ON THE HALF-HARTLEY TRANSFORM, ITS ITERATION AND COMPOSITIONS WITH FOURIER TRANSFORMS

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ABSTRACT. Employing the generalized Parseval equality for the Mellin transform and elementary trigonometric formulas, the iterated Hartley transform on the nonnegative half-axis (the iterated half-Hartley transform) is investigated in L_2. Mapping and inversion properties are discussed, its relationship with the iterated Stieltjes transform is established. Various compositions with the Fourier cosine and sine transforms are obtained. The results are applied to the uniqueness of the closed form solutions for certain new singular integral and integro-functional equations.

1. INTRODUCTION AND AUXILIARY RESULTS

The familiar reciprocal pair of the Hartley transforms [1]

$$(\mathscr{H} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\cos(xt) + \sin(xt)] f(t) dt, \ x \in \mathbb{R}, \tag{1.1}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\cos(xt) + \sin(xt)] (\mathscr{H} f)(t) dt \tag{1.2}$$

is well-known in connection with various applications in mathematical physics. Mapping and inversion properties of these transforms in L_2 as well as their multidimensional analogs were investigated, for instance, in [2], [3], [4]. These operators were treated as the so-called bilateral Watson transforms, and in some sense they are related to the Fourier cosine and Fourier sine transforms

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(xt) f(t) dt, \ x \in \mathbb{R}_+, \tag{1.3}$$

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin(xt) f(t) dt, \ x \in \mathbb{R}_+. \tag{1.4}$$

Recently [5], the author investigated the Hartley transform (1.1) with the integration over $\mathbb{R}_+$ (the half-Hartley transform)

$$(\mathscr{H}_+ f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} [\cos(xt) + \sin(xt)] f(t) dt, \ x \in \mathbb{R}_+ \tag{1.5}$$

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and proved and analog of the Plancherel theorem, establishing its reciprocal inverse operator in $L_2(\mathbb{R}_+)$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty [\sin(xt)\, S(xt)+\cos(xt)\, C(xt)]\,(\mathscr{H}_+ f)(t)dt, \tag{1.6}$$

where $S(x),\ C(x)$ are Fresnel sin- and cosine- integrals, respectively,

$$S(x)=\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{x}}\sin(t^2)dt, \quad C(x)=\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{x}}\cos(t^2)dt.$$

Our goal here is to examine the iteration of the operator (1.5) $(\mathscr{H}_+^2 f)(x)\equiv(\mathscr{H}_+\mathscr{H}_+ f)(x)$, which will be called the iterated half-Hartley transform and investigate its compositions in L_2 with the Fourier transforms (1.3), (1.4) of the form: $\mathscr{H}_+F_c,\ \mathscr{H}_+F_s,\ \mathscr{H}_+F_cF_s,\ \mathscr{H}_+^2F_c,\ \mathscr{H}_+^2F_s,\ \mathscr{H}_+^2F_cF_s$. The corresponding integral representations of these compositions will be established in L_2 and their boundedness and invertibility will be proved. Moreover, we will apply these results to establish the uniqueness of solutions in the closed form for the corresponding second kind singular integral and integro-functional equations.

Our natural approach is based on the L_2-theory of the Mellin transform [6]

$$(\mathscr{M} f)(s)=f^*(s)=\int_0^\infty f(t)t^{s-1}dt,\ s\in\sigma=\{s\in\mathbb{C},\ s=\frac{1}{2}+i\tau\}, \tag{1.7}$$

where the integral is convergent in the mean square sense with respect to the norm in $L_2(\sigma)$. Reciprocally, the inversion formula takes place

$$f(x)=\frac{1}{2\pi i}\int_\sigma f^*(s)x^{-s}ds,\ x>0 \tag{1.8}$$

with the convergence of the integral in the mean square sense with respect to the norm in $L_2(\mathbb{R}_+)$. Furthermore, for any $f_1, f_2\in L_2(\mathbb{R}_+)$ the generalized Parseval identity holds

$$\int_0^\infty f_1(xt)\, f_2(t)dt=\frac{1}{2\pi i}\int_\sigma f_1^*(s) f_2^*(1-s)x^{-s}ds,\ x>0 \tag{1.9}$$

with Parseval's equality of squares of L_2- norms

$$\int_0^\infty |f(x)|^2dx=\frac{1}{2\pi}\int_{-\infty}^\infty\left|f^*\left(\frac{1}{2}+i\tau\right)\right|^2 d\tau. \tag{1.10}$$

Finally in this section, we exhibit the known formulas [6]

$$\int_0^\infty \frac{\sin t}{t}t^{s-1}dt=\frac{\Gamma(s)}{1-s}\cos\left(\frac{\pi s}{2}\right),\ s\in\sigma, \tag{1.11}$$

$$\int_0^\infty \frac{1-\cos t}{t}t^{s-1}dt=\frac{\Gamma(s)}{1-s}\sin\left(\frac{\pi s}{2}\right),\ s\in\sigma, \tag{1.12}$$

where $\Gamma(s)$ is the Euler gamma-function, which will be used in the sequel.

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