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Roughness in Cayley graphs

M.H. Shahzamanian [a], M. Shirmohammadi [b], B. Davvaz [a,*]

[a] *Department of Mathematics, Yazd University, Yazd, Iran*
[b] *Department of Computer Engineering, Yazd University, Yazd, Iran*

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ABSTRACT

In this paper, rough approximations of Cayley graphs are studied, and rough edge Cayley graphs are introduced. Furthermore, a new algebraic definition for pseudo-Cayley graphs containing Cayley graphs is proposed, and a rough approximation is expanded to pseudo-Cayley graphs. In addition, rough vertex pseudo-Cayley graphs and rough pseudo-Cayley graphs are introduced. Some theorems are provided from which properties such as connectivity and optimal connectivity are derived. This approach opens new research fields, such as data networks.

1. Introduction

Graph theory is rapidly moving into the mainstream of mathematics, mainly due to its applications in diverse fields, including biochemistry (i.e., genomics), electrical engineering (i.e., communications networks and coding theory), computer science (i.e., algorithms and computations) and operations research (i.e., scheduling). The wide scope of these and other applications has been well documented [3,32]. The powerful combinatorial methods developed in graph theory have also been used to prove significant and well-known results in a variety of areas in mathematics. In mathematics, the Cayley graph, also known as the Cayley color graph, is a graph that encodes the structure of a discrete group. Its definition is implied by Cayley's theorem, named after Arthur Cayley, and it uses a particular, usually finite, set of generators for the discrete group. It is a central tool in combinatorial and geometric group theory.

The concept of a rough set was originally proposed by Pawlak [29] as a formal tool for modeling and processing incomplete information in information systems. Since then, the subject has been investigated in many studies. For example, see [12,13,21,27,30,31,33,35–40]. Rough set theory is an extension of set theory in which a subset of a universe is described by a pair of ordinary sets called lower and upper approximations. A key concept in the Pawlak rough set model is the equivalence relation. Equivalence classes are the building blocks for the construction of lower and upper approximations. The lower approximation of a given set is the union of all equivalence classes that are subsets of the set, and the upper approximation is the union of all equivalence classes that have a non-empty intersection with that set. It is well-known that a partition induces an equivalence relation on a set and vice versa. The properties of rough sets can be thus – examined through either partition or equivalence classes. The objects of the given universe U can be divided into three classes with respect to any subset $A \subseteq U$

* Corresponding author.
E-mail addresses: shahzamanian@stu.yazduni.ac.ir (M.H. Shahzamanian), shirmohammadi@stu.yazduni.ac.ir (M. Shirmohammadi), davvaz@yazduni.ac.ir (B. Davvaz).

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(1) the objects, which are in A;
(2) the objects, which are not in A;
(3) the objects, which are possibly in A.

The objects in class 1 form the lower approximation of A, and the objects in classes 1 and 3 together form its upper approximation. The boundary of A contains objects in class 3. Rough sets are a suitable mathematical model for vague concepts, i.e., concepts without sharp boundaries. Rough set theory is thus emerging as a powerful theory to address imperfect data. It has given rise to an expanding research area that encourages explorations in both real-world applications and theory. As such, it has enjoyed practical applications in many areas, including knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on.

Biswas and Nanda [1] introduced the rough subgroup notion. Kuroki [22] defined the rough ideal in a semigroup. Finally, Kuroki and Wang [23] studied the lower and upper approximations with respect to normal subgroups. In [5,8], Davvaz studied the relationship between rough sets and ring theory; he considered the ring as a universal set and introduced the notion of rough ideals and rough subrings with respect to the ideal of a ring. In [19], Kazanc and Davvaz introduced the notions of rough prime (respectively, primary) ideals and rough fuzzy prime (respectively, primary) ideals in a ring and provided some properties of such ideals. Rough modules have been investigated by Davvaz and Mahdavipour [11]. In [34], the notions of rough prime ideals and rough fuzzy prime ideals in a semigroup were introduced. Jun [18] discussed the roughness of Γ-subsemigroups and ideals in Γ-semigroups. In [17], the notion of rough ideals is discussed as a generalization of ideals in BCK-algebras. In [24], Leoreanu-Fotea and Davvaz introduced the concept of n-ary subpolygroups. For more information on algebraic properties of rough sets, refer to [6,7,9,10,4,20,25,26].

In this paper, rough approximations of Cayley graphs are studied and rough edge Cayley graphs are introduced. Furthermore, a new algebraic definition for pseudo-Cayley graphs, which includes Cayley graphs is proposed. Rough approximation is then expanded to pseudo-Cayley graphs. In addition, rough vertex pseudo-Cayley graphs and rough pseudo-Cayley graphs are introduced.

In distributed systems, reliability and fault tolerance are major factors that have received considerable attention across various scientific literatures [16,28]. In special cases, data networks use Cayley graphs as their backbone, as they relate to edge and vertex connectivity. Vertex connectivity (edge connectivity) is the minimum number of vertices (edges) that must be removed in order to disconnect the graph. The fault tolerance of a connected graph is the maximum number k such that if any k vertices are removed, the resulting sub graph is still connected. Reliability focuses on probabilistic edge connectivity. By computing the rough edge Cayley graphs of a modeled network, some parameters can be derived regarding edge connectivity. Also, by computing the vertex rough pseudo-Cayley graphs of a modeled networked, some parameters can be derived with respect to vertex connectivity.

2. Basic facts regarding Cayley graphs

A *graph* is a pair $X = (V(X), E(X))$ of sets satisfying $E(X) \subseteq [V(X)]^2$; thus, the elements of $E(X)$ are 2-element subsets of $V(X)$. The elements of $V(X)$ are *vertices* (or nodes) of the graph X, and the elements of $E(X)$ are its *edges*. A graph Y is a *subgraph* of X (written $Y \subseteq X$) if $V(Y) \subseteq V(X)$, $E(Y) \subseteq E(X)$. When $Y \subseteq X$ but $Y \neq X$, we write $Y \subset X$ and call Y a proper subgraph of X. If Y is a subgraph of X, then X is a supergraph of Y. A spanning subgraph (or spanning supergraph) of X is a subgraph (or supergraph) Y, with $V(Y) = V(X)$, we refer the readers to [2,14].

The *union* $X_1 \cup X_2$ of X_1 and X_2 is the supergraph with vertex set $V(X_1) \cup V(X_2)$ and edge set $E(X_1) \cup E(X_2)$. The *intersection* $X_1 \cap X_2$ of X_1 and X_2 is defined similarly, but in this case, X_1 and X_2 must have at least one vertex in common.

A *walk* of length k in a graph X is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \ldots e_{k-1} v_k$ of vertices and edges in X such that $e_i = (v_i, v_{i+1})$ for all $i < k$. If $v_0 = v_k$, the walk is closed. If the vertices in a walk are all distinct, it defines an obvious *path* in X. In general, every walk between two vertices contains a path between these vertices.

A non-empty graph X is *connected* if any two of its vertices are linked by a path in X. A connected graph X is *optimally connected* if every spanning subgraph of X is not connected.

Definition 2.1. Taking any finite group G, let $S \subset G$ be such that $1 \notin S$, where 1 represents the identity element of G, and $s \in S$ implies that $s^{-1} \in S$, where s^{-1} represents the inverse element of s. The *Cayley graph* $(G; S)$ is a graph with vertices labeled with the elements of G in which there is an edge between two vertices g and gs if and only if $s \in S$.

The exclusion of 1 from S eliminates the possibility of loops in the graph. The inclusion of the inverse of any element that is itself in S indicates that an edge is in the graph regardless of which end vertex is considered.

Let R be a subset of G that is not a group; if R contains S and $SR \subseteq R$ where $SR = \{sr | s \in S, r \in R\}$, then the *pseudo-Cayley graph* $(R; S)$ is a graph with vertices labeled with the elements of R in which there is an edge between two vertices r and rs if and only if $s \in S$.

Let G be a group, and let X be a subset of G. Let $\{H_i \mid i \in I\}$ be the family of all subgroups of G that contain X. Then $\bigcap_{i \in I} H_i$ is the *subgroup* of G *generated by the set* X and denoted by $< X >$. Obviously, if $< X > = G$, then X generates G.

Theorem 2.2 [15]. *A Cayley graph $(G; S)$ is connected if and only if S generates G.*

A subset S of G is a *minimal Cayley set* if it generates G and if $S \backslash \{s, s^{-1}\}$ generates a proper subgroup of G for all $s \in S$.

Theorem 2.3 [28]. *If S is a minimal Cayley set for the finite group G, then the Cayley graph (G; S) has optimal connectivity.*

Theorem 2.4. *If $X_1 = (G; S_1)$ and $X_2 = (G; S_2)$ are Cayley graphs, then*

(1) $X_1 \cup X_2 = (G; S_1 \cup S_2)$,
(2) $X_1 \cap X_2 = (G; S_1 \cap S_2)$.

Proof

(1) Let e be an edge of $(G; S_1 \cup S_2)$. Then there exist $g \in G$ and $s \in S_1 \cup S_2$ such that e connects two vertices g and gs. Since $s \in S_1 \cup S_2$, then $s \in S_1$ or $s \in S_2$, which means that $e \in E(X_1)$ or $e \in E(X_2)$. Therefore, by definition, $e \in E(X_1 \cup X_2)$. Conversely, in the similar way, any edge of $E(X_1 \cup X_2)$ is an edge of $(G; S_1 \cup S_2)$. This result together with $V(X_1 \cup X_2) = V((G; S_1 \cup S_2)) = G$ means that $X_1 \cup X_2 = (G; S_1 \cup S_2)$.
(2) The proof is then straightforward. □

Note that $X_1 \cup X_2$ and $X_1 \cap X_2$ are Cayley graphs.

Theorem 2.5. *If $X_1 = (H_1; S)$ and $X_2 = (H_2; S)$ ($H_1, H_2 \leqslant G$, which means that H_1 and H_2 are subgroups of G) are Cayley graphs, then*

(1) $X_1 \cup X_2 = (H_1 \cup H_2; S)$,
(2) $X_1 \cap X_2 = (H_1 \cap H_2; S)$.

Proof. The proof is similar to Theorem 2.4. □
Note that $X_1 \cap X_2$ is a Cayley graph, but $X_1 \cup X_2$ may not be a Cayley graph. $X_1 \cup X_2$ is always a pseudo-Cayley graph.

Theorem 2.6. *If $X_1 = (H_1; S_1)$ and $X_2 = (H_2; S_2)(H_1, H_2 \leqslant G)$ are Cayley graphs, then $X_1 \cap X_2 = (H_1 \cap H_2; S_1 \cap S_2)$.*

Proof. The proof is similar to 2.4. □

Note that $X_1 \cap X_2$ is a Cayley graph. $X_1 \cup X_2$ may not be a pseudo-Cayley graph.

Theorem 2.7. *If $X_1 = (G; S_1)$, $X_2 = (G; S_2)$, $Y_1 = (G_1; S)$ and $Y_1 = (G_2; S)$ are Cayley graphs, then*

(1) $X_1 \subseteq X_2$ *if and only if* $S_1 \subseteq S_2$,
(2) $Y_1 \subseteq Y_2$ *if and only if* $G_1 \subseteq G_2$.

Proof

(1) Let S_1 be a subset of $S_2 (S_1 \subseteq S_2)$. Suppose that e is an arbitrary edge of $E(X_1)$. Then there exist $g \in G$ and $s_1 \in S_1$ such that $e = (g; gs_1)$. Since $s_1 \in S_1 \subseteq S_2$, then $e \in E(X_2)$. Therefore, $E(X_1)$ is a subset of $E(X_2)$. Conversely, let $E(X_1)$ be a subset of $E(X_2)(E(X_1) \subseteq E(X_2))$. Suppose that s_1 is an element of S_1. For every $g \in G$, we have $(g; gs_1) \in E(X_1)$. Therefore, this yields $(g; gs_1) \in E(X_2)$, and as a result, $s_1 \in S_2$. Then $S_1 \subseteq S_2$. This result together with $V(X_1) = V(X_2) = G$ means that $X_1 \subseteq X_2$ if and only if $S_1 \subseteq S_2$.
(2) The proof is now straightforward. □

3. Rough groups

If N is a subgroup of a group G, then the following conditions are equivalent.

(1) The left and right congruence modulos N coincide; that is, they define the same equivalence relation on G,
(2) $aN = Na$ for all $a \in G$,
(3) for all $a \in G$, $aNa^{-1} \subseteq N$, where $aNa^{-1} = \{ana^{-1} | n \in N\}$,
(4) for all $a \in G$, $aNa^{-1} = N$.

A subgroup N of a group G that satisfies the above equivalent conditions is said to be *normal* in G or, equivalently, a *normal subgroup* of G. Let H and N be normal subgroups of group G. Then as can be easily seen, $H \cap N$ is also a normal subgroup of G.
Let G be a group with identity 1 and N be a normal subgroup of G. If A is a non-empty subset of G, then the sets

$$N_{-}(A) = \{x \in G | xN \subseteq A\} \quad \text{and} \quad N^{\wedge}(A) = \{x \in G | xN \cap A \neq \emptyset\}$$

are the *lower* and *upper approximations of the set A, respectively, with respect to the normal subgroup N.*

Theorem 3.1 [23]. *Let H and N be normal subgroups of a group G. Let A and B be any non-empty subsets of G. Then*

(1) $N\_(A) \subseteq A \subseteq N\^(A)$,
(2) $N\^(A \cup B) = N\^(A) \cup N\^(B)$,
(3) $N\_(A \cap B) = N\_(A) \cap N\_(B)$,
(4) $A \subseteq B$ *implies* $N\_(A) \subseteq N\_(B)$,
(5) $A \subseteq B$ *implies* $N\^(A) \subseteq N\^(B)$,
(6) $N\_(A \cup B) \supseteq N\_(A) \cup N\_(B)$,
(7) $N\^(A \cap B) \subseteq N\^(A) \bigcap N\^(B)$,
(8) $N \subseteq H$ *implies* $N\^(A) \subseteq H\^(A)$,
(9) $N \subseteq H$ *implies* $H\_(A) \subseteq N\_(A)$.

Theorem 3.2 [23]. *Let H and N be normal subgroups of a group G. If A is a non-empty subset of G, then*

(1) $(H \cap N)\^(A) = H\^(A) \cap N\^(A)$,
(2) $(H \cap N)\_(A) = H\_(A) \cap N\_(A)$.

$N(A) = (N\_(A), N\^(A))$ is a *rough set* of A in G. A non-empty subset A of group G is called an *$N\^$-rough (normal) subgroup* of G if the upper approximation of A is a (normal) subgroup of G. Similarly, a non-empty subset A of G is called an *$N\_$-rough (normal) subgroup* of G if the lower approximation is a (normal) subgroup of G.

Theorem 3.3 [23]. *Let N be a normal subgroup of group G.*

(1) *If A is a subgroup of G, then it is an $N\^$rough subgroup of G.*
(2) *If A is a normal subgroup of G, then it is an $N\^$-rough normal subgroup of G.*
(3) *If A is a subgroup of G such that $N \subseteq A$, then it is an $N\_$-rough subgroup of G.*
(4) *If A is a normal subgroup of G such that $N \subseteq A$, then it is an $N\_$-rough normal subgroup of G.*

4. Rough edge Cayley graphs

In this section, the concepts of lower and upper approximations edge Cayley graphs of a Cayley graph with respect to a normal subgroup is discussed; some properties of the lower and upper approximations are then introduced.

Definition 4.1. Let G be a finite group with identity 1; let N be a normal subgroup of G; and let $X = (G; S)$ be a Cayley graph. Then the following graphs:

$$\overline{X} = (G; N\^(S)^*) \quad \text{and} \quad \underline{X} = (G; N\_(S))$$

where $N\^(S)^* = N\^(S) \backslash 1$, are called *lower and upper approximations edge Cayley graphs*, respectively, of the Cayley graph $X(G; S)$ with respect to the normal subgroup N. We will now prove that these graphs are Cayley graphs.

Theorem 4.2. *The two graphs $\underline{X}$ and $\overline{X}$ are Cayley graphs.*

Proof. Theorem 3.1 (1) yields $N\_(S) \subseteq S$. Note then that $1 \notin N\_(S)$. Suppose that s is an arbitrary element of $N\_(S)$. Then $sN \subseteq S$, which implies that $sn^{-1} \in S$ for all $n \in N$. Then $(sn^{-1})^{-1} = ns^{-1} \in S$, and so either $Ns^{-1} \subseteq S$ or $s^{-1}N \subseteq S$. Therefore, $s^{-1} \in N\_(S)$.

Now, suppose that s is an arbitrary element of $N\^(S)^*$. Then $sN \cap S \neq \emptyset$ which implies that there exists $a \in sN \cap S$. Hence there exists $n \in N$ such that $a = sn \in S$, so $a^{-1} = n^{-1}s^{-1} \in S$. Alternatively, we have $n^{-1}s^{-1} \in Ns^{-1} = s^{-1}N$. Thus, $n^{-1}s^{-1} \in S \cap s^{-1}N$, which implies that $s^{-1}N \cap S \neq \emptyset$, and so $s^{-1} \in N\^(S)^*$.

Therefore, $\underline{X}$ and $\overline{X}$ are Cayley graphs. □

Example 4.3. Let G be a group congruence modulo 8 integral number Z. Let $N = \{\underline{0}, \underline{4}\}$ be a normal subgroup of G and Cayley graph $X = (G; S)$ such that S equals $\{\underline{1}, \underline{2}, \underline{6}, \underline{7}\}$. We then have $\overline{X} = (G; \{\underline{1}, \underline{2}, \underline{3}, \underline{5}, \underline{6}, \underline{7}\})$ and $\underline{X} = (G; \{\underline{2}, \underline{6}\})$; see Fig. 1.

Theorem 4.4. *Let N and H be normal subgroups of a group G. Let $X = (G; S)$, $X_1 = (G; S_1)$ and $X_2 = (G; S_2)$ be Cayley graphs. Then we have*

(1) $\underline{X} \subseteq X \subseteq \overline{X}$,
(2) $\overline{X_1 \cup X_2} = \overline{X_1} \cup \overline{X_2}$,
(3) $\underline{X_1 \cap X_2} = \underline{X_1} \cap \underline{X_2}$,

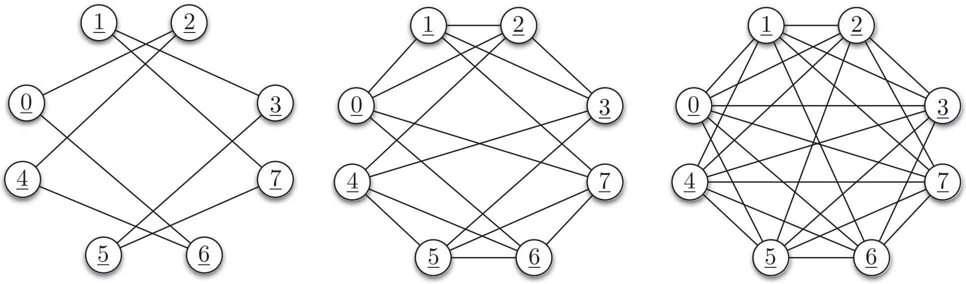

Fig. 1. The above graphs are $\underline{X}$, X and $\overline{X}$.

(4) $X_1 \subseteq X_2 \Rightarrow \underline{X_1} \subseteq \underline{X_2}$,
(5) $X_1 \subseteq X_2 \Rightarrow \overline{X_1} \subseteq \overline{X_2}$,
(6) $\underline{X_1 \cup X_2} \supseteq \underline{X_1} \cup \underline{X_2}$,
(7) $\overline{X_1 \cap X_2} \subseteq \overline{X_1} \cap \overline{X_2}$,
(8) $N \subseteq H \Rightarrow \overline{X_N} \subseteq \overline{X_H}$,
(9) $N \subseteq H \Rightarrow \underline{X_H} \subseteq \underline{X_N}$.

Proof

(1) From Theorem 3.1 (1), $N\_(S) \subseteq S \subseteq N^{\wedge}(S)$. Then $N\_(S) \subseteq S \subseteq N^{\wedge}(S)^*(1 \notin S)$. Theorem 2.7 (1) then yields $\underline{X} \subseteq X \subseteq \overline{X}$.
(2) Theorem 2.4 yields $X_1 \cup X_2 = (G; N^{\wedge}(S_1)^* \cup N^{\wedge}(S_2)^*)$. By Theorem 3.1 (5), we have $N^{\wedge}(S_1)^*$ and $N^{\wedge}(S_2)^* \subseteq N^{\wedge}(S_1 \cup S_2)^*$. Then Theorem 2.7 implies that $\overline{X_1} \cup \overline{X_2} \subseteq \overline{X_1 \cup X_2}$. Conversely, according to Theorem 3.1 (2), we have $N^{\wedge}(S_1)^* \cup N^{\wedge}(S_2)^* = N^{\wedge}(S_1 \cup S_2)^*$. Suppose that $(g; gs)$ is any edge of $E(\overline{X_1 \cup X_2})$ and $s \in N^{\wedge}(S_1 \cup S_2)^*$. Then we obtain $s \in N^{\wedge}(S_1)^* \cup N^{\wedge}(S_2)^*$, and either $s \in N^{\wedge}(S_1)^*$ or $s \in N^{\wedge}(S_2)^*$. Therefore, $(g; gs)$ is any edge of$\overline{X_1}$ or $\overline{X_2}$. Finally, we have $\overline{X_1 \cup X_2} = \overline{X_1} \cup \overline{X_2}$.
(3) By Theorem 3.1 (3), the proof is similar to (2).
(4) Assume that $X_1 \subseteq X_2$. Then $S_1 \subseteq S_2$, which implies that $N\_(S_1) \subseteq N\_(S_2)$. Hence, $\underline{X_1} \subseteq \underline{X_2}$.
(5) Using Theorem 3.1 (5), the proof is similar to (4).
(6) Theorem 3.1 (6) gives $N\_(S_1) \cup N\_(S_2) \subseteq N\_(S_1 \cup S_2)$. Then $N\_(S_1) \subseteq N\_(S_1 \cup S_2)$ and $N\_(S_2) \subseteq N\_(S_1 \cup S_2)$, therefore we have $\underline{X_1 \cup X_2} \supseteq \underline{X_1}$ and $\underline{X_1 \cup X_2} \supseteq \underline{X_2}$. And finally, $\underline{X_1 \cup X_2} \supseteq \underline{X_1} \cup \underline{X_2}$.
(7) By Theorem 3.1 (7), the proof is similar to (6).
(8) Assume that $N \subseteq H$. Theorem 3.1 (8) yields $N^{\wedge}(S) \subseteq H^{\wedge}(S)$. Then $N^{\wedge}(S)^* \subseteq H^{\wedge}(S)^*$. Now, based on Theorem 2.7 (1), we obtain $\overline{X_N} \subseteq \overline{X_H}$.
(9) Using Theorem 3.1 (9), the proof is similar to (8). □

Example 4.5. Here, we present some examples that show the contradictions that emerge from assuming the converse of the above items (4–9). Let G be a dihedral group with order 6.

(1) $X^1 \not\subseteq X^2$, $\underline{X_1} \subseteq \underline{X_2}$: $X_1 = (G; \{\varepsilon\})$, $X_2 = (G; \{P\varepsilon\})$, $N = \{1, P, P^2\}$.
(2) $X_1 \not\subseteq X_2, \overline{X_1} \subseteq \overline{X_2} : X_1 = (G; \{\varepsilon\}), X_2 = (G; \{P\varepsilon\}), N = G$.
(3) $\underline{X_1} \cup \underline{X_2} \not\subseteq \underline{X_1 \cup X_2} : X_1 = (G; \{\varepsilon, P^2\varepsilon\}), X_2 = (G; \{\varepsilon, P\varepsilon\}), N = \{1, P, P^2\}$.
(4) $\overline{X_1} \cap \overline{X_2} \not\subseteq \overline{X_1 \cap X_2} : X_1 = (G; \{P^2\varepsilon\}), X_2 = (G; \{P\varepsilon\}), N = \{1, P, P^2\}$.
(5) $N \not\subseteq H, \overline{X_N} \subseteq \overline{X_H} : X = (G; \{P, P^2, \varepsilon, P\varepsilon, P^2\varepsilon\}), N = \{1, P, P^2\}, H = \{1\}$.
(6) $N \not\subseteq H$, $\underline{X_H} \subseteq \underline{X_N}$: $X = (G; \{\varepsilon, P\varepsilon, P^2\varepsilon\})$, $N = \{1, P, P^2\}$, $H = \{1\}$.

Theorem 4.6. *Let N and H be normal subgroups of a group G. Let X = (G;S) be a Cayley graph. Then*

(1) $\overline{X}_{H \cap N} = \overline{X}_H \cap \overline{X}_N$,
(2) $\underline{X}_{H \cap N} = \underline{X}_H \cap \underline{X}_N$.

Proof

(1) We have

$$\overline{X}_{H \cap N} = (G; (H \cap N)^{\wedge}(S)) = (G; H^{\wedge}(G) \cap N^{\wedge}(S)) = (G; H^{\wedge}(S)) \cap (G; N^{\wedge}(S)) = \overline{X}_H \cap \overline{X}_N.$$

(2) Using Theorem 3.2 (2), the proof is similar to (1).

$(\underline{X}, \overline{X})$ is called a *rough edge Cayley graph* of $X = (G; S)$. A Cayley graph $X = (G; S)$ is an *$N^{\wedge}$-edge rough generating* if $N^{\wedge}(S)^{*}$ is a generating set for G. Similarly, a Cayley graph $X = (G; S)$ is an *N_{-}-edge rough generating* if $N_{-}(S)$ is a generating set for G.

A Cayley graph $X = (G; S)$ is called an *$N^{\wedge}$-edge rough optimal connected* if the $N^{\wedge}(S)^{*}$ is a minimal Cayley set for G. Similarly, a Cayley graph $X = (G; S)$ is called an *N_{-}-edge rough optimal* if $N_{-}(S)$ is a minimal Cayley set for G. □

Theorem 4.7. *Let $X = (G; S)$ be a Cayley graph. If X is $N^{\wedge}$-edge rough generating, then $\overline{X}$is connected. Similarly, If X is $N_{-}(S)$-edge rough generating, then $\underline{X}$ is connected.*

Proof. The proof is straightforward. □

Theorem 4.8. *Let $X = (G; S)$ be a Cayley graph. If X is $N^{\wedge}$-edge rough optimally connected, then $\overline{X}$is optimally connected. Similarly, if X is $N_{-}(S)$-edge rough optimally connected, then $\underline{X}$ is optimally connected.*

Proof. The proof is straightforward. □

5. Rough vertex pseudo-Cayley graphs

In this section, the concept of a lower and upper approximations vertex pseudo-Cayley graph of a pseudo-Cayley graph with respect to a normal subgroup is introduced. We prove that the lower and upper approximations are pseudo-Cayley graphs as well. Some properties of lower and upper approximations are also discussed.

Definition 5.1. Let G be a finite group with identity 1; let N be a normal subgroup; let R be a subset of G; and let $X = (R; S)$ be a pseudo-Cayley graph. Then the following graphs

$$\overline{X}' = (N^{\wedge}(R); S) \quad \text{and} \quad \underline{X}' = (N_{-}(R); S \cap N_{-}(R))$$

are called, respectively, *lower* and *upper approximations vertex pseudo-Cayley graphs* of a pseudo-Cayley graph $X(R; S)$ with respect to the normal subgroup N. We will now prove that these graphs are pseudo-Cayley graphs.

Theorem 5.2. *The two graphs $\underline{X}'$ and $\overline{X}'$ are pseudo-Cayley graphs.*

Proof. The definition of a pseudo-Cayley graph implies that $S \subseteq R$. Then $S \subseteq N^{\wedge}(R)$. If $s \in S$ and $a \in N^{\wedge}(R)$, then $aN \cap R \neq \emptyset$. Hence, there exists $n \in N$ such that $an \in R$. Since X is a pseudo-Cayley graph, then $SR \subseteq R$. So $san \in R$, which implies that $saN \cap R \neq \emptyset$, and so $sa \in N^{\wedge}(R)$. Thus $SN^{\wedge}(R) \subseteq N^{\wedge}(R)$. Then $\overline{X}'$ is a pseudo-Cayley graph.

If $s \in S$ and $a \in N_{-}(R)$, then $aN \subseteq R$. So for all $n \in N$, we find that $an \in R$. Since X is a pseudo-Cayley graph, then $SR \subseteq R$. Thus, $san \in R$, which implies that $saN \subseteq R$, and so $sa \in N_{-}(R)$. Hence, $SN_{-}(R) \subseteq N_{-}(R)$. Therefore, $S \cap N_{-}(R)$ has the necessary conditions for a pseudo-Cayley graph. Thus, $\overline{X}'$ is a pseudo-Cayley graph. □

Example 5.3. Let G be a dihedral group with order 8; let $N = \{1, P^2\}$ be a normal subgroup of G; and let $R = \{P, P^2, P^3, P\varepsilon, P^2\varepsilon, P^3\varepsilon\}$ be a subset of G. Let $X = (R; S)$ be a pseudo-Cayley graph such that S equals $\{\varepsilon\}$. We then have (see Fig. 2).

$$\underline{X}' = (\{P, P^3, P\varepsilon, P^3\varepsilon\}; S)$$

and

$$\overline{X}' = (\{P, P^2, P^3, P\varepsilon, P^2\varepsilon, P^3\varepsilon, 1, \varepsilon\}; S)$$

In order to present illustrative examples of the above analysis, software was developed using C++. Five classes and their corresponding properties and methods are defined, including Cayley Graph, pseudo-Cayley Graph, Group, NormalSubGroup, and Subsets. Since pseudo-Cayley graphs contain Cayley graphs, the Cayley Graph class inherits the properties of the pseudo-Cayley Graph class. In the main thread of the program running mode, the user inputs the number of elements of the group. The binary operation of each group is then initialized. The groups are restricted to dihedral and congruence groups. All normal subgroups and subsets that satisfy the conditions of $(G; S)$ of the Cayley graph are computed. Within the determined group using all computed Ns and Ss, the software plots all possible Cayley graphs. In addition, the lower and upper approximations of the determined group, the rough edge Cayley graphs and the rough vertex pseudo-Cayley graphs can also be computed.

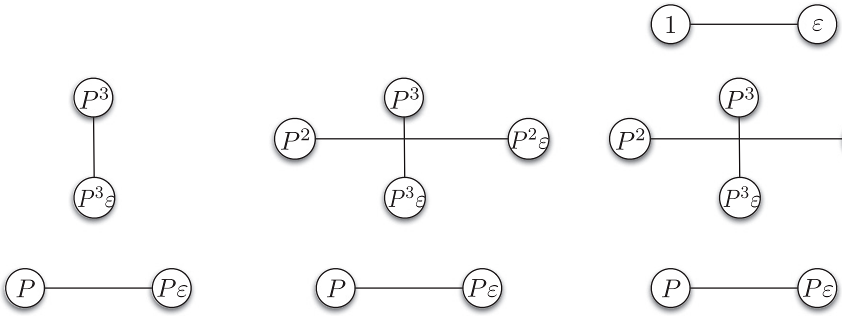

Fig. 2. The above graphs are $\underline{X}'$, X and $\overline{X}'$.

Theorem 5.4. *Let N and H be normal subgroups, and let R, R_1 and R_2 be subsets of a group G and $S \subseteq R, R_1, R_2$. Let $X = (R;S)$, $X_1 = (R_1;S)$ and $X_2 = (R_2;S)$ be pseudo-Cayley graphs. Then we have*

(1) $\underline{X}' \subseteq X \subseteq \overline{X}'$,
(2) $\overline{X_1 \cup X_2}' = \overline{X_1}' \cup \overline{X_2}'$,
(3) $\underline{X_1 \cap X_2}' = \underline{X_1}' \cap \underline{X_2}'$,
(4) $X_1 \subseteq X_2 \Rightarrow \underline{X_1}' \subseteq \underline{X_2}'$,
(5) $X_1 \subseteq X_2 \Rightarrow \overline{\overline{X_1}}' \subseteq \overline{\overline{X_2}}'$,
(6) $\underline{X_1 \cup X_2}' \supseteq \underline{X_1}' \cup \underline{X_2}'$,
(7) $\overline{\overline{X_1 \bigcap X_2}}' \subseteq \overline{\overline{X_1}}' \bigcap \overline{\overline{X_2}}'$,
(8) $N \subseteq H \Rightarrow \overline{X_N}' \subseteq \overline{X_H}'$,
(9) $N \subseteq H \Rightarrow \underline{X_H}' \subseteq \underline{X_N}'$.

Proof

(1) Note that $N\_(R) \subseteq R$. Then by Theorem 2.7 (2), we find that $(N\_(R);N\_(R) \cap S) \subseteq (R;N\_(R) \cap S)$. Theorem 2.7 (1) yields $(R;N\_(R) \cap S) \subseteq (R;S)$. And so $\underline{X}' \subseteq X$, while Theorem 3.1 (1) implies that $R \subseteq N^\wedge(R)$. Hence $(R;S) \subseteq (N^\wedge(R);S)$. Therefore $X \subseteq \overline{X}'$.
(2) Note that we have

$$\overline{X_1 \cup X_2}' = (N^\wedge(R_1 \cup R_2);S) = (N^\wedge(R_1) \cup N^\wedge(R_2);S) = (N^\wedge(R_1);S) \cup (N^\wedge(R_2);S) = \overline{X_1}' \cup \overline{X_2}'.$$

(3) Using Theorem 3.1 (3), the proof is similar to (2).
(4) Since $X_1 \subseteq X_2$, we obtain $R_1 \subseteq R_2$ according to Theorem 2.7 (2). Thus, $N\_(R_1) \subseteq N\_(R_2)$. Then $\underline{X_1}' \subseteq \underline{X_2}'$.
(5) Using Theorem 3.1 (5), the proof is similar to (4).
(6) Note that

$$\begin{aligned}\underline{X_1 \cup X_2}' &= (N\_(R_1 \cup R_2);N\_(R_1 \cup R_2) \cap S) \supseteq (N\_(R_1) \cup N\_(R_2);(N\_(R_1) \cup N\_(R_2)) \cap S) \\ &= (N\_(R_1);(N\_(R_1) \cup N\_(R_2)) \cap S) \cup (N\_(R_2);(N\_(R_1) \cup N\_(R_2)) \cap S) \\ &= (N\_(R_1);N\_(R_1) \cap S) \cup (N\_(R_2);N\_(R_2) \cap S) = \underline{X_1}' \cup \underline{X_2}'.\end{aligned}$$

(7) We have

$$\overline{X_1 \cap X_2}' = (N^\wedge(R_1) \cap N^\wedge(R_2);S) \subseteq (N^\wedge(R_1);S) \cap (N^\wedge(R_2);S) = \overline{X_1}' \cap \overline{X_2}'.$$

(8) Since $N \subseteq H$, then $N^\wedge(R) \subseteq H^\wedge(R)$. Thus,

$$\overline{X_N}' = (N^\wedge(R);S) \subseteq (H^\wedge(R);S) = \overline{X_H}'.$$

(9) Using Theorem 3.1 (9), the proof is similar to (8). □

In the following section, we present some examples that show the contradictions that emerge from the converse of the above items (4–9).

Example 5.5. Let $G = \{1, \varepsilon, P, P\varepsilon, P^2, P^2\varepsilon, P^3, P^3\varepsilon\}$ be a dihedral group with order 8.

(1) $X_1 \not\subseteq X_2, \underline{X_1}' \subseteq \underline{X_2}' : X_1 = (\{1, P^2\};\emptyset), X_2 = (\{1, P^2, \varepsilon\};\emptyset), N = \{1, P^2\}$.
(2) $X_1 \not\subseteq X_2, \overline{\overline{X_1}}' \subseteq \overline{\overline{X_2}}' : X_1 = (\{1, \varepsilon, P^2, P^2\varepsilon\};\emptyset), X_2 = (\{1, P^2, P^2\varepsilon\};\emptyset), N = \{1, P^2\}$.
(3) $\underline{X_1}' \cup \underline{X_2}' \not\subseteq \underline{X_1 \cup X_2}' : X_1 = (\{1, P, P^2\};\emptyset), X_2 = (\{1, P^3\};\emptyset), N = \{1, P^2\}$.

(4) $\overline{X_1}' \bigcap \overline{X_2}' \not\subseteq \overline{X_1 \bigcap X_2}' : X_1 = (\{1, P, P^2\}; \emptyset), X_2 = (\{1, P^2, P^3\}; \emptyset), N = \{1, P^2\}$.
(5) $N \not\subseteq H, \overline{X_N}' \subseteq \overline{X_H}' : X = (\{1, P, P^2, P^3\}; \emptyset), H = \{1, P^2\}, N = \{1, P, P^2, P^3\}$.
(6) $N \not\subseteq H, \underline{X_H}' \subseteq \underline{X_N}' : X = (\{1, P, P^2, P^3\}; \emptyset), H = \{1, P^2\}, N = \{1, P, P^2, P^3\}$.

Theorem 5.6. *Let H and N be normal subgroups of a group G, and let X = (R; S) be a pseudo-Cayley graph. Then*

(1) $\overline{X}'_{H\cap N} = \overline{X}'_H \cap \overline{X}'_N$,
(2) $\underline{X}'_{H\cap N} = \underline{X}'_H \cap \underline{X}'_N$.

Proof

(1) Note that

$$\overline{X}'_{H\cap N} = ((H\cap N)^\wedge(R); S) = (H^\wedge(R) \cap N^\wedge(R); S) = (H^\wedge(R); S) \cap (N^\wedge(R); S) = \overline{X}'_H \cap \overline{X}'_N.$$

(2) Using Theorem 3.2 (2), the proof is similar to (1). □

Theorem 5.7. *Let N be a normal subgroup; let H be a subgroup of a group G; and let $S \subseteq H$. In addition, let X = (H; S) be a Cayley graph. Then*

(1) $N \subseteq H \Longleftrightarrow \underline{X}' = X = \overline{X}'$,
(2) $N \not\subseteq H \Leftrightarrow \underline{X}' = \emptyset$.

Proof

(1) In order to prove this statement, we show that

$$N_-(H) = H = N^\wedge(H) \Longleftrightarrow N \subseteq H.$$

Suppose that $N \subseteq H$. If $h \in H$, then $hN \subseteq HN \subseteq HH(N \subseteq H) \subseteq H(H \leqslant G)$. So $h \in N_-(H)$ which implies that $H \subseteq N_-(H)$. Now, by Theorem 3.1 (1) and the above result, we obtain $H = N_-(H)$. If $g \in N^\wedge(H)$, then $gN \cap H \neq \emptyset$. Thus, there exists $n \in N$ such that $gn \in H$ which implies that $gnn^{-1} \in H(N \subseteq H$, n and $n^{-1} \in H)$. Hence, $g \in H$ and so $N^\wedge(H) \subseteq H$. Therefore, by Theorem 3.1 (1) and the above results, we find that $N^\wedge(H) = H$. Conversely, $N_-(H) = H = N^\wedge(H)$ implies that $1N = N \subseteq H$.
(2) If $g \in N_-(H)$, then $gN \subseteq H$. So for all $n \in N$, $gn \in H$. Since $N_-(H) \subseteq H$, then $g \in H$. Thus

$$g^{-1} \in H \Rightarrow g^{-1}gn \in H \Rightarrow n \in H \Rightarrow N \subseteq H,$$

which shows repugnance.

Note that, according to Theorem 3.3, $N_-(H)$ and $N^\wedge(H)$ are subgroups of G. Then $\underline{X}$ and $\overline{X}$ are Cayley graphs, and X is definable. A subset X of U is *definable* if $\underline{Apr}(X) = \overline{Apr}(X)$.

By Theorem 3.3, $N^\wedge(H)$ is a subgroup of G. Then $\overline{X}$ is a Cayley graph. □

Theorem 5.8. *Let N be a normal subgroup, R and S be subsets of a group G and $SR \subseteq R$. Let X = (R; S) be a pseudo-Cayley graph. If $N_-(R)$ is not empty and there exists $r \in R$ such that $<s>r = R$, then X is definable.*

Proof. Since $N_-(R)$ is not empty, there exist $r' \in N_-(R)$ and $s_1, s_2, \ldots s_m$ such that $r' = s_1 s_2 \ldots s_m r$. We then have

$$rN = s_1 s_2 \ldots s_m r' N \subseteq s_1 s_2 \ldots s_m R \subseteq R \quad (SR \subseteq R,\ r' \in N_-(R))$$

so $r \in N_-(R)$. Thus, for all $r'' \in R$, there exists $s'_1 s'_2 \ldots s'_{m'}$ such that $r'' = s'_1 s'_2 \ldots s'_{m'} r$. Then

$$r'' N = s'_1 s'_2 \ldots s'_{m'} rN \subseteq s'_1 s'_2 \ldots s'_{m'} R \subseteq R \quad (SR \subseteq R,\ r \in N_-(R))$$

which implies that $r'' \in N_-(R)$. Therefore we obtain $N_-(R) = R = N^\wedge(R)$. Thus X is definable.

Note that the converse of Theorem 5.8 may not be true. For example, let G be the dihedral group of order 8. Let $R = \{P, P^2, P^3, P\varepsilon, P^2\varepsilon, P^3\varepsilon\}$, $S = \{\varepsilon\}$ and $N = \{1\}$. Then $X = (R; S)$ is definable.

$(\underline{X}', \overline{X}')$ is a *rough vertex pseudo-Cayley graph* $X = (R; S)$. A pseudo-Cayley graph $X = (R; S)$ is an $N^\wedge$-*vertex rough generating* if R is $N^\wedge$-rough subgroup of G and S is a generating set for $N^\wedge(R)$. Similarly, a pseudo-Cayley graph $X = (R; S)$ is an N_--*vertex rough generating* if R is N_--rough subgroup of G and S is a generating set for $N_-(R)$.

A pseudo-Cayley graph $X = (R; S)$ is an $N^\wedge$-*vertex rough optimally connected* if R is an $N^\wedge$-rough subgroup of G and S is a minimal Cayley set for $N^\wedge(R)$. Similarly, a pseudo-Cayley graph $X = (R; S)$ is N_--*vertex rough optimal* if R is an N_--rough subgroup of G and S is a minimal Cayley set for $N_-(R)$. □

Theorem 5.9. *Let $X = (R;S)$ be a pseudo-Cayley graph. If X is $N^{\wedge}$-vertex rough generating, then $\overline{X}'$ is connected. Similarly, if X is N_{-}-vertex rough generating, then $\underline{X}'$ is connected.*

Proof. The proof is straightforward. □

Theorem 5.10. *Let $X = (R;S)$ be a pseudo-Cayley graph. If X is $N^{\wedge}$-vertex rough optimally connected, then $\overline{X}'$ is optimally connected. Similarly, if X is N_{-}-vertex rough optimally connected, then $\underline{X}'$ is optimally connected.*

Proof. The proof is straightforward. □

6. Rough pseudo-Cayley graphs

In this section, the concept of a lower and upper approximations pseudo-Cayley graph of a pseudo-Cayley graph with respect to a normal subgroup is introduced. Some properties of lower and upper approximations are then introduced.

Definition 6.1. Let G be a finite group with identity 1; let N be a normal subgroup; let R be a subset of G; and let $X = (R;S)$ be a pseudo-Cayley graph. Then the following graphs

$$\overline{X}'' = (N^{\wedge}(R); N^{\wedge}(S)^{*}) \quad \text{and} \quad \underline{X}'' = (N_{-}(R); N_{-}(S))$$

are the *lower* and *upper approximations pseudo-Cayley graphs* of a pseudo-Cayley graph $X(R;S)$, respectively, with respect to the normal subgroup N. We will now prove that these graphs are pseudo-Cayley graphs.

Theorem 6.2. *The two graphs $\underline{X}''$ and $\overline{X}''$ are pseudo-Cayley graphs.*

Proof. If $a \in N^{\wedge}(S)^{*}$ and $b \in N^{\wedge}(R)$, then $aN \cap S \neq \emptyset$ and $bN \cap R \neq \emptyset$. Thus, there exist n_1, $n_2 \in N$ such that $an_1 \in S$ and $bn_2 \in R$. Hence, $an_1bn_2 \in R$, which implies that $an_1bn_2 \in aNbN = abN$. We then obtain $abN \cap R \neq \emptyset$, which implies that $ab \in N^{\wedge}(R)$. Therefore, $N^{\wedge}(S)^{*}N^{\wedge}(R) \subseteq N^{\wedge}(R)$.

If $a \in N_{-}(S)$ and $b \in N_{-}(R)$, then $aN \subseteq S$ and $bN \subseteq R$. We then have

$$abN = aNbN \subseteq SR \subseteq R$$

so $ab \in N_{-}(R)$. Thus, $N_{-}(S)N_{-}(R) \subseteq N_{-}(R)$. Since $S \subseteq R$, then $N^{\wedge}(S)^{*} \subseteq N^{\wedge}(S) \subseteq N^{\wedge}(R)$ and $N_{-}(S) \subseteq N_{-}(R)$. Therefore $\underline{X}''$ and $\overline{X}''$ are pseudo-Cayley graphs. □

Example 6.3. Let G be a dihedral group of order 8. Let $N = \{1, P^2\}$ be a normal subgroup, and let $R = \{P, P^2, P^3, P\varepsilon, P^2\varepsilon, P^3\varepsilon\}$ be a subset of G. Let $X = (R;S)$ be a pseudo-Cayley graph such that S equals $\{\varepsilon\}$. We then have (see Fig. 3.)

$$\underline{X}'' = (\{P, P^3, P\varepsilon, P^3\varepsilon\}; \emptyset)$$

and

$$\overline{X}'' = (\{P, P^2, P^3, P\varepsilon, P^2\varepsilon, P^3\varepsilon, 1, \varepsilon\}; \{\varepsilon, P^2\varepsilon\});$$

Theorem 6.4. *Let N and H be normal subgroups, and let R, R_1 and R_2 be subsets of a group G. Let $X = (R;S)$, $X_1 = (R_1;S_1)$ and $X_2 = (R_2;S_2)$ be pseudo-Cayley graphs. Then we have*

(1) $\underline{X}'' \subseteq X \subseteq \overline{X}''$,
(2) $\underline{X_1 \cap X_2}'' = \underline{X_1}'' \cap \underline{X_2}''$,
(3) $X_1 \subseteq X_2 \Rightarrow \underline{X_1}'' \subseteq \underline{X_2}''$,
(4) $X_1 \subseteq X_2 \Rightarrow \overline{X_1}'' \subseteq \overline{X_2}''$,
(5) $\overline{X_1 \cap X_2}'' \subseteq \overline{X_1}'' \cap \overline{X_2}''$,
(6) $N \subseteq H \Rightarrow \overline{X_N}'' \subseteq \overline{X_H}''$,
(7) $N \subseteq H \Rightarrow \underline{X_H}'' \subseteq \underline{X_N}''$.

Proof

(1) We then have

$$\underline{X}'' = (N_{-}(R); N_{-}(S)) \subseteq (R; N_{-}(S)) \subseteq (R;S) \subseteq (N^{\wedge}(R); S) \subseteq (N^{\wedge}(R); N^{\wedge}(S)^{*}) = \overline{X}''.$$

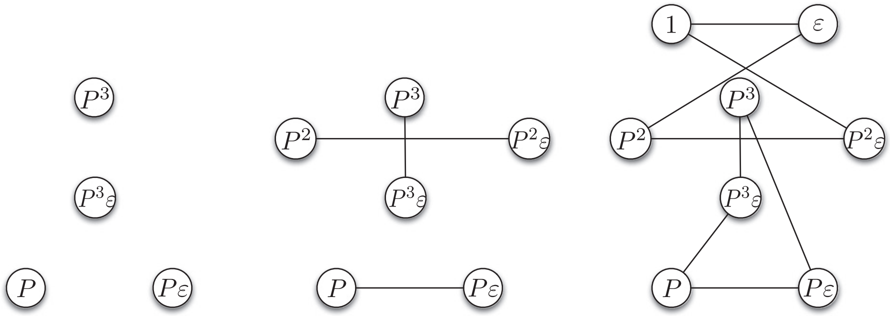

Fig. 3. The above graphs are $\underline{X}''$, X and $\overline{X}''$.

(2) We thus obtain

$$\underline{X_1 \cap X_2}'' = \underline{(R_1 \cap R_2; S_1 \cap S_2)}'' = (N^\wedge(R_1 \cap R_2); N^\wedge(S_1 \cap S_2)^*) = (N^\wedge(R_1) \cap N^\wedge(R_2); N^\wedge(S_1)^* \cap N^\wedge(S_2)^*)$$
$$= (N^\wedge(R_1); N^\wedge(S_1)^*) \cap (N^\wedge(R_2); N^\wedge(S_2)^*) = \underline{X_1}'' \cap \underline{X_2}''.$$

(3) Since $X_1 \subseteq X_2$, then $R_1 \subseteq R_2$, $S_1 \subseteq S_2$. Hence, $N\_(R_1) \subseteq N\_(R_2)$, $N\_(S_1) \subseteq N\_(S_2)$ which implies that $\underline{X_1}'' \subseteq \underline{X_2}''$.
(4) Using Theorem 3.1 (5), the proof is similar to (3).
(5) We then have

$$\overline{X_1 \cap X_2}'' = \overline{(R_1 \cap R_2; S_1 \cap S_2)}'' = (N^\wedge(R_1 \cap R_2); N^\wedge(S_1 \cap S_2)^*) \subseteq (N^\wedge(R_1) \cap N^\wedge(R_2);\ N^\wedge(S_1)^* \cap N^\wedge(S_2)^*)$$
$$= (N^\wedge(R_1); N^\wedge(S_1)^*) \cap (N^\wedge(R_2);\ N^\wedge(S_2)^*) = \overline{X_1}'' \cap \overline{X_2}''.$$

(6) Since $N \subseteq H$, then $N^\wedge(R) \subseteq H^\wedge(R)$ and $N^\wedge(S) \subseteq H^\wedge(S)$. So

$$\overline{X_N}'' = (N^\wedge(R);\ N^\wedge(S)) \subseteq (H^\wedge(R);\ H^\wedge(S)) = \overline{X_H}''.$$

(7) Using Theorem 3.1 (9), the proof is similar to (8). □

Remark 6.5. Let N be a normal subgroup, and let H be a subgroup of group G such that $N \subseteq H$ and $S \subseteq H$. In addition, let $X = (H; S)$ be a Cayley graph. Then $(\underline{X}'', \overline{X}'')$ is equivalent to a rough edge Cayley graph of X.

Proof. This is proven by simply using Theorem 5.7. □

Note that according to the previous remark and Example 4.5, the converse of the above items (4–7) is not always true.

Theorem 6.6. *Let H and N be normal subgroups of a group G. Let $X = (R; S)$ be pseudo-Cayley graph. Then*

(1) $\overline{X}''_{H\cap N} = \overline{X}''_H \cap \overline{X}''_N$,
(2) $\underline{X}''_{H\cap N} = \underline{X}''_H \cap \underline{X}''_N$.

Proof

(1) We have

$$\overline{X}''_{H\cap N} = ((H \cap N)^\wedge(R); (H \cap N)^\wedge(S)^*) = (H^\wedge(R) \cap N^\wedge(R); H^\wedge(S)^* \cap N^\wedge(S)^*) = (H^\wedge(R); H^\wedge(S)^*) \cap (N^\wedge(R); N^\wedge(S)^*) = \overline{X}''_H \cap \overline{X}''_N.$$

(2) Using Theorem 3.2 (2), the proof is similar to (1).

$(\underline{X}'', \overline{X}'')$ is a *rough pseudo-Cayley graph* $X = (R; S)$. A pseudo-Cayley graph $X = (R; S)$ is an *$N^\wedge$-rough generating* if R is an $N^\wedge$-rough subgroup of G and $N^\wedge(S)$ is a generating set for $N^\wedge(R)$. Similarly, a pseudo-Cayley graph $X = (R; S)$ is *$N\_$-rough generating* if R is $N\_$-rough subgroup of G and $N\_(S)$ is a generating set for $N\_(R)$.

A pseudo-Cayley graph $X = (R; S)$ is *$N^\wedge$-rough optimally connected* if R is an $N^\wedge$-rough subgroup of G, and $N^\wedge(S)$ is a minimal Cayley set for $N^\wedge(R)$. Similarly, a pseudo-Cayley graph $X = (R; S)$ is *$N\_$-rough optimal* if R is an $N\_$-rough subgroup of G, and $N\_(S)$ is a minimal Cayley set for $N\_(R)$. □

Theorem 6.7. *Let $X = (R; S)$ be a pseudo-Cayley graph. If X is $N^\wedge$-rough generating, then $\overline{X}''$ is connected. Similarly, If X is $N\_$-rough generating, then $\underline{X}''$ is connected.*

Proof. Using Theorem 2.2, the proof is straightforward. □

Theorem 6.8. *Let $X = (R; S)$ be a pseudo-Cayley graph. If X is $N^{\wedge}$-rough optimally connected, then $\overline{X}''$ is optimally connected. Similarly, if X is N_{-}-rough optimally connected, then $\underline{X}''$ is optimally connected.*

Proof. Using Theorem 2.3, the proof is straightforward. □

7. Conclusion

This paper has addressed a connection between two research topics, namely, rough sets and Cayley graphs, both of which have applications across a wide variety of fields. Three approximations called rough edge Cayley graphs, rough vertex pseudo-Cayley graphs and rough pseudo-Cayley graphs on Cayley graphs and pseudo-Cayley graphs have been defined. Related theorems and properties, such as connectivity, have been discussed. In order to further illustrate our analysis, a C++ software program has been developed and discussed here. These approximations can be applied to many challenging problems in distributed systems, such as reliability and fault tolerance. We leave the applications of the results presented here to future studies.

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