Rotation numbers for planar attractors of equivariant homeomorphisms

B. Alarcón*

University of Oviedo
C/ Calvo Sotelo, s/n
PC: 33007 Oviedo; SPAIN
E-mail address: alarconbegona@uniovi.es

Abstract

Given an integer $m > 1$ we consider $\mathbb{Z}_m-$ equivariant and orientation preserving homeomorphisms in $\mathbb{R}^2$ with an asymptotically stable fixed point at the origin. We present examples without periodic points and having some complicated dynamical features. The key is a preliminary construction of $\mathbb{Z}_m-$ equivariant Denjoy maps of the circle.

1 Introduction

This work is motivated by the study of the global behavior of a planar map having a fixed point which is asymptotically stable but is not a global attractor. In [3], the authors show that it can happen even when there is no periodic points different from the fixed point. Actually, they construct examples of planar dissipative homeomorphisms f such that the set $Rec(f) \setminus \{p\}$ is a Cantor set with almost automorphic dynamics, being $Rec(f)$ the set of recurrent points of f and p the asymptotically stable fixed point. Besides,

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they show that this behaviour is strongly related to the fact that these planar attractors have irrational rotation number.

As we are interested in systems with symmetry, in this work we construct symmetric and orientation preserving dissipative homeomorphisms in the plane with an asymptotically stable fixed point and irrational rotation number. These examples have no periodic points different from the fixed point and also present a complicated dynamical features. This constructions is based on the dissipative homeomorphisms given in [3],

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism with a fixed point which is not a global attractor and its basin of attraction is unbounded. In that case, Theory of Prime Ends due to Carathéodory is applied and f induces an orientation preserving homeomorphism f^* in the space of prime ends. Since this space is homeomorphic to the circle, it is possible to associate a rotation number to f being the rotation number of f^*.

The authors prove in [3] that if f is dissipative and has an irrational rotation number, then the induced map in the space of prime ends is always conjugated to a Denjoy map. Otherwise, they prove in [9] that periodic orbits different from fixed points can appear when the rotation number is rational.

Given a Lie group Γ acting on $\mathbb{R}^2$, a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be $\Gamma-$*equivariant* (or a $\Gamma-$symmetric map) if for all $x \in \mathbb{R}^2$ and $\sigma \in \Gamma$

$$f(\sigma x) = \sigma f(x).$$

In [1] the authors prove that if the group Γ is $SO(2)$ or contains any reflection, the local dynamics of the asymptotically stable fixed point implies global dynamics. That work shows that the symmetry forces the global attraction to arise from an asymptotically stable fixed point in all cases except when the map is $\mathbb{Z}_m-$equivariant.

In [2] are constructed $\mathbb{Z}_m-$equivariant homeomorphisms of the plane with an asymptotically stable fixed point and having periodic points of period m and rotation number $1/m$. So we might be led to think that the presence of the $\mathbb{Z}_m-$ symmetry implies that the rotation number of a homeomorphism should be rational. In this article we give examples which show that this is false. We prove the existence of $\mathbb{Z}_m-$ equivariant homeomorphisms with an asymptotically stable fixed point such that the induced map in the space of prime ends is conjugated to a Denjoy map, which is also $\mathbb{Z}_m-$ equivariant. The idea is to reproduce the construction given in [3] in the context of symmetry.

This work is organized as follows: In Section 2 we explain some notations and results of Denjoy maps in the circle that will be used. In Section 3 we explain the problem in the context of symmetry and construct $\mathbb{Z}_m-$equivariant Denjoy maps in the circle. In section 4 we prove the existence of homeomorphisms of the plane which induce a symmetric Denjoy map in the space of prime ends with the help of some results in [3] and Section 3.

2 Notation and Denjoy map in the circle

We introduce the same notation as in [3]:

We consider the quotient space $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ and points $\bar{\theta}=\theta+\mathbb{Z}$, with $\theta\in\mathbb{R}$. Although all figures are sketched on the unit circle $\mathbb{S}^1=\{z\in\mathbb{C}:|z|=1\}$, which is homeomorphic to $\mathbb{T}$.

The *distance between two points* $\bar{\theta}_1,\bar{\theta}_2\in\mathbb{T}$ is

$$dist_{\mathbb{T}}(\bar{\theta}_1,\bar{\theta}_2)=dist_{\mathbb{R}}(\theta_1-\theta_2,\mathbb{Z})$$

where $dist_{\mathbb{R}}$ indicates the distance from a point to a set on the real line.

A *closed counter-clockwise arc* in $\mathbb{T}$ from p to q, $p\neq q$, will be denoted by $\alpha=\widehat{pq}$ and by $\dot{\alpha}$ its corresponding *open arc*.

We define the *cyclic order* as follows: Given three different points $p_0,p_1,p_2\in\mathbb{T}$ we say that $p_0\prec p_1\prec p_2$ if $p_1\in\widehat{p_0p_2}$.

We define the *cyclic order for arcs* as follows: Given three pairwise-disjoint arcs $\alpha_0,\alpha_1,\alpha_2\subset\mathbb{T}$, we say that $\alpha_0\prec\alpha_1\prec\alpha_2$ if $p_0\prec p_1\prec p_2$ for some $p_0\in\alpha_0,\ p_1\in\alpha_1,\ p_2\in\alpha_2$.

A *Cantor set* C is any compact totally disconnected perfect subset of $\mathbb{T}$ (see [4]). C may be considered as

$$C=\mathbb{T}\setminus\bigcup_{k=0}^{\infty}\dot{\alpha}_k,$$

where $\{\alpha_k\}_{k\geq 0}$ is a family of pairwise disjoint closed arcs in $\mathbb{T}$. The set of *accessible* and *inaccessible points* will be denoted by A and I, respectively. The set A is composed by the end points of all α_k, thus

$$I = \mathbb{T} \setminus \bigcup_{k=0}^{\infty} \alpha_k.$$

Using C we define an equivalence relation on $\mathbb{T}$ by putting $\bar{\theta}_1 \sim \bar{\theta}_2$ (mod C) if $\bar{\theta}_1 = \bar{\theta}_2$ or $\bar{\theta}_1, \bar{\theta}_2 \in \alpha_k$ for some $k \geq 0$. Thus, the *Cantor function* associated to C is a continuous function $\mathcal{P} : \mathbb{T} \to \mathbb{T}$ such that

$$\mathcal{P}(\bar{\theta}_1) = \mathcal{P}(\bar{\theta}_2) \Leftrightarrow \bar{\theta}_1 \sim \bar{\theta}_2.$$

The intuitive idea of this type of maps is to collapse every arc α_k into a point in such a way that the cyclic order is preserved. See [3] and [11] for more details.

The Cantor function $\mathcal{P}$ is onto and $\mathcal{P}(A)$ is a countable and dense subset of $\mathbb{T}$. See [8] for more details.

The *rotation* $R_{\bar{\eta}} : \mathbb{T} \to \mathbb{T}$ is defined by $R_{\bar{\eta}}(\bar{\theta}) = \overline{\theta + \eta}$, where $\eta, \theta \in \mathbb{R}$. Given a homeomorphism f of $\mathbb{T}$, the $f-$ *orbit* starting at a point $\bar{\theta} \in \mathbb{T}$ is denoted by $\mathcal{O}(\bar{\theta})$. The $\omega-$ *limit* of a point $\bar{\theta} \in \mathbb{T}$ is denoted by $\omega(\bar{\theta})$. It is well known (see [11]) that an orientation preserving homeomorphism f in $\mathbb{T}$ with rational rotation number has periodic points. However, if f has irrational rotation number $\rho(f) = \bar{\tau} \notin \mathbb{Q}$, then:

(a) $\omega(\bar{\theta})$ is independent of $\bar{\theta}$.

(b) f is semi-conjugate to the rigid rotation map $R_{\bar{\tau}}$. The semi-conjugacy takes the orbits of f to orbits of $R_{\bar{\tau}}$, is at most two to one on $\omega(\bar{\theta})$ and preserves orientation.

(c) If $\omega(\bar{\theta}) = \mathbb{T}$, then f is conjugate to $R_{\bar{\tau}}$ and the minimal set of f is the whole circle $\mathbb{T}$.

(d) If $\omega(\bar{\theta}) \neq \mathbb{T}$, then the semi-conjugacy from f to $R_{\bar{\tau}}$ collapses the closure of each open interval in the complement of $\omega(\bar{\theta})$ to a point. Moreover the only minimal set of f is a Cantor set C in $\mathbb{T}$.

An orientation preserving homeomorphism $f : \mathbb{T} \to \mathbb{T}$ is said to be a Denjoy map if f has an irrational rotation number $\bar{\tau}$ and f is not conjugated to any rotation. In that case, f admits a Cantor minimal set C_f that attracts all orbits in the future and in the past —and every point in C_f is a recurrent point of f. We can associate to C_f a Cantor function $\mathcal{P}$ which is unique up to rotations and such that $\mathcal{P} \circ f = R_{\bar{\tau}} \circ \mathcal{P}$. So $\mathcal{P}$ is a semi-conjugacy from f to $R_{\bar{\tau}}$.

The construction in [11] of a Denjoy map with irrational rotation number $\bar{\tau}$ consists of choosing a point $\bar{\theta} \in \mathbb{T}$ and determining a family of pairwise disjoint open arcs in $\mathbb{T}$ with decreasing lengths whose sum is one and the complement of the union of all of them is a Cantor set. Each arc is identified with an element of the orbit of $\bar{\theta}$ via $R_{\bar{\tau}}$ which is always dense in $\mathbb{T}$. They also are put in the same order as the elements of the orbit, that is, preserving the cyclic order. These open intervals correspond to the gaps of the Cantor set and the union of the two extremes of all the intervals is the accessible set A of C_f. Next step is to define f on the union of the intervals and then extend the map to the closure.

But it is also possible to generate a Denjoy map considering the $R_{\bar{\tau}}-$orbit orbit of more than one point. Markley proved in [8] that given an irrational number $\tau \notin \mathbb{Q}$ and a countable set $D \neq \emptyset$ in $\mathbb{T}$ such that $R_{\bar{\tau}}D = D$, there exists a Denjoy map with rotation number $\bar{\tau}$ and minimal Cantor set with Cantor function verifying $\mathcal{P}(A) = D$ and being unique up to rotations. For instance, the construction in [11] corresponds to the countable set $\mathcal{P}(A)$ composed by the orbit of a unique point $\overline{\varphi} \in \mathbb{T}$ by the rotation $R_{\bar{\tau}}$, say

$$\mathcal{P}(A) = \{\overline{\varphi + n\tau} : n \in \mathbb{Z}\}$$

Figure 1 illustrates the construction of a Denjoy map considering the orbit of two different points, which is well explained in [3]. The corresponding countable set is

$$\mathcal{P}(A) = \{\overline{\varphi + n\tau} : n \in \mathbb{Z}\} \cup \{\overline{\psi + n\tau} : n \in \mathbb{Z}\}.$$

3 $\mathbb{Z}_m-$equivariant Denjoy maps in the circle

Observe that the construction in Figure 1 depends on the points $\bar{\varphi}, \bar{\psi} \in \mathbb{T}$. Since it can be made with every two points, in this section we consider

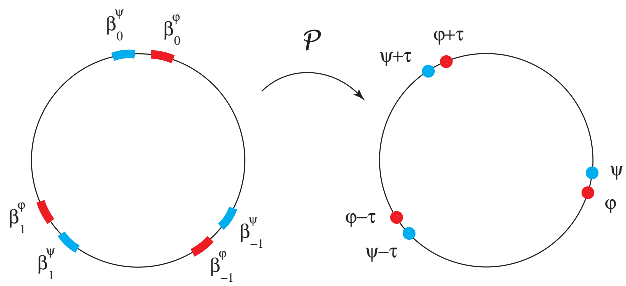

Figure 1: Construction of a Cantor set with two orbits.

$\bar{\psi} = R_{\frac{1}{2}}\bar{\varphi}$ in order to look for any symmetry of the Denjoy map (see Figure 3). This motivated us to study the more general case when the countable set D is the union of the orbits of points which are the rational rotation $R_{\frac{k}{m}}$ of a given point $\bar{\varphi} \in \mathbb{T}$ for same $k = 0, ..., m-1$. That is, given a point $\bar{\varphi} \in \mathbb{T}$ and numbers $\tau \notin \mathbb{Q}$, $m \in \mathbb{N}$ we consider the set

$$\mathcal{P}(A) = \bigcup_{k=0}^{m-1} \{\overline{\varphi^k + n\tau} : n \in \mathbb{Z}\}$$

where $\bar{\varphi}^k = R_{\frac{k}{m}}\bar{\varphi}$, for $k = 0, ..., m-1$. See Figure 2.

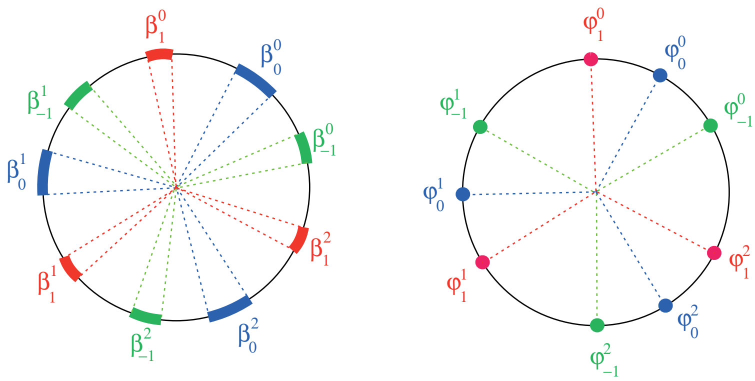

Figure 2: Construction of a Cantor set with 3 symmetric points.

Since $\mathbb{Z}_m = \{R_{\frac{k}{m}}\}_{k=0}^{m-1}$, given a point $\bar{\varphi} \in \mathbb{T}$ we define the set $\{\bar{\varphi}^k\}_{k=0}^{m-1}$ as the *orbit of the group* $\mathbb{Z}_m$ (or $\mathbb{Z}_m$−orbit) of $\bar{\varphi}$, where $\bar{\varphi}^k = R_{\frac{k}{m}}\bar{\varphi}$.

In addition, $\mathbb{Z}_m$ is a cyclic compact Lie group generated by $R_{\frac{1}{m}}$. So, in order to stay that a map f is $\mathbb{Z}_m-$equivariant, we only need to prove that

$$f(\bar{\varphi}+\frac{1}{m})=f(\bar{\varphi})+\frac{1}{m}.$$

See [5] for more details.

In this section we prove the existence of $\mathbb{Z}_m-$equivariant Denjoy maps. For simplicity, details of the proof will be explained only in case $m=2$ because the case $m>2$ is analogous. Firstly we construct a Cantor set which is invariant under the rotation $R_{\frac{1}{2}}$. Secondly we prove the existence of $\mathbb{Z}_2-$equivariant Denjoy maps in the circle with the constructed Cantor set as its minimal set. Finally, we give the keys of the proof in case $m>2$.

Lemma 3.1 (Herman [7], p. 140)**.** *Let D_1, D_2 be two dense subsets in $\mathbb{R}$, and $\phi : D_1 \to D_2$ be a strictly increasing map which is onto. Then ϕ can be extended to a monotone strictly increasing continuous map from $\mathbb{R}$ to $\mathbb{R}$.*

Lemma 3.2. *Let $\tau \notin \mathbb{Q}$. Given a point $\bar{\varphi}$ in the circle, there exists a Cantor set C such that $R_{\frac{1}{2}}C = C$ and the associated Cantor function $\mathcal{P} : \mathbb{T} \to \mathbb{T}$ verifies:*

(a) $\mathcal{P}(A) = \{\overline{\varphi+n\tau}/n \in \mathbb{Z}\} \cup \{\overline{\varphi'+n\tau}/n \in \mathbb{Z}\} \subset \mathbb{T}$, where A is the accessible set of C and $\overline{\varphi}' = R_{\frac{1}{2}}\overline{\varphi}$

(b) $\mathcal{P}$ is $\mathbb{Z}_2-$equivariant.

Proof. Given an angle $\tau \notin \mathbb{Q}$ and an orbit $\{\overline{\varphi+n\tau}/n \in \mathbb{Z}\} \subset \mathbb{T}$ we consider the countable dense set

$$\mathcal{D} = \{\overline{\varphi+n\tau}/n \in \mathbb{Z}\} \cup \{\overline{\varphi'+n\tau}/n \in \mathbb{Z}\} \subset \mathbb{T},$$

such that $\bar{\varphi}' = R_{\frac{1}{2}}\bar{\varphi}$.

We claim that there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of compact subsets of $\mathbb{T}$ such that

$$\bigcap_{n\in\mathbb{N}} A_n = \mathbb{T} \setminus \bigcup_{n\in\mathbb{N}} (\dot{\beta}_n \cup \dot{\beta'}_n \cup \dot{\beta}_{-n} \cup \dot{\beta'}_{-n}),$$

where $\{\beta'_n, \beta_n\}_{n\in\mathbb{Z}}$ is a family of pairwise disjoint closed arcs in $\mathbb{T}$ such that $\beta'_n = R_{\frac{1}{2}}\beta_n$. See Figure 3.

We construct the family $\{A_n\}_{n\in\mathbb{N}}$ by induction on $n \in \mathbb{N}$. It verifies:

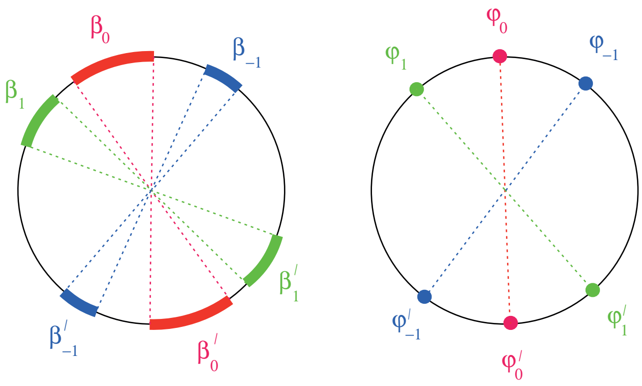

Figure 3: Construction of a Cantor set with two symmetric points.

(a) $A_n = \bigcup_{i=1}^{2k(n)} \gamma_i^n$, where $\gamma_i^n = \widehat{a_i^n b_i^n}$ is a closed arc in $\mathbb{T}$ with end points a_i^n, b_i^n. In addition $\gamma_i^n \cap \gamma_j^n = \emptyset$ for all $i \neq j$ with $i, j = 1, ..., 2k(n)$, where $k(n) = 2n - 1$. Moreover $\gamma_{k(n)+j}^n = R_{\frac{1}{2}} \gamma_j^n$ for $j = 1, ..., k(n)$.

(b) $\bigcup_{j=1}^{n} \{a_i^{j-1}, b_i^{j-1}\}_{i=1}^{2k(j-1)} \subset A_n$. That is, every extreme point of each arc γ_i^k of A_k belongs to A_n, for $k = 1, .., n-1$.

(c) $A_n \subset A_{n-1}$.

(d) $R_{\frac{1}{2}} A_n = A_n$.

(e) The correspondence which associates to each point $\bar{\varphi}_N = \overline{\varphi + N\tau}$ the arc β_N, for all $|N| \leq n-1$, preserves the cycle order. Equivalently, if $\bar{\varphi}_{N_1} \prec \bar{\varphi}_{N_2} \prec \bar{\varphi}_{N_3}$, then $\beta_{N_1} \prec \beta_{N_2} \prec \beta_{N_3}$ for all $|N_i| \leq n-1$.

Consider $A_0 = \mathbb{T}$. We associate to the points $\bar{\varphi}_0 = \overline{\varphi + 0\tau}$ and $\bar{\varphi}_0' = \overline{\varphi' + 0\tau}$ two open arcs β_0 and $\beta_0' = R_{\frac{1}{2}} \beta_0$, respectively, preserving the cyclic order such that $\beta_0 \cap \beta_0' = \emptyset$ and $\mu(\mathbb{T} \setminus \beta_0 \cup \beta_0') < \frac{1}{2}\mu(A_0) = \frac{1}{2}$. We define

$$A_1 = A_0 \setminus \beta_0 \cup \beta_0' = \gamma_1^1 \cup \gamma_2^1,$$

where γ_1^1, γ_2^1 are closed, $\gamma_1^1 \cap \gamma_2^1 = \emptyset$ and $\gamma_2^1 = R_{\frac{1}{2}} \gamma_1^1$. Clearly, $R_{\frac{1}{2}} A_1 = A_1$ and $A_1 \subset A_0$. See Figure 4.

Now we associate to the points $\bar{\varphi}_1 = \overline{\varphi + \tau}$, $\bar{\varphi}_{-1} = \overline{\varphi - \tau}$, $\varphi_1' = \overline{\varphi' + \tau}$ and $\varphi_{-1}' = \overline{\varphi' - \tau}$ four open arcs in A_1, say β_1, β_{-1}, $\beta_1' = R_{\frac{1}{2}} \beta_1$ and $\beta_{-1}' = R_{\frac{1}{2}} \beta_{-1}$,

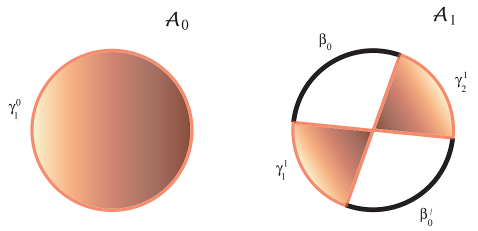

Figure 4: The subsets A_0, A_1

respectively, preserving the cycle order and being pairwise-disjoint such that

$$\mu(A_1 \setminus \beta_1 \cup \beta_1' \cup \beta_{-1} \cup \beta_{-1}') \leq \frac{1}{2}\mu(A_1).$$

We define

$$A_2 = A_1 \setminus \beta_1 \cup \beta_1' \cup \beta_{-1} \cup \beta_{-1}' = \gamma_1^2 \cup .. \cup \gamma_6^2,$$

where $\gamma_1^2, ..., \gamma_6^2$ are closed, pairwise-disjoint and $\gamma_4^2 = R_{\frac{1}{2}}\gamma_1^2, \gamma_5^2 = R_{\frac{1}{2}}\gamma_2^2$ and $\gamma_6^2 = R_{\frac{1}{2}}\gamma_3^2$. Clearly, $R_{\frac{1}{2}}A_2 = A_2$ and $\{a_1^1, b_1^1, a_2^1, b_2^1\} \subset A_2 \subset A_1$. See Figure 5.

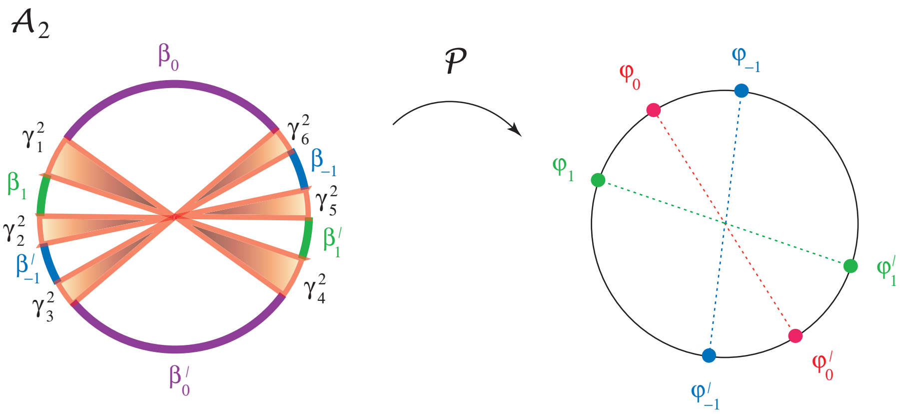

Figure 5: The subset A_2

Suppose by induction that there exists A_n verifying assumptions (a), (b), (c), (d) and (e).

As $\tau \notin \mathbb{Q}$, the orbit of φ is dense in $\mathbb{T}$ so we can associate to the points $\bar{\varphi}_n = \overline{\varphi + n\tau}$, $\bar{\varphi}_{-n} = \overline{\varphi - n\tau}$, $\bar{\varphi}_n' = \overline{\varphi' + n\tau}$ and $\bar{\varphi}_{-n}' = \overline{\varphi' - n\tau}$ four open arcs in A_{n-1}, say β_n, β_{-n}, β_n' and β_{-n}', respectively, such that

- they are pairwise-disjoint and preserving the cyclic order,
- $\beta'_n = R_{\frac{1}{2}}\beta_n$ and $\beta'_{-n} = R_{\frac{1}{2}}\beta_{-n}$,
- if $\bar{\varphi}_{N_1} \prec \bar{\varphi}_{N_2} \prec \bar{\varphi}_{N_3}$, then $\beta_{N_1} \prec \beta_{N_2} \prec \beta_{N_3}$ for all $|N_i| \leq n$.

In addition, the arcs verify

$$\mu(\gamma_i^n \setminus \beta) \leq \frac{1}{2}\mu(\gamma_i^n)$$

where β can be one arc β_n, β_{-n}, β'_n and β'_{-n} or the union of two of such arcs. The arc γ_i^n is the component of A_n containing each β.

Figures 6 and 7 show how one component of A_n could be cut by two arcs or by one. But in both cases the obtained new arcs should have less measure than the first one. That is,

Case I: Suppose the component γ_i^n is cut by only one arc β_n (respectively β'_{-n}), then there appear two new closed arcs $\gamma_j^{n+1}, \gamma_{j+1}^{n+1}$ in γ_i^n such that

$$\mu(\gamma_j^{n+1} \cup \gamma_{j+i}^{n+1}) \leq \frac{1}{2}\mu(\gamma_i^n).$$

Case II: Suppose now the component γ_i^n is cut by the two arcs β_n and β'_{-n}. Then there appear three closed arcs $\gamma_j^{n+1}, \gamma_{j+1}^{n+1}, \gamma_{j+2}^{n+1}$ in γ_i^n such that

$$\mu(\gamma_j^{n+1} \cup \gamma_{j+i}^{n+1} \cup \gamma_{j+1}^{n+1}) \leq \frac{1}{2}\mu(\gamma_i^n).$$

Observe that in the first case the arc $\gamma_i'^n = R_{\frac{1}{2}}\gamma_i^n$ has also been cut by the arcs β'_n and β_{-n} such a way the obtained new arcs are $\gamma_j'^{n+1} = R_{\frac{1}{2}}\gamma_j^{n+1}, \gamma_{j+1}'^{n+1} = R_{\frac{1}{2}}\gamma_{j+1}^{n+1}$ and $\gamma_{j+2}'^{n+1} = R_{\frac{1}{2}}\gamma_{j+2}^{n+1}$ all in $\gamma_i'^n$. The case that the arc γ_i^n had been cut only by one arc β is analogous.

We define

$$A_{n+1} = A_n \setminus \beta_n \cup \beta'_n \cup \beta_{-n} \cup \beta'_{-n} = \bigcup_{i=1}^{2k(n+1)} \gamma_i^{n+1},$$

where $k(n) = 2n-1$ and $\{\gamma_i^{n+1}\}_{i=1}^{2k(n+1)}$ is a family of pairwise-disjoint and closed arcs in $\mathbb{T}$ such that $\gamma_{k(m+1)+i}^{n+1} = R_{\frac{1}{2}}\gamma_i^{n+1}$ for all $i = 1, ..., k(n+1)$. Clearly, $R_{\frac{1}{2}}A_{n+1} = A_{n+1}$ and

$$\{a_i^1, b_i^1\}_{i=1}^{2k(1)} \cup ... \cup \{a_i^n, b_i^n\}_{i=1}^{2k(n)} \subset A_{n+1} \subset A_n \subset ... \subset A_1,$$

that is, the two extreme points of each arc γ_i^k of A_k belong to A_{n+1} for all $k = 1, ..., n$. Therefore, the family $\{A_n\}$ is well defined and

$$\bigcap_{n\in\mathbb{N}} A_n = \mathbb{T} \setminus \bigcup_{n\in\mathbb{N}} (\dot{\beta}_n \cup \dot{\beta'}_n \cup \dot{\beta}_{-n} \cup \dot{\beta'}_{-n})$$

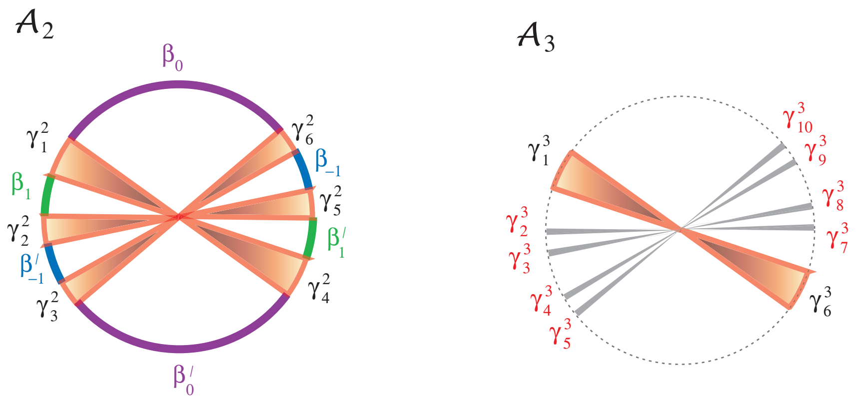

Figure 6: The subset A_3. Case I

Now we claim that $\mu(\bigcap_{n=0}^{\infty} A_n) = 0$, where μ denotes the Lebesgue measure. By construction, for each $m \geq 0$ there exists a natural number $\sigma(m) \geq m$ such that $\mu(A_{\sigma(m)}) < \frac{1}{2}\mu(A_m)$. The compact set $A_{\sigma(m)}$ is the $\sigma(m)-$th element of the family $\{A_n\}_{n=0}^{\infty}$ such that every component of A_m have been cut by an arc β_k or β'_k with $k \in \mathbb{Z}$ at least once.

Let us consider the subfamily $\{A_{\sigma_n(m)}\}_{n=0}^{\infty}$ where the compact set $A_{\sigma_{n+1}(m)}$ is an element of $\{A_n\}_{n=0}^{\infty}$ such that every component of $A_{\sigma_n(m)}$ have been cut by an arc β_k or β'_k with $k \in \mathbb{Z}$ at least once. Then,

$$\mu(A_{\sigma_{n+1}(m)}) \leq \frac{1}{2}\mu(A_{\sigma_n(m)}), \quad \forall n \geq 0$$

and $A_{\sigma_{n+1}(m)} \subseteq A_{\sigma_n(m)}$ for all $n \geq 0$. So

$$0 \leq \mu(A_{\sigma_{n+1}(m)}) \leq \frac{1}{2^n}\mu(A_m) < \frac{1}{2^{n+1}}$$

and

$$0 \leq \mu(\bigcap_{n=0}^{\infty} A_n) \leq \mu(\bigcap_{n=0}^{\infty} A_{\sigma_n(m)}) = \lim_{n\to\infty} \mu(A_{\sigma_n(m)}) = 0.$$

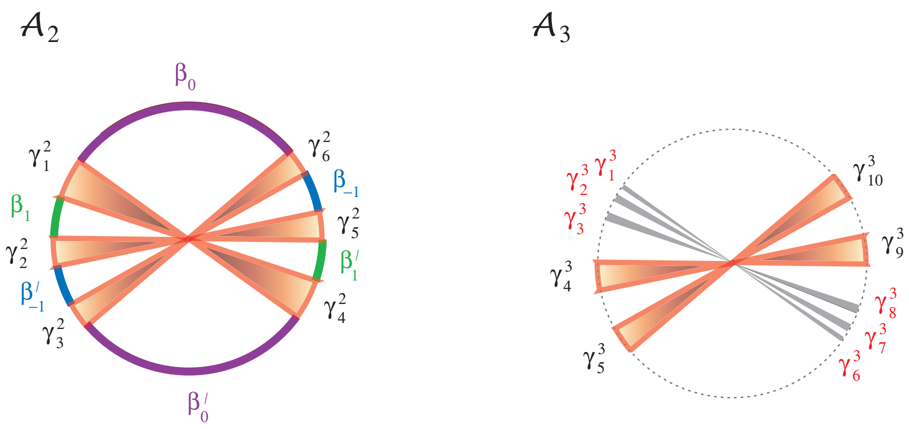

Figure 7: The subset A_3. Case II

Now we claim that

$$C = \bigcap_{n\in\mathbb{N}} A_n = \mathbb{T} \setminus \bigcup_{n\in\mathbb{N}} (\dot{\beta}_n \cup \dot{\beta}'_n \cup \dot{\beta}_{-n} \cup \dot{\beta}'_{-n})$$

is a Cantor set. C is compact because it is an intersection of compact sets. Suppose that there exists a connected component of C different from a singleton $\{p\}$, then there exists an arc belonging to C. Therefore $\mu(\bigcap_{n\in\mathbb{N}} A_n) = 0$, which is impossible. So C is totally disconnected. Now take a point $x \in C$, then $x \in A_n$ for all $n \in \mathbb{N}$. In particular, for each $n \in \mathbb{N}$, there exists a $i(n)-$th compact component of A_n such that $x \in \gamma^n_{i(n)}$. Let x_n be an extreme of $\gamma^n_{i(n)}$ such that $x_n \neq x$. As n increases the component $\gamma^n_{i(n)}$ is being cut in pieces such that the measure of the complement of the gaps is less than the half of the piece I have cut, so the distance between x_n and x is getting smaller. That is, $\forall \varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ and a component $\gamma^n_{i(n)}$ such that, if x is an extreme of $\gamma^n_{i(n)}$,

$$d(x_n, x) = L(\widehat{x_n x}) = \mu(\gamma^n_{i(n)})$$

where $L(\widehat{x_n x})$ denotes the length of the arc $\widehat{x_n x}$. Then,

$$d(x_n, x) = \mu(\gamma^n_{i(n)}) < \frac{1}{2}\mu(\gamma^{n-1}_{i(n-1)}) < ... < \frac{1}{2^n} < \varepsilon, \quad \forall n > n_0.$$

And if $x \in \dot{\gamma}^n_{i(n)}$, then

$$d(x_n, x) < d(x_n, y) < \frac{1}{2^n} < \varepsilon, \quad \forall n > n_0,$$

where y is the other extreme of $\gamma^n_{i(n)}$. Then C is perfect.

Note that, by construction, $R_{\frac{1}{2}} C = C$ and the set of accessible points of C is the union of the two extremes of all components γ^n_i for all $n \in \mathbb{N}$.

Let now to define the associated Cantor function $\mathcal{P} : \mathbb{T} \to \mathbb{T}$ of C. Consider the function

$$\mathcal{P}_* : \bigcup_{n \in \mathbb{Z}} \dot{\beta}_n \cup \dot{\beta'}_n \to \{\overline{\varphi + n\tau} : n \in \mathbb{Z}\} \cup \{\overline{\varphi' + n\tau} : n \in \mathbb{Z}\}$$

such that $\mathcal{P}_*(\dot{\beta}_n) = \overline{\varphi}_n$ and $\mathcal{P}_*(\dot{\beta'}_n) = \overline{\varphi}'_n$, for all $n \in \mathbb{Z}$. It is easy to verify by induction that it is well defined considering the functions

$$\mathcal{P}_n : \bigcup_{i=-n}^{i=n} \dot{\beta}_i \cup \dot{\beta'}_i \to \{\overline{\varphi + N\tau} : |N| \leq n\} \cup \{\overline{\varphi' + N\tau} : |N| \leq n\}$$

such that $\mathcal{P}_n(\dot{\beta}_N) = \overline{\varphi}_N$, $\mathcal{P}_n(\dot{\beta'}_N) = \overline{\varphi}'_N$, $\mathcal{P}_n(\dot{\beta}_{-N}) = \overline{\varphi}_{-N}$, $\mathcal{P}_n(\dot{\beta'}_{-N}) = \overline{\varphi}'_{-N}$, for all $|N| \leq n$ and a given $n \in \mathbb{N}$.

Observe that $\mathcal{P}_*(\dot{\beta'}_n) = \mathcal{P}_*(\dot{\beta}_n) + \frac{1}{2}$ for all $n \in \mathbb{Z}$ by construction. This implies that $\mathcal{P}_* \circ R_{\frac{1}{2}} = R_{\frac{1}{2}} \circ \mathcal{P}_*$ and $\mathcal{P}_*$ is $\mathbb{Z}_2-$equivariant. Now we extend $\mathcal{P}_*$ to the function $\mathcal{P}$ at the points in the closure of $\bigcup_{n \in \mathbb{Z}} \dot{\beta}_n \cup \dot{\beta'}_n$ applying Lemma 3.1.

Actually, $\Omega = \bigcup_{n \in \mathbb{Z}} \dot{\beta}_n \cup \dot{\beta'}_n$ is open and $\mu(\mathbb{T} \setminus \Omega) = 0$, so Ω is dense in $\mathbb{T}$. Note that $D = \{\overline{\varphi + n\tau} : n \in \mathbb{Z}\} \cup \{\overline{\varphi' + n\tau} : n \in \mathbb{Z}\}$ is also dense because $\tau \in \mathbb{R} \setminus \mathbb{Q}$. Let consider $\widehat{\Omega} \subset \mathbb{R}$ and $\widehat{D} \subset \mathbb{R}$ be the lift of Ω and D, respectively. The function $\widehat{\mathcal{P}}_* : \widehat{\Omega} \to \widehat{D}$ is onto and preserves orientation, so it is strictly increasing. By Lemma 3.1, $\widehat{\mathcal{P}}_*$ can be extended to a continuous strictly increasing function $\widehat{\mathcal{P}} : \mathbb{R} \to \mathbb{R}$ such that $\widehat{\mathcal{P}}(x+1) = \widehat{\mathcal{P}}(x) + 1$, for all $x \in \mathbb{R}$. So $\widehat{\mathcal{P}}$ can be consider as a lift of a cyclic order preserving and continuous function $\mathcal{P} : \mathbb{T} \to \mathbb{T}$ such that $\mathcal{P} \circ R_{\frac{1}{2}} = R_{\frac{1}{2}} \circ \mathcal{P}$ and $\mathcal{P}(A) = D_2$, where A is the union of the two extremes of every component γ_i^n for all $n \in \mathbb{N}$, the set of accessible points of the Cantor set C. □

Proposition 3.3. *Let $\tau \notin \mathbb{Q}$. Given a point $\bar{\varphi} \in \mathbb{T}$, there exists a Denjoy map $f : \mathbb{T} \to \mathbb{T}$ such that:*

(a) f is $\mathbb{Z}_2-$equivariant.

(b) $\rho(f) = \bar{\tau}$.

(c) f has a minimal Cantor set as in Lemma 3.2.

Proof. Let $\tau \notin \mathbb{Q}$ and consider a point $\bar{\varphi} \in \mathbb{T}$. By Lemma 3.2, we can construct a Cantor set C such that $R_{\frac{1}{2}} C = C$ and the associated Cantor function $\mathcal{P} : \mathbb{T} \to \mathbb{T}$ is Z_2-equivariant.

Let now to define the Denjoy homeomorphism with minimal set the Cantor set C. We are going to apply Lemma 3.1 in the same way as the construction of $\mathcal{P}$ in Lemma 3.2.

Consider the bijection

$$f_* : \bigcup_{n \in \mathbb{Z}} \dot{\beta}_n \cup \dot{\beta'}_n \to \bigcup_{n \in \mathbb{Z}} \dot{\beta}_n \cup \dot{\beta'}_n$$

verifying:

- For each $n \in \mathbb{Z}$, $f_*(a_n) = a_{n+1}$, $f_*(b_n) = b_{n+1}$, $f_*(a'_n) = a'_{n+1}$ and $f_*(b'_n) = b'_{n+1}$.
- $f_* \circ R_{\frac{1}{2}} = R_{\frac{1}{2}} \circ f_*$.

Analogously to the definition of the Cantor function associated to C, it is easy to verify that f_* is well defined considering for each $n \in \mathbb{N}$ the bijection

$$f_n : \bigcup_{|i| \leq n} \dot{\beta}_i \cup \dot{\beta'}_i \to \bigcup_{|i| \leq n+1} \dot{\beta}_i \cup \dot{\beta'}_i$$

verifying:

- $f_n(a_i) = a_{i+1}$, $f_n(b_i) = b_{i+1}$, $f_n(a'_i) = a'_{i+1}$ and $f_n(b'_i) = b'_{i+1}$.
- $f_n \circ R_{\frac{1}{2}} = R_{\frac{1}{2}} \circ f_n$.

Let $\Omega = \bigcup_{n\in\mathbb{Z}} \dot{\beta}_n \cup \dot{\beta'}_n$. Consider *widehat*$\Omega$ and $\widehat{f}_* : \widehat{\Omega} \to \widehat{\Omega}$ being the lift of Ω and f_*, respectively. As f_* is a bijection and preserves the cyclic order, $\widehat{f}_*$ is a strictly increasing bijection. Then, by Lemma 3.1, it can be extended to a continuous orientation preserving bijection $\widehat{f} : \mathbb{R} \to \mathbb{R}$ such that $\widehat{f}(x+1) = \widehat{f}(x) + 1$ for all $x \in \mathbb{R}$. Moreover $\widehat{f} \circ R_{\frac{1}{2}} = R_{\frac{1}{2}} \circ \widehat{f}$ and, as every continuous and one-to-one map in $\mathbb{R}$ is open, $\widehat{f}$ is an orientation preserving homeomorphism in $\mathbb{R}$. Therefore $\widehat{f}$ can be consider as a lift of an orientation preserving $\mathbb{Z}_2-$equivariant homeomorphism $f : \mathbb{T} \to \mathbb{T}$.

By construction, $R_{\bar{\tau}} \circ \mathcal{P} = \mathcal{P} \circ f$. Therefore, f is an orientation preserving $\mathbb{Z}_2-$equivariant Denjoy homeomorphisms of the circle with rotation number $\bar{\tau}$. □

Theorem 3.4. *Let $\tau \notin \mathbb{Q}$ and $m \in \mathbb{N}$. Given a point $\bar{\varphi} \in \mathbb{T}$, there exists a $\mathbb{Z}_m-$equivariant Denjoy map with rotation number $\bar{\tau}$.*

Proof. The case $m = 2$ is given in Proposition 3.3. The case $m \geq 3$ is analogous considering the dense and countable set

$$D = \bigcup_{k=0}^{m-1} \{\overline{\varphi^k + n\tau} : n \in \mathbb{Z}\}$$

If $A_0 = \mathbb{T}$, we may associate to $\{\bar{\varphi}_0^k\}_{k=0}^{m-1}$, the orbit of $\bar{\varphi}_0 = \bar{\varphi}$, the family of pairwise-disjoint open arcs $\{\beta_0^k\}_{k=0}^{m-1}$ such that $\mu(\bigcup_{k=0}^{m-1} \beta_0^k) < \frac{1}{2}\mu(A_0) = \frac{1}{2}$ and define $A_1 = A_0 \setminus \bigcup_{k=0}^{m-1} \beta_0^k$.

Analogously to case $m = 2$ we can define by induction the sets

$$A_{n+1} = A_n \setminus \bigcup_{k=0}^{m-1} \beta_n^k \cup \beta_{-n}^k, \quad \forall n \in \mathbb{N}$$

such that

(a) $A_n = \bigcup_{i=1}^{mk(n)} \gamma_i^n$, where $\gamma_i^n = \widehat{a_i^n b_i^n}$ is closed and $\gamma_i^n \cap \gamma_j^n = \emptyset$ for all $i \neq j$ and $i, j = 1, ..., mk(n)$. Moreover $k(n) = 2n - 1$ and $\gamma_{i+mj}^n = R_{\frac{j}{m}} \gamma_i^n$ for $j = 1, ..., m-1$ and $i = 1, ..., k(n)$.

(b) $\bigcup_{j=1}^{n}\{a_i^{j-1}, b_i^{j-1}\}_{i=1}^{mk(j-1)} \subset A_n$. That is, every extreme point of each arc γ_i^k of A_k belongs to A_n, for $k = 1, .., n-1$.

(c) $A_n \subset A_{n-1}$.

(d) $R_{\frac{1}{m}} A_n = A_n$.

(e) The correspondence which associates to each point $\bar{\varphi}_N = \overline{\varphi + N\tau}$ the arc β_N, for all $|N| \leq n-1$, preserves the cycle order. Equivalently, if $\bar{\varphi}_{N_1} \prec \bar{\varphi}_{N_2} \prec \bar{\varphi}_{N_3}$, then $\beta_{N_1} \prec \beta_{N_2} \prec \beta_{N_3}$ for all $|N_i| \leq n-1$.

Then, the set

$$C = \bigcap_{n\in\mathbb{N}} A_n = \mathbb{T} \setminus \bigcup_{n\in\mathbb{N}} (\bigcup_{k=0}^{m-1} \beta_n^k \cup \beta_{-n}^k)$$

is a Cantor set such that $R_{\frac{1}{m}} C = C$ and the set of accessible points of C is the union of the two extremes of all components γ_i^n for all $n \in \mathbb{N}$.

If we consider the functions

$$\mathcal{P}_* : \bigcup_{n\in\mathbb{Z}} (\bigcup_{k=0}^{m-1} \dot{\beta}_n^k) \longrightarrow \bigcup_{k=0}^{m-1} \{\overline{\varphi^k + n\tau} : n \in \mathbb{Z}\}$$

such that $\mathcal{P}_*(\dot{\beta}_n^k) = \bar{\varphi}_n^k$ for all $k = 0, ..., m-1$, $n \in \mathbb{Z}$, where $\bar{\varphi}_n^k = \overline{\varphi^k + n\tau}$. Analogously to case $m = 2$, $\mathcal{P}_*$ is well defined and for all $k = 0, ..., m-1$,

$$\mathcal{P}_*(\beta_n^k) = \mathcal{P}_*(\beta_n^0) + \frac{k}{m}.$$

By Lemma 3.1 we can extend $\mathcal{P}_*$ to a function $\mathcal{P} : \mathbb{T} \to \mathbb{T}$ which is the associated Cantor function to C such that $\mathcal{P}(A) = D_m$ and it is $\mathbb{Z}_m-$equivariant.

Now consider the bijection

$$f_* : \bigcup_{n\in\mathbb{Z}} (\bigcup_{k=0}^{m-1} \dot{\beta}_n^k) \longrightarrow \bigcup_{n\in\mathbb{Z}} (\bigcup_{k=0}^{m-1} \dot{\beta}_n^k)$$

verifying:

- For each $n \in \mathbb{Z}$ and $k = 0, ..., m-1$, $f_*(a_n^k) = a_{n+1}^k$ and $f_*(b_n^k) = b_{n+1}^k$, where a_n^k, b_n^k and a_{n+1}^k, b_{n+1}^k are the two extremes of β_n^k and β_{n+1}^k, respectively.
- $f_* \circ R_{\frac{1}{m}} = R_{\frac{1}{m}} \circ f_*$.

Analogously as in Proposition 3.3 f_* is well defined. Moreover, by Lemma 3.1, we can extend f_* to a function $f : \mathbb{T} \to \mathbb{T}$ such that:

(a) f is $\mathbb{Z}_m-$equivariant.

(b) $\rho(f) = \bar{\tau}$.

(c) f has a minimal Cantor set.

□

Observe that the construction of the $\mathbb{Z}_m-$equivariant Denjoy depend on the given point $\bar{\varphi}$. As different points in $\mathbb{T}$ define different Cantor sets we have that the construction of the $\mathbb{Z}_m-$equivariant Denjoy maps of Theorem 3.4 is not unique.

4 Main result

This work is motivated by the study of the global dynamics of an equivariant planar map with an asymptotically stable fixed point. Authors in [1] prove that the symmetry forces the existence of a global attractor in all cases except $\mathbb{Z}_m$. Moreover, they give in [2] a family of $\mathbb{Z}_m-$equivariant homeomorphisms with an asymptotically stable fixed point and rotation number $1/m$. That paper raises the question of whether there are some relationship between the rotation number and the order of the group $\mathbb{Z}_m$. In this section we construct $\mathbb{Z}_m-$equivariant homeomorphisms with an asymptotically fixed point and irrational rotation number. Consequently, there are no linkages between rotation number and the order of the group. Hence, symmetry properties does not give any extra information about global dynamics in the case $\mathbb{Z}_m$.

A dissipative and orientation preserving homeomorphism of the plane $h : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be an $U-$admissible (or admissible) map provided that has an asymptotically stable fixed point with proper and unbounded basis of attraction $U \in \mathbb{R}^2$. Note that the proper condition follows when the fixed point is not a global attractor. We can obtain automatically the unboundedness condition if we suppose that h is area contracting.

It is well studied in [10] the Theory of Prime Ends which states that an admissible map h induces an orientation preserving homeomorphism $h^* : \mathbb{P} \to \mathbb{P}$ in the space of prime ends. This topological space is homeomorphic to the cycle, that is $\mathbb{P} \simeq \mathbb{T}$, and hence the rotation number of h^* is well defined, say $\bar{\rho} \in \mathbb{T}$. The rotation number for an $U-$admissible orientation preserving maps is defined by $\rho(h, U) = \bar{\rho}$.

In [3] the authors prove that given an irrational number $\tau \notin \mathbb{Q}$ and a Denjoy homeomorphism $f : \mathbb{T} \to \mathbb{T}$, there exists an $U-$admissible map $h : \mathbb{R}^2 \to \mathbb{R}^2$ with rotation number $\rho(h, U) = \bar{\tau}$. That motivate us to prove the existence of Z_m-equivariant homeomorphisms of the plane which induce a Denjoy map in the circle of prime ends using the construction given in Section 3.

Proposition 4.1 (Corbato, Ortega and Ruiz del Portal, [3])**.** *Given a $\tau \in (0,1) \setminus \mathbb{Q}$ and a Denjoy map f, there exists an admissible map with rotation number $\rho(h, U) = \bar{\tau}$ and such that h^* is topologically conjugate to f.*

Theorem 4.2. *Given an irrational number $\tau \notin \mathbb{Q}$, there exists a $\mathbb{Z}_m-$equivariant and admissible map in $\mathbb{R}^2$ with rotation number $\bar{\tau} \in \mathbb{T}$ and such that induces a Denjoy map in the circle of prime ends which is also $\mathbb{Z}_m-$equivariant.*

Proof. Let $\tau \notin \mathbb{Q}$ an irrational number and $\bar{\varphi} \in \mathbb{T}$ be a point in the circle. By Theorem 3.4, it is possible to construct a Denjoy map $f : \mathbb{T} \to \mathbb{T}$ which is $\mathbb{Z}_m-$equivariant and has minimal Cantor set C verifying that $R_{\frac{2k\pi}{m}} C = C$, for all $k = 0, ..., m-1$.

By Proposition 4.1, there exists an admissible map with rotation number $\rho(h, U) = \bar{\tau}$ and such that the induced map in the space of prime ends h^* is topologically conjugate to f. Authors in [3] define the homeomorphism h in polar coordinates by:

$$h : \quad \theta_1 = f(\theta), \quad \rho_1 = R(\theta, \rho)$$

where $R : \mathbb{T} \times [0, +\infty) \to [0, +\infty)$ such that

$$R(\bar{\theta}, \rho) = \begin{cases} \frac{1}{2}\rho & \text{if } \rho \leq \frac{1}{2} \\ (\frac{3}{4} - \Pi(\bar{\theta}))(2\rho - 1) + \frac{1}{4} & \text{if } \frac{1}{2} < \rho \leq 1 \\ \frac{1}{2}\rho + \frac{1}{2} - \Pi(\bar{\theta}) & \text{if } \rho > 1 \end{cases}$$

and $\Pi : \mathbb{T} \to \mathbb{R}$ is such that

$$\Pi(\bar{\theta}) = \begin{cases} 0 & \text{if } \bar{\theta} \in C \\ \frac{1}{k(|n|+1)} \frac{dist_{\mathbb{T}}(\bar{\theta}, C)}{length(\beta_n^k)} & \text{if } \bar{\theta} \in \beta_n^k \end{cases}$$

We claim that h is $\mathbb{Z}_m-$equivariant. Since f is $\mathbb{Z}_m-$equivariant, we obtain that

$$f(\bar{\theta} + \frac{k}{m}) = f(\bar{\theta}) + \frac{k}{m}, \quad \forall \bar{\theta} \in \mathbb{T}$$

so we only need to verify that

$$R(\bar{\theta} + \frac{k}{m}) = R(\bar{\theta})$$

or equivalently,

$$\Pi(\bar{\theta} + \frac{k}{m}) = \Pi(\bar{\theta})$$

Indeed, if $\bar{\theta} + \frac{k}{m} \in C$ then $\bar{\theta} \in C$ and $\Pi(\bar{\theta} + \frac{k}{m}) = \Pi(\bar{\theta})$. Otherwise, there exists an arc β_n^j such that $\bar{\theta} + \frac{k}{m} \in \beta_n^j$. In this case, $\bar{\theta} \in \beta_n^{j-k}$ and $dist_{\mathbb{R}}(\bar{\theta}, C) = dist_{\mathbb{R}}(\bar{\theta} + \frac{k}{m}, C)$, so $\Pi(\bar{\theta} + \frac{k}{m}) = \Pi(\bar{\theta})$ and h is $\mathbb{Z}_m-$equivariant. □

Observe that the $\mathbb{Z}_m-$equivariant and admissible map constructed in Theorem 4.2 depends on the initial point $\bar{\varphi}$, so the constructed map is not unique.

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References

[1] B. Alarcón, S.B.S.D Castro and I. Labouriau. Global dynamics for symmetric planar maps, 2012. *Preprint* http://arxiv.org/abs/1202.0779

[2] B. Alarcón, S.B.S.D Castro and I. Labouriau. A $\mathbb{Z}_n-$symmetric local but not global attractor, 2011. *Preprint* http://arxiv.org/abs/1201.4818

[3] L.H. Corbato, R. Ortega and F. R. Ruiz del Portal. Attractors with irrational rotation number. *To appear in Math. Proc. Cambridge Phi. Soc.*

[4] J.G. Hocking and G.S. Young. *Topology*, Addison-Wesley Series in Mathematics, London, 1961.

[5] M. Golubitsky, I. Stewart and D.G. Schaeffer. Singularities and Groups in Bifurcation Theory Vol. 2. *Applied Mathematical Sciences*, 69, Springer Verlag, 1985.

[6] M. Golubitsky and I. Stewart. From Equilibrium to Chaos in Phase Space and Physical Space. The Symmetry Perspective. Birkhäuser Verlag, Basel, 2003.

[7] M. R. Herman. Sur la conjugation différentiable des difféomorphismes du cercle à des rotations. *Publications mathématiques de l'I.H.É.S.*, 49, 5-233, 1979.

[8] N. G. Markley. Homeomorphisms of the circle without periodic points *Proc. London Math. Soc.* 3 (20), 688-698, 1970.

[9] R. Ortega and F. R. Ruiz del Portal. Attractors with vanishing rotation number, *Journal European Math. Soc.*, 13, 1567-1588, 2011.

[10] Ch. Pommerenke. Boundary Behaviour of Conformal Maps. *Springer-Verlag Berlin Heiderberg*, 1992.

[11] C. Robinson. Dynamical Systems. Stability, symbolic dynamics, and chaos. *Studies in Advanced Mathematics. Second Edition.* CRC Press, Boca Raton FL, 1999.