# SYMPLECTIC RIGIDITY AND FLEXIBILITY OF ELLIPSOIDS

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ABSTRACT. Rigidity is proved for symplectic embeddings of an ellipsoid into another of the same shape type, and new flexibility results are derived from a variant of the symplectic folding process.

A volume form on a smooth $n$-dimensional manifold $M$ is a nowhere vanishing $n$-form $\Omega$. On every open set $U \subset \mathbb{R}^n$ we consider the standard volume $\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$; a smooth embedding $\varphi : U \hookrightarrow M$ is said to be volume preserving if:

$$\varphi^*\Omega = \Omega_0$$

A *symplectic manifold* is a pair $(M, \omega)$, where $M$ is a $2n-$dimensional differentiable manifold and $\omega$ is a symplectic form: a closed non degenerate 2-form. Then:

$$\Omega = \frac{1}{n!}\,\omega^n \;\text{ is a volume form, and } \mathrm{d}\omega = 0$$

A *symplectic map* is a map $\varphi : (M, \omega) \longrightarrow (M', \omega')$, such that:

$$\varphi^*\omega' = \omega$$

Let $\mathcal{D}(n)$ be the group of symplectic diffeomorphisms, or symplectomorphisms, or canonical transformations, of $\mathbb{R}^{2n}$, and $\mathrm{Sp}(n)$ its subgroup of linear isomorphisms.

On every open set $U \subset \mathbb{R}^{2n}$ we consider the standard symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n$; a smooth embedding $\varphi : U \hookrightarrow M$ is said to be symplectic if it is a symplectic map:

$$\varphi^*\omega = \omega_0, \text{ and therefore } \varphi^*\Omega = \Omega_0$$

where $\Omega$ and $\Omega_0$ are the volume forms induced by the symplectic forms.

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**Definition 1.** An open symplectic ellipsoid of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with radii $r_i = \sqrt{a_i/\pi}$ is the set:

$$E(a) = E(a_1, \ldots, a_n) = \left\{ z \,\middle|\, \frac{\pi|z_1|^2}{a_1} + \cdots + \frac{\pi|z_n|^2}{a_n} < 1 \right\},$$

where we assume $a_1 \leq \ldots \leq a_n$, and $z_j = x_j + iy_j$.

**Definition 2.** An open symplectic cylinder of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with radius $r = \sqrt{a/\pi}$ is the set:

$$\begin{aligned} Z(a) &= \{(x, y) \in \mathbb{R}^{2n} : \ \pi|(x_1, y_1)|^2 < a\} \\ &= \{z \in \mathbb{C}^n : \ \pi|z_1|^2 < a\} \end{aligned}$$

*Remark* 1. The ball of radius $r$ is denoted by $B(\pi r^2)$:

$$B(a) = E(a, a, \ldots, a), \quad Z(a) = E(a, \infty, \ldots, \infty)$$

In dimension 2, an embedding is volume preserving if and only if it is symplectic; in higher dimensions there exists symplectic rigidity, as first shown in [4]:

**Gromov Theorem.** *If there is a symplectic embedding $\varphi : B(a) \longrightarrow Z(A)$ of a ball into a symplectic cylinder, then $a \leq A$.*

The detection of embedding obstructions and the proof of the corresponding rigidity results will be based on symplectic capacities:

**Definition 3.** An *extrinsic symplectic capacity* $c$ on $(\mathbb{R}^{2n}, \omega_0)$ is a map $c$ such that, for every $A \subset \mathbb{R}^{2n}$, $c(A) \in [0, +\infty]$, satisfying the following properties:

- **Monotonicity:** $c(A) \leq c(A')$ if there exists $\varphi \in \mathcal{D}(n)$ such that $\varphi(A) \subset A'$.
- **Conformality:** $c(\alpha A) = \alpha^2 c(A)$, for any nonzero $\alpha \in \mathbb{R}$.
- **Nontriviality :** $0 < c(B(\pi)), \quad c(Z(\pi)) < \infty$

## 1. Rigidity

When considering linear symplectic embeddings, there exists symplectic rigidity:

**Theorem 1** ([7])**.** *Given two ellipsoids $E(a)$ and $E(a')$, there exists a linear symplectic map $S \in Sp(n)$ such that $S(E(a)) \subset (E(a')$ if and only if $a_i \leq a'_i$, for all $i = 1, \ldots, n$.*

Even when allowing nonlinear symplectomorphisms, symplectic rigidity can still be present:

**Theorem 2** ([3]). *Given two ellipsoids $E(a_1, a_2)$ and $E(a'_1, a'_2)$ with:*

$$\nu \leq a_1, a_2, a'_1, a'_2 \leq 1, \quad \frac{1}{2} < \nu < 1$$

*there exists a symplectic embedding $\varphi$ such that $\varphi(E(a)) \subset E(a')$ if and ony if $a_i \leq a'_i$, for $i = 1, 2$.*

In $\mathbb{C}^2 \cong \mathbb{R}^4$ it is natural to characterize the shape of a symplectic ellipsoid by:

**Definition 4.** Two ellipsoids $E(a_1, a_2)$ and $E(a'_1, a'_2)$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ have the same *shape type* if:

$$\exists k \in \mathbb{N}: \quad k \leq \frac{a_2}{a_1} < k+1, \quad k \leq \frac{a'_2}{a'_1} < k+1$$

In higher dimensions the definition will be more general.

**Definition 5.** Given an ellipsoid $E(a_1, \ldots, a_n)$, let $\{\mu_i\}$ be the sequence of the numbers $\{ka_j\}$, with $k \in \mathbb{N}$ and $j = 1, \ldots, n$, written (maybe with repetitions) in increasing order. The *Ekeland-Hofer i-capacity* for $E(a)$ is given by:

$$c_i(E(a)) = \mu_i$$

**Definition 6.** Two ellipsoids $E(a)$ and $E(a')$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ have the same *shape type* if:

$$\exists \alpha_1 = 1 < \ldots < \alpha_n: \quad \mu_{\alpha_i}(a) = a_i, \quad \mu_{\alpha_i}(a') = a'_i.$$

This is an equivalence relation if we exclude resonant ellipsoids, for which the sequence $\{\mu_i\}$ is not strictly increasing; it is easy to see that then the two definitions agree for $n = 2$.

**Example 1.** $B(a)$ and $E(a, 2a)$ have the same shape type using definition 6: their Ekeland-Hofer capacities are respectively:

$$\mu = \{a, a, 2a, 2a, 3a, 3a, 4a, 4a, \ldots\}$$
$$\mu' = \{a, 2a, 2a, 3a, 4a, 4a, 5a, 6a, \ldots\}$$

and we can choose $\alpha_1 = 1$ and $\alpha_2 = 2$. On the other hand, they have different shape types using the first definition (def. 4).

Theorem 2 considers ellipsoids with the shape type of a ball ($k = 1$), but the result can be extended to ellipsoids having the same shape type:

**Theorem 3.** *If the two ellipsoids $E(a)$ and $E(a')$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ have the same shape type, there exists a symplectic embedding $\varphi$ such that $\varphi(E(a)) \subset E(a')$ if and only if:*

$$a_i \leq a'_i, \quad i = 1, \ldots, n$$

*Proof.* If $E(a)$ embeds in $E(a')$ then it follows from the definition of capacity that:

$$c_j\left(E(a)\right) \leq c_j\left(E(a')\right)$$

for all Ekeland-Hofer capacities, in particular if they have the same shape type:

$$a_i = c_{\alpha_i}\left(E(a)\right) \leq c_{\alpha_i}\left(E(a')\right) = a'_i, \quad i = 1, \dots, n$$

□

This is a generalization of a result of F. Schlenk [11, 12]: If $a_n \leq 2a_1$, there exists no symplectic embedding of the ellipsoid $E(a) = E(a_1, \dots, a_n)$ into a ball $B(A)$ with $A < a_n$ (the shape type of the ellipsoid is that of a ball).

## 2. Flexibility

The following result shows that, if the shape type of the ellipsoids is sufficiently different, there is flexibility:

**Theorem 4** ([5, 3])**.** *For any $a > 0$, and for a sufficiently small $\varepsilon > 0$, there exists a symplectic embedding $\varphi$ such that:*

$$\varphi\left(E(\varepsilon, \dots, \varepsilon, a)\right) \subset B(\pi)$$

There are no estimates on the size of $\varepsilon$, but F. Schlenk, using symplectic folding, proved:

**Theorem 5** ([11, 12])**.** *If $\beta > 2\alpha$, there exists a symplectic embedding $\varphi$ of the ellipsoid $E(r) = E(\alpha, \dots, \alpha, \beta) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ into a ball $B(A)$ with:*

$$E(\alpha, \dots, \alpha, \beta) \hookrightarrow B(A), \quad A > \frac{\beta}{2} + \alpha$$

*Remark* 2. This theorem has been much improved in (complex) dimension 2 ([10]). But the methods used to obtain the best embedding results do not have a straightforward generalization to higher dimensions.

**Definition 7.** An open polydisk is the set:

$$\begin{aligned} P(a) = P(a_1, \dots, a_n) &= B(a_1) \times \cdots \times B(a_n) \\ &= \left\{ z \,\middle|\, \pi\frac{|z_1|^2}{a_1} < 1, \dots, \pi\frac{|z_n|^2}{a_n} < 1 \right\}, \end{aligned}$$

where we assume $a_1 \leq \ldots \leq a_n$.

A very impressive result concerning flexibility of polydisks is due to L. Guth:

**Theorem 6** ([6]). *There is a dimensional constant $C_n$ such that, given two polydisks $P(r)$ and $P(r')$, if:*

$$C_n a_1 < r_1', \quad C_n a_1 \dots a_n < a_1' \dots a_n'$$

*there exists a symplectic embedding of $P(a)$ into $P(a')$.*

This result has an obvious application to ellipsoids:

**Example 2.** In $\mathbb{C}^3 \cong \mathbb{R}^6$, there exists a constant $K > C_3 \pi$ such that:

$$E(\pi, a, a) \hookrightarrow E\left(3K, 3K, \frac{4}{K}a^2\right) \quad a > 3K$$

This follows from the embedding:

$$P(\pi, a, a) \hookrightarrow P\left(K, K, \frac{a^2}{C_3\pi}\right)$$

and the inclusions $E(\pi, a, a) \subset P(\pi, a, a)$ and:

$$P\left(K, K, \frac{a^2}{C_3\pi}\right) \subset E\left(3K, 3K, \frac{4}{K}a^2\right)$$

A similar result is valid in any dimension; it shows that if the shape type of the ellipsoid is sufficiently different from that of a ball ($a > 3K$ above) then there exists considerable flexibility and the relevant obstructions are (derived from) just the first capacity and the volume.

Capacities (in general) involve the 2-dimensional area of some object; volume can considered a generalized capacity and is $2n$-dimensional. It is natural to search for intermediate capacities that involve $2k$-dimensional volumes; it follows from the results of [6] that there are no reasonably continuous intermediate capacities.

Symplectic folding is described in [8, 9, 11, 12]; we shall use a slightly different version, but we rely on the very careful and detailed presentation in [11, 12] for all technical aspects and specially for the proofs; the adaptation to the situation described here is straightforward, but very laborious and long.

We define $T(a, b)$ as the set:

$$\begin{aligned} T(a,b) = &\{(z_1, z_2) = (u_1, v_1, u_2, v_2)\} \subset \mathbb{R}^4 \\ &(u_1, v_1) \in ]0, a[\times]0, 1[, \quad (u_2, v_2) \in ]0, b[\times]0, 1[ \\ &\frac{u_1}{a} + \frac{u_2}{b} < 1 \end{aligned}$$

and $T(a) = T(a, a)$. The projection of $T(a, b)$ on the $(u_1, u_2)$ plane is a triangle and the fibres the unit square.

**Lemma 1** ([11, 12]). *Assume $\varepsilon > 0$. Then:*

(1) *$E(a,b)$ symplectically embeds into $T(a+\varepsilon, b+\varepsilon)$*
(2) *$T(a,b)$ symplectically embeds into $E(a+\varepsilon, b+\varepsilon)$.*

*Sketch of the proof.* The main fact involved in the proof is the existence of an area preserving map $(u,v) = \sigma(z)$ in the plane [11, 12] that, outside an arbitrarily small neighbourhood of the origin, where it is a translation, essentially takes open circles of area $a$ into open rectangles $]0,a[\times]0,1[$ (fig. 1).

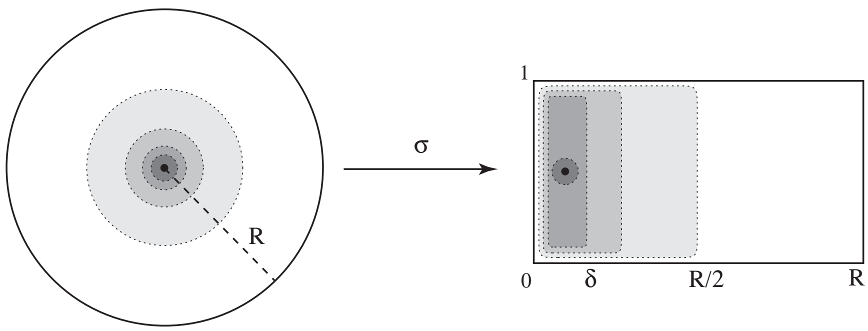


FIGURE 1. Area preserving map in the plane, $a = \pi$

Let $D(a)$ be the disk of area $a$; then:

$$E(a,b) = \left\{ z \mid z_1 \in D(a),\ z_2 \in D\left(b(1-\pi|z_1|^2/a)\right)\right\}$$

The symplectic embedding of $E$ into $T$ is then:

$$(z_1, z_2) \mapsto ((u_1,v_1),(u_2,v_2)) = (\sigma(z_1), \sigma(z_2)$$

The inverse of this map is used to embed $T$ into $E$. □

Here and subsequently we ignore everything 'small': we should consider maps $\sigma_\delta$ with sufficiently small $\delta$, but it is easier to proceed as if $\delta$ could be zero.

It follows from lemma 1 that embedding results for ellipsoids can be obtained from the corresponding results for sets of the form $T(a,b)$, and we describe symplectic folding for these sets.

Since $U$ embedding symplectically into $V$ is equivalent to $\lambda U$ embedding symplectically into $\lambda V$ for $\lambda \neq 0$, we normalize the ellipsoids $E(a)$, and therefore the sets $T$, so that $a_1 = \pi$. In the figures we really represent $T(a,\pi)$ instead of $T(\pi,a)$, as in [11].

Step 1: We separate the region $u_2 > \pi$ from the region $u_2 < \pi$, the large fibres from the small ones, extending the in-between region: here the fibres are related to the projection on the $(u_1,v_1)$ plane, and the symplectic map is the product $\varphi \times \mathrm{id}$ of an area preserving map $\varphi$ in the $(u_1,v_1)$ plane (figure 2) and the identity on the $(u_2,v_2)$ plane.

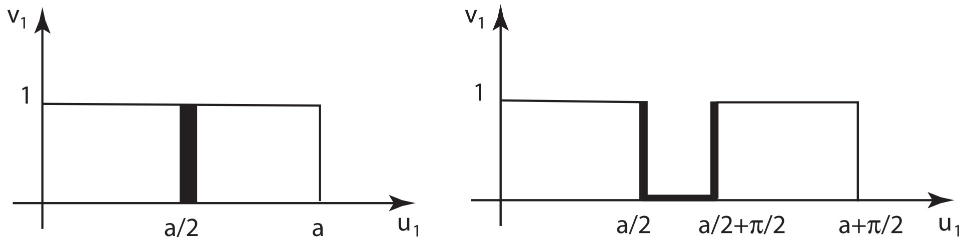


FIGURE 2. Separating the fibres: black regions have the same area

*Remark* 3. Again we should consider the regions $u_2 > b/2 + \delta$ and $u_2 < b/2 - \delta$ and deform $b/2 - \delta < u_2 < b/2 + \delta$ for a conveniently small $\delta$ (the map outside that region is the identity on the left and a translation on the right).

The result can also be seen in the $(u_1, u_2)$ plane:

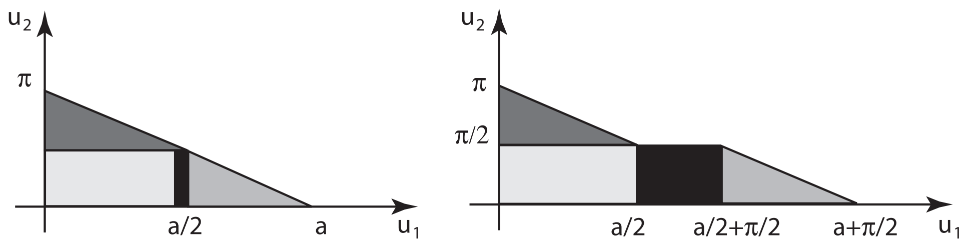


FIGURE 3. Separating the fibres, $(u_1, u_2)$ plane

Step 2: We rearrange the fibres: the symplectic map is the product of an area preserving map $\sigma_1$ in the $(u_2, v_2)$ plane (first line of figure 4), and the identity on the $(u_1, v_1)$ plane; the second line of figure 4 shows the result as seen in the $(u_1, u_2)$ plane.

Step 3: We lift the region $a/2 + \pi/2 < u_1 < a + \pi/2$ by $\pi/2$ along the $u_2$ direction. Now the symplectic map is not a product of area preserving maps: its action can be seen in the $(u_1, u_2)$ and $(u_1, v_1)$ planes (figure 5).

The grey region is the projection in $(u_1, v_1)$ of points lifted less than $\pi/2$ (and more than 0) and has area bigger than $\pi/2$.

Step 4: We contract along the $v_1$ direction, and extend along the $u_1$ direction, by $a/(a + \pi)$, keeping $(u_2, v_2)$ unchanged (figure 6).

Step 5: We now turn $T$ over $B$: we extend the grey area, then we fold twice in the base (figure 7).

The transformation of the grey area (in the $(u_1, v_1)$ plane) is as in the previous step, with a factor of $\pi/a$ now, but using the identity outside

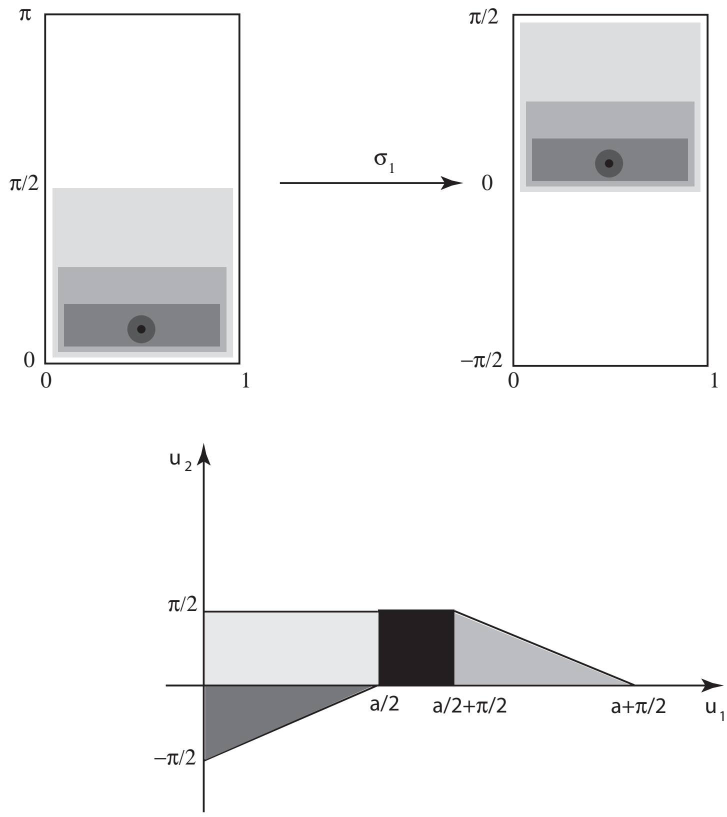


Figure 4. Rearranging the fibres in the $(u_2, v_2)$ plane

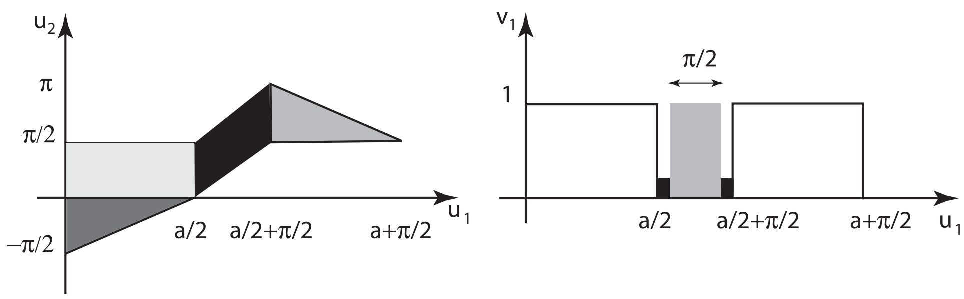


Figure 5. Lifting

that area on the left and a translation on the right. The end result in the $(u_1, u_2)$ plane is described in figure 8.

Step 6: We rearrange the fibres:

The symplectic map is the product of an area preserving map $\sigma_2$ in the $(u_2, v_2)$ plane (figure 9), and the identity on the $(u_1, v_1)$ plane.

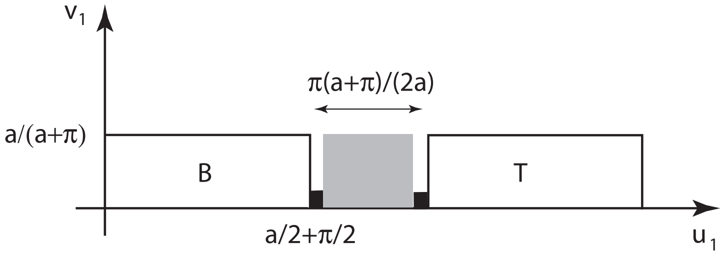


FIGURE 6. Rearranging in the $(u_1, v_1)$ plane

The symplectic folding construction is summarised in figure 10 (it should be compared to figure 3.13 in [11]): the advantage of the change

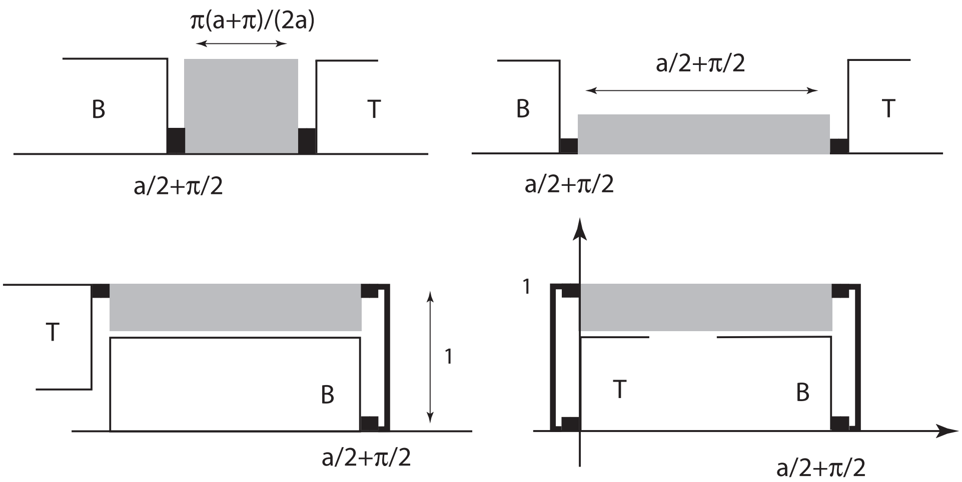


FIGURE 7. Folding in the $(u_1, v_1)$ plane

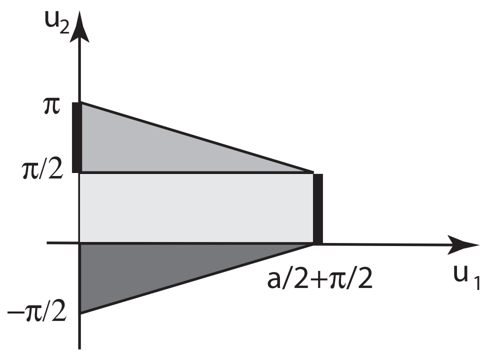


FIGURE 8. Folding in the $(u_1, u_2)$ plane

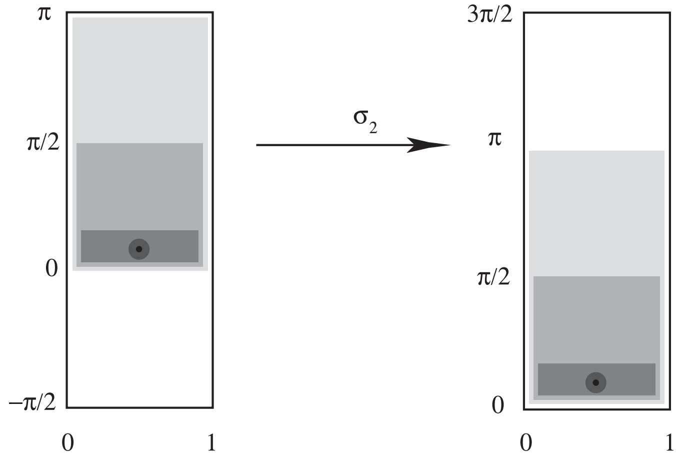


FIGURE 9. Rearranging the fibres in the $(u_2, v_2)$ plane

relative to [11, 12] is that we can get embeddings into ellipsoids, keeping the same estimates obtained for embeddings into balls.

**Theorem 7.** *If the ellipsoid $E(r) = E(r_1, r_2)$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ has shape type $k \geq 3$ with:*

$$3 \leq k < r_2/r_1 < k + 1$$

*there exists a symplectic embedding $\varphi$ such that $\varphi(E(r)) \subset E(r')$ with:*

$$r_2' < r_2 \quad \textit{and} \quad n \leq \frac{r_2'}{r_1'} < n + 1$$

*for all shape types* $n = 1, \ldots, \left[\frac{k+1}{2}\right]$.

*Proof.* We consider the normalised ellipsoid $E(\pi, a)$, with $k\pi < a < (k+1)\pi$ and $k \geq 3$. Symplectic folding gives an embedding (fig. 11):

$$T(\pi, a) \hookrightarrow T\left(\frac{a}{2} + \pi + \varepsilon\right)$$

and lines above the image of $T(\pi, a)$ in the $(u_1', u_2')-$plane correspond to sets $T(\alpha, \beta)$ into which $T(\pi, a)$ embeds; $(\alpha, 0)$ and $(0, \beta)$ are the intersections of the line with the coordinate axes.

Going from $T$-type sets to ellipsoids:

$$E(\pi, a) \hookrightarrow E\left(\frac{3}{2}\pi + \varepsilon, \frac{3}{4}(a + \pi) + \varepsilon\right), \qquad \frac{3}{4}(a + \pi) < a \Longleftrightarrow k \geq 3$$

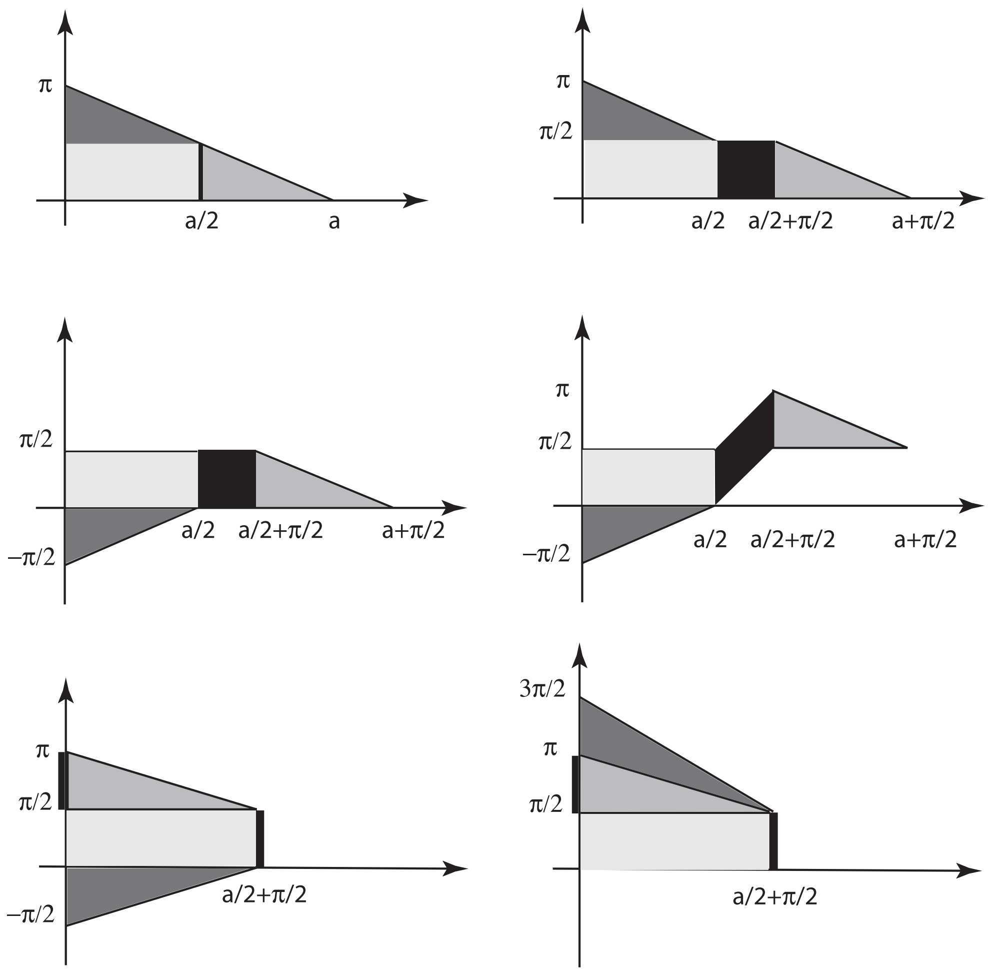


FIGURE 10. Scheme of symplectic folding in the $(u_1, u_2)$ plane

As

$$\frac{r_2'}{r_1'} = \frac{\frac{3}{4}(a+\pi)+\varepsilon}{\frac{3}{2}\pi+\varepsilon} = \frac{a+\pi}{2\pi} - \frac{a-\pi}{3\pi^2}\varepsilon + \dots$$

if $\varepsilon > 0$ is sufficiently small, then:

$$\left[\frac{k+1}{2}\right] < \frac{r_2'}{r_1'} < \left[\frac{k+1}{2}\right] + 1$$

The same construction also gives an embedding:

$$E(\pi, a) \hookrightarrow B\left(\frac{a}{2} + \pi + \varepsilon\right)$$

and clearly embeddings for all in between shape types. □

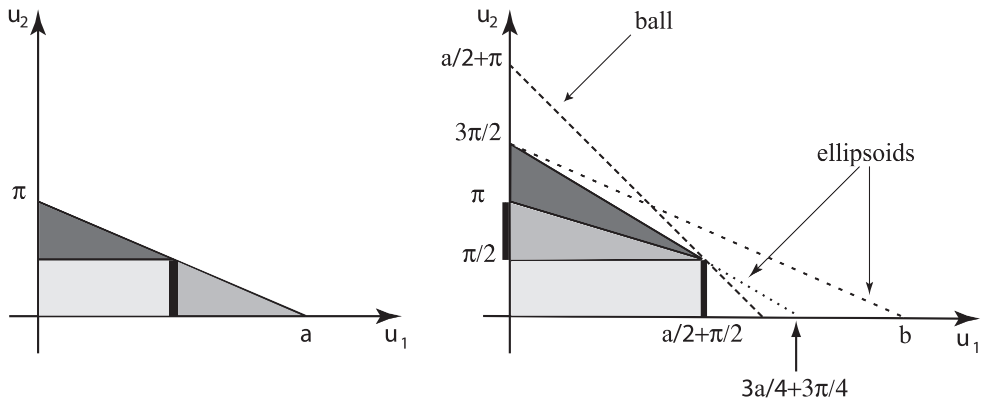


FIGURE 11. Lines correspond to ellipsoids or balls

*Remark* 4. There is a trivial embedding (again see figure 11):

$$E\left(\frac{3}{2}\pi+\varepsilon, \frac{3}{4}(a+\pi)+\varepsilon\right) \hookrightarrow E\left(\frac{3}{2}\pi+\varepsilon, b\right), \quad \frac{3}{4}(a+\pi)<b<a$$

and the shape type can thus be extended up to $\left[\frac{2k}{3}\right]$.

**Open Question** ([11, 12])**.** *Does the ellipsoid $E(a, 2a, 3a)$ symplectically embed into $B(A)$ for some $A < 3a$?*

Ekeland-Hofer capacities show that:

- $E(a, 3a, \ldots, 3a)$ does not symplectically embed into a ball $B(A)$ with $A < 3a$.
- $E(a, 2a, \ldots, 2a, 3a)$ does not symplectically embed into a ball $B(A)$ with $A < 2a$.

On the other hand, there is also some flexibility, as it follows from theorem 5 that:

$$E(a, 3a) \hookrightarrow B\left(\frac{5}{2}a+\varepsilon\right)$$

The change introduced in the symplectic folding process allows estimates (lemma 2) that are decisive in the proof of:

**Theorem 8.** *For any positive $\varepsilon$, there exists a symplectic embedding:*

$$E(\pi, b_1, \ldots, b_{n-2}=b, a) \hookrightarrow B(A+\varepsilon), \quad A<a$$

*when $a > b + \pi$, with $A$ given by:*

$$A = \frac{a + b + \pi}{2}$$

*Remark* 5. For $n = 3$, $b = 2\pi$, $a = 3\pi$:

$$E(\pi, 2\pi, 3\pi) \hookrightarrow B(A + \varepsilon), \quad A = \frac{3\pi + 2\pi}{2} + \frac{\pi}{2} = 3\pi$$

and thus $E(\pi, 2\pi, 3\pi)$ is in the boundary of (known) flexibility.

*Remark* 6. $b = \pi$ gives theorem 3.1.1 in [11] (or theorem 5):

$$E(\pi, \ldots, \pi, a) \text{ symplectically embeds into } B\left(\frac{a}{2} + \pi + \varepsilon\right), \quad \forall \varepsilon > 0$$

**Lemma 2.** *For any $\varepsilon > 0$, symplectic folding gives an embedding $\psi$:*

$$\psi : T(\pi, a) \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4, \quad \psi((u_1, v_1), (u_2, v_2)) = ((u_1', v_1'), (u_2', v_2'))$$

*such that*

$$u_1' + u_2' < A - b + \frac{b}{\pi} u_1 + \frac{b}{a} u_2 + \varepsilon, \quad A = \frac{a + b + \pi}{2}$$

*Sketch of the proof.* We assume

$$u_1' + u_2' < A - b + \frac{b}{a} u_1 + \frac{b}{\pi} u_2 + \varepsilon, \quad \forall \varepsilon > 0$$

and look for the smaller admissible $A$; this is done considering separately the four regions in $T(a, \pi)$ (figure 11).

Case I:

$$\psi : \begin{cases} u_1' = \dfrac{a + \pi}{a} u_1, & u_1 \in ]0, a/2[ \\ \\ u_2' = u_2, & u_2 \in ]0, \pi/2[ \end{cases}$$

From:

$$\frac{a + \pi}{a} u_1 + u_2 < A - b + \frac{b}{a} u_1 + \frac{b}{\pi} u_2 + \varepsilon$$

it follows that:

$$A + \varepsilon > \frac{a + \pi - b}{a} u_1 + \left(1 - \frac{b}{\pi}\right) u_2 + b$$

Taking $u_1 = a/2$, $u_2 = 0$ gives the supremum of the right hand side:

$$A + \varepsilon > \frac{a + b + \pi}{2}$$

Case II:

$$\psi : \begin{cases} u_1' = \dfrac{a+\pi}{a} u_1 - \dfrac{a+\pi}{2}, & u_1 \in ]a/2, a[ \\ \\ u_2' = u_2 + \dfrac{\pi}{2}, & u_2 \in ]0, \pi - (\pi/a)u_1[ \end{cases}$$

Now:

$$\frac{a+\pi}{a} u_1 - \frac{a+\pi}{2} + u_2 + \frac{\pi}{2} < A - b + \frac{b}{a} u_1 + \frac{b}{\pi} u_2 + \varepsilon$$

and proceeding as in case I (taking $u_1 = a$, $u_2 = 0$) leads to:

$$A+\varepsilon > \frac{a+\pi-b}{a} u_1 + \left(1 - \frac{b}{\pi}\right) u_2 - \frac{a+\pi}{2} + \frac{\pi}{2} + b, \qquad A+\varepsilon > \frac{a}{2} + \pi$$

Since $b \geq \pi$, the desired estimate is true in case II if:

$$A + \varepsilon > \frac{a+b+\pi}{2}$$

Case III:

$$\psi : \begin{cases} u_1' = \dfrac{a+\pi}{a} u_1 & u_1 \in ]0, a/2[ \\ \\ u_2' = u_2 + \dfrac{\pi}{2} - \dfrac{\pi}{a} u_1, & u_2 \in [\pi/2, \pi - (\pi/a)u_1[ \end{cases}$$

This time:

$$\frac{a+\pi}{a} u_1 + u_2 + \frac{\pi}{2} - \frac{\pi}{a} u_1 < A - b + \frac{b}{a} u_1 + \frac{b}{\pi} u_2 + \varepsilon$$

gives (taking $u_1 = a/2$, $u_2 = \pi/2$) the same estimate as in case II:

$$A+\varepsilon > \frac{a+\pi-\pi-b}{a} u_1 + \left(1 - \frac{b}{\pi}\right) u_2 + \frac{\pi}{2} + b, \qquad A+\varepsilon > \frac{a}{2} + \pi$$

Case IV:

$$\psi : \begin{cases} u_1' \in \left]0, \dfrac{a+\pi}{2}\right[, & u_1 = a/2 \\ \\ u_2' + \dfrac{\pi}{a+\pi} u_1' = u_2 + \dfrac{\pi}{2}, & u_2 \in ]0, \pi/2[ \end{cases}$$

Then:

$$u_1' + u_2' = \frac{a}{a+\pi} u_1' + u_2 + \frac{\pi}{2}$$

and $A$ should satisfy:

$$\frac{a}{a+\pi} u_1' + u_2 + \frac{\pi}{2} < A - b + \frac{b}{2} + \frac{b}{\pi} u_2 + \varepsilon$$

As above (with $u_1' = (a+\pi)/2$, $u_2 = 0$):

$$A + \varepsilon > \frac{a+b+\pi}{2}$$

In this analysis, we ignored everything 'small' in the symplectic folding process: the symplectomorphism $\psi$ is as close as wanted, but not equal, to the above map; the details of a rigourous proof (that needs a slight adaptation to the folding process presented here) can be founded in [11, 12]. Still, the conclusion is that $A$ can be chosen to be $(a+b+\pi)/2$.

□

**Lemma 3.** *If for any positive $\varepsilon$ there exists a symplectic embedding $\psi$:*

$$\psi : T(\pi, a) \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4, \quad \psi((u_1, v_1), (u_2, v_2)) = ((u_1', v_1'), (u_2', v_2'))$$

*such that:*

$$u_1' + u_2' < A - b + \frac{b}{\pi} u_1 + \frac{b}{a} u_2 + \varepsilon$$

*then there exists a a symplectic embedding $\Phi$:*

$$E(\pi, b_1, \ldots, b_{n-2} = b, a) \hookrightarrow B(A+\varepsilon)$$

*Proof.* It follows from lemma 1 and the estimate on $\psi$ that there exists a symplectic embedding $\sigma$:

$$\sigma : E(\pi, a) \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4, \qquad \sigma(z_1, z_2) = (z_1', z_2')$$

such that:

$$\pi|z_1'|^2 + \pi|z_2'|^2 < A - b + \frac{b}{\pi}\pi|z_1|^2 + \frac{b}{a}\pi|z_2|^2 + \varepsilon, \quad A = \frac{a+b+\pi}{2}$$

Then $\sigma \times \mathrm{id}_{n-2}$, after a suitable permutation $\tau$, defined by $\tau(z_1, z_2, \ldots) = (z_1, z_n, z_2, \ldots)$, gives a symplectic embedding:

$$\Phi = (\sigma \times \mathrm{id}_{n-2}) \circ \tau : E(\pi, b_1, \ldots, b_{n-2} = b, a) \hookrightarrow \mathbb{C}^n \cong \mathbb{R}^{2n}$$

The image of $\Phi$ is contained in the ball $B(A+\varepsilon)$:

$$\begin{aligned}
\pi|\Phi(z_1, \ldots, z_n)|^2 &= \pi|z_1'|^2 + \pi|z_2'|^2 + \ldots + \pi|z_n'|^2 \\
&< A - b + \frac{b}{\pi}\pi|z_1|^2 + \frac{b}{a}\pi|z_n|^2 + \pi|z_2|^2 + \ldots + \pi|z_{n-1}|^2 + \varepsilon \\
&< A - b + b\left[\frac{\pi}{\pi}|z_1|^2 + \frac{\pi}{a}|z_n|^2 + \frac{\pi}{b}|z_2|^2 + \ldots + \frac{\pi}{b}|z_{n-1}|^2\right] + \varepsilon \\
&< A - b + b\left[\frac{\pi}{\pi}|z_1|^2 + \frac{\pi}{b_1}|z_2|^2 + \ldots + \frac{\pi}{b_{n-2}}|z_{n-1}|^2 + \frac{\pi}{a}|z_n|^2\right] + \varepsilon \\
&< A - b + b + \varepsilon = A + \varepsilon
\end{aligned}$$

and therefore:

$$\Phi\left(E(\pi, b_1, \ldots, b_{n-2} = b, a)\right) \subset B\left(A + \varepsilon\right)$$

□

*Proof of theorem.* The conclusion follows from lemmas 2 and 3. □

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