H}A]$.*
2. *If z H-invariant or H is cocommutative or A/A^H is H^*-Galois and $_kH$ is finitely generated, then $A\#H$ is regular.*

Proof. (1) If we substitute $A\#H$ by $B = A\#H/\mathrm{Ann}(A)$, then ${}_AB$ is finitely generated and $c = \sum_{(t)} 1\#S(t_1) \otimes z\#t_2 \in A\#H \otimes_A A\#H$ can be lifted to $c' \in B \otimes_A B$ which still acts unitarily on A. By Corollary 3.2, A has the properties stated above.

(2) if $z \in A^H$ or H is cocommutative, then by 4.8, $A \subseteq A\#H$ is separable and hence $A\#H$ is regular as A was regular. In case A/A^H is H^*-Galois we have that $\sigma[{}_{A\#H}A] = A\#H$-Mod and by 4.8, $A \subseteq A\#H$ is a semisimple extension. Since ${}_kH$ is finitely generated, ${}_AA\#H$ is finitely generated. Hence any cyclic left ideal I of $A\#H$ is finitely generated as left A-submodule of $A\#H$. Since A is regular I is a direct summand of $A\#H$ and as $A \subseteq A\#H$ is semisimple, it is also a direct summand of $A\#H$ as left ideal. □

4.13. Locally finite Hopf algebras. Call an extension $A \subseteq B$ *locally separable* if every element of B is contained in an intermediate algebra $A \subseteq C \subseteq B$ such that C is a separable extension of A (see also A.Magid's definition [15]). Of course, if $A \subseteq B$ is locally separable and A is regular, then B is regular, because any element $x \in B$ is contained in a separable extension C of A. And if A is regular, then also C. Thus $x = xyx$ for some $y \in C \subseteq B$. Hence B is regular. The characterisation of regular group rings $k[G]$ can actually be stated as $k \subseteq k[G]$ being locally separable and k being regular. Alfaro, Ara and del Rio proved in [1, Theorem 1.3] that if G is a locally finite group acting on a regular ring A such that for every finite subgroup H there exists a central element of trace 1 with respect to H, then the skew group ring $A * G$ is regular. We will slightly generalize there result to Hopf algebra actions by showing that the hypotheses of their result imply that $A \subseteq A * G$ is locally separable.

A group is called locally finite if any finitely generated subgroup is finite.

Definition. *Let H be a Hopf algebra over k. A Hopf algebra is called locally finite if any finite set $X \subseteq H$ is contained in a Hopf subalgebra of H which contains a non-zero right integral.*

Note that any Hopf algebra H which is free as a module over k is finitely generated as k-module if and only if it contains a non-zero right integral (see [12]). A group ring $k[G]$ is of course locally finite if G is locally finite.

4.14. We are now in position to generalize [1, Theorem 1.3]:

Corollary. *Let H be a locally finite Hopf algebra over k acting on a k-algebra A such that for any Hopf subalgebra K of H that contains a non-zero right integral t, there exists a central element $z_t \in A$ with $S(t) \cdot z_t = 1$. If H is cocommutative or $z_t \in Z(A)^H$ for all right integrals t, then $A \subseteq A\#H$ is locally separable. Hence if A is regular, so is $A\#H$.*

Proof. Let $x := \sum_i^k a_i\#h_i \in A\#H$. By hypothesis $K :=< \{h_1, \ldots, h_k\} >$ contains a non-zero right integral t. By 4.8(2), $A \subseteq A\#K$ is separable, and hence regular if A was regular. □

4.15. As a consequence we have that if H is a cocommutative Hopf algebra acting on a commutative regular k-algebra having a central element of trace 1, then $A\#H$ is regular (which partly generalizes [1, Corollary 2.5]). It had been shown in [1, 2.4], that if a skew-group ring $A * G$ is regular, then is also A. This is not anymore true for smash products as it is easily seen by the fact, that for any finite dimensional Hopf algebra H over a field k, the smash product $H\#H^* \simeq M_n(k)$ is isomorphic to a semsimple artinian ring, whether H is semisimple or not. However we have that if H is an n-dimensional cosemisimple Hopf algebra over a field k acting on an algebra A such that $A\#H$ is regular, then A is regular. Simply because by the Blattner-Montgomery duality one has $(A\#H)\#H^* \simeq M_n(A)$ and since H^* is separable over k, we have $A\#H\#H^*$ being separable over $A\#H$ inducing regularity on $M_n(A)$ and hence on A.

5. Regularity and injectivity of the subring of invariants

Note that from Zelmanowitz result 2.6 we get

Corollary. *Let A be an H-module algebra that is projective in $\sigma[A]$. If A^H is large in A then A^H is left self-injective and von Neumann regular if and only if A is a self-injective left $A\#H$-module which is non-singular in $\sigma[_{A\#H}A]$. In this case A is also H-semiprime.*

Proof. The equivalence of the statements follows verbatim from 2.6. If I is an H-stable ideal of A with $I^2 = 0$, then $(I^H)^2 = 0$. But since A^H is regular, $I^H = 0$ and since A^H is large $I = 0$. □

5.1. To compare the injectivity of A and its subring of invariants, we need the following Lemma which is probably known:

Lemma. *Let $S \subseteq T$ be rings such that T_S is flat. If T is left self-injective, then so is S.*

Proof. Let I be a left ideal of S, denote the inclusion map by $f : I \longrightarrow S$ and let $g : I \longrightarrow S$ be an S-linear map. Let $\gamma : T \longrightarrow T \otimes_S S$ be the canonical isomorphism. Then γ is left T-linear and $\gamma|_{TI} : TI \longrightarrow T \otimes_S I$ is also an isomorphism of left T-modules. Let $\tilde{f} := \gamma(1 \otimes f)\gamma^{-1} : TI \longrightarrow T$. As T_S is flat, $\tilde{f}$ is injective. Also set $\tilde{g} := \gamma(1 \otimes g)\gamma^{-1} : TI \longrightarrow T$. Then we can consider the following diagram with exact rows, where $\imath : S \longrightarrow T$ denotes the inclusion map (which is of course just S-linear):

$$\begin{array}{ccccc}
0 & \longrightarrow & I & \xrightarrow{f} & S \\
 & & \imath\downarrow & & \imath\downarrow \\
0 & \longrightarrow & TI & \xrightarrow{\tilde{f}} & T \\
 & & \tilde{g}\downarrow & & \\
 & & T & &
\end{array}$$

As T is left self-injective, there there exists a T-linear map $\tilde{h} : T \longrightarrow T$ such that $\tilde{f}\tilde{h} = \tilde{g}$. Hence the outer trapezoid also commutes, i.e $f\imath\tilde{h} = \imath\tilde{g}$. Since for all $x \in I :\ (x)\imath\tilde{g} = (x)g$ we may identify $\imath\tilde{g}$ with g and take $h := \imath\tilde{h}$ to be the desired S-linear map. □

5.2. We will finish with the following result on the transfer of regularity and injectivity to the subring of invariants of a module algebra which should be compared to [9, Theorem A].

Corollary. *Let H be Hopf algebra acting on A. Suppose H has a right integral t and A has a central element z such that $S(t) \cdot z = 1$. If A is regular and left self-injective ring, then A^H is regular and left self-injective.*

Proof. Since A is $(A\#H.A)$-semisimple by 4.8, A is semi-projective as $A\#H$-module. Take any $x \in A^H$, then Ax is a direct summand in A and by relative semisimplicity also a direct summand as $A\#H$-submodule. Thus by 5.1 A^H regular. Now it follows from (1) that A^H is also left self-injective. □

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University of Porto, Department of pure Mathematics, Rua Campo Alegre 687, 4169-007 Porto (PORTUGAL), *clomp@fc.up.pt*